

Conjugating full cycles by adjacent transpositions: diameter and sorting time

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Abstract

We establish upper and lower bounds on the maximal number of steps needed to transform a cyclic permutation to the canonical cyclic permutation using conjugation by adjacent transpositions, and on the diameter of the underlying Schreier graph.

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1 Introduction

Let C_n be the set of n -cycles in the symmetric group S_n , and let $\Phi_n := \{s_i = (i, i+1) : 1 \leq i < n\}$ be the set of adjacent transpositions (Coxeter simple reflections) in S_n .

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Definition 1. Consider the undirected graph Γ_n , with vertex set C_n , in which two cycles $c_1, c_2 \in C_n$ are adjacent if $c_2 = s_i c_1 s_i$ for some adjacent transposition $s_i \in \Phi_n$.

Problem 2. Find the diameter of the graph Γ_n ,

$$D_n := \text{Diameter}(\Gamma_n) = \max_{\beta, \gamma \in C_n} \text{dist}_{\Gamma_n}(\beta, \gamma).$$

Here $\text{dist}_{\Gamma_n}(\beta, \gamma)$ is the distance (length of shortest path) in Γ_n between β and γ .

A closely related problem is the following. A *cyclic permutation* $[\pi]$ of size n is a bijective labeling of n distinct points on a circle by the elements of $[n] := \{1, 2, \dots, n\}$; two labelings are equivalent if one can be obtained from the other by a cyclic shift. The *canonical cyclic permutation* $c_n := (1, 2, \dots, n)$ is the one in which the elements are clockwise labeled by $1, \dots, n$. A *permissible step* on $[\pi]$ is a switching of two consecutive values, i and $i + 1$, for some $1 \leq i < n$.

Definition 3. The *Coxeter cyclic sorting time*, Sort_n , is the maximal number of permissible steps needed to convert a cyclic permutation $[\pi]$ of size n into the canonical cyclic permutation $c_n = (1, 2, \dots, n)$.

By Remark 7 below,

$$\text{Sort}_n := \max_{\gamma \in C_n} \text{dist}_{\Gamma_n}(c_n, \gamma).$$

Problem 4. Find the Coxeter cyclic sorting time.

For other sorting processes on cyclic permutations see [1] and references therein.

By definition, the Coxeter cyclic sorting time is not bigger than the diameter. Since the graph Γ_n is not vertex transitive for $n \geq 4$ (see, e.g., Figure 1), these two numbers are not necessarily equal. For example, for $n = 5$, $\text{Diameter}(\Gamma_5) = 5 > 4 = \text{Sort}_5$.

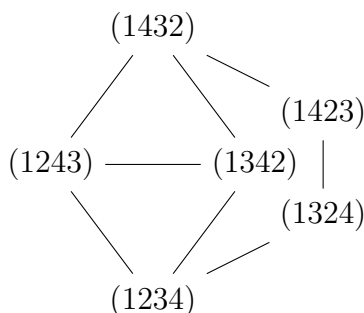


Figure 1: The graph Γ_4 .

In this paper we prove the following.

Theorem 5. For every positive integer n ,

$$\frac{8-\pi}{16} \cdot n^2 - \frac{3n}{2} \leq \text{Sort}_n \leq \frac{1}{3} \cdot n^2 - \frac{3n-1}{6}$$

and

$$\frac{8-\pi}{16} \cdot n^2 - \frac{3n}{2} \leq \text{Diameter}(\Gamma_n) \leq \frac{3}{8} \cdot n^2 - \frac{4n-1}{8}.$$

See Theorems 12, 23 and 25 below. Note that $\frac{8-\pi}{16} \sim 0.30365$.

Remark 6. Theorem 5 implies that the diameter and sorting time discussed in this paper, where permissible steps switch adjacent values, are different from those studied in [1], where steps switch adjacent positions.

2 Notation and preliminaries

Let S_n be the symmetric group on the letters $[n] := \{1, \dots, n\}$. The one-line notation of a permutation $\pi \in S_n$ is denoted by square brackets $[\pi(1), \dots, \pi(n)]$, while the cycle notation by parentheses. For example $[3, 5, 6, 1, 2, 4] = (1, 3, 6, 4)(2, 5) \in S_6$.

Letting H be a subgroup of a finite group G , and $S \subseteq G$, the *Schreier graph* $X(G/H, S)$ is the undirected graph whose vertex set consists of all left cosets $G/H := \{\pi H : \pi \in G\}$; the edges are the unordered pairs $(\pi H, s\pi H)$, for $s \in S, \pi \in G$.

3 The minv statistic

Let $c_n := (1, 2, \dots, n) = s_1 s_2 \cdots s_{n-1} \in S_n$. Denote by $\mathbb{Z}_n$ the cyclic subgroup of S_n generated by c_n . Recall the graph Γ_n from Definition 1.

Remark 7. Consider the map from C_n to the set of left cosets $S_n/\mathbb{Z}_n$, defined as follows: a cycle $\gamma = (a_1, \dots, a_n) \in C_n$ is mapped to the coset $\bar{\gamma}\mathbb{Z}_n$, where $\bar{\gamma} = [a_1, \dots, a_n]$. This map is a well-defined bijection and induces an isomorphism from the graph Γ_n to the Schreier graph $X(S_n/\mathbb{Z}_n, \Phi_n)$, where edge multiplicity is ignored.

Recall the *inversion number* of a permutation $\pi \in S_n$,

$$\text{inv}(\pi) := |\{i < j : \pi(i) > \pi(j)\}|.$$

Definition 8. Define the *minv* statistic on the left cosets of $\mathbb{Z}_n$ in S_n as follows:

$$\text{minv}(\pi\mathbb{Z}_n) := \min\{\text{inv}(\sigma) : \sigma \in \pi\mathbb{Z}_n\} \quad (\forall \pi\mathbb{Z}_n \in S_n/\mathbb{Z}_n).$$

Lemma 9. For every pair of full cycles $\gamma_1, \gamma_2 \in C_n$,

$$\text{dist}_{\Gamma_n}(\gamma_1, \gamma_2) = \min\{\text{inv}(\sigma) : \sigma \in \bar{\gamma}_1\mathbb{Z}_n\bar{\gamma}_2^{-1}\},$$

where $\bar{\gamma}_1, \bar{\gamma}_2$ are defined as in Remark 7.

Proof. By Remark 7, the distance $\text{dist}_{\Gamma_n}(\gamma_1, \gamma_2)$ is equal to the smallest nonnegative integer ℓ for which there exist simple reflections $s_{i_1}, \dots, s_{i_\ell}$ satisfying $s_{i_\ell} \cdots s_{i_1} \bar{\gamma}_1 \mathbb{Z}_n = \bar{\gamma}_2 \mathbb{Z}_n$. Equivalently,

$$\ell = \min\{\text{inv}(\tau_2 \tau_1^{-1}) : \tau_1 \in \bar{\gamma}_1 \mathbb{Z}_n, \tau_2 \in \bar{\gamma}_2 \mathbb{Z}_n\} = \min\{\text{inv}(\sigma) : \sigma \in \bar{\gamma}_2 \mathbb{Z}_n \bar{\gamma}_1^{-1}\}. \quad \square$$

Corollary 10. *For every full cycle $\gamma \in C_n$*

$$\text{dist}_{\Gamma_n}(c_n, \gamma) = \minv(\bar{\gamma} \mathbb{Z}_n),$$

where $c_n = (1, 2, \dots, n) \in C_n$ and $\bar{\gamma} \mathbb{Z}_n \in S_n/\mathbb{Z}_n$ is defined as in Remark 7. In particular, for every $n \geq 1$

$$\text{Sort}_n = \max_{\pi \in S_n} \minv(\pi \mathbb{Z}_n).$$

We conclude that Problem 4 is equivalent to the following one.

Problem 11. What is the maximal value of $\minv$ on $S_n/\mathbb{Z}_n$?

4 Upper bound on the Coxeter cyclic sorting time

Recall the Coxeter cyclic sorting time, Sort_n , from Definition 3. The main result in this section is the following.

Theorem 12. *For every $n \geq 1$,*

$$\text{Sort}_n \leq \frac{2n^2 - 3n + 1}{6}.$$

Theorem 12 is an immediate consequence of the following result.

Proposition 13. *For every $n \geq 1$, the maximum of the expected value of inv on left cosets $\pi \mathbb{Z}_n$ satisfies*

$$\max_{\pi \in S_n} E\{\text{inv}(\sigma) : \sigma \in \pi \mathbb{Z}_n\} = \frac{(2n-1)(n-1)}{6}.$$

See Remark 22 below.

Proof of Theorem 12. By Corollary 10, Definition 8 and Proposition 13,

$$\begin{aligned} \text{Sort}_n &= \max_{\pi \in S_n} \minv(\pi \mathbb{Z}_n) = \max_{\pi \in S_n} \min\{\text{inv}(\pi c_n^i) : 0 \leq i < n\} \\ &\leq \max_{\pi \in S_n} E\{\text{inv}(\pi c_n^i) : 0 \leq i < n\} = \frac{(2n-1)(n-1)}{6}. \end{aligned} \quad \square$$

In order to prove Proposition 13 we consider a weighted version of the inversion number, as well as a cyclic variant, see Definitions 14 and 17 below.

Definition 14. The *weighted inversion number* of a permutation $\pi \in S_n$ is

$$\text{winv}(\pi) := \sum_{\substack{i,j \\ i < j, \pi(i) > \pi(j)}} (\pi(i) - \pi(j)).$$

Lemma 15. For every $\pi \in S_n$,

$$\text{winv}(\pi) = \sum_{i=1}^n i^2 - \sum_{i=1}^n i \cdot \pi(i).$$

Proof. By induction on the inversion number $\text{inv}(\pi)$. If $\text{inv}(\pi) = 0$ then $\pi = id$, and the claim clearly holds. Now assume that the claim holds for all $\pi' \in S_n$ with $\text{inv}(\pi') \leq k$, and let $\pi \in S_n$ have $\text{inv}(\pi) = k + 1$. Since $\pi \neq id$, there exists $1 \leq j < n$ such that $\pi(j) > \pi(j + 1)$, so that $\pi' := \pi s_j$ satisfies $\text{inv}(\pi') = k$. By the induction hypothesis,

$$\text{winv}(\pi') = \sum_{i=1}^n i^2 - \sum_{i=1}^n i \cdot \pi'(i).$$

By Definition 14, $\text{winv}(\pi) - \text{winv}(\pi') = \pi(j) - \pi(j + 1)$. Also,

$$\left(\sum_{i=1}^n i^2 - \sum_{i=1}^n i \cdot \pi(i) \right) - \left(\sum_{i=1}^n i^2 - \sum_{i=1}^n i \cdot \pi'(i) \right) = \sum_{i=1}^n i \cdot (\pi'(i) - \pi(i)) = \pi(j) - \pi(j + 1),$$

and thus the claim holds for π as well. This completes the induction step and the proof. $\square$

Denote $w_0 := [n, n - 1, \dots, 1] \in S_n$. Recall that w_0 has the largest inversion number in S_n .

Observation 16. For any $\pi \in S_n$,

$$\text{winv}(\pi) + \text{winv}(\pi w_0) = \sum_{1 \leq k < \ell \leq n} (\ell - k) = \sum_{\ell=2}^n \binom{\ell}{2} = \binom{n+1}{3}$$

and

$$0 = \text{winv}(id) \leq \text{winv}(\pi) \leq \text{winv}(w_0) = \binom{n+1}{3}.$$

The following cyclic variant of winv plays a key role in the proof.

Definition 17. The *cyclic weighted inversion number* of a permutation $\pi \in S_n$ is

$$\text{cwinv}(\pi) := n \cdot \text{inv}(\pi) - 2 \cdot \text{winv}(\pi) = \sum_{\substack{i,j \\ i < j, \pi(i) > \pi(j)}} (n - 2\pi(i) + 2\pi(j)).$$

The following observation justifies the adjective *cyclic* in Definition 17.

Observation 18. *The cyclic weighted inversion number is invariant under cyclic shifts; namely,*

$$\text{cwinv}(\pi c_n) = \text{cwinv}(\pi) \quad (\forall \pi \in S_n).$$

Proof. Denote $m := \pi(1)$. Then $m = \pi c_n(n)$, so that

$$\text{inv}(\pi c_n) - \text{inv}(\pi) = (n - m) - (m - 1) = n + 1 - 2m$$

and

$$\text{winv}(\pi c_n) - \text{winv}(\pi) = \sum_{m < j \leq n} (j - m) - \sum_{1 \leq j < m} (m - j) = \binom{n - m + 1}{2} - \binom{m}{2}.$$

Thus

$$\text{cwinv}(\pi c_n) - \text{cwinv}(\pi) = n(n + 1 - 2m) - (n - m + 1)(n - m) + m(m - 1) = 0. \quad \square$$

Observation 19. *For every $\pi \in S_n$*

$$\text{cwinv}(\pi) + \text{cwinv}(\pi w_0) = n \binom{n}{2} - 2 \binom{n+1}{3} = \binom{n}{3}.$$

Corollary 20. *For every $\pi \in S_n$*

$$0 = \text{cwinv}(id) \leq \text{cwinv}(\pi) \leq \text{cwinv}(w_0) = \binom{n}{3}.$$

Proof. By Observation 18, we may assume that $\pi(n) = n$. Let $\bar{\pi} = [\pi(1), \dots, \pi(n-1)] \in S_{n-1}$. By definition,

$$\text{cwinv}(\pi) = n \cdot \text{inv}(\pi) - 2 \cdot \text{winv}(\pi) = n \cdot \text{inv}(\bar{\pi}) - 2 \cdot \text{winv}(\bar{\pi}) = \text{cwinv}(\bar{\pi}) + \text{inv}(\bar{\pi}).$$

Hence,

$$\max_{\pi \in S_n} \text{cwinv}(\pi) \leq \max_{\bar{\pi} \in S_{n-1}} \text{cwinv}(\bar{\pi}) + \max_{\pi \in S_{n-1}} \text{inv}(\pi).$$

By iteration,

$$\begin{aligned} \max_{\pi \in S_n} \text{cwinv}(\pi) &\leq \max_{\bar{\pi} \in S_{n-1}} \text{cwinv}(\bar{\pi}) + \max_{\pi \in S_{n-1}} \text{inv}(\pi) \\ &\leq \sum_{j=1}^{n-1} \max_{\pi \in S_j} \text{inv}(\pi) = \sum_{j=1}^{n-1} \binom{j}{2} = \binom{n}{3} = \text{cwinv}(w_0). \end{aligned}$$

Combining this upper bound with Observation 19,

$$\min_{\pi \in S_n} \text{cwinv}(\pi) = \binom{n}{3} - \max_{\pi \in S_n} \text{cwinv}(\pi) \geq 0 = \text{cwinv}(id). \quad \square$$

Lemma 21. For every $\pi \in S_n$

$$n \cdot E\{\text{inv}(\sigma) : \sigma \in \pi\mathbb{Z}_n\} = \sum_{i=1}^n \text{inv}(\pi c_n^i) = \text{cwinv}(\pi) + \binom{n+1}{3}.$$

Proof. As in the proof of Observation 18,

$$\text{inv}(\pi c_n) = \text{inv}(\pi) + n + 1 - 2\pi(1).$$

Thus, by iteration

$$\text{inv}(\pi c_n^i) = \text{inv}(\pi) + \sum_{j=1}^i (n + 1 - 2\pi(j)).$$

Summing over $1 \leq i \leq n$ we obtain

$$\begin{aligned} \sum_{i=1}^n \text{inv}(\pi c_n^i) &= n \cdot \text{inv}(\pi) + \sum_{i=1}^n \sum_{j=1}^i (n + 1 - 2\pi(j)) \\ &= n \cdot \text{inv}(\pi) + \sum_{j=1}^n (n + 1 - j)(n + 1 - 2\pi(j)) \\ &= n \cdot \text{inv}(\pi) + \sum_{j=1}^n j \cdot 2\pi(j) - (n + 1) \sum_{j=1}^n (2\pi(j) + j) + n(n + 1)^2 \\ &= n \cdot \text{inv}(\pi) + 2 \sum_{j=1}^n j \cdot \pi(j) - \frac{1}{2}n(n + 1)^2 \end{aligned}$$

By Lemma 15 and Definition 17, this is equal to

$$n \cdot \text{inv}(\pi) + 2 \sum_{j=1}^n j^2 - 2 \cdot \text{winv}(\pi) - \frac{1}{2}n(n + 1)^2 = \text{cwinv}(\pi) + \binom{n+1}{3}. \quad \square$$

Proof of Proposition 13. By Lemma 21 and Corollary 20,

$$\begin{aligned} \max_{\pi \in S_n} E\{\text{inv}(\sigma) : \sigma \in \pi\mathbb{Z}_n\} &= \frac{1}{n} \left(\max_{\pi \in S_n} \text{cwinv}(\pi) + \binom{n+1}{3} \right) \\ &= \frac{1}{n} \left(\binom{n}{3} + \binom{n+1}{3} \right) = \frac{(2n-1)(n-1)}{6}. \quad \square \end{aligned}$$

Remark 22. By Corollary 20 and Lemma 21, the maximum of the expected value of the inversion number on a $\mathbb{Z}_n$ -coset is attained at $w_0\mathbb{Z}_n$. However, surprisingly, the maximal value of the statistic minv (which, by Corollary 10, is equal to Sort_n) is not attained at this coset, since $\text{minv}(w_0\mathbb{Z}_n) = \binom{\lceil n/2 \rceil}{2} + \binom{\lfloor n/2 \rfloor}{2} = \left\lfloor \frac{(n-1)^2}{4} \right\rfloor$ is strictly less than the lower bound proved in the next section (Theorem 23).

5 Lower bound on the Coxeter cyclic sorting time

In this section we prove the following lower bound on Sort_n , and thus on $\text{Diameter}(\Gamma_n)$ as well.

Theorem 23. *For every $n \geq 1$*

$$\text{Sort}_n \geq \left(\frac{1}{2} - \frac{\pi}{16}\right) n^2 - \frac{3}{2}n.$$

The proof of this theorem uses the following characterization.

Lemma 24. *For every $\pi \in S_n$, $\text{inv}(\pi) = \text{minv}(\pi\mathbb{Z}_n)$ if and only if π is “heavy tailed,” namely if, for all $1 \leq k \leq n$, the sum of the last k entries of π is at least $\frac{k(n+1)}{2}$; equivalently if, for all $1 \leq k \leq n$, the sum of the first k entries of π is at most $\frac{k(n+1)}{2}$:*

$$\sum_{j=1}^k \pi(j) \leq \frac{k(n+1)}{2} \quad (\forall k).$$

Proof. For every $1 \leq k \leq n$, $\text{inv}(\pi) \leq \text{inv}(\pi c_n^k)$ if and only if

$$\begin{aligned} 0 \leq \text{inv}(\pi c_n^k) - \text{inv}(\pi) &= \sum_{j=1}^k (\text{inv}(\pi c_n^j) - \text{inv}(\pi c_n^{j-1})) \\ &= \sum_{j=1}^k ((n - \pi c_n^{j-1}(1)) - (\pi c_n^{j-1}(1) - 1)) \\ &= \sum_{j=1}^k (n + 1 - 2\pi(j)) = k(n + 1) - 2 \sum_{j=1}^k \pi(j). \quad \square \end{aligned}$$

Proof of Theorem 23. By Corollary 10, if $\pi \in S_n$ satisfied the conditions of Lemma 24 then

$$\text{Sort}_n \geq \text{minv}(\pi\mathbb{Z}_n) = \text{inv}(\pi).$$

To get a good lower bound, we shall construct such a permutation $\pi \in S_n$ with a large $\text{inv}(\pi)$.

Recall the permutation $w_0 = [n, \dots, 1]$ and consider, as a first approximation, its cyclic shift $w_0 c_n^m = [n - m, \dots, 1, n, \dots, n - m + 1]$, where $m := \lfloor n/2 \rfloor$. Noting that $w_0 c_n^m$ satisfies the conditions of Lemma 24, one concludes that

$$\text{minv}(w_0\mathbb{Z}_n) = \text{inv}(w_0 c_n^m) = \binom{n-m}{2} + \binom{m}{2} = \binom{\lceil n/2 \rceil}{2} + \binom{\lfloor n/2 \rfloor}{2}.$$

Now, take $w_0 c_n^m$ and push the letter n to the left as much as possible, say to position $k_1 + 1$, so that the conditions of Lemma 24 are still satisfied. Then push the letter $n - 1$ to

the left as much as possible, say to position $k_2 + 2$, and so on. We obtain the permutation

$$\begin{aligned} \pi_0 := [n - m, n - m - 1, \dots, n - m - (k_1 - 1), n, n - m - k_1, \dots, n - m - (k_2 - 1), \\ n - 1, n - m - k_2, \dots, n - m - (k_t - 1), n + 1 - t, n - m - k_t, \dots] \in S_n, \end{aligned} \quad (5.1)$$

where $0 \leq k_1 \leq \dots \leq k_m \leq n - m$ have the minimal possible values such that π_0 still satisfies the conditions of Lemma 24. In the sequel we shall obtain explicit expressions for these minimal values, for each n ; for example, for $n = 12$, $\pi_0 = [6, 5, 4, 3, 12, 2, 11, 1, 10, 9, 8, 7]$.

Let us compute $\text{inv}(\pi_0)$, which will serve as a lower bound for Sort_n . Considering separately the small values $1 \leq i \leq m$ and the large values $m + 1 \leq n + 1 - t \leq n$,

$$\text{minv}(\pi_0 \mathbb{Z}_n) = \text{inv}(\pi_0) = \sum_{i=1}^m (i - 1) + \sum_{t=1}^m (n - t - k_t) = \binom{n}{2} - \sum_{t=1}^m k_t.$$

First, assume that $n = 2m$ is even. By the conditions of Lemma 24, for any $1 \leq t \leq m$,

$$\frac{(k_t + t)(n + 1)}{2} \geq \sum_{j=1}^{k_t+t} \pi_0(j) = \sum_{i=0}^{k_t-1} (m - i) + \sum_{j=0}^{t-1} (n - j) = \frac{k_t(n + 1 - k_t)}{2} + \frac{t(2n + 1 - t)}{2}.$$

Simplifying, this yields

$$k_t^2 \geq t(n - t) \quad (\forall 1 \leq t \leq m).$$

If these inequalities are satisfied, then the inequalities of Lemma 24 for indices other than $k_1 + 1, \dots, k_m + m$ obviously follow. Minimizing the parameters k_t , for all $1 \leq t \leq m$, we obtain approximately

$$\sum_{t=1}^m k_t \sim \sum_{t=1}^{n/2} (t(n - t))^{1/2} \sim \int_0^{n/2} (x(n - x))^{1/2} dx.$$

Letting $x = \frac{n}{2}(1 - \cos \theta)$, we obtain

$$\int_0^{n/2} (x(n - x))^{1/2} dx = \frac{n^2}{4} \int_0^{\pi/2} (1 - \cos^2 \theta)^{1/2} \sin \theta d\theta = \frac{\pi}{16} n^2.$$

This yields

$$\text{Sort}_n = \max_{\pi \in S_n} \text{minv}(\pi \mathbb{Z}_n) \geq \text{minv}(\pi_0 \mathbb{Z}_n) = \binom{n}{2} - \sum_{t=1}^m k_t \sim \binom{n}{2} - \frac{\pi}{16} n^2.$$

For a more precise lower bound, let us bound the error in this approximation. Since

$$k_t = \lceil (t(n - t))^{1/2} \rceil \quad (1 \leq t \leq m),$$

we conclude that

$$0 \leq \sum_{t=1}^{n/2} k_t - \sum_{t=1}^{n/2} (t(n-t))^{1/2} \leq n/2. \quad (5.2)$$

On the other hand, $f(x) := (x(n-x))^{1/2}$ is a monotone increasing function in the range $0 \leq x \leq n/2$, and therefore

$$f(t-1) \cdot 1 \leq \int_{t-1}^t f(x) dx \leq f(t) \cdot 1 \quad (1 \leq t \leq n/2).$$

Hence

$$0 \leq \sum_{t=1}^{n/2} f(t) - \int_0^{n/2} f(x) dx \leq \sum_{t=1}^{n/2} f(t) - \sum_{t=1}^{n/2} f(t-1) = f\left(\frac{n}{2}\right) - f(0) = \frac{n}{2}$$

or, explicitly,

$$0 \leq \sum_{t=1}^{n/2} (t(n-t))^{1/2} - \frac{\pi}{16} n^2 \leq \frac{n}{2}. \quad (5.3)$$

Adding inequalities (5.2) and (5.3) yields

$$0 \leq \sum_{t=1}^{n/2} k_t - \frac{\pi}{16} n^2 \leq n.$$

Thus, for even n :

$$\text{Sort}_n \geq \text{inv}(\pi_0) = \binom{n}{2} - \sum_{t=1}^{n/2} k_t \geq \left(\frac{1}{2} - \frac{\pi}{16}\right) n^2 - \frac{3}{2} n.$$

Now assume that $n = 2m + 1$ is odd. The computations are similar to those for even n , but a little more complicated. By the conditions of Lemma 24, for every $1 \leq t \leq m$

$$\begin{aligned} \frac{(k_t + t)(n + 1)}{2} &\geq \sum_{j=1}^{k_t+t} \pi_0(j) = \sum_{i=0}^{k_t-1} (m + 1 - i) + \sum_{j=0}^{t-1} (n - j) \\ &= \frac{k_t(n + 2 - k_t)}{2} + \frac{t(2n + 1 - t)}{2}. \end{aligned}$$

Simplifying, this yields

$$k_t^2 - k_t \geq t(n - t) \quad (\forall 1 \leq t \leq m)$$

or, equivalently,

$$k_t \geq (t(n - t) + 1/4)^{1/2} + 1/2 \quad (\forall 1 \leq t \leq m).$$

Again,

$$\min v(\pi_0 \mathbb{Z}_n) = \inf(\pi_0) = \sum_{i=1}^{m+1} (i-1) + \sum_{t=1}^m (n-t-k_t) = \binom{n}{2} - \sum_{t=1}^m k_t.$$

Minimizing the parameters k_t , for all $1 \leq t \leq m$, we obtain

$$\sum_{t=1}^m k_t \sim \sum_{t=1}^{(n-1)/2} ((t(n-t) + 1/4)^{1/2} + 1/2) \sim \int_0^{(n-1)/2} ((x(n-x) + 1/4)^{1/2} + 1/2) dx.$$

Denoting $g(x) := (x(n-x) + 1/4)^{1/2} + 1/2$ and letting $x = \frac{n}{2} - \sqrt{\frac{n^2+1}{4}} \cos \theta$, we get

$$\begin{aligned} \int_0^{(n-1)/2} g(x) dx &= \sqrt{\frac{n^2+1}{4}} \int_{\arccos \frac{n}{\sqrt{n^2+1}}}^{\arccos \frac{1}{\sqrt{n^2+1}}} \left(\frac{n^2+1}{4} - \frac{n^2+1}{4} \cos^2 \theta \right)^{1/2} \sin \theta d\theta + \frac{n-1}{4} \\ &= \frac{n^2+1}{4} \left(\frac{\pi - 4 \arctan(1/n)}{4} \right) + \frac{n-1}{4}. \end{aligned}$$

Since $\arctan(x) > x - x^3/3$ for $x > 0$, we conclude that

$$\int_0^{(n-1)/2} g(x) dx < \frac{\pi}{16} n^2 + \frac{\pi-4}{16} + \frac{1-2n^2}{12n^3} < \frac{\pi}{16} n^2.$$

Now

$$k_t = \lceil g(t) \rceil \quad (1 \leq t \leq m)$$

implies that

$$0 \leq \sum_{t=1}^{(n-1)/2} k_t - \sum_{t=1}^{(n-1)/2} g(t) \leq \frac{n-1}{2}. \quad (5.4)$$

On the other hand, $g(x)$ is a monotone increasing function for $0 \leq x \leq n/2$, and therefore

$$g(t-1) \cdot 1 \leq \int_{t-1}^t g(x) dx \leq g(t) \cdot 1 \quad (1 \leq t \leq (n-1)/2).$$

Hence

$$\begin{aligned} 0 \leq \sum_{t=1}^{(n-1)/2} g(t) - \int_0^{(n-1)/2} g(x) dx &\leq \sum_{t=1}^{(n-1)/2} g(t) - \sum_{t=1}^{(n-1)/2} g(t-1) \\ &= g\left(\frac{n-1}{2}\right) - g(0) = \frac{n+1}{2} - 1 = \frac{n-1}{2}. \end{aligned} \quad (5.5)$$

Adding inequalities (5.4) and (5.5) yields

$$0 \leq \sum_{t=1}^{(n-1)/2} k_t - \int_0^{(n-1)/2} g(x) dx \leq n-1.$$

Altogether, we obtain for odd n :

$$\text{Sort}_n \geq \text{inv}(\pi_0) = \binom{n}{2} - \sum_{t=1}^m k_t > \binom{n}{2} - \frac{\pi}{16}n^2 - (n-1) > \left(\frac{1}{2} - \frac{\pi}{16}\right)n^2 - \frac{3}{2}n.$$

This completes the proof of the lower bound for all values of n . $\square$

6 Upper bound on the diameter

In this section we prove the following.

Theorem 25. *For every $n \geq 1$,*

$$\text{Diameter}(\Gamma_n) \leq \frac{3n^2 - 4n + 1}{8}.$$

The proof of Theorem 25 requires several lemmas, which may be of independent interest.

The following expression for the diameter of Γ_n follows immediately from Lemma 9.

Corollary 26. *For every $n \geq 1$,*

$$\text{Diameter}(\Gamma_n) = \max_{\pi_1, \pi_2 \in S_n} \min\{\text{inv}(\tau) : \tau \in \pi_1 \mathbb{Z}_n \pi_2^{-1}\},$$

where $\mathbb{Z}_n$ is the subgroup of S_n generated by the n -cycle $c_n = (1, 2, \dots, n)$.

Lemma 27. *For every permutation $\tau \in S_n$ and $0 \leq k \leq n$,*

$$\text{inv}(\tau) \leq \binom{n-k}{2} - k + \sum_{i=1}^k \tau(i).$$

Proof. Fix $0 \leq k \leq n$, and evaluate

$$|\{(i, j) : i \leq k < j, \tau(i) > \tau(j)\}|.$$

We can assume, without loss of generality, that $\tau(1) < \tau(2) < \dots < \tau(k)$. Then, for every $1 \leq i \leq k$,

$$|\{j : k < j, \tau(i) > \tau(j)\}| = \tau(i) - i.$$

Thus

$$|\{(i, j) : i \leq k < j, \tau(i) > \tau(j)\}| = \sum_{i=1}^k (\tau(i) - i) = \sum_{i=1}^k \tau(i) - \binom{k+1}{2}.$$

Note that, obviously,

$$|\{(i, j) : i < j \leq k, \tau(i) > \tau(j)\}| + |\{(i, j) : k < i < j, \tau(i) > \tau(j)\}| \leq \binom{k}{2} + \binom{n-k}{2}.$$

Altogether, for every fixed $0 \leq k \leq n$,

$$\begin{aligned} \text{inv}(\tau) = |\{(i, j) : i < j, \tau(i) > \tau(j)\}| &\leq \sum_{i=1}^k \tau(i) - \binom{k+1}{2} + \binom{k}{2} + \binom{n-k}{2} \\ &= \sum_{i=1}^k \tau(i) - k + \binom{n-k}{2}. \end{aligned} \quad \square$$

The key lemma in the proof of Theorem 25 is the following.

Lemma 28. *For any $\pi_1, \pi_2 \in S_n$ and any $0 \leq k \leq n$, there exists $\tau \in \pi_1 \mathbb{Z}_n \pi_2^{-1}$, such that*

$$\sum_{i=1}^k \tau(i) \leq \frac{k(n+1)}{2}.$$

Proof. First notice that, for any $\pi_1, \pi_2 \in S_n$ and any $1 \leq i \leq n$,

$$\{\tau(i) : \tau \in \pi_1 \mathbb{Z}_n \pi_2^{-1}\} = \{\pi_1 c_n^j \pi_2^{-1}(i) : 1 \leq j \leq n\} = \{\pi_1(t) : 1 \leq t \leq n\} = \{1, \dots, n\}.$$

It follows that, for any $1 \leq i \leq n$, the expected value, in position i , of a random permutation in $\pi_1 \mathbb{Z}_n \pi_2^{-1}$ is

$$E_{\tau \in \pi_1 \mathbb{Z}_n \pi_2^{-1}} [\tau(i)] = \frac{1}{n} \sum_{j=1}^n \pi_1 c_n^j \pi_2^{-1}(i) = \frac{1}{n} (1 + \dots + n) = \frac{n+1}{2}.$$

Hence, the expected sum of values of $\tau \in \pi_1 \mathbb{Z}_n \pi_2^{-1}$ on a prefix of length k is

$$E_{\tau \in \pi_1 \mathbb{Z}_n \pi_2^{-1}} \left[\sum_{i=1}^k \tau(i) \right] = \sum_{i=1}^k E_{\tau \in \pi_1 \mathbb{Z}_n \pi_2^{-1}} [\tau(i)] = k \cdot \frac{n+1}{2}.$$

There must exist $\tau \in \pi_1 \mathbb{Z}_n \pi_2^{-1}$ for which $\sum_{i=1}^k \tau(i)$ is smaller or equal to its expected value, and this completes the proof. $\square$

Proof of Theorem 25. By Lemmas 27 and 28, for any $\pi_1, \pi_2 \in S_n$ and any $0 \leq k \leq n$,

$$\begin{aligned} \min_{\tau \in \pi_1 \mathbb{Z}_n \pi_2^{-1}} \text{inv}(\tau) &\leq \binom{n-k}{2} - k + \min_{\tau \in \pi_1 \mathbb{Z}_n \pi_2^{-1}} \sum_{i=1}^k \tau(i) \\ &\leq \binom{n-k}{2} - k + \frac{k(n+1)}{2} = \binom{n}{2} - \frac{k(n-k)}{2}. \end{aligned}$$

The RHS is minimized for $k = n/2$ (if n is even) or $k = (n \pm 1)/2$ (if n is odd). It follows that

$$\min_{\tau \in \pi_1 \mathbb{Z}_n \pi_2^{-1}} \text{inv}(\tau) \leq \binom{n}{2} - \frac{n^2 - \chi(n \text{ odd})}{8} = \frac{3n^2 - 4n + \chi(n \text{ odd})}{8},$$

where $\chi(n \text{ odd})$ is 1 if n is odd, and 0 otherwise. Corollary 26 completes the proof. $\square$

7 Final remarks and open problems

A strengthening of Theorem 5 is desired.

Conjecture 29. The following asymptotic equality holds:

$$\text{Sort}_n = \frac{8 - \pi}{16} \cdot n^2 + O(n).$$

Recall the permutation π_0 from (5.1). Computer experiments suggest that, furthermore, for every $n \geq 1$

$$\text{Sort}_n = \text{inv}(\pi_0). \quad (7.1)$$

This was verified for every $n \leq 12$.

Recall that the maximal value of the minv statistics on $S_n/\mathbb{Z}_n$ is equal to the Coxeter sorting time Sort_n (Corollary 10). Hence, Theorem 5 provides bounds on this maximal value. The actual distribution of minv deserves further study. We conjecture the following.

Conjecture 30. The generating function of minv on $S_n/\mathbb{Z}_n$ is unimodal.

This conjecture was verified for every $n \leq 12$.

The winv statistic, from Definition 14, plays a crucial role in the proof of the upper bound on Sort_n . Lemma 15 implies the following geometric interpretation of winv:

$$\frac{\text{winv}(\pi)}{\sum_{i=1}^n i^2} = 1 - \cos \alpha_\pi,$$

where α_π is the angle between the vectors $v_\pi := (\pi(1), \dots, \pi(n))$ and $v_{id} := (1, 2, \dots, n)$ in $\mathbb{R}^n$. The distribution of winv is mysterious. Note that, by Observation 16, the winv generating function is palindromic.

The set of adjacent transpositions $\Phi_n := \{(i, i+1) : 1 \leq i < n\}$ may be augmented by the transposition $(n, 1)$, to yield a cyclically symmetric set:

$$\Phi_n^{cyc} := \{(i, i+1 \bmod n) : 1 \leq i \leq n\} = \Phi_n \sqcup \{(n, 1)\}.$$

It is natural to sort cyclic permutations of size n by conjugation with elements of this cyclically symmetric set. Let Γ_n^{cyc} be the undirected graph whose vertex set is the class C_n of n -cycles in S_n , and two cycles $c_1, c_2 \in C_n$ are adjacent if there exists a transposition $s \in \Phi_n^{cyc}$ such that $c_2 = sc_1s$. Note that Γ_n^{cyc} is different from the underlying graph of [1]. Actually, even for $n = 4$ these graphs are non-isomorphic. However, their sorting times are always equal, as shown below.

Proposition 31. *The sorting time for the graph Γ_n^{cyc} is*

$$\text{Sort}(\Gamma_n^{cyc}) := \max_{c \in C_n} \text{dist}_{\Gamma_n^{cyc}}(c, c_n) = \left\lfloor \frac{(n-1)^2}{4} \right\rfloor.$$

The diameter of this graph satisfies

$$\left\lfloor \frac{(n-1)^2}{4} \right\rfloor \leq \text{Diameter}(\Gamma_n^{cyc}) \leq \left\lfloor \frac{n^2}{4} \right\rfloor.$$

Proof. Recall, from Remark 7, the bijection between C_n and $S_n/\mathbb{Z}_n$ defined by mapping a cycle $c = (a_1, \dots, a_n) \in C_n$ to the coset $\bar{c}\mathbb{Z}_n \in S_n/\mathbb{Z}_n$, where $\bar{c} = [a_1, \dots, a_n] \in S_n$. For any $s \in \Phi_n^{cyc}$, transforming c to scs amounts to interchanging two cyclically-adjacent values in the one-line notation of $\bar{c}$ (where n and 1 are considered cyclically-adjacent values). This, in turn, is equivalent to swapping two cyclically-adjacent *positions* in the one-line notation of $\bar{c}^{-1}$. It follows that the sorting time for Γ_n^{cyc} is the same as the sorting time for the graph considered in [1], which is $\lfloor (n-1)^2/4 \rfloor$. The lower bound on the diameter of Γ_n^{cyc} follows.

In the other direction: by [8], the diameter of $X(S_n, \Phi_n^{cyc})$, the Cayley graph of the symmetric group S_n with respect to the generating set Φ_n^{cyc} , is $\lfloor n^2/4 \rfloor$. Since Γ_n^{cyc} is a quotient of $X(S_n, \Phi_n^{cyc})$, the upper bound follows. $\square$

A natural generalization of the graph Γ_n from Definition 1 is the following. Let μ be an integer partition of n , and let C_μ be the conjugacy class consisting all permutations in S_n of cycle type μ .

Definition 32. The graph Γ_μ is the undirected graph with vertex set C_μ , in which two permutations $\pi, \sigma \in C_\mu$ are adjacent if $\sigma = s_i \pi s_i$ for some adjacent transposition $s_i \in \Phi_n$.

Problem 33. Find the diameter of the graph Γ_μ .

Using properties of the involutive weak order [2, Definition 1.3] and the Bruhat order on fixed-point-free involutions [7, 5, 6, 4], Avni and Bagno [3] proved that for $\mu = (2^n)$, the diameter of $\Gamma_{(2^n)}$ is equal to $(2n-1)!!$.

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References

- [1] R. M. Adin, N. Alon and Y. Roichman, *Circular sorting*, Israel J. Math., to appear, [arXiv:2502.14398](#).
- [2] R. M. Adin, A. Postnikov and Y. Roichman, *Combinatorial Gelfand models*, J. Algebra **320** (2008), 1311–1325.
- [3] A. Avni and E. Bagno, *On the weak order on fixed-point free involutions*, in preparation. See also: A. Avni, *Matchings and permutations*, M.Sc. Thesis, Bar-Ilan University, 2013.
- [4] M. B. Can, Y. Cherniavsky and T. Twelbeck, *Lexicographic shellability of the Bruhat-Chevalley order on fixed-point-free involutions*, Israel J. Math. **207** (2015), 281–299.
- [5] R. S. Deodhar and M. K. Srinivasan, *A statistic on involutions*, J. Algebraic Combinatorics **13** (2001), 187–198.

- [6] A. Hultman, *Fixed points of involutive automorphisms of the Bruhat order*, Advances in Math. **195** (2005), 283–296.
- [7] R. W. Richardson and T. A. Springer, *The Bruhat order on symmetric varieties*, Geometriae Dedicata **35** (1990), 389–436.
- [8] A. van Zuylen, J. Bieron, F. Schalekamp and G. Yu, *A tight upper bound on the number of cyclically adjacent transpositions to sort a permutation*, Information Processing Letters **116** (2016), 718–722.