

Perfect codes in Cayley graphs of Hajós groups

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Abstract

A perfect code in a graph Γ is an independent set C such that every vertex of Γ outside C has exactly one neighbour in C . A perfect code in a directed graph can be defined similarly by requiring no arc between vertices of C and for every vertex v outside C there exists exactly one vertex u in C such that the arc from u to v exists in Γ . A subset X of an abelian group G is said to be periodic if there exists a non-identity element g of G such that $g + X = X$. A factorization of G is a pair of nonempty subsets (A, B) of G such that every element g of G can be expressed uniquely as $g = a + b$ with $a \in A$ and $b \in B$. If for every factorization (A, B) of an abelian group G at least one of A and B is periodic, then G is said to be a Hajós group. In this paper we classify all Cayley graphs (directed or undirected) of Hajós groups which admit perfect codes, and moreover we determine all perfect codes in such Cayley graphs.

Mathematics Subject Classifications: 05C25, 05C69

1 Introduction

All groups considered in this paper are finite, and all graphs considered are finite and simple. A q -ary code of length n is a set of words of length n over an alphabet of size q , where $q \geq 2$ and $n \geq 1$ are integers. Equivalently, a q -ary code of length n is a subset C of $\mathbb{Z}_q^n$ if the alphabet is set to be $\mathbb{Z}_q = \{0, 1, \dots, q-1\}$. The covering radius $\rho(C)$ of C is the least integer r such that every word of length n is at distance no more than r to at least one codeword of C , and the minimum distance $d(C)$ of C is the minimum distance between any two distinct codewords of C . In coding theory, the distance used is usually the Hamming distance or Lee distance, which is the graph distance in the Hamming graph $H(n, q)$ or in the graph $L(n, q)$, respectively, where $H(n, q) = K_q \square \dots \square K_q$ is the n -fold Cartesian product of the complete graph K_q of order q , and $L(n, q) = C_q \square \dots \square C_q$ is the n -fold Cartesian product of the cycle C_q of length $q \geq 3$. A code C is said to be

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t -error-correcting if $d(C) \geq 2t + 1$, and is called a perfect t -code if $d(C) = 2t + 1$ and $\rho(C) = t$, where $t \geq 1$ is an integer. Equivalently, C is a *perfect t -code* if every word in $\mathbb{Z}_q^n$ is at distance no more than t to exactly one codeword of C . Ever since the beginning of information theory, perfect codes play an important role in coding theory due to their maximum capacity of error correction without ambiguity. The Hamming codes, the Golay code G_{23} , and the Golay code G_{11} are well known examples of perfect codes, and they are the only nontrivial linear perfect codes according to a famous result [24] which settled a long-standing open problem in coding theory, where a *linear code* is a subspace of a linear space over a finite field. The reader is referred to [25] for an early survey on perfect codes and to [11] for a more recent survey on binary perfect 1-codes.

The notion of perfect t -codes can be generalized to graphs [1, 13] simply by replacing the Hamming or Lee distance by the distance in a graph. More specifically, a subset C of vertices of a graph Γ is called a *perfect t -code* if every vertex of Γ is at distance no more than t to exactly one vertex in C . Equivalently, C is a perfect t -code in Γ if the closed t -neighbourhoods $N_t[v], v \in C$, form a partition of the vertex set of Γ , where $N_t[v]$ is the set of vertices of Γ at distance no more than t to v . This definition applies to directed graphs Γ when $N_t[v]$ is interpreted as the set of vertices of Γ at (directed) distance no more than t from v . In graph theory, a perfect 1-code in Γ is also called an efficient dominating set of Γ [5, 6, 7, 16, 18]. It is readily seen that perfect t -codes in $H(n, q)$ or $L(n, q)$ are precisely perfect t -codes under the Hamming distance or Lee distance in the classical setting.

Perfect codes in Cayley graphs are especially interesting and significant objects of study due to their connections to coding theory and factorizations of groups. See [9, Section 1] and [12, Section 1] for short reviews of related results and background information. Given a group G with identity element e and a subset S of $G \setminus \{e\}$, define $\text{Cay}(G, S)$ to be the directed graph with vertex set G such that for $x, y \in G$ there is an arc from x to y if and only if $yx^{-1} \in S$. It is known that $\text{Cay}(G, S)$ is connected if and only if S generates G . In the case when S is inverse-closed (that is, $g^{-1} \in S$ whenever $g \in S$), $\text{Cay}(G, S)$ is regarded as an undirected graph as the arc from x to y exists if and only if the arc from y to x exists. For brevity, we call $\text{Cay}(G, S)$ the *Cayley graph* of G with *connection set* S even when it is a directed graph (that is, S is not inverse-closed). It can be verified that both $H(n, q)$ and $L(n, q)$ are Cayley graphs of $\mathbb{Z}_q^n$. As such perfect codes in Cayley graphs are natural generalizations of perfect codes in the classical setting. They can also be viewed as factorizations of the underlying group. A *factorization* of G (into two factors) [22] is a pair of subsets (A, B) of G such that every element of G can be uniquely expressed as ab with $a \in A$ and $b \in B$ (so that $G = AB$). If in addition $e \in A \cap B$, then this factorization is a *tiling* of G [8]. As noted in [12], if (A, B) is a tiling of G with $A^{-1} = A$, then B is a perfect 1-code in $\text{Cay}(G, A \setminus \{e\})$, and conversely any normalized perfect 1-code C in a Cayley graph $\text{Cay}(G, S)$ gives rise to the tiling (S_0, C) of G , where $S_0 = S \cup \{e\}$. A similar statement holds for perfect t -codes in Cayley graphs if one considers S_0^t instead of S_0 .

Perfect codes in Cayley graphs have received considerable attention in recent years (see, for example, [5, 12, 30]). In particular, the question about when a given subgroup

of a group can be a perfect 1-code in some Cayley graph of the group has been studied extensively. See [3, 14, 23, 26, 28, 29] for recent results in this line of research. In view of potential applications, perfect codes in Cayley graphs of abelian groups are the most important. This is because for an abelian group $G = \mathbb{Z}_{q_1} \times \cdots \times \mathbb{Z}_{q_n}$ the codewords of a perfect code in a Cayley graph of G are strings in the usual sense with i^{th} bit from alphabet $\mathbb{Z}_{q_i}$. Such a perfect code is a mixed perfect code [11] if not all $q_1, \dots, q_n$ are equal. Unfortunately, even for Cayley graphs of cyclic groups, called *circulant graphs*, we do not have a complete characterization of perfect 1-codes in them, though a number of partial results exist in the literature (see [6, 7, 9, 10, 15, 16, 18, 27, 30]).

In this paper we study perfect 1-codes in Cayley graphs of Hajós groups, where a group G is *Hajós* [20] if G is abelian and for every factorization (A, B) of G , either A or B is periodic in the sense that it is fixed under the translation by some non-identity element of G . We focus on infinite families of Hajós groups and thus perfect 1-codes in Cayley graphs of the following Hajós groups are not considered:

$$\mathbb{Z}_3^3, \quad \mathbb{Z}_4 \times \mathbb{Z}_2^2, \quad \mathbb{Z}_4 \times \mathbb{Z}_4, \quad \mathbb{Z}_8 \times \mathbb{Z}_2^2, \quad \mathbb{Z}_4 \times \mathbb{Z}_2^3, \quad \text{and} \quad \mathbb{Z}_2^5. \quad (1)$$

The classification of all Hajós groups was completed by Sands in 1962 [20] (see Lemma 2.4). With the help of this result we classify all Cayley graphs of Hajós groups which admit perfect 1-codes, and moreover we determine all perfect 1-codes in such Cayley graphs. As far as we know, this is the first large class of abelian groups for which we have complete knowledge of perfect 1-codes in their Cayley graphs. Since we will only consider perfect 1-codes, for short we will call them *perfect codes* from now on. Since each connected component of a Cayley graph $\text{Cay}(G, S)$ is isomorphic to a Cayley graph of the subgroup of G generated by S , and subgroups of Hajós groups are also Hajós, it suffices to consider connected Cayley graphs of Hajós abelian groups. Also, we do not need to consider complete graphs as each of their perfect codes consists of a single vertex.

Our main result can be summarized as follows.

Theorem 1.1. *Let G be a Hajós group not listed in (1) and $\text{Cay}(G, S)$ a connected non-complete Cayley graph of G . Then a subset C of G is a normalized perfect code in $\text{Cay}(G, S)$ if and only if the triple (G, S_0, C) lies in Table 1.1, where $S_0 = S \cup \{0\}$, k is a positive integer and p, q, r , and s are different primes.*

We would like to emphasize that this result holds for both directed and undirected Cayley graphs $\text{Cay}(G, S)$ of Hajós groups. More details on (S_0, C) will be given in later sections as indicated in Table 1.1. Theorem 1.1 implies that $\text{Cay}(G, S)$ admits a perfect code if and only if there exists a subset C of G such that (G, S_0, C) is as shown in Table 1.1, and moreover we obtain all normalized perfect codes C in $\text{Cay}(G, S)$ in this manner.

Table 1.1: Perfect codes in Cayley graphs of Hajós groups

Row	G	S_0 and C	Page	Condition
1	$\mathbb{Z}_p \times \mathbb{Z}_p$	Theorem 3.1	10	$p \neq 2$
2	$\mathbb{Z}_p \times \mathbb{Z}_q$	Table 3.1	12	
3	$\mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_q$	Table 3.2	12	$q \leq 3$

4	$\mathbb{Z}_{p^k}$	Theorem 4.1	14	
5	$\mathbb{Z}_{p^k} \times \mathbb{Z}_q$	Theorem 4.2	14	
6	$\mathbb{Z}_{2^k} \times \mathbb{Z}_2$	Theorem 4.3	15	
7	$\mathbb{Z}_{p^2}$	Theorem 5.1	40	
8	$\mathbb{Z}_{p^3}$	Table 5.1	40	
9	$\mathbb{Z}_{p^2} \times \mathbb{Z}_2$	Table 5.2	41	$p \neq 2$
10	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	Lemma 5.5	43	
11	$\mathbb{Z}_{p^3} \times \mathbb{Z}_2$	Table 5.5	43	$p \neq 2$
12	$\mathbb{Z}_{p^2} \times \mathbb{Z}_2 \times \mathbb{Z}_2$	Table 5.8	47	$p \neq 2$
13	$\mathbb{Z}_p \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	Table 5.11	51	$p \neq 2$
14	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	Table 5.14	53	
15	$\mathbb{Z}_{p^3} \times \mathbb{Z}_2 \times \mathbb{Z}_2$	Table 5.15	54	$p \neq 2$
16	$\mathbb{Z}_{p^2} \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	Table 5.18	63	$p \neq 2$
17	$\mathbb{Z}_p \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	Table 5.21	71	$p \neq 2$
18	$\mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r$	Table 6.1	76	
19	$\mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_2 \times \mathbb{Z}_2$	Table 6.2	76	$p, q \neq 2$
20	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p$	Table 6.5	81	$p \leq 3$
21	$\mathbb{Z}_p \times \mathbb{Z}_4$	Table 6.6	82	$p \neq 2$
22	$\mathbb{Z}_p \times \mathbb{Z}_4 \times \mathbb{Z}_2$	Table 6.7	84	$p \neq 2$
23	$\mathbb{Z}_{p^2} \times \mathbb{Z}_{q^2}$	[6, 7] & Table 6.10	90	
24	$\mathbb{Z}_{p^2} \times \mathbb{Z}_q \times \mathbb{Z}_r$	[6, 7] & Table 6.13	94	
25	$\mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r \times \mathbb{Z}_s$	[6, 7]	—	

There are 14 families of Hajós groups, of which four families consist of cyclic groups (items 9-12 in Lemma 2.4). So Cayley graphs of these four families of Hajós groups are circulant graphs. Circulant graphs having perfect codes with size a prime or relatively prime to the size of the connection set plus one in these circulant graphs have been classified in [6, 7]. In these two papers, they restrict the Cayley graphs to be undirected graphs, that is requiring $S = -S$. Although they have this restriction, the authors of [6, 7] do not use the property $S = -S$ to get their results. This means the characterization they got also works for perfect codes in directed circulant graphs on Hajós groups. We will determine perfect codes in these circulant graphs with other sizes. Of course, we will also determine perfect codes in Cayley graphs of the other families of Hajós abelian groups.

We divide the proof of Theorem 1.1 into a number of theorems. In Section 2, we provide some background information and lemmas that will be used to prove our theorems. In Section 3, we characterize Cayley graphs of $\mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_q$ and its subgroups that admit a perfect code. Section 4 focuses on Hajós groups that contain $\mathbb{Z}_{p^k}$ as a subgroup for a prime p . In Section 5, we consider Hajós groups of the form $\mathbb{Z}_{p^k} \times \mathbb{Z}_2^l$. The last section, Section 6, covers the remaining Hajós groups, that is, $\mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_p \times \mathbb{Z}_4 \times \mathbb{Z}_2$, $\mathbb{Z}_9 \times \mathbb{Z}_3$, $\mathbb{Z}_{p^2} \times \mathbb{Z}_{q^2}$, $\mathbb{Z}_{p^2} \times \mathbb{Z}_q \times \mathbb{Z}_r$ and their subgroups.

Throughout this paper we reserve the word *theorem* for a characterization of the perfect codes in the Cayley graphs of an infinite family of Hajós groups, and use *lemma* both for auxiliary results and for characterizations of specific groups, namely $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_2^3$, $\mathbb{Z}_2^4$, and $\mathbb{Z}_{p^2} \times \mathbb{Z}_p$ for $p \leq 3$. Each of these lemmas is used in the proof of at least one theorem, and their characterizations are no less complete than those stated as theorems.

2 Preliminaries

In this section we fix our notation and collect the results that will be used throughout the paper. The conventions governing the statements and tables of Sections 3–6 are gathered in Section 2.3 and apply throughout those sections.

2.1 Notation, factorizations and Hajós groups

Additive notation will be used in the rest of the paper: the group operation will be $+$, the identity 0 , the inverse of g will be $-g$, and mg will be used instead of g^m . For a positive integer n , let $\mathbb{Z}_n$ be the cyclic group of integers modulo n and set

$$[n] = \{0, 1, \dots, n-1\}.$$

For any abelian group written as a direct sum of cyclic groups, we use $\mathbf{e}_i$ to denote its element with 1 in the i^{th} coordinate and 0 in all other coordinates.

Let G be a group, and let A and B be nonempty subsets of G . The *sum* of A and B is defined as

$$A + B = \{a + b \mid a \in A, b \in B\}.$$

In particular, we write $g + A, A + g$ in place of $\{g\} + A, A + \{g\}$, respectively. If every element of $A + B$ can be written uniquely as $a + b$ with $a \in A$ and $b \in B$, we say that the sum is a *direct sum* and denote it by $A \oplus B$. If $G = A \oplus B$, we call this expression a *factorization* of G into *factors* A and B . Since we use additive notation, in a Cayley graph $\text{Cay}(G, S)$ there is an arc from x to y if and only if $y - x \in S$. Set

$$S_0 = S \cup \{0\}.$$

The following lemma says that $\text{Cay}(G, S)$ admits a perfect code if and only if G admits a factorization with S_0 as one of the factors.

Lemma 2.1 ([6, 12, Lemma 2.3]). *Let G be a group, S a subset of $G \setminus \{0\}$, and C a subset of G . Then C is a perfect code in $\text{Cay}(G, S)$ if and only if $G = S_0 \oplus C$.*

In particular, $\text{Cay}(G, S)$ admits a perfect code if and only if there exists a subset C of G such that $G = S_0 \oplus C$. A perfect code in $\text{Cay}(G, S)$ is said to be *normalized* if it contains 0 . By Lemma 2.1, one can see that any translation of a perfect code in a Cayley graph is also a perfect code. Hence every perfect code in a Cayley graph is a translation of some normalized perfect code. Therefore, to classify all perfect codes in a Cayley graph it suffices to consider normalized perfect codes.

Lemma 2.2 ([22]). *Let A and B be nonempty subsets of an abelian group G . The following statements are equivalent:*

- (a) $G = A \oplus B$;
- (b) $G = A + B$ and $|G| = |A||B|$;

- (c) $|G| = |A||B|$ and $(A - A) \cap (B - B) = \{0\}$;
 (d) $A + B$ is a direct sum and $|G| = |A||B|$.

Let A be a nonempty subset of an abelian group G . Define

$$L_A = \{g \in G \mid g + A = A\}.$$

Then $L_A \neq \emptyset$ as $0 \in L_A$. Note that L_A is the setwise stabilizer of A in G in the action of G on itself by addition and hence is a subgroup of G . For any subset X of G , set

$$(X + L_A)/L_A = \{x + L_A \mid x \in X\}.$$

Lemma 2.3 ([22]). *Let G be an abelian group and A a nonempty subset of G . Then L_A is a subgroup of G , A is the union of some cosets of L_A (that is, $A = L_A \oplus D$ for some $D \subseteq G$), and $L_A = \bigcap_{a \in A} (A - a)$.*

We call L_A the *subgroup of periods* of A [22]. If L_A contains at least one non-zero element (called a *period* of A in G), then A is called a *periodic* subset of G ; otherwise, A is called an *aperiodic* subset of G . In other words, A is periodic or aperiodic depending on whether L_A is a nontrivial or trivial subgroup of G . If in every factorization of an abelian group G at least one factor is periodic, then G is called a *Hajós group*.

Lemma 2.4 ([20]). *All Hajós groups are precisely the following groups and their subgroups (where p, q, r , and s are different primes, and k is a positive integer):*

- | | | |
|--|--|---|
| 1. $\mathbb{Z}_p \times \mathbb{Z}_p$ [17] | 6. $\mathbb{Z}_{p^2} \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ [20] | 11. $\mathbb{Z}_{p^2} \times \mathbb{Z}_q \times \mathbb{Z}_r$ [19] |
| 2. $\mathbb{Z}_p \times \mathbb{Z}_3 \times \mathbb{Z}_3$ [20] | 7. $\mathbb{Z}_p \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ [20] | 12. $\mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r \times \mathbb{Z}_s$ [19] |
| 3. $\mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_2 \times \mathbb{Z}_2$ [20] | 8. $\mathbb{Z}_{2^k} \times \mathbb{Z}_2$ [21] | |
| 4. $\mathbb{Z}_p \times \mathbb{Z}_4 \times \mathbb{Z}_2$ [20] | 9. $\mathbb{Z}_{p^k} \times \mathbb{Z}_q$ [4] | 13. $\mathbb{Z}_9 \times \mathbb{Z}_3$ [21] |
| 5. $\mathbb{Z}_{p^3} \times \mathbb{Z}_2 \times \mathbb{Z}_2$ [20] | 10. $\mathbb{Z}_{p^2} \times \mathbb{Z}_{q^2}$ [19] | 14. $\mathbb{Z}_4 \times \mathbb{Z}_4$ [21] |

2.2 Lemmas on quotients and perfect codes

We now present four lemmas that will be used in the proof of our main result. The following lemma is a combination of Lemma 2.6 and Corollary 2.2 in [22] with some new results.

Lemma 2.5. *Let G be an abelian group, and let $G = A \oplus B$ be a factorization of G such that A is periodic. Then for any subset D of G with $A = L_A \oplus D$,*

$$G/L_A = ((D + L_A)/L_A) \oplus ((B + L_A)/L_A)$$

is a factorization of G/L_A . Moreover, $(D + L_A)/L_A$ is aperiodic in G/L_A , $|(D + L_A)/L_A| = |D|$, and $|(B + L_A)/L_A| = |B|$.

Proof. Since $G = A \oplus B = L_A \oplus D \oplus B$, for any $g + L_A \in G/L_A$, we have $g + L_A = l + d + b + L_A = (d + l + L_A) + (b + L_A) = (d + L_A) + (b + L_A) \in ((D + L_A)/L_A) + ((B + L_A)/L_A)$ for some $l \in L_A$, $d \in D$ and $b \in B$. If $g + L_A = (d_1 + L_A) + (b_1 + L_A) = (d_2 + L_A) + (b_2 + L_A) \in ((D + L_A)/L_A) + ((B + L_A)/L_A)$ for some $d_1, d_2 \in D$ and $b_1, b_2 \in B$, then $d_1 + b_1 + l_1 = d_2 + b_2 + l_2$ for some $l_1, l_2 \in L_A$. Since $G = L_A \oplus D \oplus B$, we must have $b_1 = b_2$ and $d_1 = d_2$. Since this holds for any $g + L_A \in G/L_A$, it follows that $G/L_A = ((D + L_A)/L_A) \oplus ((B + L_A)/L_A)$.

We now prove $(D + L_A)/L_A$ is aperiodic in G/L_A . Suppose $(g + L_A) + (D + L_A)/L_A = (D + L_A)/L_A$. We will prove $g \in L_A$, which means $(D + L_A)/L_A$ has no period in G/L_A and so is aperiodic. Set $D = \{d_1, d_2, \dots, d_k\}$. Then $(D + L_A)/L_A = \{d_1 + L_A, \dots, d_k + L_A\}$ and $(g + L_A) + (D + L_A)/L_A = \{g + d_1 + L_A, \dots, g + d_k + L_A\}$. Now,

$$g + A = g + D + L_A = \bigcup_{i=1}^k (g + d_i + L_A) = \bigcup_{i=1}^k (d_i + L_A) = D + L_A = A.$$

So $g \in L_A$ as desired.

Since $A = L_A \oplus D$, we have $(L_A \cap (D - D)) \subseteq ((L_A - L_A) \cap (D - D)) = \{0\}$. For $d_1, d_2 \in D$, we have

$$d_1 + L_A = d_2 + L_A \Leftrightarrow d_1 - d_2 \in L_A \Leftrightarrow d_1 - d_2 \in (L_A \cap (D - D)) = \{0\} \Leftrightarrow d_1 = d_2.$$

So $|(D + L_A)/L_A| = |D|$. Now,

$$|(B + L_A)/L_A| = \frac{|G/L_A|}{|(D + L_A)/L_A|} = \frac{|G|}{|L_A| \cdot |D|} = \frac{|G|}{|A|} = |B|.$$

This completes the proof. $\square$

Lemma 2.6. Let $G = A \oplus B$ and $A = L_A \oplus D$ be as in Lemma 2.5. Let $|(D + L_A)/L_A| = n$ and $|(B + L_A)/L_A| = m$. If

$$L_A = \{l_i \mid i \in [t]\}, (D + L_A)/L_A = \{d_j + L_A \mid j \in [n]\}$$

and

$$(B + L_A)/L_A = \{b_k + L_A \mid k \in [m]\},$$

then

$$A = \{l_i + d_j \mid i \in [t], j \in [n]\} \quad \text{and} \quad B = \{b_k + \ell_k \mid k \in [m]\}$$

for some $\ell_k \in L_A$.

Proof. First we prove the lemma for B . From Lemma 2.5 we have $|B| = |(B + L_A)/L_A| = m$, so let $B = \{x_k \mid k \in [m]\}$. By definition $(B + L_A)/L_A = \{x_k + L_A \mid k \in [m]\}$. This means there is a permutation $\sigma \in \text{sym}(m)$ such that $b_k + L_A = x_{\sigma(k)} + L_A$ for all k in $[m]$. Hence $x_{\sigma(k)} = b_k + \ell_k$ for some ℓ_k in L_A . Therefore $B = \{x_k \mid k \in [m]\} = \{x_{\sigma(k)} \mid k \in [m]\} = \{b_k + \ell_k \mid k \in [m]\}$ for some ℓ_k in L_A .

Now we prove the lemma for A . By following the same arguments for B , we will get $D = \{d_j + \ell_j \mid j \in [n]\}$ for some ℓ_j in L_A . As $L_A + \ell_j = L_A$, we have

$$\begin{aligned} A &= L_A \oplus D \\ &= \bigcup_{j \in [n]} (L_A + d_j + \ell_j) \\ &= \bigcup_{j \in [n]} (L_A + d_j) \\ &= L_A \oplus \{d_j \mid j \in [n]\} \\ &= \{l_i + d_j \mid i \in [t], j \in [n]\}. \end{aligned} \quad \square$$

Lemma 2.7. *Let $G = A \oplus B$ and $A = L_A \oplus D$ be as in Lemma 2.5. The following two statements hold.*

- (a) A generates G if and only if $(D + L_A)/L_A$ generates G/L_A .
- (b) $0 \in A$ if and only if $L_A \in (D + L_A)/L_A$.

Proof. Suppose A generates G and let $g + L_A \in G/L_A$. Since A generates G , we have $g = \sum_{i=1}^n a_i$ for some integer n and all $a_i \in A$. For every i , write $a_i = l_i + d_i$ where $l_i \in L_A$ and $d_i \in D$. Then $g + L_A = \sum_{i=1}^n (l_i + d_i) + L_A = \sum_{i=1}^n d_i + L_A$. This means $(D + L_A)/L_A$ generates G/L_A .

Now suppose $(D + L_A)/L_A$ generates G/L_A . Let $g \in G$ and write $g + L_A = \sum_{i=1}^m d_i + L_A$ for some integer m and $d_i \in D$. Let $l = g - \sum_{i=1}^m d_i \in L_A$. Then $g = \sum_{i=1}^{m-1} (d_i + 0) + (d_m + l)$. Since both $d_i + 0$ and $d_m + l$ belong to $D \oplus L_A = A$ and g is an arbitrary element of G , it follows that A generates G . This proves (a).

Suppose $0 \in A$. Then $0 \in A = D \oplus L_A$ and hence $0 = d + l$ for some $d \in D$ and $l \in L_A$. So $d = -l \in L_A$ and hence $L_A = d + L_A \in (D + L_A)/L_A$. Now suppose $L_A \in (D + L_A)/L_A$. This implies $L_A = d + L_A$ for some d in D , and hence $d \in L_A$. As L_A is a subgroup of G by Lemma 2.3, $-d$ is in L_A . It follows that $0 = d + (-d) \in D \oplus L_A = A$. This proves (b). $\square$

Lemma 2.8. *Let G be a Hajós group. Let $\text{Cay}(G, S)$ be a connected but non-complete Cayley graph of G which admits a perfect code C .*

- (a) If G is not a cyclic group, then $|S_0| > 2$.
- (b) If S_0 is periodic, then $1 < |L_{S_0}| < |S_0|$ and $|L_{S_0}|$ divides $|S_0|$.
- (c) If $|S_0|$ is a prime, then S_0 is aperiodic and $C = g + L_C$ for some element g of G .

Part (c) in Lemma 2.8 is an extension of Theorem 3.1 part (b) in [2]. In [2], they consider undirected graph and restrict $|S_0|$ to be an odd prime. The result still holds for directed graphs and when $|S_0| = 2$. The authors of [2] also prove that for every Hajós group G , every coset of a subgroup of order $|G|/p$ where p is a proper odd prime divisor of $|G|$ is a perfect code for some Cayley graph of G .

Proof of Lemma 2.8. Since $\text{Cay}(G, S)$ is connected, S is a generating set of G . If $|S_0| = 2$, then $|S| = 1$ and hence G is a cyclic group. This proves part (a). Suppose S_0 is periodic. Then $|L_{S_0}| > 1$ and $S_0 = L_{S_0} \oplus D$ for some subset D . So $|L_{S_0}|$ divides $|S_0|$. If $|L_{S_0}| = |S_0|$, then S_0 is a coset of L_{S_0} . Since both S_0 and L_{S_0} contain 0, we must have $S_0 = L_{S_0}$. Hence S_0 is a subgroup of G that generates G , which means $S_0 = G$, but this contradicts the assumption that $\text{Cay}(G, S)$ is not a complete graph. Therefore, $|L_{S_0}| < |S_0|$ and (b) is proved.

Now assume $|S_0| = p$ is a prime. Suppose S_0 is periodic. Then by Lemma 2.3 we have $|L_{S_0}| > 1$ and $|L_{S_0}|$ divides $|S_0| = p$. So $|L_{S_0}| = |S_0|$. Since by Lemma 2.3, S_0 is a union of some cosets of L_{S_0} , it follows that $S_0 = L_{S_0}$ and hence S_0 is a subgroup of G . Since S_0 also generates G , we have $S_0 = G$ and so $\text{Cay}(G, S)$ is a complete graph, but this contradicts our assumption. Therefore, S_0 is aperiodic. Since G is a Hajós group and $G = C \oplus S_0$ (as C is a perfect code in $\text{Cay}(G, S)$), C must be periodic. Since $G = C \oplus S_0$ and C is periodic, by Lemma 2.5, for any subset D of G with $C = L_C \oplus D$ we have

$$G/L_C = ((D + L_C)/L_C) \oplus ((S_0 + L_C)/L_C).$$

Since G is abelian, G/L_C is isomorphic to a subgroup of G . Thus G/L_C is also a Hajós group as G is Hajós. Since, by Lemma 2.5, $(D + L_C)/L_C$ is aperiodic in G/L_C , $(S_0 + L_C)/L_C$ in the factorization above must be periodic in the Hajós abelian group G/L_C . Since $|(S_0 + L_C)/L_C| = |S_0|$, $(S_0 + L_C)/L_C$ is equal to its subgroup of periods in G/L_C . Since S_0 generates G , $(S_0 + L_C)/L_C$ generates G/L_C . Now that $(S_0 + L_C)/L_C$ is a subgroup of G/L_C which generates G/L_C , we must have $(S_0 + L_C)/L_C = G/L_C$. Hence

$$|L_C| = \frac{|G|}{|G/L_C|} = \frac{|G|}{|(S_0 + L_C)/L_C|} = \frac{|G|}{|S_0|} = |C|.$$

Therefore, $|D| = 1$ and so $C = g + L_C$ for some $g \in G$. □

2.3 Conventions

The classifications in Sections 3–6 are all presented in the same format. We collect here the conventions governing them; they are in force in every theorem and lemma.

(C1) *Free parameters.* The symbols α, β, γ and δ , carrying any subscripts, denote integers and are called *free parameters*. A free parameter occurring in a coordinate whose cyclic factor has order m is read modulo m ; if the same symbol occurs in two coordinates whose factors have orders m and m' , it ranges over a complete residue system modulo $\text{lcm}(m, m')$. A free parameter in S_0 is chosen independently of the free parameter denoted by the same symbol in C , unless stated otherwise. Finally, every free parameter all of whose subscripts are 0 is itself 0, that is,

$$\alpha_0 = \alpha_{00} = \alpha_{000} = \alpha_{0000} = \beta_0 = \beta_{00} = \beta_{000} = \gamma_0 = \gamma_{00} = \delta_0 = 0,$$

which guarantees $0 \in S_0$ and $0 \in C$.

- (C2) *Choice of generators.* Each table is written with respect to the displayed decomposition of G into cyclic factors, $\mathbf{e}_i$ denoting a generator of the i^{th} factor. All the results that allow change of generators are followed by a remark naming an elementary abelian direct factor H of G , all isomorphic to $\mathbb{Z}_p^l$ for a prime p and $l \geq 2$. Such a result is to be read as quantified over the choice of a basis of H : a subset C of G is a perfect code in $\text{Cay}(G, S)$ if and only if there is a basis of H for which (S_0, C) is one of the listed pairs, the generators of H occurring in the table being replaced by that basis. No generality is lost in this way: a change of basis of H extends to an automorphism φ of G fixing the remaining generators, and φ is an isomorphism from $\text{Cay}(G, S)$ to $\text{Cay}(G, \varphi(S))$ that maps perfect codes to perfect codes, since $G = S_0 \oplus C$ if and only if $G = \varphi(S_0) \oplus \varphi(C)$.
- (C3) *Sufficiency.* Each classification asserts an equivalence, and the two directions are established differently. In every section we prove, case by case, that (S_0, C) must be one of the listed pairs. Conversely, in Sections 3 and 4 we verify in full that each listed pair satisfies $G = S_0 \oplus C$, in the proof of Theorem 3.6 and in Lemma 4.4 respectively. In Sections 5 and 6 the corresponding calculation is of the same kind and is omitted; there we exhibit the perfect code C for each pair. In each case the reader may verify $G = S_0 \oplus C$ directly from the displayed forms of S_0 and C , as in the proof of Theorem 3.6.

3 Perfect codes in Cayley graphs of $\mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_q$ and their subgroups

In this section, we characterize perfect codes in Cayley graphs on Hajós groups of the form $\mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_q$ and their subgroups, where p and q are distinct primes. These Hajós groups are $\mathbb{Z}_p \times \mathbb{Z}_p$ with $p \geq 3$, $\mathbb{Z}_p \times \mathbb{Z}_q$ with $p \neq q$, and $\mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_q$ with $q \leq 3$ and $p \neq q$. See Lemma 5.2 for perfect codes in Cayley graphs of $\mathbb{Z}_2 \times \mathbb{Z}_2$. The main results in this section are the following four theorems.

Theorem 3.1. *Let $G = \mathbb{Z}_p \times \mathbb{Z}_p$ for an odd prime p , let S be a proper subset of $G \setminus \{0\}$ that generates G , and let $S_0 = S \cup \{(0, 0)\}$. Then a subset C of G is a perfect code in $\text{Cay}(G, S)$ if and only if $|S| = p - 1$ and there exist $a, b \in \mathbb{Z}_p$ such that $\{as_1 - bs_2 \mid (s_1, s_2) \in S_0\} = \mathbb{Z}_p$ and $C = \{(bn + c, an + d) \mid n \in \mathbb{Z}_p\}$ for some $c, d \in \mathbb{Z}_p$.*

Proof. Suppose that $|S| = p - 1$ and there exist $a, b \in \mathbb{Z}_p$ such that $\{as_1 - bs_2 \mid (s_1, s_2) \in S_0\} = \mathbb{Z}_p$ and $C = \{(bn + c, an + d) \mid n \in \mathbb{Z}_p\}$. Then $|\{as_1 - bs_2 \mid (s_1, s_2) \in S_0\}| = |S_0|$. To prove that C is a perfect code in $\text{Cay}(G, S)$, suppose for a contradiction that $(bn_1 + c, an_1 + d) + (s_1, s_2) = (bn_2 + c, an_2 + d) + (s_3, s_4)$ for distinct $(s_1, s_2), (s_3, s_4) \in S_0$. Then $b(n_1 - n_2) = s_3 - s_1$ and $a(n_1 - n_2) = s_4 - s_2$. It follows that $a(s_3 - s_1) = ab(n_1 - n_2) = b(s_4 - s_2)$, which implies $as_3 - bs_4 = as_1 - bs_2$, contradicting $|\{as_1 - bs_2 \mid (s_1, s_2) \in S_0\}| = |S_0|$. Hence $C + S_0$ is a direct sum. Since $|S_0| \cdot |C| = p^2 = |G|$, we conclude that $G = S_0 \oplus C$.

Conversely, suppose that $\text{Cay}(G, S)$ admits a perfect code C , then $G = S_0 \oplus C$. From Lemma 2.2, we have $|S| = p - 1$. Since $|S_0| = p$ is a prime, Lemma 2.8 implies

that $C = L_C + g$ for some $g \in G$. Note that L_C is a proper subgroup of G , so $C = \langle (b, a) \rangle + (c, d) = \{(bn + c, an + d) \mid n \in \mathbb{Z}_p\}$ for some non-zero $(b, a) \in G$ and $c, d \in \mathbb{Z}_p$. It remains to prove that $\{as_1 - bs_2 \mid (s_1, s_2) \in S_0\} = \mathbb{Z}_p$, which is equivalent to that $as_1 - bs_2$ is pairwise distinct for all $(s_1, s_2) \in S_0$. Suppose there are distinct $(s_1, s_2), (s_3, s_4) \in S_0$ such that $as_1 - bs_2 = as_3 - bs_4$, so that $a(s_1 - s_3) = b(s_2 - s_4)$. If $a \neq 0$, then

$$\begin{aligned}(s_1, s_2) + (c, d) &= (s_3, s_4) + (s_1 - s_3 + c, s_2 - s_4 + d) \\ &= (s_3, s_4) + (b(a^{-1}s_2 - a^{-1}s_4) + c, a(a^{-1}s_2 - a^{-1}s_4) + d)\end{aligned}$$

are two representations of $(s_1 + c, s_2 + d)$ as the sum of an element of S_0 and an element of C . If $a = 0$, then $b \neq 0$ and

$$\begin{aligned}(s_1, s_2) + (c, d) &= (s_3, s_4) + (s_1 - s_3 + c, s_2 - s_4 + d) \\ &= (s_3, s_4) + (b(b^{-1}s_1 - b^{-1}s_3) + c, a(b^{-1}s_1 - b^{-1}s_3) + d)\end{aligned}$$

are two representations of $(s_1 + c, s_2 + d)$ as the sum of an element of S_0 and an element of C . This contradicts $G = S_0 \oplus C$, so $\{as_1 - bs_2 \mid (s_1, s_2) \in S_0\} = \mathbb{Z}_p$, which completes the proof. $\square$

We give an equivalent form of Theorem 3.1 that requires less calculation to check whether $\text{Cay}(G, S)$ admits a perfect code.

Theorem 3.2. *Let $G = \mathbb{Z}_p \times \mathbb{Z}_p$ for an odd prime p and let S be a proper subset of $G \setminus \{0\}$ that generates G . Then $\text{Cay}(G, S)$ admits a perfect code if and only if $|S| = p - 1$ and S satisfies: (1) $\{s_1 \mid (s_1, s_2) \in S_0\} = \mathbb{Z}_p$, or (2) there is an element $a \in \mathbb{Z}_p$ such that $\{as_1 - s_2 \mid (s_1, s_2) \in S_0\} = \mathbb{Z}_p$. Moreover, the perfect codes are $\{(b, n) \mid n \in \mathbb{Z}_p\}$ for any $b \in \mathbb{Z}_p$ if S satisfies (1), or $\{(n, an + b) \mid n \in \mathbb{Z}_p\}$ for any $b \in \mathbb{Z}_p$ if S satisfies (2).*

Proof. This theorem follows from Theorem 3.1 and the fact that $\{as_1 - bs_2 \mid (s_1, s_2) \in S_0\} = \mathbb{Z}_p$ if and only if $\{\lambda as_1 - \lambda bs_2 \mid (s_1, s_2) \in S_0\} = \mathbb{Z}_p$ for any non-zero $\lambda \in \mathbb{Z}_p$. So, instead of checking if (a, b) satisfies $\{as_1 - bs_2 \mid (s_1, s_2) \in S_0\} = \mathbb{Z}_p$ for all non-zero $(a, b) \in \mathbb{Z}_p^2$, we only need to check (a, b) for $b = 1$ and for $(a, b) = (1, 0)$. $\square$

Remark 3.3. In Theorem 3.2, the two conditions of S for $\text{Cay}(\mathbb{Z}_p^2, S)$ to have a perfect code are not mutually exclusive. One example is $\Gamma = \text{Cay}(\mathbb{Z}_5^2, \{(1, 0), (2, 2), (3, 2), (4, 1)\})$, as Γ satisfies both (1) $\{s_1 \mid (s_1, s_2) \in S_0\} = \mathbb{Z}_5$, and (2) $\{3s_1 - s_2 \mid (s_1, s_2) \in S_0\} = \mathbb{Z}_5$. In this example, both (1) $\{(b, n) \mid n \in \mathbb{Z}_5\}$ and (2) $\{(n, 3n + b) \mid n \in \mathbb{Z}_5\}$ are perfect codes in Γ for any $b \in \mathbb{Z}_5$.

Theorem 3.1 seems simpler compared to Theorem 3.2 for characterizing $\text{Cay}(\mathbb{Z}_p^2, S)$ that admits a perfect code, as there is only one condition for S . But the condition in Theorem 3.1 has $O(p^2)$ candidates for (a, b) , and the condition in Theorem 3.2 only has $O(p)$ candidates. For this reason, we argue that Theorem 3.2 is a better characterization for $\text{Cay}(\mathbb{Z}_p^2, S)$ to have a perfect code.

Remark 3.4. Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be two generators of $G = \mathbb{Z}_p \times \mathbb{Z}_p$. If C is a normalized perfect code in Theorem 3.1, then without loss of generality $C = L_C = \langle (a, b) \rangle = \langle \mathbf{v}_1 \rangle$. By setting $D = \{(0, 0)\}$ in the factorization $G/L_C = ((S_0 + L_C)/L_C) \oplus ((D + L_C)/L_C)$ given by Lemma 2.5, we have $(S_0 + L_C)/L_C = G/L_C = \{i\mathbf{v}_2 + L_C \mid i \in [p]\}$. This implies $S_0 = \{\alpha_i \mathbf{v}_1 + i\mathbf{v}_2 \mid i \in [p]\}$, for some $\alpha_i \in \mathbb{Z}$.

Theorem 3.5. Let $G = \mathbb{Z}_p \times \mathbb{Z}_q$ for distinct primes p and q , and let S be a proper subset of $G \setminus \{0\}$ that generates G . Let $0 \in C \subseteq G$. Then C is a perfect code in $\text{Cay}(G, S)$ if and only if (S_0, C) is one of the pairs in Table 3.1.

Table 3.1: Perfect codes in $\text{Cay}(\mathbb{Z}_p \times \mathbb{Z}_q, S)$

Row	S_0	C
1	$\{(i, \alpha_i) \mid i \in [p]\}$	$\{(0, i) \mid i \in [q]\}$
2	$\{(\alpha_i, i) \mid i \in [q]\}$	$\{(i, 0) \mid i \in [p]\}$

Proof. Suppose $\text{Cay}(G, S)$ has a perfect code C . Since $\text{Cay}(G, S)$ is not a trivial graph, we have $|S_0| = p$ or q . If $|S_0| = p$, then C is a subgroup of order q from Lemma 2.8, so $C = L_C = \langle (0, 1) \rangle$. Recall the factorization $G/L_C = ((S_0 + L_C)/L_C) \oplus ((D + L_C)/L_C)$ given in Lemma 2.5, where $C = D \oplus L_C$. By setting $D = \{(0, 0)\}$, we have $C = D \oplus L_C$ and $(S_0 + L_C)/L_C = G/L_C = (\mathbb{Z}_p \times \mathbb{Z}_q)/\langle (0, 1) \rangle = \{(i, 0) + L_C \mid i \in [p]\}$. This implies $S_0 = \{(i, \alpha_i) \mid i \in [p]\}$, for some $\alpha_i \in \mathbb{Z}$, which gives the first pair in Table 3.1. Now if (S_0, C) is the first pair in Table 3.1, then $|S_0| \cdot |C| = |G|$, and for any $(x, y) \in G$, $(x, y) = (x, \alpha_x) + (0, y - \alpha_x) \in S_0 + C$. Therefore, $G = S_0 \oplus C$. The same reason applies for $|S_0| = q$ to get the second pair in Table 3.1. $\square$

Theorem 3.6. Let $G = \mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_q$ for distinct primes p and q , where $q \leq 3$, and let S be a proper subset of $G \setminus \{0\}$ that generates G . Let $0 \in C \subseteq G$. Then C is a perfect code in $\text{Cay}(G, S)$ if and only if (S_0, C) is one of the pairs in Table 3.2.

Table 3.2: Perfect codes in $\text{Cay}(\mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_q, S)$

Row	S_0	C	Condition
1	$\{i\mathbf{e}_1 + \alpha_i\mathbf{e}_2 + \beta_i\mathbf{e}_3 \mid i \in [p]\}$	$\{i\mathbf{e}_2 + j\mathbf{e}_3 \mid i, j \in [q]\}$	$q = 3$
2	$\{\alpha_i\mathbf{e}_1 + \alpha_i\mathbf{e}_2 + i\mathbf{e}_3 \mid i \in [3]\}$	$\{i\mathbf{e}_1 + j\mathbf{e}_2 \mid i \in [p], j \in [3]\}$	
3	$\{j\mathbf{e}_1 + \alpha_i\mathbf{e}_2 + i\mathbf{e}_3 \mid i \in [3], j \in [p]\}$	$\{\alpha_i\mathbf{e}_1 + i\mathbf{e}_2 \mid i \in [3]\}$	$q = 3$
4	$\{i\mathbf{e}_1 + j\mathbf{e}_2 + \alpha_i\mathbf{e}_3 \mid i \in [p], j \in [q]\}$	$\{\alpha_i\mathbf{e}_2 + i\mathbf{e}_3 \mid i \in [q]\}$	
5	$\{i\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + j\mathbf{e}_3 \mid i \in [p], j \in [q]\}$	$\{i\mathbf{e}_2 \mid i \in [q]\}$	
6	$\{\alpha_i\mathbf{e}_1 + j\mathbf{e}_2 + i\mathbf{e}_3 \mid i, j \in [q]\}$	$\{i\mathbf{e}_1 + \alpha_i\mathbf{e}_2 \mid i \in [p]\}$	
7	$\{\alpha_{ij}\mathbf{e}_1 + i\mathbf{e}_2 + j\mathbf{e}_3 \mid i, j \in [q]\}$	$\{i\mathbf{e}_1 \mid i \in [p]\}$	

Proof of Theorem 3.6. Suppose $\text{Cay}(G, S)$ has a perfect code C . From Lemma 2.8, we have $|S_0| > 2$.

If $|S_0| = p$, then from Lemma 2.8, S_0 is aperiodic and C is a subgroup of order q^2 , so $C = L_C = \{0\} \times \mathbb{Z}_q \times \mathbb{Z}_q$. From the factorization of G/L_C given in Lemma 2.5,

$(S_0 + L_C)/L_C = G/L_C = (\mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_q)/(\{0\} \times \mathbb{Z}_q \times \mathbb{Z}_q) = \{i\mathbf{e}_1 + L_C \mid i \in [p]\}$, and hence $S_0 = \{i\mathbf{e}_1 + \alpha_i\mathbf{e}_2 + \beta_i\mathbf{e}_3 \mid i \in [p]\}$, $\alpha_i, \beta_i \in \mathbb{Z}$, which is pair 1 in Table 3.2.

If $|S_0| = q = 3$, then from Lemma 2.8, S_0 is aperiodic and C is a subgroup of order pq , so $C = L_C = \langle \mathbf{e}_1 + \mathbf{e}_2 \rangle$. From the factorization of G/L_C given in Lemma 2.5, $(S_0 + L_C)/L_C = G/L_C = \{i\mathbf{e}_3 + L_C \mid i \in [q]\}$, and hence $S_0 = \{\alpha_i\mathbf{e}_1 + \alpha_i\mathbf{e}_2 + i\mathbf{e}_3 \mid i \in [q]\}$, $\alpha_i \in \mathbb{Z}$, which is pair 2 in Table 3.2. If $q = 2$, then S_0 does not generate G .

If $|S_0| = pq$ and S_0 is periodic, then $|L_{S_0}| = q$ or p by Lemma 2.8. Let $S_0 = D \oplus L_{S_0}$ for some subset D . From the factorization of G/L_{S_0} given in Lemma 2.5, we have $G/L_{S_0} = ((D + L_{S_0})/L_{S_0}) \oplus ((C + L_{S_0})/L_{S_0})$. If $|L_{S_0}| = p$, then $L_{S_0} = \langle \mathbf{e}_1 \rangle$. Note that $(D + L_{S_0})/L_{S_0}$ is an aperiodic set of order q and $G/L_{S_0} \cong \mathbb{Z}_q \times \mathbb{Z}_q$. If $q = 2$, then from Lemmas 2.7, $(D + L_{S_0})/L_{S_0}$ is an aperiodic set of order 2 that generates $\mathbb{Z}_2 \times \mathbb{Z}_2$, which is not possible by Lemma 2.8. If $q = 3$, from Theorem 3.1 and Remark 3.4, we have $(C + L_{S_0})/L_{S_0} = \{i\mathbf{e}_2 + L_{S_0} \mid i \in [3]\}$, and hence $(D + L_{S_0})/L_{S_0} = \{\alpha_i\mathbf{e}_2 + i\mathbf{e}_3 \mid i \in [3]\}$. Therefore $S_0 = D \oplus L_{S_0} = \{j\mathbf{e}_1 + \alpha_i\mathbf{e}_2 + i\mathbf{e}_3 \mid i \in [3], j \in [p]\}$ and $C = \{\alpha_i\mathbf{e}_1 + i\mathbf{e}_2 \mid i \in [3]\}$, which is pair 3 in Table 3.2.

If $|S_0| = pq$, S_0 is periodic, and $|L_{S_0}| = q$, then $L_{S_0} = \langle \mathbf{e}_2 \rangle$. Note that $(D + L_{S_0})/L_{S_0}$ is an aperiodic set of order p and $G/L_{S_0} \cong \mathbb{Z}_p \times \mathbb{Z}_q$. From the case of $|S_0| = p$ in Theorem 3.5 (pair 1), we have $(D + L_{S_0})/L_{S_0} = \{i\mathbf{e}_1 + \alpha_i\mathbf{e}_3 + L_{S_0} \mid i \in [p]\}$, and $(C + L_{S_0})/L_{S_0} = \{i\mathbf{e}_3 + L_{S_0} \mid i \in [q]\}$. Therefore $S_0 = D \oplus L_{S_0} = \{i\mathbf{e}_1 + j\mathbf{e}_2 + \alpha_i\mathbf{e}_3 \mid i \in [p], j \in [q]\}$ and $C = \{\alpha_i\mathbf{e}_2 + i\mathbf{e}_3 \mid i \in [q]\}$, which is pair 4 in Table 3.2.

If $|S_0| = pq$ and S_0 is aperiodic, then $|C| = q$ and C is periodic, and hence $C = L_C = \langle \mathbf{e}_2 \rangle$. From the factorization of G/L_C given in Lemma 2.5, $(S_0 + L_C)/L_C = G/L_C = \{i\mathbf{e}_1 + j\mathbf{e}_3 + L_C \mid i \in [p], j \in [q]\}$, and hence $S_0 = \{i\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + j\mathbf{e}_3 \mid i \in [p], j \in [q]\}$, $\alpha_{ij} \in \mathbb{Z}$, which is pair 5 in Table 3.2.

If $|S_0| = q^2$ and S_0 is periodic, then $|L_{S_0}| = q$ from Lemma 2.8 and hence $L_{S_0} = \langle \mathbf{e}_2 \rangle$. Let $S_0 = D \oplus L_{S_0}$ for some subset D . From the factorization of G/L_{S_0} given in Lemma 2.5, we have $G/L_{S_0} = ((D + L_{S_0})/L_{S_0}) \oplus ((C + L_{S_0})/L_{S_0})$. Note that $(D + L_{S_0})/L_{S_0}$ is an aperiodic set of order q and $G/L_{S_0} \cong \mathbb{Z}_p \times \mathbb{Z}_q$. From the case of $|S_0| = q$ in Theorem 3.5 (pair 2), we have $(D + L_{S_0})/L_{S_0} = \{\alpha_i\mathbf{e}_1 + i\mathbf{e}_3 + L_{S_0} \mid i \in [q]\}$, and $(C + L_{S_0})/L_{S_0} = \{i\mathbf{e}_1 + L_{S_0} \mid i \in [p]\}$. Therefore $S_0 = D \oplus L_{S_0} = \{\alpha_i\mathbf{e}_1 + j\mathbf{e}_2 + i\mathbf{e}_3 \mid i, j \in [q]\}$ and $C = \{i\mathbf{e}_1 + \alpha_i\mathbf{e}_2 \mid i \in [p]\}$, which is pair 6 in Table 3.2.

If $|S_0| = q^2$ and S_0 is aperiodic, then $|C| = p$ and C is periodic, and hence $C = L_C = \langle \mathbf{e}_1 \rangle$. From the factorization of G/L_C given in Lemma 2.5, we have $(S_0 + L_C)/L_C = G/L_C = \{i\mathbf{e}_2 + j\mathbf{e}_3 + L_C \mid i, j \in [q]\}$, and hence $S_0 = \{\alpha_{ij}\mathbf{e}_1 + i\mathbf{e}_2 + j\mathbf{e}_3 \mid i, j \in [q]\}$, $\alpha_{ij} \in \mathbb{Z}$, which is pair 7 in Table 3.2.

Now, suppose (S_0, C) is one of the pairs in Table 3.2. For every pair (S_0, C) , we have $|S_0| \cdot |C| = pq^2 = |G|$. In every row in the table, we can see that for every i , exactly one of S_0 or C has an independent variable (i or j) before $\mathbf{e}_i$, and possibly a fixed integer (α_i, β_i , or α_{ij}) in the other set. This means that for every $\mathbf{x} = (x, y, z) \in G$, if we want $\mathbf{x} = \mathbf{s} + \mathbf{c}$ for $\mathbf{s} \in S_0$ and $\mathbf{c} \in C$, we can set the independent variable to be x or $x - \alpha_i$ (similarly for y and z). For example, take the fourth row, we have

$$(x, y, z) = (x\mathbf{e}_1 + (y - \alpha'_{z-\alpha_x})\mathbf{e}_2 + \alpha_x\mathbf{e}_3) + (\alpha'_{z-\alpha_x}\mathbf{e}_2 + (z - \alpha_x)\mathbf{e}_3).$$

This means $G = S_0 \oplus C$ and we are done. $\square$

Remark 3.7. In Theorem 3.6, the generators $\mathbf{e}_2$ and $\mathbf{e}_3$ may be replaced by any basis of $H = \{0\} \times \mathbb{Z}_q \times \mathbb{Z}_q$, as in (C2).

4 Perfect codes in Cayley graphs of Hajós groups containing $\mathbb{Z}_{p^k}$

In this section, we will focus on Hajós groups that contain $\mathbb{Z}_{p^k}$ as a subgroup, where p is a prime and k is a positive integer. These Hajós groups are $\mathbb{Z}_{p^k}$, $\mathbb{Z}_{p^k} \times \mathbb{Z}_q$, and $\mathbb{Z}_{2^k} \times \mathbb{Z}_2$.

Theorem 4.1. *Let $G = \mathbb{Z}_{p^k}$ for a prime number p and a positive integer k . Let S be a proper subset of $G \setminus \{0\}$ that generates G . Let $0 \in C \subseteq G$. Then C is a perfect code in $\text{Cay}(G, S)$ if and only if there are integers $0 = m_0 < m_1 < \dots < m_n = k$ and $1 \leq l_t \leq m_{t+1} - m_t$ for $t \in [n]$, such that*

$$S_0 = \left\{ \sum_{t=0}^{n-1} (i_t p^{m_t} + \alpha_{i_0, \dots, i_t} p^{m_t + l_t}) \mid i_t \in [p^{l_t}] \right\} \quad (2)$$

and

$$C = \left\{ \sum_{t=0}^{n-1} (i_t p^{m_t + l_t} + \beta_{i_0, \dots, i_t} p^{m_{t+1}}) \mid i_t \in [p^{m_{t+1} - m_t - l_t}] \right\}. \quad (3)$$

Theorem 4.2. *Let $G = \mathbb{Z}_{p^k} \times \mathbb{Z}_q$ for distinct primes p, q , and a positive integer k . Let S be a proper subset of $G \setminus \{0\}$ that generates G . Let $0 \in C \subseteq G$. Then C is a perfect code in $\text{Cay}(G, S)$ if and only if there are integers $r \in [n]$, $0 = m_0 < m_1 < \dots < m_r \leq m_{r+1} < \dots < m_n = k$, and $1 \leq l_t \leq m_{t+1} - m_t$ for $t \in [n] \setminus \{r\}$ and $0 \leq l_r \leq m_{r+1} - m_r$, such that (S_0, C) is one of the following pairs. In (c) and (d) below, $\beta_{i_0, \dots, i_{r-1}} = 0$ when $r = 0$.*

$$\begin{aligned} \text{(a)} \quad S_0 &= \left\{ \left(\sum_{t=0}^{n-1} i_t p^{m_t} + \sum_{t \in [n] \setminus \{r\}} \alpha_{i_0, \dots, i_t} p^{m_t + l_t}, \alpha_{i_0, \dots, i_r} \right) \mid i_t \in [p^{l_t}] \right\} \\ C &= \left\{ \left(\sum_{t \in [n] \setminus \{r\}} i_t p^{m_t + l_t} + \sum_{t=0}^{n-1} \beta_{i_0, \dots, i_t} p^{m_{t+1}}, i_r \right) \mid i_t \in [p^{m_{t+1} - m_t - l_t}] \text{ for } t \neq r, i_r \in [q] \right\} \\ &\text{where } m_r + l_r = m_{r+1}. \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad S_0 &= \left\{ \left(\sum_{t=0}^{n-1} (i_t p^{m_t} + \alpha_{i_0, \dots, i_t} p^{m_t + l_t}), \alpha_{i_0, \dots, i_r} \right) \mid i_t \in [p^{l_t}] \right\} \\ C &= \left\{ \left(\sum_{t=0}^{n-1} i_t p^{m_t + l_t} + \sum_{t=0}^{r-1} \beta_{i_0, \dots, i_t} p^{m_{t+1}} + \sum_{t=r}^{n-1} \beta_{i_0, \dots, i_t, j} p^{m_{t+1}}, j \right) \mid i_t \in [p^{m_{t+1} - m_t - l_t}], \right. \\ &\quad \left. j \in [q] \right\}. \end{aligned}$$

$$(c) \quad S_0 = \left\{ \left(\sum_{t \in [n] \setminus \{r\}} i_t p^{m_t} + \sum_{t=0}^{n-1} \alpha_{i_0, \dots, i_t} p^{m_t + l_t}, i_r \right) \middle| i_t \in [p^{l_t}] \text{ for } t \neq r, i_r \in [q] \right\}$$

$$C = \left\{ \left(\sum_{t=0}^{n-1} i_t p^{m_t + l_t} + \sum_{t \in [n] \setminus \{r-1\}} \beta_{i_0, \dots, i_t} p^{m_{t+1}}, \beta_{i_0, \dots, i_{r-1}} \right) \middle| i_t \in [p^{m_{t+1} - m_t - l_t}] \right\}$$

where $l_r = 0$.

$$(d) \quad S_0 = \left\{ \left(\sum_{t=0}^{n-1} i_t p^{m_t} + \sum_{t=0}^{r-1} \alpha_{i_0, \dots, i_t} p^{m_t + l_t} + \sum_{t=r}^{n-1} \alpha_{i_0, \dots, i_t, j} p^{m_t + l_t}, j \right) \middle| i_t \in [p^{l_t}], j \in [q] \right\}$$

$$C = \left\{ \left(\sum_{t=0}^{n-1} (i_t p^{m_t + l_t} + \beta_{i_0, \dots, i_t} p^{m_{t+1}}), \beta_{i_0, \dots, i_{r-1}} \right) \middle| i_t \in [p^{m_{t+1} - m_t - l_t}] \right\}.$$

Theorem 4.3. Let $G = \mathbb{Z}_{2^k} \times \mathbb{Z}_2$ for a positive integer k and S be a proper subset of $G \setminus \{0\}$ that generates G . Let $0 \in C \subseteq G$. Then C is a perfect code in $\text{Cay}(G, S)$ if and only if there are integers $r \in [n]$, $0 = m_0 < m_1 < \dots < m_r \leq m_{r+1} < \dots < m_n = k$, and $1 \leq l_t \leq m_{t+1} - m_t$ for $t \in [n] \setminus \{r\}$ and $0 \leq l_r \leq m_{r+1} - m_r$, such that (S_0, C) is one of the following pairs. In (d), (e) and (f) below, $\beta_{i_0, \dots, i_{r-1}} = \gamma_{i_0, \dots, i_{r-1}} = 0$ when $r = 0$.

$$(a) \quad S_0 = \left\{ \left(\sum_{t=0}^{n-1} i_t 2^{m_t} + \sum_{t \in [n] \setminus \{r\}} \alpha_{i_0, \dots, i_t} 2^{m_t + l_t}, \alpha_{i_0, \dots, i_r} \right) \middle| i_t \in [2^{l_t}] \right\}$$

$$C = \left\{ \left(\sum_{t \in [n] \setminus \{r\}} i_t 2^{m_t + l_t} + \sum_{t=0}^{n-1} \beta_{i_0, \dots, i_t} 2^{m_{t+1}}, i_r \right) \middle| i_t \in [2^{m_{t+1} - m_t - l_t}] \text{ for } t \neq r, i_r \in [2] \right\}$$

where $m_r + l_r = m_{r+1}$.

$$(b) \quad S_0 = \left\{ \left(\sum_{t=0}^{n-1} i_t 2^{m_t} + \sum_{t \in [n] \setminus \{r\}} \alpha_{i_0, \dots, i_t} 2^{m_t + l_t} + \alpha_{i_0, \dots, i_r} 2^{m_r + l_r - 1}, \alpha_{i_0, \dots, i_r} \right) \middle| i_t \in [2^{l_t}] \right\}$$

$$C = \left\{ \left(\sum_{t \in [n] \setminus \{r\}} i_t 2^{m_t + l_t} + \sum_{t=0}^{n-1} \beta_{i_0, \dots, i_t} 2^{m_{t+1}} + i_r 2^{m_r + l_r - 1}, i_r \right) \middle| i_t \in [2^{m_{t+1} - m_t - l_t}] \right.$$

$$\left. \text{for } t \neq r, i_r \in [2^{m_{r+1} - m_r - l_r + 1}] \right\}$$

where $m_r + l_r \geq 1$.

$$(c) \quad S_0 = \left\{ \left(\sum_{t=0}^{n-1} (i_t 2^{m_t} + \alpha_{i_0, \dots, i_t} 2^{m_t + l_t}), \gamma_{i_0, \dots, i_r} \right) \middle| i_t \in [2^{l_t}] \right\}$$

$$C = \left\{ \left(\sum_{t=0}^{n-1} i_t 2^{m_t + l_t} + \sum_{t=0}^{r-1} \beta_{i_0, \dots, i_t} 2^{m_{t+1}} + \sum_{t=r}^{n-1} \beta_{i_0, \dots, i_t, j} 2^{m_{t+1}}, j \right) \middle| i_t \in [2^{m_{t+1} - m_t - l_t}], \right.$$

$$\left. j \in [2] \right\}$$

$$(d) \quad S_0 = \left\{ \left(\sum_{t \in [n] \setminus \{r\}} i_t 2^{m_t} + \sum_{t=0}^{n-1} \alpha_{i_0, \dots, i_t} 2^{m_t+l_t}, i_r \right) \middle| i_t \in [2^{l_t}] \text{ for } t \neq r, i_r \in [2] \right\}$$

$$C = \left\{ \left(\sum_{t=0}^{n-1} i_t 2^{m_t+l_t} + \sum_{t \in [n] \setminus \{r-1\}} \beta_{i_0, \dots, i_t} 2^{m_{t+1}}, \beta_{i_0, \dots, i_{r-1}} \right) \middle| i_t \in [2^{m_{t+1}-m_t-l_t}] \right\}$$

where $l_r = 0$.

$$(e) \quad S_0 = \left\{ \left(\sum_{t \in [n] \setminus \{r\}} i_t 2^{m_t} + \sum_{t=0}^{n-1} \alpha_{i_0, \dots, i_t} 2^{m_t+l_t} + i_r 2^{m_{r-1}}, i_r \right) \middle| i_t \in [2^{l_t}] \text{ for } t \neq r, \right.$$

$$\left. i_r \in [2^{l_r+1}] \right\}$$

$$C = \left\{ \left(\sum_{t=0}^{n-1} i_t 2^{m_t+l_t} + \sum_{t \in [n] \setminus \{r-1\}} \beta_{i_0, \dots, i_t} 2^{m_{t+1}} + \beta_{i_0, \dots, i_{r-1}} 2^{m_{r-1}}, \beta_{i_0, \dots, i_{r-1}} \right) \middle| \right.$$

$$\left. i_t \in [2^{m_{t+1}-m_t-l_t}] \right\}$$

where $m_r \geq 1$.

$$(f) \quad S_0 = \left\{ \left(\sum_{t=0}^{n-1} i_t 2^{m_t} + \sum_{t=0}^{r-1} \alpha_{i_0, \dots, i_t} 2^{m_t+l_t} + \sum_{t=r}^{n-1} \alpha_{i_0, \dots, i_t, j} 2^{m_t+l_t}, j \right) \middle| i_t \in [2^{l_t}], j \in [2] \right\}$$

$$C = \left\{ \left(\sum_{t=0}^{n-1} (i_t 2^{m_t+l_t} + \beta_{i_0, \dots, i_t} 2^{m_{t+1}}), \gamma_{i_0, \dots, i_{r-1}} \right) \middle| i_t \in [2^{m_{t+1}-m_t-l_t}] \right\}$$

The rest of this section is devoted to the proof of Theorems 4.1, 4.2, and 4.3.

4.1 Sufficiency of Theorems 4.1, 4.2, and 4.3

Lemma 4.4. *Let (S_0, C) be one of the pairs in Theorem 4.1, 4.2 or 4.3 and G the corresponding group. Then $G = S_0 \oplus C$.*

Proof. First, note that

$$\prod_{t=0}^{n-1} p^{l_t} \cdot \prod_{t=0}^{n-1} p^{m_{t+1}-m_t-l_t} = p^{m_n-m_0} = p^k,$$

and hence $|S_0| \cdot |C| = |G|$ for every pair. Let $s_1, s_2 \in S_0$ and $c_1, c_2 \in C$ such that $s_1 + c_1 = s_2 + c_2$. We will prove $s_1 = s_2$ and $c_1 = c_2$.

Case 1. (S_0, C) is the pair in Theorem 4.1.

Let $s_1 = \sum_{t=0}^{n-1} (i_t p^{m_t} + \alpha_{i_0, \dots, i_t} p^{m_t+l_t})$, $s_2 = \sum_{t=0}^{n-1} (j_t p^{m_t} + \alpha_{j_0, \dots, j_t} p^{m_t+l_t})$ for one $i_t, j_t \in [p^{l_t}]$, and $c_1 = \sum_{t=0}^{n-1} (u_t p^{m_t+l_t} + \beta_{u_0, \dots, u_t} p^{m_{t+1}})$, $c_2 = \sum_{t=0}^{n-1} (v_t p^{m_t+l_t} + \beta_{v_0, \dots, v_t} p^{m_{t+1}})$ for one

$u_t, v_t \in [p^{m_{t+1}-m_t-l_t}]$. Since $s_1 + c_1 = s_2 + c_2$,

$$\begin{aligned} & i_0 + \alpha_{i_0} p^{l_0} + u_0 p^{l_0} + \beta_{u_0} p^{m_1} + \sum_{t=1}^{n-1} \left(i_t p^{m_t} + \alpha_{i_0, \dots, i_t} p^{m_t+l_t} \right) + \sum_{t=1}^{n-1} \left(u_t p^{m_t+l_t} + \beta_{u_0, \dots, u_t} p^{m_{t+1}} \right) \\ &= j_0 + \alpha_{j_0} p^{l_0} + v_0 p^{l_0} + \beta_{v_0} p^{m_1} + \sum_{t=1}^{n-1} \left(j_t p^{m_t} + \alpha_{j_0, \dots, j_t} p^{m_t+l_t} \right) + \sum_{t=1}^{n-1} \left(v_t p^{m_t+l_t} + \beta_{v_0, \dots, v_t} p^{m_{t+1}} \right). \end{aligned}$$

Note that $0 = m_0 < m_0 + l_0 \leq m_1 < m_1 + l_1 \leq m_2 < m_2 + l_2 \leq \dots \leq m_n = k$. By taking the last equation in modulo p^{l_0} , we have $i_0 \equiv j_0 \pmod{p^{l_0}}$. Since $i_0, j_0 \in [p^{l_0}]$, $i_0 = j_0$ in $\mathbb{Z}_{p^k}$. If $l_0 < m_1$, take the equation in modulo p^{m_1} to get $u_0 p^{l_0} \equiv v_0 p^{l_0} \pmod{p^{m_1}}$. Since $u_0, v_0 \in [p^{m_1-l_0}]$, $u_0 = v_0$ in $\mathbb{Z}_{p^k}$. If $l_0 = m_1$, then $u_0, v_0 \in [p^{m_1-l_0}] = \{0\}$. Therefore, we have $i_0 = j_0$ and $u_0 = v_0$. Using a similar argument, by taking the equation in modulo $p^{m_x+l_x}$ and $p^{m_{x+1}}$ for $x = 0, \dots, t$ in order, we will get $i_t = j_t$ and $u_t = v_t$, and hence $s_1 = s_2$ and $c_1 = c_2$.

Case 2. (S_0, C) is pair (a) in Theorem 4.2, or pair (a) in Theorem 4.3.

We use the same argument in Case 1 to get that i_t in s_1 and s_2 are the same for $t \in [r+1]$ and i_t in c_1 and c_2 are the same for $t \in [r]$. By looking at the second component of $s_1 + c_1 = s_2 + c_2$, we get i_r in c_1 and c_2 are equal. Since we already have i_t in s_1 and s_2 and i_t in c_1 and c_2 are the same for $t \in [r+1]$, the remaining equation in $s_1 + c_1 = s_2 + c_2$ is the same as in Case 1, and this leads to $s_1 = s_2$ and $c_1 = c_2$.

Case 3. (S_0, C) is pair (c) in Theorem 4.2, or pair (d) in Theorem 4.3.

Similar to Case 2, but i_t in s_1 and s_2 and i_t in c_1 and c_2 are the same for $t \in [r]$, and from the second component of $s_1 + c_1 = s_2 + c_2$, we get i_r in s_1 and s_2 are equal. The remaining arguments follow.

Case 4. (S_0, C) is pair (b) or (d) in Theorem 4.2, or pair (c) or (f) in Theorem 4.3.

Similar to Cases 2 and 3, but from the second component of $s_1 + c_1 = s_2 + c_2$, we get j in s_1 and s_2 or j in c_1 and c_2 are equal. The remaining arguments follow.

Case 5. (S_0, C) is pair (b) in Theorem 4.3.

Let

$$\begin{aligned} s_1 &= \left(\sum_{t=0}^{n-1} i_t 2^{m_t} + \sum_{t \in [n] \setminus \{r\}} \alpha_{i_0, \dots, i_t} 2^{m_t+l_t} + \alpha_{i_0, \dots, i_r} 2^{m_r+l_r-1}, \alpha_{i_0, \dots, i_r} \right) \in S_0, \\ s_2 &= \left(\sum_{t=0}^{n-1} j_t 2^{m_t} + \sum_{t \in [n] \setminus \{r\}} \alpha_{j_0, \dots, j_t} 2^{m_t+l_t} + \alpha_{j_0, \dots, j_r} 2^{m_r+l_r-1}, \alpha_{j_0, \dots, j_r} \right) \in S_0, \\ c_1 &= \left(\sum_{t \in [n] \setminus \{r\}} u_t 2^{m_t+l_t} + \sum_{t=0}^{n-1} \beta_{u_0, \dots, u_t} 2^{m_{t+1}} + u_r 2^{m_r+l_r-1}, u_r \right) \in C, \text{ and} \\ c_2 &= \left(\sum_{t \in [n] \setminus \{r\}} v_t 2^{m_t+l_t} + \sum_{t=0}^{n-1} \beta_{v_0, \dots, v_t} 2^{m_{t+1}} + v_r 2^{m_r+l_r-1}, v_r \right) \in C, \end{aligned}$$

for one $i_t, j_t \in [2^t]$, one $u_t, v_t \in [2^{m_{t+1}-m_t-l_t}]$ for $t \neq r$, and one $u_r, v_r \in [2^{m_{r+1}-m_r-l_r+1}]$. Let $s_1 + c_1 = s_2 + c_2$, we will prove $s_1 = s_2$ and $c_1 = c_2$. As the second component of $s_1 + c_1 = s_2 + c_2$ is in $\mathbb{Z}_2$, we know that $(\alpha_{i_0, \dots, i_r} + u_r)$ and $(\alpha_{j_0, \dots, j_r} + v_r)$ have the same parity. This implies

$$(\alpha_{i_0, \dots, i_r} + u_r) 2^{m_r+l_r-1} \equiv (\alpha_{j_0, \dots, j_r} + v_r) 2^{m_r+l_r-1} \pmod{2^{m_r+l_r}}. \quad (4)$$

If $r \geq 1$, similar to Case 1, taking $s_1 + c_1 = s_2 + c_2$ in modulo $2^{m_x+l_x}$ for $x \in [r]$ and $2^{m_{x+1}}$ for $x \in [r-1]$ in ascending order, we will get $i_x = j_x$ for $x \in [r]$ and $u_x = v_x$ for $x \in [r-1]$. By taking $s_1 + c_1 = s_2 + c_2$ in modulo 2^{m_r} we get

$$\begin{aligned} & (\alpha_{i_0, \dots, i_r} 2^{m_r+l_r-1}, \alpha_{i_0, \dots, i_r}) + (u_{r-1} 2^{m_{r-1}+l_{r-1}} + u_r 2^{m_r+l_r-1}, u_r) \\ & \equiv (\alpha_{j_0, \dots, j_r} 2^{m_r+l_r-1}, \alpha_{j_0, \dots, j_r}) + (v_{r-1} 2^{m_{r-1}+l_{r-1}} + v_r 2^{m_r+l_r-1}, v_r) \pmod{2^{m_r}}. \end{aligned} \quad (5)$$

If $l_r \geq 1$, then $2^{m_r+l_r-1} \equiv 0 \pmod{2^{m_r}}$ and we have

$$u_{r-1} 2^{m_{r-1}+l_{r-1}} \equiv v_{r-1} 2^{m_{r-1}+l_{r-1}} \pmod{2^{m_r}}.$$

If $l_r = 0$, then using (4) on the first component of (5) gives the same equality in modulo 2^{m_r} . We conclude $u_{r-1} = v_{r-1}$ similar to the previous cases.

If $r = 0$, there are no earlier indices to recover. In both cases, we proceed as follows. Take $s_1 + c_1 = s_2 + c_2$ in modulo $2^{m_r+l_r}$, we have

$$\begin{aligned} & (i_r 2^{m_r} + \alpha_{i_0, \dots, i_r} 2^{m_r+l_r-1}, \alpha_{i_0, \dots, i_r}) + (u_r 2^{m_r+l_r-1}, u_r) \\ & \equiv (j_r 2^{m_r} + \alpha_{j_0, \dots, j_r} 2^{m_r+l_r-1}, \alpha_{j_0, \dots, j_r}) + (v_r 2^{m_r+l_r-1}, v_r) \pmod{2^{m_r+l_r}}. \end{aligned} \quad (6)$$

Using (4) on the first component of (6) gives

$$i_r 2^{m_r} \equiv j_r 2^{m_r} \pmod{2^{m_r+l_r}}.$$

The rest of the arguments follow Case 1.

Case 6. (S_0, C) is pair (e) in Theorem 4.3.

Similar to previous cases, by taking $s_1 + c_1 = s_2 + c_2$ in modulo $2^{m_x+l_x}$ and $2^{m_{x+1}}$ for $x \in [r-1]$ in ascending order, we will get $i_x = j_x$ and $u_x = v_x$ for $x \in [r-1]$. Similar to Case 5, take $s_1 + c_1 = s_2 + c_2$ in modulo $2^{m_{r-1}+l_{r-1}}$ and 2^{m_r} and use the fact that the second component of $s_1 + c_1 = s_2 + c_2$ have the same parity to get $i_{r-1} = j_{r-1}$ and $u_{r-1} = v_{r-1}$, respectively. The rest of the arguments follow. $\square$

4.2 Proof of Theorem 4.1

Proof of Theorem 4.1. Suppose S_0 and C are as in (2) and (3), respectively. Then $G = S_0 \oplus C$ by Lemma 4.4. Suppose $\text{Cay}(G, S)$ has a perfect code C . Then $G = \mathbb{Z}_{p^k} = S_0 \oplus C$ and $|S_0| = p^r$. We will prove (2) and (3) using induction on r . First, for $|S_0| = p$, by Lemma 2.8, S_0 is aperiodic and $C = p\mathbb{Z}_{p^k} = \{ip + \beta_i p^k \mid i \in [p^{k-1}]\}$. In this case, $s_1 \not\equiv s_2 \pmod{p}$ for $s_1, s_2 \in S_0$, which means we can write S_0 as $\{i + \alpha_i p \mid i \in [p]\}$ as desired.

Now, suppose (2) and (3) are true for $|S_0| < p^r$.

Case 1. S_0 is periodic.

Let $S_0 = L_{S_0} \oplus D$ and $|L_{S_0}| = p^s$. Then $0 < s < r$ because S_0 generates G . From Lemma 2.5, $G/L_{S_0} = ((D + L_{S_0})/L_{S_0}) \oplus ((C + L_{S_0})/L_{S_0})$ and $|(D + L_{S_0})/L_{S_0}| = |D| < |S_0|$. We also have $G/L_{S_0} \cong \mathbb{Z}_{p^{k-s}}$. From the induction hypothesis, there are integers $0 = m_0 < m_1 < \dots < m_n = \log_p(|G/L_{S_0}|) = k - s$ and $1 \leq l_t \leq m_{t+1} - m_t$ for $t \in [n]$ such that

$$\begin{aligned} (D + L_{S_0})/L_{S_0} &= \left\{ \sum_{t=0}^{n-1} (i_t p^{m_t} + \alpha_{i_0, \dots, i_t} p^{m_t + l_t}) + L_{S_0} \mid i_t \in [p^{l_t}] \right\}, \text{ and} \\ (C + L_{S_0})/L_{S_0} &= \left\{ \sum_{t=0}^{n-1} (i_t p^{m_t + l_t} + \beta_{i_0, \dots, i_t} p^{m_{t+1}}) + L_{S_0} \mid i_t \in [p^{m_{t+1} - m_t - l_t}] \right\}. \end{aligned}$$

Now $L_{S_0} = p^{k-s} \mathbb{Z}_{p^k} = \{h p^{k-s} \mid h \in [p^s]\}$. Therefore,

$$\begin{aligned} S_0 &= L_{S_0} \oplus D \\ &= \left\{ \sum_{t=0}^{n-1} (i_t p^{m_t} + \alpha_{i_0, \dots, i_t} p^{m_t + l_t}) + h p^{k-s} \mid i_t \in [p^{l_t}], h \in [p^s] \right\} \\ &= \left\{ \sum_{t=0}^{n-1} (i_t p^{m_t} + \alpha_{i_0, \dots, i_t} p^{m_t + l_t}) + h p^{k-s} + \alpha_{i_0, \dots, i_{n-1}, h} p^k \mid i_t \in [p^{l_t}], h \in [p^s] \right\} \\ &= \left\{ \sum_{t=0}^n (i_t p^{m_t} + \alpha_{i_0, \dots, i_t} p^{m_t + l_t}) \mid i_t \in [p^{l_t}] \right\}, \end{aligned}$$

where $m_n = k - s$, $l_n = s$, and $i_n = h$. Moreover, $m_{n+1} = k > m_n$ and $1 \leq l_n = m_{n+1} - m_n$. Since $G/L_{S_0} \cong \mathbb{Z}_{p^{k-s}}$, $p^{k-s} + L_{S_0} = L_{S_0}$, so we have

$$\begin{aligned} &(C + L_{S_0})/L_{S_0} \\ &= \left\{ \sum_{t=0}^{n-2} (i_t p^{m_t + l_t} + \beta_{i_0, \dots, i_t} p^{m_{t+1}}) + i_{n-1} p^{m_{n-1} + l_{n-1}} + \beta_{i_0, \dots, i_{n-1}} p^{m_n} + L_{S_0} \mid i_t \in [p^{m_{t+1} - m_t - l_t}] \right\} \\ &= \left\{ \sum_{t=0}^{n-2} (i_t p^{m_t + l_t} + \beta_{i_0, \dots, i_t} p^{m_{t+1}}) + i_{n-1} p^{m_{n-1} + l_{n-1}} + L_{S_0} \mid i_t \in [p^{m_{t+1} - m_t - l_t}] \right\}. \end{aligned}$$

Therefore,

$$\begin{aligned} C &= \left\{ \sum_{t=0}^{n-2} (i_t p^{m_t + l_t} + \beta_{i_0, \dots, i_t} p^{m_{t+1}}) + i_{n-1} p^{m_{n-1} + l_{n-1}} + \beta_{i_0, \dots, i_{n-1}} p^{k-s} \mid i_t \in [p^{m_{t+1} - m_t - l_t}] \right\} \\ &= \left\{ \sum_{t=0}^{n-1} (i_t p^{m_t + l_t} + \beta_{i_0, \dots, i_t} p^{m_{t+1}}) \mid i_t \in [p^{m_{t+1} - m_t - l_t}] \right\}. \end{aligned}$$

Note that in C we can also set the sum from $t = 0$ to n because $m_n + l_n = m_{n+1} = k$.

Case 2. S_0 is aperiodic.

In this case C is periodic, so let $C = L_C \oplus D$. Consider the direct sum $G/L_C = ((S_0 + L_C)/L_C) \oplus ((D + L_C)/L_C)$. If $|L_C| = |C|$, then $L_C = C = p^r \mathbb{Z}_{p^k} = \{ip^r \mid i \in [p^{k-r}]\}$. Since $C = L_C$, we can take $D = \{0\}$ and we have $G/L_C = (S_0 + L_C)/L_C$. This means $(S_0 + L_C)/L_C = \{i + L_C \mid i \in [p^r]\}$ and hence $S_0 = \{i + \alpha_i p^r \mid i \in [p^r]\}$.

If $|L_C| < |C|$, then by Lemma 2.5, $(D + L_C)/L_C$ is aperiodic. Therefore $(S_0 + L_C)/L_C$ is periodic and $|(S_0 + L_C)/L_C| = |S_0|$. Let $|L_C| = p^s$. From Case 1, there are integers $0 = m_0 < m_1 < \dots < m_{n+1} = k - s$ and $1 \leq l_t \leq m_{t+1} - m_t$ for $t \in [n + 1]$ such that

$$(S_0 + L_C)/L_C = \left\{ \sum_{t=0}^n (i_t p^{m_t} + \alpha_{i_0, \dots, i_t} p^{m_t + l_t}) + L_C \mid i_t \in [p^{l_t}] \right\}, \text{ and}$$

$$(D + L_C)/L_C = \left\{ \sum_{t=0}^{n-1} (i_t p^{m_t + l_t} + \beta_{i_0, \dots, i_t} p^{m_{t+1}}) + L_C \mid i_t \in [p^{m_{t+1} - m_t - l_t}] \right\}.$$

From the previous case, we also have $m_n + l_n = m_{n+1} = k - s$. Since $|G/L_C| = p^{k-s}$, $p^{k-s} + L_C = L_C$, so we have

$$(S_0 + L_C)/L_C = \left\{ \sum_{t=0}^{n-1} (i_t p^{m_t} + \alpha_{i_0, \dots, i_t} p^{m_t + l_t}) + i_n p^{m_n} + L_C \mid i_t \in [p^{l_t}] \right\}.$$

Since $|L_C| = p^s$, $L_C = p^{k-s} \mathbb{Z}_{p^k} = \{hp^{k-s} \mid h \in [p^s]\}$. Therefore,

$$S_0 = \left\{ \sum_{t=0}^{n-1} (i_t p^{m_t} + \alpha_{i_0, \dots, i_t} p^{m_t + l_t}) + i_n p^{m_n} + \alpha_{i_0, \dots, i_n} p^{k-s} \mid i_t \in [p^{l_t}] \right\}$$

$$= \left\{ \sum_{t=0}^n (i_t p^{m_t} + \alpha_{i_0, \dots, i_t} p^{m_t + l_t}) \mid i_t \in [p^{l_t}] \right\} \text{ and}$$

$$C = D \oplus L_C$$

$$= \left\{ \sum_{t=0}^{n-1} (i_t p^{m_t + l_t} + \beta_{i_0, \dots, i_t} p^{m_{t+1}}) + hp^{k-s} \mid i_t \in [p^{m_{t+1} - m_t - l_t}], h \in [p^s] \right\}$$

$$= \left\{ \sum_{t=0}^{n-1} (i_t p^{m_t + l_t} + \beta_{i_0, \dots, i_t} p^{m_{t+1}}) + hp^{k-s} + \beta_{i_0, \dots, i_{n-1}, h} p^k \mid i_t \in [p^{m_{t+1} - m_t - l_t}], h \in [p^s] \right\}.$$

By setting $i_n = h$ and a new value for m_{n+1} to be equal to k , we have

$$C = \left\{ \sum_{t=0}^n (i_t p^{m_t + l_t} + \beta_{i_0, \dots, i_t} p^{m_{t+1}}) \mid i_t \in [p^{m_{t+1} - m_t - l_t}] \right\}. \quad \square$$

4.3 Proof of Theorem 4.2

Proof of Theorem 4.2. Suppose (S_0, C) is one of the pairs in Theorem 4.2. Then $G = S_0 \oplus C$ by Lemma 4.4. Suppose $\text{Cay}(G, S)$ has a perfect code C . Then $G = \mathbb{Z}_{p^k} \times \mathbb{Z}_q = S_0 \oplus C$. We may write $|S_0| = p^e q^f$ as $|S_0|$ divides $|G| = p^k q$, where $f = 0$ or 1 and $0 < e + f < k + 1$. We will prove (S_0, C) is one of the four pairs in Theorem 4.2 using induction on $e + f$.

First, consider the case $|S_0| = p$. From Lemma 2.8, S_0 is aperiodic and $C = p\mathbb{Z}_{p^k} \times \mathbb{Z}_q = \{(ip, j) \mid i \in [p^{k-1}], j \in [q]\}$. Write $C = L_C \oplus D$ where $D = \{0\}$, by Lemma 2.5, we have a direct sum $G/L_C = ((S_0 + L_C)/L_C) \oplus ((D + L_C)/L_C)$. Since $D = \{0\}$, we have $G/L_C = (S_0 + L_C)/L_C$. This means $(S_0 + L_C)/L_C = \{(i, 0) + L_C \mid i \in [p]\}$ and hence $S_0 = \{(i, 0) + \alpha_i(p, 1) \mid i \in [p]\} = \{(i + \alpha_i p, \alpha_i) \mid i \in [p]\}$. Note that the pair (S_0, C) here is pair (b) in Theorem 4.2 where $n = 1$ and $r = 0$.

Now, for $|S_0| = q$. From Lemma 2.8, S_0 is aperiodic and $C = \mathbb{Z}_{p^k} \times \{0\} = \{(i, 0) \mid i \in [p^k]\}$. Using the same arguments as in the case of $|S_0| = p$, $(S_0 + L_C)/L_C = G/L_C = \{(0, i) + L_C \mid i \in [q]\}$ and hence $S_0 = \{(0, i) + \alpha_i(1, 0) \mid i \in [q]\} = \{(\alpha_i, i) \mid i \in [q]\}$. Note that the pair (S_0, C) here is pair (c) in Theorem 4.2 where $n = 1$ and $r = 0$.

Suppose Theorem 4.2 is true for $e + f < m$. We will prove that (S_0, C) is one of the four pairs in Theorem 4.2 for $|S_0| = p^m$ and $|S_0| = p^{m-1}q$.

Case 1. $|S_0| = p^m$ and S_0 is periodic.

Let $S_0 = L_{S_0} \oplus D$ and $|L_{S_0}| = p^s$, then $0 < s < m$ because S_0 generates G . From Lemma 2.5, $G/L_{S_0} = ((D + L_{S_0})/L_{S_0}) \oplus ((C + L_{S_0})/L_{S_0})$. Since $G/L_{S_0} \cong \mathbb{Z}_{p^{k-s}} \times \mathbb{Z}_q$ and $|(D + L_{S_0})/L_{S_0}| = |D| = p^{m-s}$, from the induction hypothesis, there are integers $0 = m_0 < m_1 < \dots < m_r \leq m_{r+1} < \dots < m_n = k - s$ and $1 \leq l_t \leq m_{t+1} - m_t$ for $t \in [n] \setminus \{r\}$ and $0 \leq l_r \leq m_{r+1} - m_r$, such that $(D + L_{S_0})/L_{S_0}$ and $(C + L_{S_0})/L_{S_0}$ are one of the four pairs in Theorem 4.2. Since $|D| = p^{m-s}$, they can only be pair (a) or (b).

If $(D + L_{S_0})/L_{S_0}$ and $(C + L_{S_0})/L_{S_0}$ are pair (b), then

$$\begin{aligned} (D + L_{S_0})/L_{S_0} &= \left\{ \left(\sum_{t=0}^{n-1} (i_t p^{m_t} + \alpha_{i_0, \dots, i_t} p^{m_t + l_t}), \alpha_{i_0, \dots, i_r} \right) + L_{S_0} \mid i_t \in [p^{l_t}] \right\}, \text{ and} \\ (C + L_{S_0})/L_{S_0} &= \left\{ \left(\sum_{t=0}^{n-1} i_t p^{m_t + l_t} + \sum_{t=0}^{r-1} \beta_{i_0, \dots, i_t} p^{m_{t+1}} + \sum_{t=r}^{n-1} \beta_{i_0, \dots, i_t, j} p^{m_{t+1}}, j \right) + L_{S_0} \mid \right. \\ &\quad \left. i_t \in [p^{m_{t+1} - m_t - l_t}], j \in [q] \right\}. \end{aligned}$$

Now, $L_{S_0} = p^{k-s} \mathbb{Z}_{p^k} \times \{0\} = \{(hp^{k-s}, 0) \mid h \in [p^s]\}$. Therefore,

$$\begin{aligned} S_0 &= L_{S_0} \oplus D \\ &= \left\{ \left(\sum_{t=0}^{n-1} (i_t p^{m_t} + \alpha_{i_0, \dots, i_t} p^{m_t + l_t}), \alpha_{i_0, \dots, i_r} \right) + (hp^{k-s}, 0) \mid i_t \in [p^{l_t}], h \in [p^s] \right\} \end{aligned}$$

$$\begin{aligned}
&= \left\{ \left(\sum_{t=0}^{n-1} (i_t p^{m_t} + \alpha_{i_0, \dots, i_t} p^{m_t+l_t}), \alpha_{i_0, \dots, i_r} \right) + (h p^{k-s} + \alpha_{i_0, \dots, i_{n-1}, h} p^k, 0) \mid i_t \in [p^{l_t}], \right. \\
&\quad \left. h \in [p^s] \right\} \\
&= \left\{ \left(\sum_{t=0}^n (i_t p^{m_t} + \alpha_{i_0, \dots, i_t} p^{m_t+l_t}), \alpha_{i_0, \dots, i_r} \right) \mid i_t \in [p^{l_t}], h \in [p^s] \right\}
\end{aligned}$$

where $m_n = k-s$, $l_n = s$, and $i_n = h$. Moreover, $m_{n+1} = k > m_n$ and $1 \leq l_n = m_{n+1} - m_n$. Since $G/L_{S_0} \cong \mathbb{Z}_{p^{k-s}} \times \mathbb{Z}_q$, $(p^{k-s}, 0) + L_{S_0} = L_{S_0}$, so we have

$$\begin{aligned}
(C + L_{S_0})/L_{S_0} = & \left\{ \left(\sum_{t=0}^{n-2} i_t p^{m_t+l_t} + \sum_{t=0}^{r-1} \beta_{i_0, \dots, i_t} p^{m_{t+1}} + \sum_{t=r}^{n-2} \beta_{i_0, \dots, i_t, j} p^{m_{t+1}}, j \right) \right. \\
& \left. + (i_{n-1} p^{m_{n-1}+l_{n-1}}, 0) + L_{S_0} \mid i_t \in [p^{m_{t+1}-m_t-l_t}], j \in [q] \right\}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
C = & \left\{ \left(\sum_{t=0}^{n-2} i_t p^{m_t+l_t} + \sum_{t=0}^{r-1} \beta_{i_0, \dots, i_t} p^{m_{t+1}} + \sum_{t=r}^{n-2} \beta_{i_0, \dots, i_t, j} p^{m_{t+1}}, j \right) + (i_{n-1} p^{m_{n-1}+l_{n-1}} + \right. \\
& \left. \beta_{i_0, \dots, i_{n-1}, j} p^{k-s}, 0) \mid i_t \in [p^{m_{t+1}-m_t-l_t}], j \in [q] \right\} \\
= & \left\{ \left(\sum_{t=0}^{n-1} i_t p^{m_t+l_t} + \sum_{t=0}^{r-1} \beta_{i_0, \dots, i_t} p^{m_{t+1}} + \sum_{t=r}^{n-1} \beta_{i_0, \dots, i_t, j} p^{m_{t+1}}, j \right) \mid i_t \in [p^{m_{t+1}-m_t-l_t}], j \in [q] \right\}.
\end{aligned}$$

Note that in C we can also set the sum from $t = 0$ to n because $m_n + l_n = m_{n+1} = k$. The pair (S_0, C) here is also in the form of pair (b).

If $(D + L_{S_0})/L_{S_0}$ and $(C + L_{S_0})/L_{S_0}$ are pair (a), the same algebraic operation we did for pair (b) is still applicable and we will get (S_0, C) to be another pair in the form of pair (a).

Case 2. $|S_0| = p^m$ and S_0 is aperiodic.

In this case, C is periodic and $|C| = p^{k-m}q$. Let $C = L_C \oplus D$. Then $|L_C| = q, p^s$, or $p^s q$, for $0 < s \leq k - m$. Consider the factorization given in Lemma 2.5, we have $G/L_C = ((S_0 + L_C)/L_C) \oplus ((D + L_C)/L_C)$.

Subcase 2.1. $|L_C| = q$. In this case, $G/L_C \cong \mathbb{Z}_{p^k}$ and $(S_0 + L_C)/L_C$ is periodic. From the case when S_0 is periodic in Theorem 4.1, we have

$$\begin{aligned}
(S_0 + L_C)/L_C = & \left\{ \left(\sum_{t=0}^n (i_t p^{m_t} + \alpha_{i_0, \dots, i_t} p^{m_t+l_t}), 0 \right) + L_C \mid i_t \in [p^{l_t}] \right\}, \text{ and} \\
(D + L_C)/L_C = & \left\{ \left(\sum_{t=0}^{n-1} (i_t p^{m_t+l_t} + \beta_{i_0, \dots, i_t} p^{m_{t+1}}), 0 \right) + L_C \mid i_t \in [p^{m_{t+1}-m_t-l_t}] \right\}.
\end{aligned}$$

From the proof of Theorem 4.1, we also have $m_n + l_n = m_{n+1} = k$. Since $(p^k, 0) + L_C = L_C$,

$$(S_0 + L_C)/L_C = \left\{ \left(\sum_{t=0}^{n-1} (i_t p^{m_t} + \alpha_{i_0, \dots, i_t} p^{m_t+l_t}), 0 \right) + (i_n p^{m_n}, 0) + L_C \mid i_t \in [p^{l_t}] \right\}.$$

Since $|L_C| = q$, $L_C = \{(0, j) \mid j \in [q]\} = \langle (0, 1) \rangle$. Therefore,

$$\begin{aligned} S_0 &= \left\{ \left(\sum_{t=0}^{n-1} (i_t p^{m_t} + \alpha_{i_0, \dots, i_t} p^{m_t + l_t}), 0 \right) + (i_n p^{m_n}, 0) + \alpha_{i_0, \dots, i_n} (0, 1) \mid i_t \in [p^{l_t}] \right\} \\ &= \left\{ \left(\sum_{t=0}^n i_t p^{m_t} + \sum_{t=0}^{n-1} \alpha_{i_0, \dots, i_t} p^{m_t + l_t}, \alpha_{i_0, \dots, i_n} \right) \mid i_t \in [p^{l_t}] \right\}. \end{aligned}$$

By setting $r = n$, we obtain

$$S_0 = \left\{ \left(\sum_{t=0}^n i_t p^{m_t} + \sum_{t \in [n+1] \setminus \{r\}} \alpha_{i_0, \dots, i_t} p^{m_t + l_t}, \alpha_{i_0, \dots, i_r} \right) \mid i_t \in [p^{l_t}] \right\}.$$

Now, we will prove that C is also in the form of C in pair (a). We have

$$\begin{aligned} C &= D \oplus L_C \\ &= \left\{ \left(\sum_{t=0}^{n-1} (i_t p^{m_t + l_t} + \beta_{i_0, \dots, i_t} p^{m_{t+1}}), 0 \right) + (0, j) \mid i_t \in [p^{m_{t+1} - m_t - l_t}], j \in [q] \right\} \\ &= \left\{ \left(\sum_{t=0}^{n-1} (i_t p^{m_t + l_t} + \beta_{i_0, \dots, i_t} p^{m_{t+1}}), 0 \right) + (\beta_{i_0, \dots, i_{n-1}, j} p^k, j) \mid i_t \in [p^{m_{t+1} - m_t - l_t}], j \in [q] \right\}. \end{aligned}$$

Recall that we set $r = n$. Let $j = i_r$ and set a new value for m_{n+1} to be equal to k . Then

$$C = \left\{ \left(\sum_{t \in [n+1] \setminus \{r\}} i_t p^{m_t + l_t} + \sum_{t=0}^n \beta_{i_0, \dots, i_t} p^{m_{t+1}}, i_r \right) \mid i_t \in [p^{m_{t+1} - m_t - l_t}], \text{ for } t \neq r, i_r \in [q] \right\}.$$

The pair (S_0, C) here is in the form of pair (a).

Subcase 2.2. $|L_C| = p^s$. In this case, $G/L_C \cong \mathbb{Z}_{p^{k-s}} \times \mathbb{Z}_q$, $|(S_0 + L_C)/L_C| = |S_0| = p^m$, and $(S_0 + L_C)/L_C$ is periodic. From Case 1, $(S_0 + L_C)/L_C$ is either in the form of pair (a) or (b) in Theorem 4.2. We calculate $(S_0 + L_C)/L_C$ in the form of pair (a). The calculation of $(S_0 + L_C)/L_C$ in the form of pair (b) will follow similarly.

From Case 1, there are integers $0 = m_0 < m_1 < \dots < m_r \leq m_{r+1} < \dots < m_{n+1} = k - s$ and $1 \leq l_t \leq m_{t+1} - m_t$ for $t \in [n+1] \setminus \{r\}$ and $0 \leq l_r \leq m_{r+1} - m_r$, such that

$$\begin{aligned} (S_0 + L_C)/L_C &= \left\{ \left(\sum_{t=0}^n i_t p^{m_t} + \sum_{t \in [n+1] \setminus \{r\}} \alpha_{i_0, \dots, i_t} p^{m_t + l_t}, \alpha_{i_0, \dots, i_r} \right) + L_C \mid i_t \in [p^{l_t}] \right\}, \text{ and} \\ (D + L_C)/L_C &= \left\{ \left(\sum_{t \in [n] \setminus \{r\}} i_t p^{m_t + l_t} + \sum_{t=0}^{n-1} \beta_{i_0, \dots, i_t} p^{m_{t+1}}, i_r \right) + L_C \mid i_t \in [p^{m_{t+1} - m_t - l_t}] \right. \\ &\quad \left. \text{for } t \neq r, i_r \in [q] \right\}. \end{aligned}$$

We also have $m_n + l_n = m_{n+1} = k - s$ from Case 1. Since $G/L_C \cong \mathbb{Z}_{p^{k-s}} \times \mathbb{Z}_q$, we have $(p^{k-s}, 0) + L_C = L_C$ and

$$(S_0 + L_C)/L_C = \left\{ \left(\sum_{t=0}^{n-1} i_t p^{m_t} + \sum_{t \in [n] \setminus \{r\}} \alpha_{i_0, \dots, i_t} p^{m_t + l_t}, \alpha_{i_0, \dots, i_r} \right) + (i_n p^{m_n}, 0) + L_C \mid i_t \in [p^{l_t}] \right\}.$$

Since $|L_C| = p^s$, we have $L_C = p^{k-s} \mathbb{Z}_{p^k} \times \{0\} = \{(hp^{k-s}, 0) \mid h \in [p^s]\}$. Therefore,

$$\begin{aligned} S_0 &= \left\{ \left(\sum_{t=0}^{n-1} i_t p^{m_t} + \sum_{t \in [n] \setminus \{r\}} \alpha_{i_0, \dots, i_t} p^{m_t + l_t}, \alpha_{i_0, \dots, i_r} \right) + (i_n p^{m_n} + \alpha_{i_0, \dots, i_n} p^{k-s}, 0) \mid i_t \in [p^{l_t}] \right\} \\ &= \left\{ \left(\sum_{t=0}^n i_t p^{m_t} + \sum_{t \in [n+1] \setminus \{r\}} \alpha_{i_0, \dots, i_t} p^{m_t + l_t}, \alpha_{i_0, \dots, i_r} \right) \mid i_t \in [p^{l_t}] \right\}, \text{ and} \end{aligned}$$

$$\begin{aligned} C &= D \oplus L_C \\ &= \left\{ \left(\sum_{t \in [n] \setminus \{r\}} i_t p^{m_t + l_t} + \sum_{t=0}^{n-1} \beta_{i_0, \dots, i_t} p^{m_{t+1}}, i_r \right) + (hp^{k-s}, 0) \mid i_t \in [p^{m_{t+1} - m_t - l_t}] \text{ for } t \neq r, \right. \\ &\quad \left. i_r \in [q], h \in [p^s] \right\} \\ &= \left\{ \left(\sum_{t \in [n] \setminus \{r\}} i_t p^{m_t + l_t} + \sum_{t=0}^{n-1} \beta_{i_0, \dots, i_t} p^{m_{t+1}}, i_r \right) + (hp^{k-s} + \beta_{i_0, \dots, i_{n-1}, h} p^k, 0) \mid \right. \\ &\quad \left. i_t \in [p^{m_{t+1} - m_t - l_t}] \text{ for } t \neq r, i_r \in [q], h \in [p^s] \right\}. \end{aligned}$$

By setting $i_n = h$ and a new value for m_{n+1} to be equal to k , we have

$$C = \left\{ \left(\sum_{t \in [n+1] \setminus \{r\}} i_t p^{m_t + l_t} + \sum_{t=0}^n \beta_{i_0, \dots, i_t} p^{m_{t+1}}, i_r \right) \mid i_t \in [p^{m_{t+1} - m_t - l_t}] \text{ for } t \neq r, i_r \in [q] \right\}.$$

The pair (S_0, C) here is in the form of pair (a).

Subcase 2.3. $|L_C| = p^{k-m}q$. In this case, $C = L_C = \{(ip^m, j) \mid i \in [p^{k-m}], j \in [q]\} = \langle (p^m, 1) \rangle$ and $G/L_C \cong \mathbb{Z}_{p^m}$. Since $C = L_C$, we can take $D = \{0\}$ and we have $G/L_C = (S_0 + L_C)/L_C$. This means $(S_0 + L_C)/L_C = \{(i, 0) + L_C \mid i \in [p^m]\}$ and hence $S_0 = \{(i, 0) + \alpha_i(p^m, 1) \mid i \in [p^m]\} = \{(i + \alpha_i p^m, \alpha_i) \mid i \in [p^m]\}$. The pair (S_0, C) here is in the form of pair (b).

Subcase 2.4. $|L_C| = p^s q$ for $s < k - m$. In this case, $G/L_C \cong \mathbb{Z}_{p^{k-s}}$, $|(S_0 + L_C)/L_C| = |S_0| = p^m$, and $(S_0 + L_C)/L_C$ is periodic. From the case when S_0 is periodic in Theorem 4.1, we have

$$\begin{aligned} (S_0 + L_C)/L_C &= \left\{ \left(\sum_{t=0}^n (i_t p^{m_t} + \alpha_{i_0, \dots, i_t} p^{m_t + l_t}), 0 \right) + L_C \mid i_t \in [p^{l_t}] \right\}, \text{ and} \\ (D + L_C)/L_C &= \left\{ \left(\sum_{t=0}^{n-1} (i_t p^{m_t + l_t} + \beta_{i_0, \dots, i_t} p^{m_{t+1}}), 0 \right) + L_C \mid i_t \in [p^{m_{t+1} - m_t - l_t}] \right\}. \end{aligned}$$

We also have $m_n + l_n = m_{n+1} = k - s$ from the same case. Since $(p^{k-s}, 0) + L_C = L_C$,

$$(S_0 + L_C)/L_C = \left\{ \left(\sum_{t=0}^{n-1} (i_t p^{m_t} + \alpha_{i_0, \dots, i_t} p^{m_t + l_t}), 0 \right) + (i_n p^{m_n}, 0) + L_C \mid i_t \in [p^{l_t}] \right\}.$$

Since $|L_C| = p^s q$, $L_C = p^{k-s} \mathbb{Z}_{p^k} \times \mathbb{Z}_q = \{(hp^{k-s}, j) \mid h \in [p^s], j \in [q]\} = \langle (p^{k-s}, 1) \rangle$. Therefore,

$$\begin{aligned} S_0 &= \left\{ \left(\sum_{t=0}^{n-1} (i_t p^{m_t} + \alpha_{i_0, \dots, i_t} p^{m_t + l_t}), 0 \right) + (i_n p^{m_n}, 0) + \alpha_{i_0, \dots, i_n} (p^{k-s}, 1) \mid i_t \in [p^{l_t}] \right\} \\ &= \left\{ \left(\sum_{t=0}^n (i_t p^{m_t} + \alpha_{i_0, \dots, i_t} p^{m_t + l_t}), \alpha_{i_0, \dots, i_n} \right) \mid i_t \in [p^{l_t}] \right\} \text{ and} \end{aligned}$$

$$C = D \oplus L_C$$

$$\begin{aligned} &= \left\{ \left(\sum_{t=0}^{n-1} (i_t p^{m_t + l_t} + \beta_{i_0, \dots, i_t} p^{m_{t+1}}), 0 \right) + (hp^{k-s}, j) \mid i_t \in [p^{m_{t+1} - m_t - l_t}], h \in [p^s], j \in [q] \right\} \\ &= \left\{ \left(\sum_{t=0}^{n-1} (i_t p^{m_t + l_t} + \beta_{i_0, \dots, i_t} p^{m_{t+1}}), 0 \right) + (hp^{k-s} + \beta_{i_0, \dots, i_{n-1}, h, j} p^k, j) \mid i_t \in [p^{m_{t+1} - m_t - l_t}], \right. \\ &\quad \left. h \in [p^s], j \in [q] \right\}. \end{aligned}$$

By setting $i_n = h$, $r = n$, and a new value for m_{n+1} to be equal to k , we have

$$C = \left\{ \left(\sum_{t=0}^n i_t p^{m_t + l_t} + \sum_{t=0}^{n-1} \beta_{i_0, \dots, i_t} p^{m_{t+1}} + \sum_{t=r}^n \beta_{i_0, \dots, i_t, j} p^{m_{t+1}}, j \right) \mid i_t \in [p^{m_{t+1} - m_t - l_t}], j \in [q] \right\}.$$

The pair (S_0, C) here is in the form of pair (b).

Case 3. $|S_0| = p^{m-1}q$ ($m > 1$) and S_0 is periodic.

Let $S_0 = L_{S_0} \oplus D$. Then $|L_{S_0}| = q$, p^s ($0 < s \leq m-1$), or $p^s q$ ($0 < s < m-1$). Consider the factorization $G/L_{S_0} = ((D + L_{S_0})/L_{S_0}) \oplus ((C + L_{S_0})/L_{S_0})$ given in Lemma 2.5.

Subcase 3.1. $|L_{S_0}| = q$. In this case, $G/L_{S_0} \cong \mathbb{Z}_{p^k}$, $(D + L_{S_0})/L_{S_0}$ is aperiodic, and $|(D + L_{S_0})/L_{S_0}| = |D| = p^{m-1}$. From Theorem 4.1, there are integers $0 = m_0 < m_1 < \dots < m_n = k$ and $1 \leq l_t \leq m_{t+1} - m_t$ for $t \in [n]$ such that

$$\begin{aligned} (D + L_{S_0})/L_{S_0} &= \left\{ \left(\sum_{t=0}^{n-1} (i_t p^{m_t} + \alpha_{i_0, \dots, i_t} p^{m_t + l_t}), 0 \right) + L_{S_0} \mid i_t \in [p^{l_t}] \right\}, \text{ and} \\ (C + L_{S_0})/L_{S_0} &= \left\{ \left(\sum_{t=0}^{n-1} (i_t p^{m_t + l_t} + \beta_{i_0, \dots, i_t} p^{m_{t+1}}), 0 \right) + L_{S_0} \mid i_t \in [p^{m_{t+1} - m_t - l_t}] \right\}. \end{aligned}$$

Now $L_{S_0} = \{(0, j) \mid j \in [q]\}$. Therefore,

$$\begin{aligned} S_0 &= L_{S_0} \oplus D \\ &= \left\{ \left(\sum_{t=0}^{n-1} (i_t p^{m_t} + \alpha_{i_0, \dots, i_t} p^{m_t + l_t}), 0 \right) + (0, j) \mid i_t \in [p^{l_t}], j \in [q] \right\} \\ &= \left\{ \left(\sum_{t=0}^{n-1} (i_t p^{m_t} + \alpha_{i_0, \dots, i_t} p^{m_t + l_t}), j \right) \mid i_t \in [p^{l_t}], j \in [q] \right\}. \end{aligned}$$

By setting $r = n$, $i_n = j$, and $m_r = m_r + l_r = m_{r+1} = k$, we have

$$S_0 = \left\{ \left(\sum_{t \in [n+1] \setminus \{r\}} i_t p^{m_t} + \sum_{t=0}^n \alpha_{i_0, \dots, i_t} p^{m_t + l_t}, i_r \right) \mid i_t \in [p^{l_t}] \text{ for } t \neq r, i_r \in [q] \right\}.$$

Now,

$$(C + L_{S_0})/L_{S_0} = \left\{ \left(\sum_{t=0}^{n-2} (i_t p^{m_t + l_t} + \beta_{i_0, \dots, i_t} p^{m_{t+1}}) + i_{n-1} p^{m_{n-1} + l_{n-1}}, 0 \right) + L_{S_0} \mid i_t \in [p^{m_{t+1} - m_t - l_t}] \right\}.$$

Therefore,

$$\begin{aligned} C &= \left\{ \left(\sum_{t=0}^{n-2} (i_t p^{m_t + l_t} + \beta_{i_0, \dots, i_t} p^{m_{t+1}}) + i_{n-1} p^{m_{n-1} + l_{n-1}}, 0 \right) + \beta_{i_0, \dots, i_{n-1}}(0, 1) \mid i_t \in [p^{m_{t+1} - m_t - l_t}] \right\} \\ &= \left\{ \left(\sum_{t=0}^{n-1} i_t p^{m_t + l_t} + \sum_{t=0}^{n-2} \beta_{i_0, \dots, i_t} p^{m_{t+1}}, \beta_{i_0, \dots, i_{n-1}} \right) \mid i_t \in [p^{m_{t+1} - m_t - l_t}] \right\}. \end{aligned}$$

Recall that $r = n$, $i_n = j$, and $m_r = m_r + l_r = m_{r+1} = k$. So we have

$$C = \left\{ \left(\sum_{t=0}^n i_t p^{m_t + l_t} + \sum_{t \in [n+1] \setminus \{r-1\}} \beta_{i_0, \dots, i_t} p^{m_{t+1}}, \beta_{i_0, \dots, i_{r-1}} \right) \mid i_t \in [p^{m_{t+1} - m_t - l_t}] \right\}.$$

The pair (S_0, C) here is in the form of pair (c).

Subcase 3.2. $|L_{S_0}| = p^s$ ($0 < s \leq m - 1$). In this case, $G/L_{S_0} \cong \mathbb{Z}_{p^{k-s}} \times \mathbb{Z}_q$, $(D + L_{S_0})/L_{S_0}$ is aperiodic, and $|(D + L_{S_0})/L_{S_0}| = |D| = p^{m-s-1}q$. From the induction hypothesis, there are integers $0 = m_0 < m_1 < \dots < m_r \leq m_{r+1} < \dots < m_n = k - s$ and $1 \leq l_t \leq m_{t+1} - m_t$ for $t \in [n] \setminus \{r\}$ and $0 \leq l_r \leq m_{r+1} - m_r$, such that $(D + L_{S_0})/L_{S_0}$ and $(C + L_{S_0})/L_{S_0}$ are one of the four pairs in Theorem 4.2. Since $|(D + L_{S_0})/L_{S_0}| = p^{m-s-1}q$, they can only be pair (c) or (d).

If $(D + L_{S_0})/L_{S_0}$ and $(C + L_{S_0})/L_{S_0}$ are pair (c), then

$$(D + L_{S_0})/L_{S_0} = \left\{ \left(\sum_{t \in [n] \setminus \{r\}} i_t p^{m_t} + \sum_{t=0}^{n-1} \alpha_{i_0, \dots, i_t} p^{m_t + l_t}, i_r \right) + L_{S_0} \mid i_t \in [p^{l_t}] \text{ for } t \neq r, \right. \\ \left. i_r \in [q] \right\},$$

and

$$(C + L_{S_0})/L_{S_0} = \left\{ \left(\sum_{t=0}^{n-1} i_t p^{m_t + l_t} + \sum_{t \in [n] \setminus \{r-1\}} \beta_{i_0, \dots, i_t} p^{m_{t+1}}, \beta_{i_0, \dots, i_{r-1}} \right) + L_{S_0} \mid \right. \\ \left. i_t \in [p^{m_{t+1} - m_t - l_t}] \right\}.$$

Now, $L_{S_0} = p^{k-s} \mathbb{Z}_{p^k} \times \{0\} = \{(hp^{k-s}, 0) \mid h \in [p^s]\}$. Therefore,

$$\begin{aligned} S_0 &= L_{S_0} \oplus D \\ &= \left\{ \left(\sum_{t \in [n] \setminus \{r\}} i_t p^{m_t} + \sum_{t=0}^{n-1} \alpha_{i_0, \dots, i_t} p^{m_t + l_t}, i_r \right) + (hp^{k-s}, 0) \mid i_t \in [p^{l_t}] \text{ for } t \neq r, i_r \in [q], \right. \\ &\quad \left. h \in [p^s] \right\} \\ &= \left\{ \left(\sum_{t \in [n] \setminus \{r\}} i_t p^{m_t} + \sum_{t=0}^{n-1} \alpha_{i_0, \dots, i_t} p^{m_t + l_t}, i_r \right) + (hp^{k-s} + \alpha_{i_0, \dots, i_{n-1}, h} p^k, 0) \mid i_t \in [p^{l_t}] \right. \\ &\quad \left. \text{for } t \neq r, i_r \in [q], h \in [p^s] \right\} \\ &= \left\{ \left(\sum_{t \in [n+1] \setminus \{r\}} i_t p^{m_t} + \sum_{t=0}^n \alpha_{i_0, \dots, i_t} p^{m_t + l_t}, i_r \right) \mid i_t \in [p^{l_t}] \text{ for } t \neq r, i_r \in [q] \right\}, \end{aligned}$$

where $m_n = k - s$, $l_n = s$, and $i_n = h$. Since $G/L_{S_0} \cong \mathbb{Z}_{p^{k-s}} \times \mathbb{Z}_q$, $(p^{k-s}, 0) + L_{S_0} = L_{S_0}$, so we have

$$(C + L_{S_0})/L_{S_0} = \left\{ \left(\sum_{t=0}^{n-2} i_t p^{m_t + l_t} + \sum_{t \in [n-1] \setminus \{r-1\}} \beta_{i_0, \dots, i_t} p^{m_{t+1}}, \beta_{i_0, \dots, i_{r-1}} \right) + (i_{n-1} p^{m_{n-1} + l_{n-1}}, 0) \right. \\ \left. + L_{S_0} \mid i_t \in [p^{m_{t+1} - m_t - l_t}] \right\}.$$

Therefore,

$$\begin{aligned} C &= \left\{ \left(\sum_{t=0}^{n-2} i_t p^{m_t + l_t} + \sum_{t \in [n-1] \setminus \{r-1\}} \beta_{i_0, \dots, i_t} p^{m_{t+1}}, \beta_{i_0, \dots, i_{r-1}} \right) + (i_{n-1} p^{m_{n-1} + l_{n-1}} \right. \\ &\quad \left. + \beta_{i_0, \dots, i_{n-1}} p^{k-s}, 0) \mid i_t \in [p^{m_{t+1} - m_t - l_t}] \right\} \\ &= \left\{ \left(\sum_{t=0}^{n-1} i_t p^{m_t + l_t} + \sum_{t \in [n] \setminus \{r-1\}} \beta_{i_0, \dots, i_t} p^{m_{t+1}}, \beta_{i_0, \dots, i_{r-1}} \right) \mid i_t \in [p^{m_{t+1} - m_t - l_t}] \right\}. \end{aligned}$$

Note that in C we can also set the sum from $t = 0$ to n because $m_n + l_n = m_{n+1} = k$. The pair (S_0, C) here is also in the form of pair (c).

If $(D + L_{S_0})/L_{S_0}$ and $(C + L_{S_0})/L_{S_0}$ are pair (d), the same algebraic operation we did for pair (c) is still applicable and we will get (S_0, C) to be another pair in the form of pair (d).

Subcase 3.3. $|L_{S_0}| = p^s q$ ($0 < s < m - 1$). In this case, $G/L_{S_0} \cong \mathbb{Z}_{p^{k-s}}$, $(D + L_{S_0})/L_{S_0}$ is aperiodic, and $|(D + L_{S_0})/L_{S_0}| = |D| = p^{m-s-1}$. From Theorem 4.1, there are integers $0 = m_0 < m_1 < \dots < m_n = \log_p(|G/L_{S_0}|) = k - s$ and $1 \leq l_t \leq m_{t+1} - m_t$ for $t \in [n]$ such that

$$(D + L_{S_0})/L_{S_0} = \left\{ \left(\sum_{t=0}^{n-1} (i_t p^{m_t} + \alpha_{i_0, \dots, i_t} p^{m_t + l_t}), 0 \right) + L_{S_0} \mid i_t \in [p^{l_t}] \right\}, \text{ and}$$

$$(C + L_{S_0})/L_{S_0} = \left\{ \left(\sum_{t=0}^{n-1} (i_t p^{m_t + l_t} + \beta_{i_0, \dots, i_t} p^{m_{t+1}}), 0 \right) + L_{S_0} \mid i_t \in [p^{m_{t+1} - m_t - l_t}] \right\}.$$

Now $L_{S_0} = \{ (hp^{k-s}, j) \mid h \in [p^s], j \in [q] \} = \langle (p^{k-s}, 1) \rangle$. Therefore,

$$\begin{aligned} S_0 &= L_{S_0} \oplus D \\ &= \left\{ \left(\sum_{t=0}^{n-1} (i_t p^{m_t} + \alpha_{i_0, \dots, i_t} p^{m_t + l_t}), 0 \right) + (hp^{k-s}, j) \mid i_t \in [p^{l_t}], h \in [p^s], j \in [q] \right\} \\ &= \left\{ \left(\sum_{t=0}^{n-1} (i_t p^{m_t} + \alpha_{i_0, \dots, i_t} p^{m_t + l_t}), 0 \right) + (hp^{k-s} + \alpha_{i_0, \dots, i_{n-1}, h, j} p^k, j) \mid i_t \in [p^{l_t}], \right. \\ &\quad \left. h \in [p^s], j \in [q] \right\}. \end{aligned}$$

By setting $i_n = h$, $r = n$, $m_n = k - s$, and $l_r = s$, we have

$$S_0 = \left\{ \left(\sum_{t=0}^n i_t p^{m_t} + \sum_{t=0}^{r-1} \alpha_{i_0, \dots, i_t} p^{m_t + l_t} + \sum_{t=r}^n \alpha_{i_0, \dots, i_t, j} p^{m_t + l_t}, j \right) \mid i_t \in [p^{l_t}], j \in [q] \right\}.$$

Since $G/L_{S_0} \cong \mathbb{Z}_{p^{k-s}}$, $(p^{k-s}, 0) + L_{S_0} = L_{S_0}$, so we have

$$\begin{aligned} (C + L_{S_0})/L_{S_0} &= \left\{ \left(\sum_{t=0}^{n-2} (i_t p^{m_t + l_t} + \beta_{i_0, \dots, i_t} p^{m_{t+1}}), 0 \right) + (i_{n-1} p^{m_{n-1} + l_{n-1}}, 0) + L_{S_0} \mid i_t \in [p^{m_{t+1} - m_t - l_t}] \right\}. \end{aligned}$$

Therefore,

$$\begin{aligned} C &= \left\{ \left(\sum_{t=0}^{n-2} (i_t p^{m_t + l_t} + \beta_{i_0, \dots, i_t} p^{m_{t+1}}), 0 \right) + (i_{n-1} p^{m_{n-1} + l_{n-1}}, 0) + \beta_{i_0, \dots, i_{n-1}} (p^{k-s}, 1) \mid \right. \\ &\quad \left. i_t \in [p^{m_{t+1} - m_t - l_t}] \right\} \\ &= \left\{ \left(\sum_{t=0}^{n-1} (i_t p^{m_t + l_t} + \beta_{i_0, \dots, i_t} p^{m_{t+1}}), \beta_{i_0, \dots, i_{r-1}} \right) \mid i_t \in [p^{m_{t+1} - m_t - l_t}] \right\}. \end{aligned}$$

Note that in C we can also set the sum from $t = 0$ to n because $m_n + l_n = m_{n+1} = k$. The pair (S_0, C) here is in the form of pair (d).

Case 4. $|S_0| = p^{m-1}q$ and S_0 is aperiodic. In this case, C is periodic and $|C| = p^{k-m+1}$. Let $C = L_C \oplus D$. Then $|L_C| = p^s$, for $0 < s \leq k - m + 1$. Consider the factorization $G/L_C = ((S_0 + L_C)/L_C) \oplus ((D + L_C)/L_C)$ given in Lemma 2.5.

Subcase 4.1. $|L_C| = p^{k-m+1}$. In this case, $C = L_C = \{(ip^{m-1}, 0) \mid i \in [p^{k-m+1}]\} = \langle (p^{m-1}, 0) \rangle$ and $G/L_C \cong \mathbb{Z}_{p^{m-1}} \times \mathbb{Z}_q$. Since $C = L_C$, we can take $D = \{0\}$ and we have $G/L_C = (S_0 + L_C)/L_C$. This means $(S_0 + L_C)/L_C = \{(i, j) + L_C \mid i \in [p^{m-1}], j \in [q]\}$ and hence $S_0 = \{(i, j) + \alpha_{ij}(p^{m-1}, 0) \mid i \in [p^{m-1}], j \in [q]\} = \{(i + \alpha_{ij}p^{m-1}, j) \mid i \in [p^{m-1}], j \in [q]\}$. The pair (S_0, C) here is in the form of pair (d).

Subcase 4.2. $|L_C| = p^s$ for $s < k - m + 1$. In this case, $G/L_C \cong \mathbb{Z}_{p^{k-s}} \times \mathbb{Z}_q$, $|(S_0 + L_C)/L_C| = |S_0| = p^{m-1}q$, and $(S_0 + L_C)/L_C$ is periodic. From Case 3, $(S_0 + L_C)/L_C$ is either in the form of pair (c) or (d) in Theorem 4.2. We will give the calculation for $(S_0 + L_C)/L_C$ in the form of pair (c). The calculation when $(S_0 + L_C)/L_C$ is in the form of pair (d) will follow similarly.

From Cases 3.1 and 3.2,

$$\begin{aligned} (S_0 + L_C)/L_C &= \left\{ \left(\sum_{t \in [n+1] \setminus \{r\}} i_t p^{m_t} + \sum_{t=0}^n \alpha_{i_0, \dots, i_t} p^{m_t + l_t}, i_r \right) + L_C \mid i_t \in [p^{l_t}] \text{ for } t \neq r, i_r \in [q] \right\}, \\ (D + L_C)/L_C &= \left\{ \left(\sum_{t=0}^{n-1} i_t p^{m_t + l_t} + \sum_{t \in [n] \setminus \{r-1\}} \beta_{i_0, \dots, i_t} p^{m_{t+1}}, \beta_{i_0, \dots, i_{r-1}} \right) + L_C \mid i_t \in [p^{m_{t+1} - m_t - l_t}] \right\}. \end{aligned}$$

We also have $m_n + l_n = m_{n+1} = k - s$ from Subcase 3.1 and 3.2. Since $G/L_C \cong \mathbb{Z}_{p^{k-s}} \times \mathbb{Z}_q$, $(p^{k-s}, 0) + L_C = L_C$, and

$$(S_0 + L_C)/L_C = \left\{ \left(\sum_{t \in [n] \setminus \{r\}} i_t p^{m_t} + \sum_{t=0}^{n-1} \alpha_{i_0, \dots, i_t} p^{m_t + l_t}, i_r \right) + (i_n p^{m_n}, 0) + L_C \mid i_t \in [p^{l_t}] \text{ for } t \neq r, i_r \in [q] \right\}.$$

Since $|L_C| = p^s$, $L_C = p^{k-s} \mathbb{Z}_{p^k} \times \{0\} = \{(hp^{k-s}, 0) \mid h \in [p^s]\}$. Therefore,

$$\begin{aligned} S_0 &= \left\{ \left(\sum_{t \in [n] \setminus \{r\}} i_t p^{m_t} + \sum_{t=0}^{n-1} \alpha_{i_0, \dots, i_t} p^{m_t + l_t}, i_r \right) + (i_n p^{m_n} + \alpha_{i_0, \dots, i_n} p^{k-s}, 0) \mid i_t \in [p^{l_t}] \text{ for } t \neq r, i_r \in [q] \right\} \\ &= \left\{ \left(\sum_{t \in [n+1] \setminus \{r\}} i_t p^{m_t} + \sum_{t=0}^n \alpha_{i_0, \dots, i_t} p^{m_t + l_t}, i_r \right) \mid i_t \in [p^{l_t}] \text{ for } t \neq r, i_r \in [q] \right\}, \end{aligned}$$

and

$$\begin{aligned}
C &= D \oplus L_C \\
&= \left\{ \left(\sum_{t=0}^{n-1} i_t p^{m_t+l_t} + \sum_{t \in [n] \setminus \{r-1\}} \beta_{i_0, \dots, i_t} p^{m_{t+1}}, \beta_{i_0, \dots, i_{r-1}} \right) + (hp^{k-s}, 0) \mid i_t \in [p^{m_{t+1}-m_t-l_t}], \right. \\
&\quad \left. h \in [p^s] \right\} \\
&= \left\{ \left(\sum_{t=0}^{n-1} i_t p^{m_t+l_t} + \sum_{t \in [n] \setminus \{r-1\}} \beta_{i_0, \dots, i_t} p^{m_{t+1}}, \beta_{i_0, \dots, i_{r-1}} \right) + (hp^{k-s} + \beta_{i_0, \dots, i_{n-1}, h} p^k, 0) \mid \right. \\
&\quad \left. i_t \in [p^{m_{t+1}-m_t-l_t}], h \in [p^s] \right\}.
\end{aligned}$$

By setting $i_n = h$ and a new value for m_{n+1} to be equal to k , we have

$$C = \left\{ \left(\sum_{t=0}^n i_t p^{m_t+l_t} + \sum_{t \in [n+1] \setminus \{r-1\}} \beta_{i_0, \dots, i_t} p^{m_{t+1}}, \beta_{i_0, \dots, i_{r-1}} \right) \mid i_t \in [p^{m_{t+1}-m_t-l_t}] \right\}.$$

The pair (S_0, C) here is in the form of pair (c). $\square$

4.4 Proof of Theorem 4.3

Observation 4.5. For $1 \leq s \leq k$, there are three subgroups of $G = \mathbb{Z}_{2^k} \times \mathbb{Z}_2$ of order 2^s , namely $\langle (2^{k-s}, 0) \rangle$, $\langle (2^{k-s}, 1) \rangle$, and $\langle (2^{k-s+1}, 0), (0, 1) \rangle$. Their quotient groups are $G / \langle (2^{k-s}, 0) \rangle \cong \mathbb{Z}_{2^{k-s}} \times \mathbb{Z}_2$, $G / \langle (2^{k-s}, 1) \rangle \cong \mathbb{Z}_{2^{k-s+1}}$, and $G / \langle (2^{k-s+1}, 0), (0, 1) \rangle \cong \mathbb{Z}_{2^{k-s+1}}$.

Proof of Theorem 4.3. Suppose (S_0, C) is one of the pairs in Theorem 4.3. Then $G = S_0 \oplus C$ by Lemma 4.4. Suppose $\text{Cay}(G, S)$ has a perfect code C . Then $|S_0| = 2^m$ for some positive integer m and $G = \mathbb{Z}_{2^k} \times \mathbb{Z}_2 = S_0 \oplus C$. We will prove (S_0, C) is one of the six pairs in Theorem 4.3 using induction on m .

The group $G = \mathbb{Z}_{2^k} \times \mathbb{Z}_2$ is not a cyclic group, and since S_0 generates G , there are at least two non-zero elements in S_0 . So we start the basis step at $m = 2$. First, we give an outline of the method to be used. When S_0 is periodic, we will prove the basis step at $m = 2$, and for $m > 2$, we will use induction by using Theorem 4.1 and when S_0 is aperiodic and $|S_0| < 2^m$. When S_0 is aperiodic, we will use Theorem 4.1 and the result when S_0 is periodic and $|S_0| = 2^m$.

Suppose $|S_0| = 4$ and S_0 is periodic. From Lemma 2.8, $|L_{S_0}| = 2$, so $L_{S_0} = \langle (2^{k-1}, 0) \rangle$, $\langle (2^{k-1}, 1) \rangle$, or $\langle (0, 1) \rangle$. Consider the factorization $G/L_{S_0} = ((D + L_{S_0})/L_{S_0}) \oplus ((C + L_{S_0})/L_{S_0})$ given by Lemma 2.5. If $L_{S_0} = \langle (2^{k-1}, 0) \rangle$, then $G/L_{S_0} \cong \mathbb{Z}_{2^{k-1}} \times \mathbb{Z}_2$ and $|(D + L_{S_0})/L_{S_0}| = 2$. Since $\mathbb{Z}_{2^{k-1}} \times \mathbb{Z}_2$ is not a cyclic group, we know that no set S_0 of order 2 generates $\mathbb{Z}_{2^{k-1}} \times \mathbb{Z}_2$.

If $L_{S_0} = \langle (2^{k-1}, 1) \rangle$, then $G/L_{S_0} \cong \mathbb{Z}_{2^k}$ and $(D + L_{S_0})/L_{S_0}$ is an aperiodic set of order 2. From Lemma 2.8 and Theorem 4.1, $(C + L_{S_0})/L_{S_0} = \{(2i, 0) + L_{S_0} \mid i \in [2^{k-1}]\}$ and $(D + L_{S_0})/L_{S_0} = \{(i + 2\alpha_i, 0) + L_{S_0} \mid i \in [2]\}$. This means $S_0 = D \oplus L_{S_0} =$

$\{(i + 2\alpha_i, 0) + j(2^{k-1}, 1) \mid i, j \in [2]\} = \{(i + 2\alpha_i + j2^{k-1}, j) \mid i, j \in [2]\}$. By setting $i_0 = i$ and $i_1 = j$, we have $S_0 = \{(i_0 + 2\alpha_{i_0} + i_1 2^{k-1}, i_1) \mid i_0, i_1 \in [2]\}$. Note that $C = \{(2i, 0) + \beta_i(2^{k-1}, 1) \mid i \in [2^{k-1}]\} = \{(2i_0 + \beta_{i_0} 2^{k-1}, \beta_{i_0}) \mid i_0 \in [2^{k-1}]\}$. The pair (S_0, C) here is pair (e) in Theorem 4.3 where $r = 1$, $m_0 = 0$, $m_1 = k$, $m_2 = k$, $l_0 = 1$, and $l_1 = 0$.

Similarly, if $L_{S_0} = \langle(0, 1)\rangle$, then $G/L_{S_0} \cong \mathbb{Z}_{2^k}$, and hence $(C + L_{S_0})/L_{S_0} = \{(2i, 0) + L_{S_0} \mid i \in [2^{k-1}]\}$ and $(D + L_{S_0})/L_{S_0} = \{(i + 2\alpha_i, 0) + L_{S_0} \mid i \in [2]\}$. This means $S_0 = D \oplus L_{S_0} = \{(i_0 + 2\alpha_{i_0}, i_1) \mid i_0, i_1 \in [2]\}$ and $C = \{(2i_0, \beta_{i_0}) \mid i_0 \in [2^{k-1}]\}$. The pair (S_0, C) here is pair (d) in Theorem 4.3 where $r = 1$, $m_0 = 0$, $m_1 = k$, $m_2 = k$, $l_0 = 1$, and $l_1 = 0$.

Suppose Theorem 4.3 is true for $|S_0| < 2^m$. We will prove that (S_0, C) is one of the six pairs in Theorem 4.3 for $|S_0| = 2^m$.

Case 1. S_0 is periodic.

Let $S_0 = L_{S_0} \oplus D$ and $|L_{S_0}| = 2^s$. Then $0 < s < m$ because S_0 generates G , and $L_{S_0} = \langle(2^{k-s}, 0)\rangle$, $\langle(2^{k-s}, 1)\rangle$, or $\langle(2^{k-s+1}, 0), (0, 1)\rangle$. Consider the factorization $G/L_{S_0} = ((D + L_{S_0})/L_{S_0}) \oplus ((C + L_{S_0})/L_{S_0})$ given by Lemma 2.5.

Subcase 1.1. $L_{S_0} = \langle(2^{k-s}, 0)\rangle$. In this case, $G/L_{S_0} \cong \mathbb{Z}_{2^{k-s}} \times \mathbb{Z}_2$ and $|(D + L_{S_0})/L_{S_0}| = |D| = 2^{m-s}$. From the induction hypothesis, there are integers $0 = m_0 < m_1 < \dots < m_r \leq m_{r+1} < \dots < m_n = k - s$ and $1 \leq l_t \leq m_{t+1} - m_t$ for $t \in [n] \setminus \{r\}$ and $0 \leq l_r \leq m_{r+1} - m_r$, such that $(D + L_{S_0})/L_{S_0}$ and $(C + L_{S_0})/L_{S_0}$ are one of the six pairs in Theorem 4.3.

We will give the calculation for $(D + L_{S_0})/L_{S_0}$ and $(C + L_{S_0})/L_{S_0}$ in the form of pair (c). The calculation for the other pairs will follow similarly. Since $(D + L_{S_0})/L_{S_0}$ and $(C + L_{S_0})/L_{S_0}$ are pair (c), we have

$$\begin{aligned} (D + L_{S_0})/L_{S_0} &= \left\{ \left(\sum_{t=0}^{n-1} (i_t 2^{m_t} + \alpha_{i_0, \dots, i_t} 2^{m_t + l_t}), \gamma_{i_0, \dots, i_r} \right) + L_{S_0} \mid i_t \in [2^{l_t}] \right\}, \text{ and} \\ (C + L_{S_0})/L_{S_0} &= \left\{ \left(\sum_{t=0}^{n-1} i_t 2^{m_t + l_t} + \sum_{t=0}^{r-1} \beta_{i_0, \dots, i_t} 2^{m_{t+1}} + \sum_{t=r}^{n-1} \beta_{i_0, \dots, i_t, j} 2^{m_{t+1}}, j \right) + L_{S_0} \right. \\ &\quad \left. \mid i_t \in [2^{m_{t+1} - m_t - l_t}], j \in [2] \right\}. \end{aligned}$$

Now, $L_{S_0} = 2^{k-s} \mathbb{Z}_{2^k} \times \{0\} = \{(h 2^{k-s}, 0) \mid h \in [2^s]\}$. Therefore,

$$\begin{aligned} S_0 &= L_{S_0} \oplus D \\ &= \left\{ \left(\sum_{t=0}^{n-1} (i_t 2^{m_t} + \alpha_{i_0, \dots, i_t} 2^{m_t + l_t}), \gamma_{i_0, \dots, i_r} \right) + (h 2^{k-s}, 0) \mid i_t \in [2^{l_t}], h \in [2^s] \right\} \\ &= \left\{ \left(\sum_{t=0}^{n-1} (i_t 2^{m_t} + \alpha_{i_0, \dots, i_t} 2^{m_t + l_t}), \gamma_{i_0, \dots, i_r} \right) + (h 2^{k-s} + \alpha_{i_0, \dots, i_{n-1}, h} 2^k, 0) \mid i_t \in [2^{l_t}], \right. \\ &\quad \left. h \in [2^s] \right\} \\ &= \left\{ \left(\sum_{t=0}^n (i_t 2^{m_t} + \alpha_{i_0, \dots, i_t} 2^{m_t + l_t}), \gamma_{i_0, \dots, i_r} \right) \mid i_t \in [2^{l_t}], h \in [2^s] \right\} \end{aligned}$$

where $m_n = k - s$, $l_n = s$, and $i_n = h$. Moreover, $m_{n+1} = k > m_n$ and $1 \leq l_n = m_{n+1} - m_n$. Since $G/L_{S_0} \cong \mathbb{Z}_{2^{k-s}} \times \mathbb{Z}_2$, we have $(2^{k-s}, 0) + L_{S_0} = L_{S_0}$ and

$$(C + L_{S_0})/L_{S_0} = \left\{ \left(\sum_{t=0}^{n-2} i_t 2^{m_t+l_t} + \sum_{t=0}^{r-1} \beta_{i_0, \dots, i_t} 2^{m_{t+1}} + \sum_{t=r}^{n-2} \beta_{i_0, \dots, i_t, j} 2^{m_{t+1}}, j \right) + (i_{n-1} 2^{m_{n-1}+l_{n-1}}, 0) + L_{S_0} \mid i_t \in [2^{m_{t+1}-m_t-l_t}], j \in [2] \right\}.$$

Therefore,

$$\begin{aligned} C &= \left\{ \left(\sum_{t=0}^{n-2} i_t 2^{m_t+l_t} + \sum_{t=0}^{r-1} \beta_{i_0, \dots, i_t} 2^{m_{t+1}} + \sum_{t=r}^{n-2} \beta_{i_0, \dots, i_t, j} 2^{m_{t+1}}, j \right) + (i_{n-1} 2^{m_{n-1}+l_{n-1}} + \beta_{i_0, \dots, i_{n-1}, j} 2^{k-s}, 0) \mid i_t \in [2^{m_{t+1}-m_t-l_t}], j \in [2] \right\} \\ &= \left\{ \left(\sum_{t=0}^{n-1} i_t 2^{m_t+l_t} + \sum_{t=0}^{r-1} \beta_{i_0, \dots, i_t} 2^{m_{t+1}} + \sum_{t=r}^{n-1} \beta_{i_0, \dots, i_t, j} 2^{m_{t+1}}, j \right) \mid i_t \in [2^{m_{t+1}-m_t-l_t}], j \in [2] \right\}. \end{aligned}$$

Note that in C we can also set the sum from $t = 0$ to n because $m_n + l_n = m_{n+1} = k$. The pair (S_0, C) here is also in the form of pair (c).

Subcase 1.2. $L_{S_0} = \langle (2^{k-s}, 1) \rangle$. In this case, $G/L_{S_0} \cong \mathbb{Z}_{2^{k-s+1}}$ and $|(D + L_{S_0})/L_{S_0}| = |D| = 2^{m-s}$.

From Theorem 4.1, there are integers $0 = m_0 < m_1 < \dots < m_n = k - s + 1$ and $1 \leq l_t \leq m_{t+1} - m_t$ for $t \in [n]$ such that

$$\begin{aligned} (D + L_{S_0})/L_{S_0} &= \left\{ \left(\sum_{t=0}^{n-1} (i_t 2^{m_t} + \alpha_{i_0, \dots, i_t} 2^{m_t+l_t}), 0 \right) + L_{S_0} \mid i_t \in [2^{l_t}] \right\}, \text{ and} \\ (C + L_{S_0})/L_{S_0} &= \left\{ \left(\sum_{t=0}^{n-1} (i_t 2^{m_t+l_t} + \beta_{i_0, \dots, i_t} 2^{m_{t+1}}), 0 \right) + L_{S_0} \mid i_t \in [2^{m_{t+1}-m_t-l_t}] \right\}. \end{aligned}$$

Now $L_{S_0} = \langle (2^{k-s}, 1) \rangle = \{h(2^{k-s}, 1) \mid h \in [2^s]\}$. Therefore,

$$\begin{aligned} S_0 &= L_{S_0} \oplus D \\ &= \left\{ \left(\sum_{t=0}^{n-1} (i_t 2^{m_t} + \alpha_{i_0, \dots, i_t} 2^{m_t+l_t}), 0 \right) + (h 2^{k-s}, h) \mid i_t \in [2^{l_t}], h \in [2^s] \right\} \\ &= \left\{ \left(\sum_{t=0}^{n-1} (i_t 2^{m_t} + \alpha_{i_0, \dots, i_t} 2^{m_t+l_t}), 0 \right) + (h 2^{k-s} + \alpha_{i_0, \dots, i_{n-1}, h} 2^k, h) \mid i_t \in [2^{l_t}], h \in [2^s] \right\}. \end{aligned}$$

By setting $i_n = h$, $r = n$, $m_n = k - s + 1$, and $l_n = s - 1$, and $m_{n+1} = m_n + l_n = k$, we have

$$S_0 = \left\{ \left(\sum_{t \in [n+1] \setminus \{r\}} i_t 2^{m_t} + \sum_{t=0}^n \alpha_{i_0, \dots, i_t} 2^{m_t+l_t} + i_r 2^{m_r-1}, i_r \right) \mid i_t \in [2^{l_t}] \text{ for } t \neq r, i_r \in [2^{l_r+1}] \right\}.$$

Since $G/L_{S_0} \cong \mathbb{Z}_{2^{k-s+1}}$, we have $(2^{m_n}, 0) + L_{S_0} = L_{S_0}$ and hence

$$(C + L_{S_0})/L_{S_0} = \left\{ \left(\sum_{t=0}^{n-2} (i_t 2^{m_t+l_t} + \beta_{i_0, \dots, i_t} 2^{m_{t+1}}), 0 \right) + (i_{n-1} 2^{m_{n-1}+l_{n-1}}, 0) + L_{S_0} \mid i_t \in [2^{m_{t+1}-m_t-l_t}] \right\}.$$

Therefore,

$$\begin{aligned} C &= \left\{ \left(\sum_{t=0}^{n-2} (i_t 2^{m_t+l_t} + \beta_{i_0, \dots, i_t} 2^{m_{t+1}}), 0 \right) + (i_{n-1} 2^{m_{n-1}+l_{n-1}}, 0) + \beta_{i_0, \dots, i_{n-1}} (2^{k-s}, 1) \mid i_t \in [2^{m_{t+1}-m_t-l_t}] \right\} \\ &= \left\{ \left(\sum_{t=0}^{n-1} i_t 2^{m_t+l_t} + \sum_{t \in [n] \setminus \{r-1\}} \beta_{i_0, \dots, i_t} 2^{m_{t+1}} + \beta_{i_0, \dots, i_{r-1}} 2^{m_{r-1}}, \beta_{i_0, \dots, i_{r-1}} \right) \mid i_t \in [2^{m_{t+1}-m_t-l_t}] \right\}. \end{aligned}$$

Note that in C we can also set the sum from $t = 0$ to n because $m_n + l_n = m_{n+1} = k$. The pair (S_0, C) here is in the form of pair (e).

Subcase 1.3. $L_{S_0} = \langle (2^{k-s+1}, 0), (0, 1) \rangle$ for $1 < s < m$. In this case, $G/L_{S_0} \cong \mathbb{Z}_{2^{k-s+1}}$ and $|(D + L_{S_0})/L_{S_0}| = |D| = 2^{m-s}$. From Theorem 4.1, there are integers $0 = m_0 < m_1 < \dots < m_n = k - s + 1$ and $1 \leq l_t \leq m_{t+1} - m_t$ for $t \in [n]$ such that

$$\begin{aligned} (D + L_{S_0})/L_{S_0} &= \left\{ \left(\sum_{t=0}^{n-1} (i_t 2^{m_t} + \alpha_{i_0, \dots, i_t} 2^{m_t+l_t}), 0 \right) + L_{S_0} \mid i_t \in [2^{l_t}] \right\}, \text{ and} \\ (C + L_{S_0})/L_{S_0} &= \left\{ \left(\sum_{t=0}^{n-1} (i_t 2^{m_t+l_t} + \beta_{i_0, \dots, i_t} 2^{m_{t+1}}), 0 \right) + L_{S_0} \mid i_t \in [2^{m_{t+1}-m_t-l_t}] \right\}. \end{aligned}$$

Now $L_{S_0} = \{(h 2^{k-s+1}, j) \mid h \in [2^{s-1}], j \in [2]\}$. Therefore,

$$\begin{aligned} S_0 &= L_{S_0} \oplus D \\ &= \left\{ \left(\sum_{t=0}^{n-1} (i_t 2^{m_t} + \alpha_{i_0, \dots, i_t} 2^{m_t+l_t}), 0 \right) + (h 2^{k-s+1}, j) \mid i_t \in [2^{l_t}], h \in [2^{s-1}], j \in [2] \right\} \\ &= \left\{ \left(\sum_{t=0}^{n-1} (i_t 2^{m_t} + \alpha_{i_0, \dots, i_t} 2^{m_t+l_t}), 0 \right) + (h 2^{k-s+1} + \alpha_{i_0, \dots, i_{n-1}, h, j} 2^k, j) \mid i_t \in [2^{l_t}], h \in [2^{s-1}], j \in [2] \right\}. \end{aligned}$$

By setting $i_n = h$, $r = n$, $m_n = k - s + 1$, and $l_n = s - 1$, and $m_{n+1} = m_n + l_n = k$, we have

$$S_0 = \left\{ \left(\sum_{t=0}^n i_t 2^{m_t} + \sum_{t=0}^{r-1} \alpha_{i_0, \dots, i_t} 2^{m_t+l_t} + \sum_{t=r}^n \alpha_{i_0, \dots, i_t, j} 2^{m_t+l_t}, j \right) \mid i_t \in [2^{l_t}], j \in [2] \right\}.$$

Since $G/L_{S_0} \cong \mathbb{Z}_{2^{k-s+1}}$, we have $(2^{m_n}, 0) + L_{S_0} = L_{S_0}$ and hence

$$(C + L_{S_0})/L_{S_0} = \left\{ \left(\sum_{t=0}^{n-2} (i_t 2^{m_t+l_t} + \beta_{i_0, \dots, i_t} 2^{m_{t+1}}), 0 \right) + (i_{n-1} 2^{m_{n-1}+l_{n-1}}, 0) + L_{S_0} \mid i_t \in [2^{m_{t+1}-m_t-l_t}] \right\}.$$

Therefore,

$$\begin{aligned} C &= \left\{ \left(\sum_{t=0}^{n-2} (i_t 2^{m_t+l_t} + \beta_{i_0, \dots, i_t} 2^{m_{t+1}}), 0 \right) + (i_{n-1} 2^{m_{n-1}+l_{n-1}}, 0) + \beta_{i_0, \dots, i_{n-1}} (2^{k-s+1}, 0) \right. \\ &\quad \left. + \gamma_{i_0, \dots, i_{n-1}} (0, 1) \mid i_t \in [2^{m_{t+1}-m_t-l_t}] \right\} \\ &= \left\{ \left(\sum_{t=0}^{n-1} (i_t 2^{m_t+l_t} + \beta_{i_0, \dots, i_t} 2^{m_{t+1}}), \gamma_{i_0, \dots, i_{n-1}} \right) \mid i_t \in [2^{m_{t+1}-m_t-l_t}] \right\}. \end{aligned}$$

Note that in C we can also set the sum from $t = 0$ to n because $m_n + l_n = m_{n+1} = k$. The pair (S_0, C) here is in the form of pair (f).

Subcase 1.4. $L_{S_0} = \langle (2^{k-s+1}, 0), (0, 1) \rangle$ for $s = 1$. In this case, $L_{S_0} = \langle (0, 1) \rangle$ and $G/L_{S_0} \cong \mathbb{Z}_{2^k}$, $(D + L_{S_0})/L_{S_0}$ is aperiodic, and $|(D + L_{S_0})/L_{S_0}| = |D| = 2^{m-1}$. From Theorem 4.1, there are integers $0 = m_0 < m_1 < \dots < m_n = k$ and $1 \leq l_t \leq m_{t+1} - m_t$ for $t \in [n]$ such that

$$\begin{aligned} (D + L_{S_0})/L_{S_0} &= \left\{ \left(\sum_{t=0}^{n-1} (i_t 2^{m_t} + \alpha_{i_0, \dots, i_t} 2^{m_t+l_t}), 0 \right) + L_{S_0} \mid i_t \in [2^{l_t}] \right\}, \text{ and} \\ (C + L_{S_0})/L_{S_0} &= \left\{ \left(\sum_{t=0}^{n-1} (i_t 2^{m_t+l_t} + \beta_{i_0, \dots, i_t} 2^{m_{t+1}}), 0 \right) + L_{S_0} \mid i_t \in [2^{m_{t+1}-m_t-l_t}] \right\}. \end{aligned}$$

Now $L_{S_0} = \{(0, j) \mid j \in [2]\}$. Therefore,

$$\begin{aligned} S_0 &= L_{S_0} \oplus D \\ &= \left\{ \left(\sum_{t=0}^{n-1} (i_t 2^{m_t} + \alpha_{i_0, \dots, i_t} 2^{m_t+l_t}), 0 \right) + (0, j) \mid i_t \in [2^{l_t}], j \in [2] \right\} \\ &= \left\{ \left(\sum_{t=0}^{n-1} (i_t 2^{m_t} + \alpha_{i_0, \dots, i_t} 2^{m_t+l_t}), j \right) \mid i_t \in [2^{l_t}], j \in [2] \right\}. \end{aligned}$$

By setting $r = n$, $i_n = j$, and $m_r = m_r + l_r = m_{r+1} = k$, we have

$$S_0 = \left\{ \left(\sum_{t \in [n+1] \setminus \{r\}} i_t 2^{m_t} + \sum_{t=0}^n \alpha_{i_0, \dots, i_t} 2^{m_t+l_t}, i_r \right) \mid i_t \in [2^{l_t}] \text{ for } t \neq r, i_r \in [2] \right\}.$$

Now,

$$(C + L_{S_0})/L_{S_0} = \left\{ \left(\sum_{t=0}^{n-2} (i_t 2^{m_t+l_t} + \beta_{i_0, \dots, i_t} 2^{m_{t+1}}) + i_{n-1} 2^{m_{n-1}+l_{n-1}}, 0 \right) + L_{S_0} \mid i_t \in [2^{m_{t+1}-m_t-l_t}] \right\}.$$

Therefore,

$$\begin{aligned} C &= \left\{ \left(\sum_{t=0}^{n-2} (i_t 2^{m_t+l_t} + \beta_{i_0, \dots, i_t} 2^{m_{t+1}}) + i_{n-1} 2^{m_{n-1}+l_{n-1}}, 0 \right) + \beta_{i_0, \dots, i_{n-1}}(0, 1) \mid i_t \in [2^{m_{t+1}-m_t-l_t}] \right\} \\ &= \left\{ \left(\sum_{t=0}^{n-1} i_t 2^{m_t+l_t} + \sum_{t=0}^{n-2} \beta_{i_0, \dots, i_t} 2^{m_{t+1}}, \beta_{i_0, \dots, i_{n-1}} \right) \mid i_t \in [2^{m_{t+1}-m_t-l_t}] \right\}. \end{aligned}$$

Recall that $r = n$, $i_n = j$, and $m_r = m_r + l_r = m_{r+1} = k$. So we have

$$C = \left\{ \left(\sum_{t=0}^n i_t 2^{m_t+l_t} + \sum_{t \in [n+1] \setminus \{r-1\}} \beta_{i_0, \dots, i_t} 2^{m_{t+1}}, \beta_{i_0, \dots, i_{r-1}} \right) \mid i_t \in [2^{m_{t+1}-m_t-l_t}] \right\}.$$

The pair (S_0, C) here is in the form of pair (d).

Case 2. S_0 is aperiodic.

In this case, C is periodic and $|C| = 2^{k-m+1}$. Let $C = L_C \oplus D$, $|L_C| = 2^s$ for $0 < s \leq k - m + 1$. Consider the factorization $G/L_C = ((S_0 + L_C)/L_C) \oplus ((D + L_C)/L_C)$ given by Lemma 2.5.

Subcase 2.1. $|L_C| = 2^{k-m+1}$. In this case $C = L_C$, and by taking $D = \{0\}$ we have $G/L_C = (S_0 + L_C)/L_C$.

If $L_C = \langle (2^{m-1}, 0) \rangle = \{(i 2^{m-1}, 0) \mid i \in [2^{k-m+1}]\}$, then $(S_0 + L_C)/L_C = \{(i, j) + L_C \mid i \in [2^{m-1}], j \in [2]\}$. This means $S_0 = \{(i, j) + \alpha_{ij}(2^{m-1}, 0) \mid i \in [2^{m-1}], j \in [2]\} = \{(i + \alpha_{ij} 2^{m-1}, j) \mid i \in [2^{m-1}], j \in [2]\}$. The pair (S_0, C) here is in the form of pair (f) where $n = 1$ and $r = 0$.

If $L_C = \langle (2^{m-1}, 1) \rangle = \{(i 2^{m-1}, i) \mid i \in [2^{k-m+1}]\}$, then $(S_0 + L_C)/L_C = \{(i, 0) + L_C \mid i \in [2^m]\}$. This means $S_0 = \{(i, 0) + \alpha_i(2^{m-1}, 1) \mid i \in [2^m]\} = \{(i + \alpha_i 2^{m-1}, \alpha_i) \mid i \in [2^m]\}$. The pair (S_0, C) here is in the form of pair (b) where $n = 1$, $r = 0$, and $l_0 = m$.

If $L_C = \langle (2^m, 0), (0, 1) \rangle$ for $m < k$, then $L_C = \{(i 2^m, j) \mid i \in [2^{k-m}], j \in [2]\}$ and $(S_0 + L_C)/L_C = \{(i, 0) + L_C \mid i \in [2^m]\}$. This means $S_0 = \{(i, 0) + \alpha_i(2^m, 0) + \gamma_i(0, 1) \mid i \in [2^m]\} = \{(i + \alpha_i 2^m, \gamma_i) \mid i \in [2^m]\}$. The pair (S_0, C) here is in the form of pair (c) where $n = 1$ and $r = 0$.

If $L_C = \langle (2^m, 0), (0, 1) \rangle$ for $m = k$, then $L_C = \langle (0, 1) \rangle = \{(0, i) \mid i \in [2]\}$ and $(S_0 + L_C)/L_C = \{(i, 0) + L_C \mid i \in [2^k]\}$. This means $S_0 = \{(i, 0) + \alpha_i(0, 1) \mid i \in [2^k]\} = \{(i, \alpha_i) \mid i \in [2^k]\}$. The pair (S_0, C) here is in the form of pair (a) where $n = 1$ and $r = 0$.

Subcase 2.2. $L_C = \langle (2^{k-s}, 0) \rangle$ for $s < k - m + 1$. In this case, $G/L_C \cong \mathbb{Z}_{2^{k-s}} \times \mathbb{Z}_2$, $|(S_0 + L_C)/L_C| = |S_0| = 2^m$, and $(S_0 + L_C)/L_C$ is periodic. From Case 1, $(S_0 + L_C)/L_C$

is one of the six pairs in Theorem 4.3. We will calculate $(S_0 + L_C)/L_C$ and $(D + L_C)/L_C$ in the form of pair (a). The calculation for the other forms will follow similarly.

From Case 1, there are integers $0 = m_0 < m_1 < \cdots < m_r \leq m_{r+1} < \cdots < m_{n+1} = k-s$ and $1 \leq l_t \leq m_{t+1} - m_t$ for $t \in [n+1] \setminus \{r\}$ and $0 \leq l_r \leq m_{r+1} - m_r$, such that

$$(S_0 + L_C)/L_C = \left\{ \left(\sum_{t=0}^n i_t 2^{m_t} + \sum_{t \in [n+1] \setminus \{r\}} \alpha_{i_0, \dots, i_t} 2^{m_t + l_t}, \alpha_{i_0, \dots, i_r} \right) + L_C \mid i_t \in [2^{l_t}] \right\}, \text{ and}$$

$$(D + L_C)/L_C = \left\{ \left(\sum_{t \in [n] \setminus \{r\}} i_t 2^{m_t + l_t} + \sum_{t=0}^{n-1} \beta_{i_0, \dots, i_t} 2^{m_{t+1}}, i_r \right) + L_C \mid i_t \in [2^{m_{t+1} - m_t - l_t}] \right. \\ \left. \text{for } t \neq r, i_r \in [2] \right\}.$$

We also have $m_n + l_n = m_{n+1} = k-s$ from Case 1. Since $G/L_C \cong \mathbb{Z}_{2^{k-s}} \times \mathbb{Z}_2$, $(2^{m_n + l_n}, 0) + L_C = L_C$, and

$$(S_0 + L_C)/L_C = \left\{ \left(\sum_{t=0}^{n-1} i_t 2^{m_t} + \sum_{t \in [n] \setminus \{r\}} \alpha_{i_0, \dots, i_t} 2^{m_t + l_t}, \alpha_{i_0, \dots, i_r} \right) + (i_n 2^{m_n}, 0) + L_C \mid i_t \in [2^{l_t}] \right\}.$$

Since $L_C = \langle (2^{k-s}, 0) \rangle = \{ (h 2^{k-s}, 0) \mid h \in [2^s] \}$, we have

$$S_0 = \left\{ \left(\sum_{t=0}^{n-1} i_t 2^{m_t} + \sum_{t \in [n] \setminus \{r\}} \alpha_{i_0, \dots, i_t} 2^{m_t + l_t}, \alpha_{i_0, \dots, i_r} \right) + (i_n 2^{m_n} + \alpha_{i_0, \dots, i_n} 2^{k-s}, 0) \mid i_t \in [2^{l_t}] \right\} \\ = \left\{ \left(\sum_{t=0}^n i_t 2^{m_t} + \sum_{t \in [n+1] \setminus \{r\}} \alpha_{i_0, \dots, i_t} 2^{m_t + l_t}, \alpha_{i_0, \dots, i_r} \right) \mid i_t \in [2^{l_t}] \right\}$$

and

$$C = D \oplus L_C \\ = \left\{ \left(\sum_{t \in [n] \setminus \{r\}} i_t 2^{m_t + l_t} + \sum_{t=0}^{n-1} \beta_{i_0, \dots, i_t} 2^{m_{t+1}}, i_r \right) + (h 2^{k-s}, 0) \mid i_t \in [2^{m_{t+1} - m_t - l_t}] \text{ for } t \neq r, \right. \\ \left. i_r \in [2], h \in [2^s] \right\} \\ = \left\{ \left(\sum_{t \in [n] \setminus \{r\}} i_t 2^{m_t + l_t} + \sum_{t=0}^{n-1} \beta_{i_0, \dots, i_t} 2^{m_{t+1}}, i_r \right) + (h 2^{k-s} + \beta_{i_0, \dots, i_{n-1}, h} 2^k, 0) \mid \right. \\ \left. i_t \in [2^{m_{t+1} - m_t - l_t}] \text{ for } t \neq r, i_r \in [2], h \in [2^s] \right\}.$$

By setting $i_n = h$ and a new value for m_{n+1} to be equal to k , we have

$$C = \left\{ \left(\sum_{t \in [n+1] \setminus \{r\}} i_t 2^{m_t + l_t} + \sum_{t=0}^n \beta_{i_0, \dots, i_t} 2^{m_{t+1}}, i_r \right) \mid i_t \in [2^{m_{t+1} - m_t - l_t}] \text{ for } t \neq r, i_r \in [2] \right\}.$$

The pair (S_0, C) here is in the form of pair (a).

Subcase 2.3. $L_C = \langle (2^{k-s}, 1) \rangle$ for $s < k - m + 1$. In this case, $G/L_C \cong \mathbb{Z}_{2^{k-s+1}}$, $|(S_0 + L_C)/L_C| = |S_0| = 2^m$, and $(S_0 + L_C)/L_C$ is periodic. From the case when S_0 is periodic in Theorem 4.1, we have

$$(S_0 + L_C)/L_C = \left\{ \left(\sum_{t=0}^n (i_t 2^{m_t} + \alpha_{i_0, \dots, i_t} 2^{m_t+l_t}), 0 \right) + L_C \mid i_t \in [2^{l_t}] \right\}, \text{ and}$$

$$(D + L_C)/L_C = \left\{ \left(\sum_{t=0}^{n-1} (i_t 2^{m_t+l_t} + \beta_{i_0, \dots, i_t} 2^{m_{t+1}}), 0 \right) + L_C \mid i_t \in [2^{m_{t+1}-m_t-l_t}] \right\}.$$

We also have $m_n + l_n = m_{n+1} = k - s + 1$ from the same case. Since $(2^{m_n+l_n}, 0) + L_C = L_C$,

$$(S_0 + L_C)/L_C = \left\{ \left(\sum_{t=0}^n i_t 2^{m_t} + \sum_{t=0}^{n-1} \alpha_{i_0, \dots, i_t} 2^{m_t+l_t}, 0 \right) + L_C \mid i_t \in [2^{l_t}] \right\}.$$

Since $L_C = \langle (2^{k-s}, 1) \rangle$,

$$S_0 = \left\{ \left(\sum_{t=0}^n i_t 2^{m_t} + \sum_{t=0}^{n-1} \alpha_{i_0, \dots, i_t} 2^{m_t+l_t}, 0 \right) + \alpha_{i_0, \dots, i_n} (2^{k-s}, 1) \mid i_t \in [2^{l_t}] \right\}.$$

By setting $r = n$, we have

$$S_0 = \left\{ \left(\sum_{t=0}^n i_t 2^{m_t} + \sum_{t \in [n+1] \setminus \{r\}} \alpha_{i_0, \dots, i_t} 2^{m_t+l_t} + \alpha_{i_0, \dots, i_r} 2^{m_r+l_r-1}, \alpha_{i_0, \dots, i_r} \right) \mid i_t \in [2^{l_t}] \right\}.$$

Now $L_C = \{(h 2^{k-s}, h) \mid h \in [2^s]\}$. Therefore,

$$\begin{aligned} C &= D \oplus L_C \\ &= \left\{ \left(\sum_{t=0}^{n-1} (i_t 2^{m_t+l_t} + \beta_{i_0, \dots, i_t} 2^{m_{t+1}}), 0 \right) + (h 2^{k-s}, h) \mid i_t \in [2^{m_{t+1}-m_t-l_t}], h \in [2^s] \right\} \\ &= \left\{ \left(\sum_{t=0}^{n-1} (i_t 2^{m_t+l_t} + \beta_{i_0, \dots, i_t} 2^{m_{t+1}}), 0 \right) + (h 2^{m_r+l_r-1} + \beta_{i_0, \dots, i_{n-1}, h} 2^k, h) \mid \right. \\ &\quad \left. i_t \in [2^{m_{t+1}-m_t-l_t}], h \in [2^{m_{r+1}-m_r-l_r+1}] \right\}. \end{aligned}$$

By setting $i_r = h$, and a new value for m_{n+1} to be equal to k , we have

$$C = \left\{ \left(\sum_{t \in [n+1] \setminus \{r\}} i_t 2^{m_t+l_t} + \sum_{t=0}^n \beta_{i_0, \dots, i_t} 2^{m_{t+1}} + i_r 2^{m_r+l_r-1}, i_r \right) \mid i_t \in [2^{m_{t+1}-m_t-l_t}] \right.$$

for $t \neq r, i_r \in [2^{m_{r+1}-m_r-l_r+1}] \left. \right\}.$

The pair (S_0, C) here is in the form of pair (b).

Subcase 2.4. $L_C = \langle (2^{k-s+1}, 0), (0, 1) \rangle$ for $1 < s < k - m + 1$. In this case, $G/L_C \cong \mathbb{Z}_{2^{k-s+1}}$ and $(S_0 + L_C)/L_C$ is periodic. From the case when S_0 is periodic in Theorem 4.1, we have

$$(S_0 + L_C)/L_C = \left\{ \left(\sum_{t=0}^n (i_t 2^{m_t} + \alpha_{i_0, \dots, i_t} 2^{m_t+l_t}), 0 \right) + L_C \mid i_t \in [2^{l_t}] \right\}, \text{ and}$$

$$(D + L_C)/L_C = \left\{ \left(\sum_{t=0}^{n-1} (i_t 2^{m_t+l_t} + \beta_{i_0, \dots, i_t} 2^{m_{t+1}}), 0 \right) + L_C \mid i_t \in [2^{m_{t+1}-m_t-l_t}] \right\}.$$

We also have $m_n + l_n = m_{n+1} = k - s + 1$ from the same case. Since $(2^{m_n+l_n}, 0) + L_C = L_C$,

$$(S_0 + L_C)/L_C = \left\{ \left(\sum_{t=0}^n i_t 2^{m_t} + \sum_{t=0}^{n-1} \alpha_{i_0, \dots, i_t} 2^{m_t+l_t}, 0 \right) + L_C \mid i_t \in [2^{l_t}] \right\}.$$

Since $L_C = \langle (2^{k-s+1}, 0), (0, 1) \rangle$,

$$\begin{aligned} S_0 &= \left\{ \left(\sum_{t=0}^n i_t 2^{m_t} + \sum_{t=0}^{n-1} \alpha_{i_0, \dots, i_t} 2^{m_t+l_t}, 0 \right) + \alpha_{i_0, \dots, i_n} (2^{k-s+1}, 0) + \gamma_{i_0, \dots, i_n} (0, 1) \mid i_t \in [2^{l_t}] \right\} \\ &= \left\{ \left(\sum_{t=0}^n (i_t 2^{m_t} + \alpha_{i_0, \dots, i_t} 2^{m_t+l_t}), \gamma_{i_0, \dots, i_n} \right) \mid i_t \in [2^{l_t}] \right\}. \end{aligned}$$

Now $L_C = \{(h 2^{k-s+1}, j) \mid h \in [2^{s-1}], j \in [2]\}$. Therefore,

$$\begin{aligned} C &= D \oplus L_C \\ &= \left\{ \left(\sum_{t=0}^{n-1} (i_t 2^{m_t+l_t} + \beta_{i_0, \dots, i_t} 2^{m_{t+1}}), 0 \right) + (h 2^{k-s+1}, j) \mid i_t \in [2^{m_{t+1}-m_t-l_t}], \right. \\ &\quad \left. h \in [2^{s-1}], j \in [2] \right\} \\ &= \left\{ \left(\sum_{t=0}^{n-1} (i_t 2^{m_t+l_t} + \beta_{i_0, \dots, i_t} 2^{m_{t+1}}), 0 \right) + (h 2^{m_r+l_r} + \beta_{i_0, \dots, i_{n-1}, h, j} 2^k, j) \mid i_t \in [2^{m_{t+1}-m_t-l_t}], \right. \\ &\quad \left. h \in [2^{s-1}], j \in [2] \right\}. \end{aligned}$$

By setting $r = n$, $i_n = h$, and a new value for m_{n+1} to be equal to k , we have

$$C = \left\{ \left(\sum_{t=0}^n i_t 2^{m_t+l_t} + \sum_{t=0}^{r-1} \beta_{i_0, \dots, i_t} 2^{m_{t+1}} + \sum_{t=r}^n \beta_{i_0, \dots, i_t, j} 2^{m_{t+1}}, j \right) \mid i_t \in [2^{m_{t+1}-m_t-l_t}], \right. \\ \left. j \in [2] \right\}.$$

The pair (S_0, C) here is in the form of pair (c).

Subcase 2.5. $L_C = \langle (2^{k-s+1}, 0), (0, 1) \rangle$ for $s = 1$. In this case $L_C = \langle (0, 1) \rangle$, $G/L_C \cong \mathbb{Z}_{2^k}$, and $(S_0 + L_C)/L_C$ is periodic. From the case when S_0 is periodic in Theorem 4.1, we have

$$(S_0 + L_C)/L_C = \left\{ \left(\sum_{t=0}^n (i_t 2^{m_t} + \alpha_{i_0, \dots, i_t} 2^{m_t+l_t}), 0 \right) + L_C \mid i_t \in [2^{l_t}] \right\}, \text{ and}$$

$$(D + L_C)/L_C = \left\{ \left(\sum_{t=0}^{n-1} (i_t 2^{m_t+l_t} + \beta_{i_0, \dots, i_t} 2^{m_{t+1}}), 0 \right) + L_C \mid i_t \in [2^{m_{t+1}-m_t-l_t}] \right\}.$$

From the proof of Theorem 4.1, we also have $m_n + l_n = m_{n+1} = k$. Since $(2^{m_n+l_n}, 0) + L_C = L_C$,

$$(S_0 + L_C)/L_C = \left\{ \left(\sum_{t=0}^{n-1} (i_t 2^{m_t} + \alpha_{i_0, \dots, i_t} 2^{m_t+l_t}), 0 \right) + (i_n 2^{m_n}, 0) + L_C \mid i_t \in [2^{l_t}] \right\}.$$

Since $L_C = \langle (0, 1) \rangle = \{(0, j) \mid j \in [2]\}$,

$$\begin{aligned} S_0 &= \left\{ \left(\sum_{t=0}^{n-1} (i_t 2^{m_t} + \alpha_{i_0, \dots, i_t} 2^{m_t+l_t}), 0 \right) + (i_n 2^{m_n}, 0) + \alpha_{i_0, \dots, i_n} (0, 1) \mid i_t \in [2^{l_t}] \right\} \\ &= \left\{ \left(\sum_{t=0}^n i_t 2^{m_t} + \sum_{t=0}^{n-1} \alpha_{i_0, \dots, i_t} 2^{m_t+l_t}, \alpha_{i_0, \dots, i_n} \right) \mid i_t \in [2^{l_t}] \right\}. \end{aligned}$$

By setting $r = n$, we get

$$S_0 = \left\{ \left(\sum_{t=0}^n i_t 2^{m_t} + \sum_{t \in [n+1] \setminus \{r\}} \alpha_{i_0, \dots, i_t} 2^{m_t+l_t}, \alpha_{i_0, \dots, i_r} \right) \mid i_t \in [2^{l_t}] \right\}.$$

Now, we will prove that C is also in the form of C in pair (a). We have

$$\begin{aligned} C &= D \oplus L_C \\ &= \left\{ \left(\sum_{t=0}^{n-1} (i_t 2^{m_t+l_t} + \beta_{i_0, \dots, i_t} 2^{m_{t+1}}), 0 \right) + (0, j) \mid i_t \in [2^{m_{t+1}-m_t-l_t}], j \in [2] \right\} \\ &= \left\{ \left(\sum_{t=0}^{n-1} (i_t 2^{m_t+l_t} + \beta_{i_0, \dots, i_t} 2^{m_{t+1}}), 0 \right) + (\beta_{i_0, \dots, i_{n-1}, j} 2^k, j) \mid i_t \in [2^{m_{t+1}-m_t-l_t}], j \in [2] \right\}. \end{aligned}$$

Recall that we set $r = n$. Let $j = i_r$ and set a new value for m_{n+1} to be equal to k . Then we have

$$C = \left\{ \left(\sum_{t \in [n+1] \setminus \{r\}} i_t 2^{m_t+l_t} + \sum_{t=0}^n \beta_{i_0, \dots, i_t} 2^{m_{t+1}}, i_r \right) \mid i_t \in [2^{m_{t+1}-m_t-l_t}], \text{ for } t \neq r, i_r \in [2] \right\}.$$

The pair (S_0, C) here is in the form of pair (a). □

5 Perfect codes in Cayley graphs of $\mathbb{Z}_{p^k} \times \mathbb{Z}_2^l$

In this section, we characterize perfect codes in Cayley graphs $\text{Cay}(G, S)$ of Hajós groups of the form $G = \mathbb{Z}_{p^k} \times \mathbb{Z}_2^l$. These Hajós groups are $\mathbb{Z}_{p^k}$, $\mathbb{Z}_{p^k} \times \mathbb{Z}_2$, $\mathbb{Z}_{p^3} \times \mathbb{Z}_2^2$, $\mathbb{Z}_{p^2} \times \mathbb{Z}_2^3$, $\mathbb{Z}_p \times \mathbb{Z}_2^4$ and all their subgroups. The first two groups, $\mathbb{Z}_{p^k}$, $\mathbb{Z}_{p^k} \times \mathbb{Z}_2$, and their subgroups are covered in the previous section. We are left to consider $\mathbb{Z}_{p^3} \times \mathbb{Z}_2^2$, $\mathbb{Z}_{p^2} \times \mathbb{Z}_2^3$, $\mathbb{Z}_p \times \mathbb{Z}_2^4$ and their subgroups.

The calculation required to prove $G = S_0 \oplus C$ for given (S_0, C) in this section is similar to the calculation in the proof of Theorem 3.6. So from this point onward, to prove that $\text{Cay}(G, S)$ admits a perfect code, we will provide the perfect code C but omit the calculation leading to $G = S_0 \oplus C$.

Theorem 5.1. *Let $G = \mathbb{Z}_{p^2}$ for a prime p and let S be a proper subset of $G \setminus \{0\}$ that generates G . Let $0 \in C \subseteq G$. Then C is a perfect code in $\text{Cay}(G, S)$ if and only if $S_0 = \{i + \alpha_i p \mid i \in [p]\}$ and $C = \{ip \mid i \in [p]\}$.*

Proof. Suppose $\text{Cay}(G, S)$ admits a perfect code C . Since $|G| = p^2$ and S is a proper subset of $G \setminus \{0\}$, we have $|S_0| = p$. From Lemma 2.8, C is a subgroup of order p , so $C = L_C = p\mathbb{Z}_{p^2}$. From the factorization of G/L_C given in Lemma 2.5, $(S_0 + L_C)/L_C = G/L_C = \mathbb{Z}_{p^2}/p\mathbb{Z}_{p^2} = \{i + L_C \mid i \in [p]\}$, and hence $S_0 = \{i + \alpha_i p \mid i \in [p]\}$, $\alpha_i \in \mathbb{Z}$. Since $0 \in S_0$, we have $\alpha_0 = 0$. $\square$

Lemma 5.2. *Let $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ and let S be a proper subset of $G \setminus \{0\}$ that generates G . Then $\text{Cay}(G, S)$ does not admit a perfect code.*

Proof. Suppose $\text{Cay}(G, S)$ admits a perfect code C . From Lemma 2.8, $|S_0| \geq 3$. Since $G = S_0 \oplus C$, $|S_0|$ divides $|G| = 4$. This means $S_0 = G$, contradicting the assumption that S is a proper subset of $G \setminus \{0\}$. $\square$

Theorem 5.3. *Let $G = \mathbb{Z}_{p^3}$ for a prime p and let S be a proper subset of $G \setminus \{0\}$ that generates G . Let $0 \in C \subseteq G$. Then C is a perfect code in $\text{Cay}(G, S)$ if and only if (S_0, C) is one of the pairs in Table 5.1.*

Table 5.1: Perfect codes in Cayley graphs of $\mathbb{Z}_{p^3}$

Row	S_0	C
1	$\{i + \alpha_i p \mid i \in [p]\}$	$\{ip \mid i \in [p^2]\}$
2	$\{i + \alpha_i p + jp^2 \mid i, j \in [p]\}$	$\{ip + \alpha_i p^2 \mid i \in [p]\}$
3	$\{i + \alpha_i p^2 \mid i \in [p^2]\}$	$\{ip^2 \mid i \in [p]\}$

Proof. Suppose $\text{Cay}(G, S)$ admits a perfect code C . Then $G = S_0 \oplus C$ and so $|S_0| \cdot |C| = |G| = p^3$. Since S is a proper subset of $G \setminus \{0\}$, $\text{Cay}(G, S)$ is not a complete graph. Hence $|C| \geq 2$ and therefore $2 \leq |S_0|, |C| < p^3$.

If $|S_0| = p$, then from Lemma 2.8, S_0 is aperiodic and C is a subgroup of G with order p^2 , so $C = L_C = p\mathbb{Z}_{p^3}$. From the factorization of G/L_C given in Lemma 2.5,

$(S_0 + L_C)/L_C = G/L_C = \mathbb{Z}_{p^3}/p\mathbb{Z}_{p^3} = \{i + L_C \mid i \in [p]\}$, and hence $S_0 = \{i + \alpha_i p \mid i \in [p]\}$, $\alpha_i \in \mathbb{Z}$. This gives row 1 in Table 5.1.

If $|S_0| = p^2$ and S_0 is periodic, then $|L_{S_0}| = p$ by Lemma 2.8, so $L_{S_0} = p^2\mathbb{Z}_{p^3}$. Let $S_0 = D \oplus L_{S_0}$ for some subset D . From the factorization of G/L_{S_0} given in Lemma 2.5, we have $G/L_{S_0} = ((D + L_{S_0})/L_{S_0}) \oplus ((C + L_{S_0})/L_{S_0})$. Note that $(D + L_{S_0})/L_{S_0}$ is aperiodic and $G/L_{S_0} \cong \mathbb{Z}_{p^2}$. From Theorem 5.1, we have $(D + L_{S_0})/L_{S_0} = \{i + \alpha_i p + L_{S_0} \mid i \in [p]\}$ and $(C + L_{S_0})/L_{S_0} = \{ip + L_{S_0} \mid i \in [p]\}$. Therefore, $S_0 = D \oplus L_{S_0} = \{i + \alpha_i p + jp^2 \mid i, j \in [p]\}$ and $C = \{ip + \alpha_i p^2 \mid i \in [p]\}$. This gives row 2 in Table 5.1.

If $|S_0| = p^2$ and S_0 is aperiodic, then $|C| = p$ and C is periodic. Hence $C = L_C = p^2\mathbb{Z}_{p^3} = \{ip^2 \mid i \in [p]\}$. From the factorization of G/L_C given in Lemma 2.5, $(S_0 + L_C)/L_C = G/L_C = \mathbb{Z}_{p^3}/p^2\mathbb{Z}_{p^3} = \{i + L_C \mid i \in [p^2]\}$, and hence $S_0 = \{i + \alpha_i p^2 \mid i \in [p^2]\}$, $\alpha_i \in \mathbb{Z}$. This gives row 3 in Table 5.1. $\square$

Theorem 5.4. Let $G = \mathbb{Z}_{p^2} \times \mathbb{Z}_q$ for distinct primes p and q , and let S be a proper subset of $G \setminus \{0\}$ that generates G . Let $0 \in C \subseteq G$. Then C is a perfect code in $\text{Cay}(G, S)$ if and only if (S_0, C) is one of the pairs in Table 5.2.

Table 5.2: Perfect codes in Cayley graphs of $\mathbb{Z}_{p^2} \times \mathbb{Z}_q$

Row	S_0	C
1	$\{(i + \alpha_i p, \alpha_i) \mid i \in [p]\}$	$\{(ip, j) \mid i \in [p], j \in [q]\}$
2	$\{(\alpha_i, i) \mid i \in [q]\}$	$\{(i, 0) \mid i \in [p^2]\}$
3	$\{(i + jp, \alpha_i) \mid i, j \in [p]\}$	$\{(\alpha_i p, i) \mid i \in [q]\}$
4	$\{(i, \alpha_i) \mid i \in [p^2]\}$	$\{(0, i) \mid i \in [q]\}$
5	$\{(i + \alpha_i p, j) \mid i \in [p], j \in [q]\}$	$\{(ip, \alpha_i) \mid i \in [p]\}$
6	$\{(\alpha_i + jp, i) \mid i \in [q], j \in [p]\}$	$\{(i + \alpha_i p, 0) \mid i \in [p]\}$
7	$\{(i + \alpha_{ij} p, j) \mid i \in [p], j \in [q]\}$	$\{(ip, 0) \mid i \in [p]\}$

Proof. Suppose C is a perfect code in $\text{Cay}(G, S)$. Then $G = S_0 \oplus C$ and so $|S_0| \cdot |C| = |G| = p^2q$. Since S is a proper subset of $G \setminus \{0\}$, $\text{Cay}(G, S)$ is not a complete graph. Hence $|C| \geq 2$ and therefore $2 \leq |S_0|, |C| < p^2q$.

Case 1. S_0 is aperiodic.

Since G is Hajós and $G = S_0 \oplus C$, in this case C must be periodic, that is, L_C is nontrivial. Write $C = L_C \oplus D$ for some subset D of G whose existence is ensured by Lemma 2.3. Then $|L_C| \cdot |D| = |C|$ and $2 \leq |L_C| \leq |C|$. If $|S_0|$ is a prime, then $|L_C| = |C|$ by Lemma 2.8 part (c). If $|S_0|$ is not a prime, then $|S_0|$ must be a multiple of two primes (either p^2 or pq). This means that $|C| = p^2q/|S_0|$ is a prime, and hence $|L_C| = |C|$. Set

$$A = (S_0 + L_C)/L_C \quad \text{and} \quad B = (D + L_C)/L_C.$$

Since C is periodic, by Lemma 2.5, we have

$$G/L_C = A \oplus B.$$

Since $|L_C| = |C|$, we have $C = L_C$ as $0 \in C$. This means we can take $D = \{0\}$ and hence $B = L_C/L_C$ and $G/L_C = A$. We use C to get A and then use Lemma 2.6 to obtain S_0 .

All possible values for $(|S_0|, |C|, |L_C|)$ are given in the first column of Table 5.3. In the second column, we list C , A , and S_0 . The last column indicates which row in Table 5.2 the result corresponds to.

Table 5.3: The case when $G = \mathbb{Z}_{p^2} \times \mathbb{Z}_q$ and S_0 is aperiodic

(S_0 , C , L_C)	$C, A, \text{ and } S_0$	Result
(p, pq, pq)	$C = \{(ip, j) \mid i \in [p], j \in [q]\}$ $A = \{(i, 0) + L_C \mid i \in [p]\}$ $S_0 = \{(i + \alpha_i p, \alpha_i) \mid i \in [p]\}$	row 1
(q, p^2, p^2)	$C = \{(i, 0) \mid i \in [p^2]\}$ $A = \{(0, i) + L_C \mid i \in [q]\}$ $S_0 = \{(\alpha_i, i) \mid i \in [q]\}$	row 2
(p^2, q, q)	$C = \{(0, i) \mid i \in [q]\}$ $A = \{(i, 0) + L_C \mid i \in [p^2]\}$ $S_0 = \{(i, \alpha_i) \mid i \in [p^2]\}$	row 4
(pq, p, p)	$C = \{(ip, 0) \mid i \in [p]\}$ $A = \{(i, j) + L_C \mid i \in [p], j \in [q]\}$ $S_0 = \{(i + \alpha_{ij} p, j) \mid i \in [p], j \in [q]\}$	row 7

Case 2. S_0 is periodic.

In this case we have $|L_{S_0}| \geq 2$ and $S_0 = D \oplus L_{S_0}$ for some subset D of G by Lemma 2.3. So $|D| \cdot |L_{S_0}| = |S_0|$ and $1 < |L_{S_0}| < |S_0|$ by Lemma 2.8 part (b). Set

$$X = (D + L_{S_0})/L_{S_0} \quad \text{and} \quad Y = (C + L_{S_0})/L_{S_0}.$$

Since S_0 is periodic, by Lemma 2.5, we have

$$G/L_{S_0} = X \oplus Y$$

and X is aperiodic in G/L_{S_0} . We also know that X contains L_{S_0} and X generates G/L_{S_0} by Lemma 2.7. This factorization enables us to make use of Lemma 2.6 and previously established theorems to obtain the pair (S_0, C) via (X, Y) .

All possible values for $(|S_0|, |C|, |L_{S_0}|)$ are given in the first column of Table 5.4. In the second column we list L_{S_0} , X , Y , S_0 , and C . The third column shows which previous theorem is used to obtain (X, Y) . The last column indicates which row in Table 5.2 the result corresponds to. $\square$

Table 5.4: The case when $G = \mathbb{Z}_{p^2} \times \mathbb{Z}_q$ and S_0 is periodic

(S_0 , C , L_{S_0})	$L_{S_0}, X, Y, S_0, \text{ and } C$	Reference	Result
	$L_{S_0} = \langle (p, 0) \rangle$		
(p^2, q, p)	$X = \{(i, \alpha_i) + L_{S_0} \mid i \in [p]\}$ $Y = \{(0, i) + L_{S_0} \mid i \in [q]\}$ $S_0 = \{(i + jp, \alpha_i) \mid i, j \in [p]\}$	3.5 (row 1)	row 3

	$C = \{(\alpha_i p, i) \mid i \in [q]\}$		
	$L_{S_0} = \langle (0, 1) \rangle$		
	$X = \{(i + \alpha_i p, 0) + L_{S_0} \mid i \in [p]\}$		
(pq, p, q)	$Y = \{(ip, 0) + L_{S_0} \mid i \in [p]\}$	5.1	row 5
	$S_0 = \{(i + \alpha_i p, j) \mid i \in [p], j \in [q]\}$		
	$C = \{(ip, \alpha_i) \mid i \in [p]\}$		
	$L_{S_0} = \langle (p, 0) \rangle$		
	$X = \{(\alpha_i, i) + L_{S_0} \mid i \in [q]\}$		
(pq, p, p)	$Y = \{(i, 0) + L_{S_0} \mid i \in [p]\}$	3.5 (row 2)	row 6
	$S_0 = \{(\alpha_i + jp, i) \mid i \in [q], j \in [p]\}$		
	$C = \{(i + \alpha_i p, 0) \mid i \in [p]\}$		

Lemma 5.5. *Let $G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and let S be a proper subset of $G \setminus \{0\}$ that generates G . Let $0 \in C \subseteq G$. Then C is a perfect code in $\text{Cay}(G, S)$ if and only if $S_0 = \{\alpha_{ij} \mathbf{e}_1 + i\mathbf{e}_2 + j\mathbf{e}_3 \mid i, j \in [2]\}$ and $C = \{i\mathbf{e}_1 \mid i \in [2]\}$.*

Proof. Suppose $\text{Cay}(G, S)$ admits a perfect code C . Then $G = S_0 \oplus C$ and so $|S_0| \cdot |C| = |G| = 8$. From Lemma 2.8, we have $|S_0| > 2$ and hence $(|S_0|, |C|) = (4, 2)$ as $|S_0| \cdot |C| = 8$ and $|C| \geq 2$.

If S_0 is periodic, then $|L_{S_0}| = 2$ and $L_{S_0} = \langle \mathbf{e}_1 \rangle$. Let $S_0 = D \oplus L_{S_0}$ for some subset D . From the factorization of G/L_{S_0} given in Lemma 2.5, we have $G/L_{S_0} = ((D + L_{S_0})/L_{S_0}) \oplus ((C + L_{S_0})/L_{S_0})$. From Lemma 2.7, we know that $(D + L_{S_0})/L_{S_0}$ is a generating set of $G/L_{S_0} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and $L_{S_0} \in (D + L_{S_0})/L_{S_0}$. This case is not possible by Lemma 5.2.

If S_0 is aperiodic, then $|C| = 2$ and C is periodic, and hence $C = L_C = \langle \mathbf{e}_1 \rangle$. From the factorization of G/L_C given in Lemma 2.5, $(S_0 + L_C)/L_C = G/L_C = \{i\mathbf{e}_2 + j\mathbf{e}_3 + L_C \mid i, j \in [2]\}$, and therefore $S_0 = \{i\mathbf{e}_2 + j\mathbf{e}_3 + \alpha_{ij}\mathbf{e}_1 \mid i, j \in [2]\}$, $\alpha_{ij} \in \mathbb{Z}$. $\square$

Remark 5.6. In Lemma 5.5, the generators $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ may be replaced by any basis of $H = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, as in (C2).

Theorem 5.7. *Let $G = \mathbb{Z}_{p^3} \times \mathbb{Z}_2$ for an odd prime p and let S be a proper subset of $G \setminus \{0\}$ that generates G . Let $0 \in C \subseteq G$. Then C is a perfect code in $\text{Cay}(G, S)$ if and only if (S_0, C) is one of the pairs in Table 5.5.*

Table 5.5: Perfect codes in Cayley graphs of $\mathbb{Z}_{p^3} \times \mathbb{Z}_2$

Row	S_0	C
1	$\{(i + \alpha_i p, \alpha_i) \mid i \in [p]\}$	$\{(ip, j) \mid i \in [p^2], j \in [2]\}$
2	$\{(\alpha_i, i) \mid i \in [2]\}$	$\{(i, 0) \mid i \in [p^3]\}$
3	$\{(i + \alpha_i p + jp^2, \alpha_i) \mid i, j \in [p]\}$	$\{(ip + \alpha_{ij} p^2, j) \mid i \in [p], j \in [2]\}$
4	$\{(i + \alpha_i p + jp^2, \alpha_{ij}) \mid i, j \in [p]\}$	$\{(ip + \alpha_i p^2, j) \mid i \in [p], j \in [2]\}$
5	$\{(i + jp + \alpha_{ij} p^2, \alpha_i) \mid i, j \in [p]\}$	$\{(\alpha_i p + jp^2, i) \mid i \in [2], j \in [p]\}$
6	$\{(i + \alpha_i p^2, \alpha_i) \mid i \in [p^2]\}$	$\{(ip^2, j) \mid i \in [p], j \in [2]\}$
7	$\{(i + \alpha_i p, j) \mid i \in [p], j \in [2]\}$	$\{(ip, \alpha_i) \mid i \in [p^2]\}$
8	$\{(\alpha_i + jp^2, i) \mid i \in [2], j \in [p]\}$	$\{(i + \alpha_i p^2, 0) \mid i \in [p^2]\}$

9	$\{(i + \alpha_i p + \alpha_{ij} p^2, j) \mid i \in [p], j \in [2]\}$	$\{(ip + jp^2, \alpha_i) \mid i, j \in [p]\}$
10	$\{(\alpha_i + jp + \alpha_{ij} p^2, i) \mid i \in [2], j \in [p]\}$	$\{(i + \alpha_i p + jp^2, 0) \mid i, j \in [p]\}$
11	$\{(i + \alpha_{ij} p, j) \mid i \in [p], j \in [2]\}$	$\{(ip, 0) \mid i \in [p^2]\}$
12	$\{(i + jp^2, \alpha_i) \mid i \in [p^2], j \in [p]\}$	$\{(\alpha_i p^2, i) \mid i \in [2]\}$
13	$\{(i + jp, \alpha_i) \mid i \in [p], j \in [p^2]\}$	$\{(\alpha_i p, i) \mid i \in [2]\}$
14	$\{(i, \alpha_i) \mid i \in [p^3]\}$	$\{(0, i) \mid i \in [2]\}$
15	$\{(i + \alpha_i p^2, j) \mid i \in [p^2], j \in [2]\}$	$\{(ip^2, \alpha_i) \mid i \in [p]\}$
16	$\{(i + \alpha_{ik} p + jp^2, k) \mid i, j \in [p], k \in [2]\}$	$\{(ip + \alpha_i p^2, 0) \mid i \in [p]\}$
17	$\{(i + \alpha_i p + jp^2, k) \mid i, j \in [p], k \in [2]\}$	$\{(ip + \alpha_i p^2, \alpha_i) \mid i \in [p]\}$
18	$\{(\alpha_i + jp, i) \mid i \in [2], j \in [p^2]\}$	$\{(i + \alpha_i p, 0) \mid i \in [p]\}$
19	$\{(i + \alpha_{ij} p^2, j) \mid i \in [p^2], j \in [2]\}$	$\{(ip^2, 0) \mid i \in [p]\}$

Proof. Suppose C is a perfect code in $\text{Cay}(G, S)$. Then $G = S_0 \oplus C$ and so $|S_0| \cdot |C| = |G| = 2p^3$. Since S is a proper subset of $G \setminus \{0\}$, $\text{Cay}(G, S)$ is not a complete graph. Hence $|C| \geq 2$ and therefore $2 \leq |S_0|, |C| < 2p^3$.

Case 1. S_0 is aperiodic.

Since G is Hajós and $G = S_0 \oplus C$, in this case C must be periodic, that is, L_C is nontrivial. Write $C = L_C \oplus D$ for some subset D of G whose existence is ensured by Lemma 2.3. Then $|L_C| \cdot |D| = |C|$ and $2 \leq |L_C| \leq |C|$. If $|S_0|$ is a prime, then $|L_C| = |C|$ by Lemma 2.8 part (c). If $|S_0|$ is not a prime, then $|L_C|$ could be any divisor of $|C|$. Set

$$A = (S_0 + L_C)/L_C \quad \text{and} \quad B = (D + L_C)/L_C.$$

Since C is periodic, by Lemma 2.5, we have

$$G/L_C = A \oplus B$$

and B is aperiodic in G/L_C . Since G/L_C is isomorphic to a subgroup of G and subgroups of Hajós group are also Hajós groups, A is periodic in G/L_C . We also know that A contains L_C and A generates G/L_C by Lemma 2.7. Furthermore, $|A| = |S_0|$ and $|B| = |D|$. Note that if $|L_C| = |C|$, then $C = L_C$ as $0 \in C$. This means we can take $D = \{0\}$ and hence $B = L_C/L_C$ and $G/L_C = A$. We use C to get A and then use Lemma 2.6 to obtain S_0 . If $|L_C| < |C|$, then this factorization enables us to make use of previously established theorems to obtain the pair (S_0, C) via (A, B) .

The first column of Table 5.6 represent all possible values for $(|S_0|, |C|, |L_C|)$. In the second column, we list C , A , and S_0 when $|L_C| = |C|$ and we list L_C , A , B , S_0 , and C when $|L_C| < |C|$. The third column shows which previous theorem is used to obtain (A, B) when $|L_C| < |C|$. The last column indicates which row in Table 5.5 the result corresponds to.

Table 5.6: The case when $G = \mathbb{Z}_{p^3} \times \mathbb{Z}_2$ and S_0 is aperiodic

(S_0 , C , L_C)	L_C, A, B, S_0 , and C	Reference	Result
	$C = \{(ip, j) \mid i \in [p^2], j \in [2]\}$		
$(p, 2p^2, 2p^2)$	$A = \{(i, 0) + L_C \mid i \in [p]\}$		row 1

	$S_0 = \{(i + \alpha_i p, \alpha_i) \mid i \in [p]\}$		
$(2, p^3, p^3)$	$C = \{(i, 0) \mid i \in [p^3]\}$ $A = \{(0, i) + L_C \mid i \in [2]\}$ $S_0 = \{(\alpha_i, i) \mid i \in [2]\}$		row 2
$(p^2, 2p, 2)$	$L_C = \langle (0, 1) \rangle$ $A = \{(i + \alpha_i p + jp^2, 0) + L_C \mid i, j \in [p]\}$ $B = \{(ip + \alpha_i p^2, 0) + L_C \mid i \in [p]\}$ $S_0 = \{(i + \alpha_i p + jp^2, \alpha_{ij}) \mid i, j \in [p]\}$ $C = \{(ip + \alpha_i p^2, j) \mid i \in [p], j \in [2]\}$	5.3 (row 2)	row 4
$(p^2, 2p, p)$	$L_C = \langle (p^2, 0) \rangle$ $A = \{(i + jp, \alpha_i) + L_C \mid i, j \in [p]\}$ $B = \{(\alpha_i p, i) + L_C \mid i \in [2]\}$ $S_0 = \{(i + jp + \alpha_{ij} p^2, \alpha_i) \mid i, j \in [p]\}$ $C = \{(\alpha_i p + jp^2, i) \mid i \in [2], j \in [p]\}$	5.4 (row 3)	row 5
$(p^2, 2p, 2p)$	$C = \{(ip^2, j) \mid i \in [p], j \in [2]\}$ $A = \{(i, 0) + L_C \mid i \in [p^2]\}$ $S_0 = \{(i + \alpha_i p^2, \alpha_i) \mid i \in [p^2]\}$		row 6
$(2p, p^2, p)$	$L_C = \langle (p^2, 0) \rangle$ $A = \{(i + \alpha_i p, j) + L_C \mid i \in [p], j \in [2]\}$ $B = \{(ip, \alpha_i) + L_C \mid i \in [p]\}$ $S_0 = \{(i + \alpha_i p + \alpha_{ij} p^2, j) \mid i \in [p], j \in [2]\}$ $C = \{(ip + jp^2, \alpha_i) \mid i, j \in [p]\}$	5.4 (row 5)	row 9
$(2p, p^2, p)$	$L_C = \langle (p^2, 0) \rangle$ $A = \{(\alpha_i + jp, i) + L_C \mid i \in [2], j \in [p]\}$ $B = \{(i + \alpha_i p, 0) + L_C \mid i \in [p]\}$ $S_0 = \{(\alpha_i + jp + \alpha_{ij} p^2, i) \mid i \in [2], j \in [p]\}$ $C = \{(i + \alpha_i p + jp^2, 0) \mid i, j \in [p]\}$	5.4 (row 6)	row 10
$(2p, p^2, p^2)$	$C = \{(ip, 0) \mid i \in [p^2]\}$ $A = \{(i, j) + L_C \mid i \in [p], j \in [2]\}$ $S_0 = \{(i + \alpha_{ij} p, j) \mid i \in [p], j \in [2]\}$		row 11
$(p^3, 2, 2)$	$C = \{(0, i) \mid i \in [2]\}$ $A = \{(i, 0) + L_C \mid i \in [p^3]\}$ $S_0 = \{(i, \alpha_i) \mid i \in [p^3]\}$		row 14
$(2p^2, p, p)$	$C = \{(ip^2, 0) \mid i \in [p]\}$ $A = \{(i, j) + L_C \mid i \in [p^2], j \in [2]\}$ $S_0 = \{(i + \alpha_{ij} p^2, j) \mid i \in [p^2], j \in [2]\}$		row 19

Case 2. S_0 is periodic.

In this case we have $|L_{S_0}| \geq 2$ and $S_0 = D \oplus L_{S_0}$ for some subset D of G by Lemma 2.3. So $|D| \cdot |L_{S_0}| = |S_0|$ and $1 < |L_{S_0}| < |S_0|$ by Lemma 2.8 part (b). Set

$$X = (D + L_{S_0})/L_{S_0} \quad \text{and} \quad Y = (C + L_{S_0})/L_{S_0}.$$

Since S_0 is periodic, by Lemma 2.5, we have

$$G/L_{S_0} = X \oplus Y$$

and X is aperiodic in G/L_{S_0} . We also know that X contains L_{S_0} and X generates G/L_{S_0} by Lemma 2.7. This factorization enables us to make use of Lemma 2.6 and previously established theorems to obtain the pair (S_0, C) via (X, Y) .

The first column of Table 5.7 represent all possible values for $(|S_0|, |C|, |L_{S_0}|)$. In the second column we list L_{S_0} , X , Y , S_0 , and C . The third column shows which previous theorem is used to obtain (X, Y) . The last column indicates which row in Table 5.5 the result corresponds to. $\square$

Table 5.7: The case when $G = \mathbb{Z}_{p^3} \times \mathbb{Z}_2$ and S_0 is periodic

(S_0 , C , L_{S_0})	$L_{S_0}, X, Y, S_0, \text{ and } C$	Reference	Result
$(p^2, 2p, p)$	$L_{S_0} = \langle (p^2, 0) \rangle$		
	$X = \{(i + \alpha_i p, \alpha_i) + L_{S_0} \mid i \in [p]\}$		
	$Y = \{(ip, j) + L_{S_0} \mid i \in [p], j \in [2]\}$	5.4 (row 1)	row 3
	$S_0 = \{(i + \alpha_i p + jp^2, \alpha_i) \mid i, j \in [p]\}$		
	$C = \{(ip + \alpha_{ij} p^2, j) \mid i \in [p], j \in [2]\}$		
$(2p, p^2, 2)$	$L_{S_0} = \langle (0, 1) \rangle$		
	$X = \{(i + \alpha_i p, 0) + L_{S_0} \mid i \in [p]\}$		
	$Y = \{(ip, 0) + L_{S_0} \mid i \in [p^2]\}$	5.3 (row 1)	row 7
	$S_0 = \{(i + \alpha_i p, j) \mid i \in [p], j \in [2]\}$		
	$C = \{(ip, \alpha_i) \mid i \in [p^2]\}$		
$(2p, p^2, p)$	$L_{S_0} = \langle (p^2, 0) \rangle$		
	$X = \{(\alpha_i, i) + L_{S_0} \mid i \in [2]\}$		
	$Y = \{(i, 0) + L_{S_0} \mid i \in [p^2]\}$	5.4 (row 2)	row 8
	$S_0 = \{(\alpha_i + jp^2, i) \mid i \in [2], j \in [p]\}$		
	$C = \{(i + \alpha_i p^2, 0) \mid i \in [p^2]\}$		
$(p^3, 2, p)$	$L_{S_0} = \langle (p^2, 0) \rangle$		
	$X = \{(i, \alpha_i) + L_{S_0} \mid i \in [p^2]\}$		
	$Y = \{(0, i) + L_{S_0} \mid i \in [2]\}$	5.4 (row 4)	row 12
	$S_0 = \{(i + jp^2, \alpha_i) \mid i \in [p^2], j \in [p]\}$		
	$C = \{(\alpha_i p^2, i) \mid i \in [2]\}$		
$(p^3, 2, p^2)$	$L_{S_0} = \langle (p, 0) \rangle$		
	$X = \{(i, \alpha_i) + L_{S_0} \mid i \in [p]\}$		
	$Y = \{(0, i) + L_{S_0} \mid i \in [2]\}$	3.5 (row 1)	row 13
	$S_0 = \{(i + jp, \alpha_i) \mid i \in [p], j \in [p^2]\}$		
	$C = \{(\alpha_i p, i) \mid i \in [2]\}$		
$(2p^2, p, 2)$	$L_{S_0} = \langle (0, 1) \rangle$		
	$X = \{(i + \alpha_i p^2, 0) + L_{S_0} \mid i \in [p^2]\}$		
	$Y = \{(ip^2, 0) + L_{S_0} \mid i \in [p]\}$	5.3 (row 3)	row 15
	$S_0 = \{(i + \alpha_i p^2, j) \mid i \in [p^2], j \in [2]\}$		
	$C = \{(ip^2, \alpha_i) \mid i \in [p]\}$		
$(2p^2, p, p)$	$L_{S_0} = \langle (p^2, 0) \rangle$		
	$X = \{(i + \alpha_{ij} p, j) + L_{S_0} \mid i \in [p], j \in [2]\}$		
	$Y = \{(ip, 0) + L_{S_0} \mid i \in [p]\}$	5.4 (row 7)	row 16
	$S_0 = \{(i + \alpha_{ik} p + jp^2, k) \mid i, j \in [p], k \in [2]\}$		

	$C = \{(ip + \alpha_i p^2, 0) \mid i \in [p]\}$		
	$L_{S_0} = \langle (p^2, 1) \rangle$		
$(2p^2, p, 2p)$	$X = \{(i + \alpha_i p, 0) + L_{S_0} \mid i \in [p]\}$		
	$Y = \{(ip, 0) + L_{S_0} \mid i \in [p]\}$	5.1	row 17
	$S_0 = \{(i + \alpha_i p + jp^2, k) \mid i, j \in [p], k \in [2]\}$		
	$C = \{(ip + \alpha_i p^2, \alpha_i) \mid i \in [p]\}$		
	$L_{S_0} = \langle (p, 0) \rangle$		
$(2p^2, p, p^2)$	$X = \{(\alpha_i, i) + L_{S_0} \mid i \in [2]\}$		
	$Y = \{(i, 0) + L_{S_0} \mid i \in [p]\}$	3.5 (row 2)	row 18
	$S_0 = \{(\alpha_i + jp, i) \mid i \in [2], j \in [p^2]\}$		
	$C = \{(i + \alpha_i p, 0) \mid i \in [p]\}$		

Theorem 5.8. Let $G = \mathbb{Z}_{p^2} \times \mathbb{Z}_2 \times \mathbb{Z}_2$ for an odd prime p and let S be a proper subset of $G \setminus \{0\}$ that generates G . Let $0 \in C \subseteq G$. Then C is a perfect code in $\text{Cay}(G, S)$ if and only if (S_0, C) is one of the pairs in Table 5.8.

Table 5.8: Perfect codes in Cayley graphs of $\mathbb{Z}_{p^2} \times \mathbb{Z}_2 \times \mathbb{Z}_2$

Row	S_0	C
1	$\{(i + \alpha_i p)\mathbf{e}_1 + \alpha_i \mathbf{e}_2 + \beta_i \mathbf{e}_3 \mid i \in [p]\}$	$\{i p \mathbf{e}_1 + j \mathbf{e}_2 + k \mathbf{e}_3 \mid i \in [p], j, k \in [2]\}$
2	$\{(i + jp)\mathbf{e}_1 + \alpha_i \mathbf{e}_2 + \beta_i \mathbf{e}_3 \mid i, j \in [p]\}$	$\{\alpha_{ij} p \mathbf{e}_1 + i \mathbf{e}_2 + j \mathbf{e}_3 \mid i, j \in [2]\}$
3	$\{(i + jp)\mathbf{e}_1 + \alpha_{ij} \mathbf{e}_2 + \alpha_i \mathbf{e}_3 \mid i, j \in [p]\}$	$\{\alpha_i p \mathbf{e}_1 + j \mathbf{e}_2 + i \mathbf{e}_3 \mid i, j \in [2]\}$
4	$\{i \mathbf{e}_1 + \alpha_i \mathbf{e}_2 + \beta_i \mathbf{e}_3 \mid i \in [p^2]\}$	$\{i \mathbf{e}_2 + j \mathbf{e}_3 \mid i, j \in [2]\}$
5	$\{(i + \alpha_i p)\mathbf{e}_1 + j \mathbf{e}_2 + \alpha_i \mathbf{e}_3 \mid i \in [p], j \in [2]\}$	$\{i p \mathbf{e}_1 + \alpha_{ij} \mathbf{e}_2 + j \mathbf{e}_3 \mid i \in [p], j \in [2]\}$
6	$\{(i + \alpha_{ij} p)\mathbf{e}_1 + j \mathbf{e}_2 + \alpha_i \mathbf{e}_3 \mid i \in [p], j \in [2]\}$	$\{j p \mathbf{e}_1 + \alpha_i \mathbf{e}_2 + i \mathbf{e}_3 \mid i \in [2], j \in [p]\}$
7	$\{(i + \alpha_i p)\mathbf{e}_1 + \alpha_{ij} \mathbf{e}_2 + j \mathbf{e}_3 \mid i \in [p], j \in [2]\}$	$\{i p \mathbf{e}_1 + j \mathbf{e}_2 + \alpha_i \mathbf{e}_3 \mid i \in [p], j \in [2]\}$
8	$\{(\alpha_i + jp)\mathbf{e}_1 + \alpha_{ij} \mathbf{e}_2 + i \mathbf{e}_3 \mid i \in [2], j \in [p]\}$	$\{(i + \alpha_i p)\mathbf{e}_1 + j \mathbf{e}_2 \mid i \in [p], j \in [2]\}$
9	$\{(i + \alpha_{ij} p)\mathbf{e}_1 + \alpha_{ij} \mathbf{e}_2 + j \mathbf{e}_3 \mid i \in [p], j \in [2]\}$	$\{i p \mathbf{e}_1 + j \mathbf{e}_2 \mid i \in [p], j \in [2]\}$
10	$\{\alpha_i \mathbf{e}_1 + j \mathbf{e}_2 + i \mathbf{e}_3 \mid i, j \in [2]\}$	$\{i \mathbf{e}_1 + \alpha_i \mathbf{e}_2 \mid i \in [p^2]\}$
11	$\{(\alpha_i + \alpha_{ij} p)\mathbf{e}_1 + j \mathbf{e}_2 + i \mathbf{e}_3 \mid i, j \in [2]\}$	$\{(i + jp)\mathbf{e}_1 + \alpha_i \mathbf{e}_2 \mid i, j \in [p]\}$
12	$\{\alpha_{ij} \mathbf{e}_1 + i \mathbf{e}_2 + j \mathbf{e}_3 \mid i, j \in [2]\}$	$\{i \mathbf{e}_1 \mid i \in [p^2]\}$
13	$\{i \mathbf{e}_1 + j \mathbf{e}_2 + \alpha_i \mathbf{e}_3 \mid i \in [p^2], j \in [2]\}$	$\{\alpha_i \mathbf{e}_2 + i \mathbf{e}_3 \mid i \in [2]\}$
14	$\{(i + kp)\mathbf{e}_1 + \alpha_{ij} \mathbf{e}_2 + j \mathbf{e}_3 \mid i, k \in [p], j \in [2]\}$	$\{\alpha_i p \mathbf{e}_1 + i \mathbf{e}_2 \mid i \in [2]\}$
15	$\{(i + jp)\mathbf{e}_1 + k \mathbf{e}_2 + \alpha_i \mathbf{e}_3 \mid i, j \in [p], k \in [2]\}$	$\{\alpha_i p \mathbf{e}_1 + \alpha_i \mathbf{e}_2 + i \mathbf{e}_3 \mid i \in [2]\}$
16	$\{i \mathbf{e}_1 + \alpha_{ij} \mathbf{e}_2 + j \mathbf{e}_3 \mid i \in [p^2], j \in [2]\}$	$\{i \mathbf{e}_2 \mid i \in [2]\}$
17	$\{(i + \alpha_{ij} p)\mathbf{e}_1 + k \mathbf{e}_2 + j \mathbf{e}_3 \mid i \in [p], j, k \in [2]\}$	$\{i p \mathbf{e}_1 + \alpha_i \mathbf{e}_2 \mid i \in [p]\}$
18	$\{(\alpha_{ij} + kp)\mathbf{e}_1 + i \mathbf{e}_2 + j \mathbf{e}_3 \mid i, j \in [2], k \in [p]\}$	$\{(i + \alpha_i p)\mathbf{e}_1 \mid i \in [p]\}$
19	$\{(\alpha_i + jp)\mathbf{e}_1 + k \mathbf{e}_2 + i \mathbf{e}_3 \mid i, k \in [2], j \in [p]\}$	$\{(i + \alpha_i p)\mathbf{e}_1 + \alpha_i \mathbf{e}_2 \mid i \in [p]\}$
20	$\{(i + \alpha_i p)\mathbf{e}_1 + j \mathbf{e}_2 + k \mathbf{e}_3 \mid i \in [p], j, k \in [2]\}$	$\{i p \mathbf{e}_1 + \alpha_i \mathbf{e}_2 + \beta_i \mathbf{e}_3 \mid i \in [p]\}$
21	$\{(i + \alpha_{ijk} p)\mathbf{e}_1 + j \mathbf{e}_2 + k \mathbf{e}_3 \mid i \in [p], j, k \in [2]\}$	$\{i p \mathbf{e}_1 \mid i \in [p]\}$

Proof. Suppose C is a perfect code in $\text{Cay}(G, S)$. Then $G = S_0 \oplus C$ and so $|S_0| \cdot |C| = |G| = 4p^2$. Since G is not a cyclic group, $|S_0| > 2$ by Lemma 2.8. Therefore, $2 < |S_0| < 4p^2$.

Case 1. S_0 is aperiodic.

Since G is Hajós and $G = S_0 \oplus C$, in this case C must be periodic, that is, L_C is nontrivial. Write $C = L_C \oplus D$ for some subset D of G whose existence is ensured by

Lemma 2.3. Then $|L_C| \cdot |D| = |C|$ and $2 \leq |L_C| \leq |C|$. If $|S_0|$ is a prime, then $|L_C| = |C|$ by Lemma 2.8 part (c). If $|S_0|$ is not a prime, then $|L_C|$ can be any divisor of $|C|$. Set

$$A = (S_0 + L_C)/L_C \quad \text{and} \quad B = (D + L_C)/L_C.$$

Since C is periodic, by Lemma 2.5, we have

$$G/L_C = A \oplus B$$

and B is aperiodic in G/L_C . Since G/L_C is isomorphic to a subgroup of G and subgroups of Hajós group are also Hajós groups, A is periodic in G/L_C . We also know that A contains L_C and A generates G/L_C by Lemma 2.7. Furthermore, $|A| = |S_0|$ and $|B| = |D|$. Note that if $|L_C| = |C|$, then $C = L_C$ as $0 \in C$. This means we can take $D = \{0\}$ and hence $B = L_C/L_C$ and $G/L_C = A$. We use C to get A and then use Lemma 2.6 to obtain S_0 . If $|L_C| < |C|$, then this factorization enables us to make use of previously established theorems to obtain the pair (S_0, C) via (A, B) .

The columns of Table 5.9 represent all possible values for $(|S_0|, |C|, |L_C|)$; C , A , and S_0 when $|L_C| = |C|$ or L_C , A , B , S_0 , and C when $|L_C| < |C|$; which previous theorem is used to obtain (A, B) when $|L_C| < |C|$; and which row in Table 5.8 the result corresponds to.

Table 5.9: The case when $G = \mathbb{Z}_{p^2} \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and S_0 is aperiodic

(S_0 , C , L_C)	L_C, A, B, S_0 , and C	Reference	Result
$(p, 4p, 4p)$	$C = \langle p\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_3 \rangle$ $A = \{i\mathbf{e}_1 + L_C \mid i \in [p]\}$ $S_0 = \{(i + \alpha_i p)\mathbf{e}_1 + \alpha_i \mathbf{e}_2 + \beta_i \mathbf{e}_3 \mid i \in [p]\}$		row 1
$(p^2, 4, 2)$	$L_C = \langle \mathbf{e}_2 \rangle$ $A = \{(i + jp)\mathbf{e}_1 + \alpha_i \mathbf{e}_3 + L_C \mid i, j \in [p]\}$ $B = \{\alpha_i p\mathbf{e}_1 + i\mathbf{e}_3 + L_C \mid i \in [2]\}$ $S_0 = \{(i + jp)\mathbf{e}_1 + \alpha_{ij} \mathbf{e}_2 + \alpha_i \mathbf{e}_3 \mid i, j \in [p]\}$ $C = \{\alpha_i p\mathbf{e}_1 + j\mathbf{e}_2 + i\mathbf{e}_3 \mid i, j \in [2]\}$	5.4 (row 3)	row 3
$(p^2, 4, 4)$	$C = \langle \mathbf{e}_2, \mathbf{e}_3 \rangle$ $A = \{i\mathbf{e}_1 + L_C \mid i \in [p^2]\}$ $S_0 = \{i\mathbf{e}_1 + \alpha_i \mathbf{e}_2 + \beta_i \mathbf{e}_3 \mid i \in [p^2]\}$		row 4
$(2p, 2p, p)$	$L_C = \langle p\mathbf{e}_1 \rangle$ $A = \{i\mathbf{e}_1 + j\mathbf{e}_2 + \alpha_i \mathbf{e}_3 + L_C \mid i \in [p], j \in [2]\}$ $B = \{\alpha_i \mathbf{e}_2 + i\mathbf{e}_3 + L_C \mid i \in [2]\}$ $S_0 = \{(i + \alpha_{ij} p)\mathbf{e}_1 + j\mathbf{e}_2 + \alpha_i \mathbf{e}_3 \mid i \in [p], j \in [2]\}$ $C = \{jp\mathbf{e}_1 + \alpha_i \mathbf{e}_2 + i\mathbf{e}_3 \mid i \in [2], j \in [p]\}$	3.6 (row 4)	row 6
$(2p, 2p, 2)$	$L_C = \langle \mathbf{e}_2 \rangle$ $A = \{(i + \alpha_i p)\mathbf{e}_1 + j\mathbf{e}_3 + L_C \mid i \in [p], j \in [2]\}$ $B = \{ip\mathbf{e}_1 + \alpha_i \mathbf{e}_3 + L_C \mid i \in [p]\}$ $S_0 = \{(i + \alpha_i p)\mathbf{e}_1 + \alpha_{ij} \mathbf{e}_2 + j\mathbf{e}_3 \mid i \in [p], j \in [2]\}$ $C = \{ip\mathbf{e}_1 + j\mathbf{e}_2 + \alpha_i \mathbf{e}_3 \mid i \in [p], j \in [2]\}$	5.4 (row 5)	row 7
$(2p, 2p, 2)$	$L_C = \langle \mathbf{e}_2 \rangle$ $A = \{(\alpha_i + jp)\mathbf{e}_1 + i\mathbf{e}_3 + L_C \mid i \in [2], j \in [p]\}$ $B = \{(i + \alpha_i p)\mathbf{e}_1 + L_C \mid i \in [p]\}$ $S_0 = \{(\alpha_i + jp)\mathbf{e}_1 + \alpha_{ij} \mathbf{e}_2 + i\mathbf{e}_3 \mid i \in [2], j \in [p]\}$ $C = \{(i + \alpha_i p)\mathbf{e}_1 + j\mathbf{e}_2 \mid i \in [p], j \in [2]\}$	5.4 (row 6)	row 8

$(2p, 2p, 2p)$	$C = \langle p\mathbf{e}_1 + \mathbf{e}_2 \rangle$ $A = \{i\mathbf{e}_1 + j\mathbf{e}_3 + L_C \mid i \in [p], j \in [2]\}$ $S_0 = \{(i + \alpha_{ij}p)\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + j\mathbf{e}_3 \mid i \in [p], j \in [2]\}$		row 9
$(4, p^2, p)$	$L_C = \langle p\mathbf{e}_1 \rangle$ $A = \{\alpha_i\mathbf{e}_1 + j\mathbf{e}_2 + i\mathbf{e}_3 + L_C \mid i, j \in [2]\}$ $B = \{i\mathbf{e}_1 + \alpha_i\mathbf{e}_2 + L_C \mid i \in [p]\}$ $S_0 = \{(\alpha_i + \alpha_{ij}p)\mathbf{e}_1 + j\mathbf{e}_2 + i\mathbf{e}_3 \mid i, j \in [2]\}$ $C = \{(i + jp)\mathbf{e}_1 + \alpha_i\mathbf{e}_2 \mid i, j \in [p]\}$	3.6 (row 6)	row 11
$(4, p^2, p^2)$	$C = \langle \mathbf{e}_1 \rangle$ $A = \{i\mathbf{e}_2 + j\mathbf{e}_3 + L_C \mid i, j \in [2]\}$ $S_0 = \{\alpha_{ij}\mathbf{e}_1 + i\mathbf{e}_2 + j\mathbf{e}_3 \mid i, j \in [2]\}$		row 12
$(2p^2, 2, 2)$	$C = \langle \mathbf{e}_2 \rangle$ $A = \{i\mathbf{e}_1 + j\mathbf{e}_3 + L_C \mid i \in [p^2], j \in [2]\}$ $S_0 = \{i\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + j\mathbf{e}_3 \mid i \in [p^2], j \in [2]\}$		row 16
$(4p, p, p)$	$C = \langle p\mathbf{e}_1 \rangle$ $A = \{i\mathbf{e}_1 + j\mathbf{e}_2 + k\mathbf{e}_3 + L_C \mid i \in [p], j, k \in [2]\}$ $S_0 = \{(i + \alpha_{ijk}p)\mathbf{e}_1 + j\mathbf{e}_2 + k\mathbf{e}_3 \mid i \in [p], j, k \in [2]\}$		row 21

Case 2. S_0 is periodic.

In this case we have $|L_{S_0}| \geq 2$ and $S_0 = D \oplus L_{S_0}$ for some subset D of G by Lemma 2.3. So $|D| \cdot |L_{S_0}| = |S_0|$ and $1 < |L_{S_0}| < |S_0|$ by Lemma 2.8 part (b). Set

$$X = (D + L_{S_0})/L_{S_0} \quad \text{and} \quad Y = (C + L_{S_0})/L_{S_0}.$$

Since S_0 is periodic, by Lemma 2.5, we have

$$G/L_{S_0} = X \oplus Y$$

and X is aperiodic in G/L_{S_0} . We also know that X contains L_{S_0} and X generates G/L_{S_0} by Lemma 2.7. This factorization enables us to make use of Lemma 2.6 and previously established theorems to obtain the pair (S_0, C) via (X, Y) .

The columns of Table 5.10 represent all possible values of $(|S_0|, |C|, |L_{S_0}|)$; L_{S_0} , X , Y , S_0 , and C ; which previous theorem or lemma is used to obtain (X, Y) ; and which row in Table 5.8 the result corresponds to. Note that in some cases, for example when $(|S_0|, |C|, |L_{S_0}|) = (2p, 2p, p)$, no S_0 and C exist since no X exist from the previous theorem given in the third column. $\square$

Table 5.10: The case when $G = \mathbb{Z}_{p^2} \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and S_0 is periodic

(S_0 , C , L_{S_0})	L_{S_0} , X , Y , S_0 , and C	Reference	Result
$(p^2, 4, p)$	$L_{S_0} = \langle p\mathbf{e}_1 \rangle$ $X = \{i\mathbf{e}_1 + \alpha_i\mathbf{e}_2 + \beta_i\mathbf{e}_3 + L_{S_0} \mid i \in [p]\}$ $Y = \{i\mathbf{e}_2 + j\mathbf{e}_3 + L_{S_0} \mid i, j \in [2]\}$ $S_0 = \{(i + jp)\mathbf{e}_1 + \alpha_i\mathbf{e}_2 + \beta_i\mathbf{e}_3 \mid i, j \in [p]\}$ $C = \{\alpha_{ij}p\mathbf{e}_1 + i\mathbf{e}_2 + j\mathbf{e}_3 \mid i, j \in [2]\}$	3.6 (row 1)	row 2
	$L_{S_0} = \langle \mathbf{e}_2 \rangle$ $X = \{(i + \alpha_i p)\mathbf{e}_1 + \alpha_i\mathbf{e}_3 + L_{S_0} \mid i \in [p]\}$		

$(2p, 2p, 2)$	$Y = \{ipe_1 + je_3 + L_{S_0} \mid i \in [p], j \in [2]\}$ $S_0 = \{(i + \alpha_i p)e_1 + je_2 + \alpha_i e_3 \mid i \in [p], j \in [2]\}$ $C = \{ipe_1 + \alpha_{ij}e_2 + je_3 \mid i \in [p], j \in [2]\}$	5.4 (row 1)	row 5
$(2p, 2p, p)$	Not possible since no X exists	3.6	
$(4, p^2, 2)$	$L_{S_0} = \langle e_2 \rangle$ $X = \{\alpha_i e_1 + ie_3 + L_{S_0} \mid i \in [2]\}$ $Y = \{ie_1 + L_{S_0} \mid i \in [p^2]\}$ $S_0 = \{\alpha_i e_1 + je_2 + ie_3 \mid i, j \in [2]\}$ $C = \{ie_1 + \alpha_i e_2 \mid i \in [p^2]\}$	5.4 (row 2)	row 10
$(2p^2, 2, 2)$	$L_{S_0} = \langle e_2 \rangle$ $X = \{ie_1 + \alpha_i e_3 + L_{S_0} \mid i \in [p^2]\}$ $Y = \{ie_3 + L_{S_0} \mid i \in [2]\}$ $S_0 = \{ie_1 + je_2 + \alpha_i e_3 \mid i \in [p^2], j \in [2]\}$ $C = \{\alpha_i e_2 + ie_3 \mid i \in [2]\}$	5.4 (row 4)	row 13
$(2p^2, 2, p)$	$L_{S_0} = \langle pe_1 \rangle$ $X = \{ie_1 + \alpha_{ij}e_2 + je_3 + L_{S_0} \mid i \in [p], j \in [2]\}$ $Y = \{ie_2 + L_{S_0} \mid i \in [2]\}$ $S_0 = \{(i + kp)e_1 + \alpha_{ij}e_2 + je_3 \mid i, k \in [p], j \in [2]\}$ $C = \{\alpha_i pe_1 + ie_2 \mid i \in [2]\}$	3.6 (row 5)	row 14
$(2p^2, 2, 2p)$	$L_{S_0} = \langle pe_1 + e_2 \rangle$ $X = \{ie_1 + \alpha_i e_3 + L_{S_0} \mid i \in [p]\}$ $Y = \{ie_3 + L_{S_0} \mid i \in [2]\}$ $S_0 = \{(i + jp)e_1 + ke_2 + \alpha_i e_3 \mid i, j \in [p], k \in [2]\}$ $C = \{\alpha_i pe_1 + \alpha_i e_2 + ie_3 \mid i \in [2]\}$	3.5 (row 1)	row 15
$(2p^2, 2, p^2)$	Not possible since no X exists	5.2	
$(4p, p, 2)$	$L_{S_0} = \langle e_2 \rangle$ $X = \{(i + \alpha_{ij}p)e_1 + je_3 + L_{S_0} \mid i \in [p], j \in [2]\}$ $Y = \{ipe_1 + L_{S_0} \mid i \in [p]\}$ $S_0 = \{(i + \alpha_{ij}p)e_1 + ke_2 + je_3 \mid i \in [p], j, k \in [2]\}$ $C = \{ipe_1 + \alpha_i e_2 \mid i \in [p]\}$	5.4 (row 7)	row 17
$(4p, p, p)$	$L_{S_0} = \langle pe_1 \rangle$ $X = \{\alpha_{ij}e_1 + ie_2 + je_3 + L_{S_0} \mid i, j \in [2]\}$ $Y = \{ie_1 + L_{S_0} \mid i \in [p]\}$ $S_0 = \{(\alpha_{ij} + kp)e_1 + ie_2 + je_3 \mid i, j \in [2], k \in [p]\}$ $C = \{(i + \alpha_i p)e_1 \mid i \in [p]\}$	3.6 (row 7)	row 18
$(4p, p, 2p)$	$L_{S_0} = \langle pe_1 + e_2 \rangle$ $X = \{\alpha_i e_1 + ie_3 + L_{S_0} \mid i \in [2]\}$ $Y = \{ie_1 + L_{S_0} \mid i \in [p]\}$ $S_0 = \{(\alpha_i + jp)e_1 + ke_2 + ie_3 \mid i, k \in [2], j \in [p]\}$ $C = \{(i + \alpha_i p)e_1 + \alpha_i e_2 \mid i \in [p]\}$	3.5 (row 2)	row 19
$(4p, p, 4)$	$L_{S_0} = \langle e_2, e_3 \rangle$ $X = \{(i + \alpha_i p)e_1 + L_{S_0} \mid i \in [p]\}$ $Y = \{ipe_1 + L_{S_0} \mid i \in [p]\}$ $S_0 = \{(i + \alpha_i p)e_1 + je_2 + ke_3 \mid i \in [p], j, k \in [2]\}$ $C = \{ipe_1 + \alpha_i e_2 + \beta_i e_3 \mid i \in [p]\}$	5.1	row 20

Remark 5.9. In Theorem 5.8, the generators e_2 and e_3 may be replaced by any basis of $H = \{0\} \times \mathbb{Z}_2 \times \mathbb{Z}_2$, as in (C2).

Theorem 5.10. *Let $G = \mathbb{Z}_p \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ for an odd prime p and let S be a proper subset of $G \setminus \{0\}$ that generates G . Let $0 \in C \subseteq G$. Then C is a perfect code in $\text{Cay}(G, S)$*

if and only if (S_0, C) is one of the pairs in Table 5.11.

Table 5.11: Perfect codes in Cayley graphs of $\mathbb{Z}_p \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

Row	S_0	C
1	$\{i\mathbf{e}_1 + \alpha_i\mathbf{e}_2 + \beta_i\mathbf{e}_3 + \gamma_i\mathbf{e}_4 \mid i \in [p]\}$	$\{i\mathbf{e}_2 + j\mathbf{e}_3 + k\mathbf{e}_4 \mid i, j, k \in [2]\}$
2	$\{i\mathbf{e}_1 + j\mathbf{e}_2 + \alpha_i\mathbf{e}_3 + \beta_i\mathbf{e}_4 \mid i \in [p], j \in [2]\}$	$\{\alpha_{ij}\mathbf{e}_2 + i\mathbf{e}_3 + j\mathbf{e}_4 \mid i, j \in [2]\}$
3	$\{i\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + j\mathbf{e}_3 + \alpha_i\mathbf{e}_4 \mid i \in [p], j \in [2]\}$	$\{j\mathbf{e}_2 + \alpha_i\mathbf{e}_3 + i\mathbf{e}_4 \mid i, j \in [2]\}$
4	$\{i\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + \beta_{ij}\mathbf{e}_3 + j\mathbf{e}_4 \mid i \in [p], j \in [2]\}$	$\{i\mathbf{e}_2 + j\mathbf{e}_3 \mid i, j \in [2]\}$
5	$\{\alpha_i\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + j\mathbf{e}_3 + i\mathbf{e}_4 \mid i, j \in [2]\}$	$\{i\mathbf{e}_1 + j\mathbf{e}_2 + \alpha_i\mathbf{e}_3 \mid i \in [p], j \in [2]\}$
6	$\{\alpha_{ij}\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + i\mathbf{e}_3 + j\mathbf{e}_4 \mid i, j \in [2]\}$	$\{i\mathbf{e}_1 + j\mathbf{e}_2 \mid i \in [p], j \in [2]\}$
7	$\{i\mathbf{e}_1 + k\mathbf{e}_2 + \alpha_{ij}\mathbf{e}_3 + j\mathbf{e}_4 \mid i \in [p], j, k \in [2]\}$	$\{\alpha_i\mathbf{e}_2 + i\mathbf{e}_3 \mid i \in [2]\}$
8	$\{k\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + i\mathbf{e}_3 + j\mathbf{e}_4 \mid i, j \in [2], k \in [p]\}$	$\{\alpha_i\mathbf{e}_1 + i\mathbf{e}_2 \mid i \in [2]\}$
9	$\{i\mathbf{e}_1 + j\mathbf{e}_2 + k\mathbf{e}_3 + \alpha_i\mathbf{e}_4 \mid i \in [p], j, k \in [2]\}$	$\{\alpha_i\mathbf{e}_2 + \beta_i\mathbf{e}_3 + i\mathbf{e}_4 \mid i \in [2]\}$
10	$\{i\mathbf{e}_1 + \alpha_{ijk}\mathbf{e}_2 + j\mathbf{e}_3 + k\mathbf{e}_4 \mid i \in [p], j, k \in [2]\}$	$\{i\mathbf{e}_2 \mid i \in [2]\}$
11	$\{\alpha_{ij}\mathbf{e}_1 + k\mathbf{e}_2 + i\mathbf{e}_3 + j\mathbf{e}_4 \mid i, j, k \in [2]\}$	$\{i\mathbf{e}_1 + \alpha_i\mathbf{e}_2 \mid i \in [p]\}$
12	$\{\alpha_i\mathbf{e}_1 + j\mathbf{e}_2 + k\mathbf{e}_3 + i\mathbf{e}_4 \mid i, j, k \in [2]\}$	$\{i\mathbf{e}_1 + \alpha_i\mathbf{e}_2 + \beta_i\mathbf{e}_3 \mid i \in [p]\}$
13	$\{\alpha_{ijk}\mathbf{e}_1 + i\mathbf{e}_2 + j\mathbf{e}_3 + k\mathbf{e}_4 \mid i, j, k \in [2]\}$	$\{i\mathbf{e}_1 \mid i \in [p]\}$

Proof. Suppose C is a perfect code in $\text{Cay}(G, S)$. Then $G = S_0 \oplus C$ and so $|S_0| \cdot |C| = |G| = 8p$. Since G is not a cyclic group, $|S_0| > 2$ by Lemma 2.8. Therefore, $2 < |S_0| < 8p$.

Case 1. S_0 is aperiodic.

Since G is Hajós and $G = S_0 \oplus C$, in this case C must be periodic, that is, L_C is nontrivial. Write $C = L_C \oplus D$ for some subset D of G whose existence is ensured by Lemma 2.3. Then $|L_C| \cdot |D| = |C|$ and $2 \leq |L_C| \leq |C|$. If $|S_0|$ is a prime, then $|L_C| = |C|$ by Lemma 2.8 part (c). If $|S_0|$ is not a prime, then $|L_C|$ can be any divisor of $|C|$. Set

$$A = (S_0 + L_C)/L_C \quad \text{and} \quad B = (D + L_C)/L_C.$$

Since C is periodic, by Lemma 2.5, we have

$$G/L_C = A \oplus B$$

and B is aperiodic in G/L_C . Since G/L_C is isomorphic to a subgroup of G and subgroups of Hajós group are also Hajós groups, A is periodic in G/L_C . We also know that A contains L_C and A generates G/L_C by Lemma 2.7. Furthermore, $|A| = |S_0|$ and $|B| = |D|$. Note that if $|L_C| = |C|$, then $C = L_C$ as $0 \in C$. This means we can take $D = \{0\}$ and hence $B = L_C/L_C$ and $G/L_C = A$. We use C to get A and then use Lemma 2.6 to obtain S_0 . If $|L_C| < |C|$, then this factorization enables us to make use of previously established theorems to obtain the pair (S_0, C) via (A, B) .

Similar to Table 5.9, the columns of Table 5.12 represent all possible values for $(|S_0|, |C|, |L_C|)$; C , A , and S_0 when $|L_C| = |C|$ or L_C , A , B , S_0 , and C when $|L_C| < |C|$; which previous theorem or lemma is used to obtain (A, B) when $|L_C| < |C|$; and which row in Table 5.11 the result corresponds to.

Table 5.12: The case when $G = \mathbb{Z}_p \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and S_0 is aperiodic

(S_0 , C , L_C)	$L_C, A, B, S_0, \text{ and } C$	Reference	Result
$(p, 8, 8)$	$C = \langle \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4 \rangle$ $A = \{i\mathbf{e}_1 + L_C \mid i \in [p]\}$ $S_0 = \{i\mathbf{e}_1 + \alpha_i\mathbf{e}_2 + \beta_i\mathbf{e}_3 + \gamma_i\mathbf{e}_4 \mid i \in [p]\}$		row 1
$(2p, 4, 2)$	$L_C = \langle \mathbf{e}_2 \rangle$ $A = \{i\mathbf{e}_1 + j\mathbf{e}_3 + \alpha_i\mathbf{e}_4 + L_C \mid i \in [p], j \in [2]\}$ $B = \{\alpha_i\mathbf{e}_3 + i\mathbf{e}_4 + L_C \mid i \in [2]\}$ $S_0 = \{i\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + j\mathbf{e}_3 + \alpha_i\mathbf{e}_4 \mid i \in [p], j \in [2]\}$ $C = \{j\mathbf{e}_2 + \alpha_i\mathbf{e}_3 + i\mathbf{e}_4 \mid i, j \in [2]\}$	3.6 (row 4)	row 3
$(2p, 4, 4)$	$C = \langle \mathbf{e}_2, \mathbf{e}_3 \rangle$ $A = \{i\mathbf{e}_1 + j\mathbf{e}_4 + L_C \mid i \in [p], j \in [2]\}$ $S_0 = \{i\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + \beta_{ij}\mathbf{e}_3 + j\mathbf{e}_4 \mid i \in [p], j \in [2]\}$		row 4
$(4, 2p, p)$	Not possible since no A exists	5.5	
$(4, 2p, 2)$	$L_C = \langle \mathbf{e}_2 \rangle$ $A = \{\alpha_i\mathbf{e}_1 + j\mathbf{e}_3 + i\mathbf{e}_4 + L_C \mid i, j \in [2]\}$ $B = \{i\mathbf{e}_1 + \alpha_i\mathbf{e}_3 + L_C \mid i \in [p]\}$ $S_0 = \{\alpha_i\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + j\mathbf{e}_3 + i\mathbf{e}_4 \mid i, j \in [2]\}$ $C = \{i\mathbf{e}_1 + j\mathbf{e}_2 + \alpha_i\mathbf{e}_3 \mid i \in [p], j \in [2]\}$	3.6 (row 6)	row 5
$(4, 2p, 2p)$	$C = \langle \mathbf{e}_1 + \mathbf{e}_2 \rangle$ $A = \{i\mathbf{e}_3 + j\mathbf{e}_4 + L_C \mid i, j \in [2]\}$ $S_0 = \{\alpha_{ij}\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + i\mathbf{e}_3 + j\mathbf{e}_4 \mid i, j \in [2]\}$		row 6
$(4p, 2, 2)$	$C = \langle \mathbf{e}_2 \rangle$ $A = \{i\mathbf{e}_1 + j\mathbf{e}_3 + k\mathbf{e}_4 + L_C \mid i \in [p], j, k \in [2]\}$ $S_0 = \{i\mathbf{e}_1 + \alpha_{ijk}\mathbf{e}_2 + j\mathbf{e}_3 + k\mathbf{e}_4 \mid i \in [p], j, k \in [2]\}$		row 10
$(8, p, p)$	$C = \langle \mathbf{e}_1 \rangle$ $A = \{i\mathbf{e}_2 + j\mathbf{e}_3 + k\mathbf{e}_4 + L_C \mid i, j, k \in [2]\}$ $S_0 = \{\alpha_{ijk}\mathbf{e}_1 + i\mathbf{e}_2 + j\mathbf{e}_3 + k\mathbf{e}_4 \mid i, j, k \in [2]\}$		row 13

Case 2. S_0 is periodic.

In this case we have $|L_{S_0}| \geq 2$ and $S_0 = D \oplus L_{S_0}$ for some subset D of G by Lemma 2.3. So $|D| \cdot |L_{S_0}| = |S_0|$ and $1 < |L_{S_0}| < |S_0|$ by Lemma 2.8 part (b). Set

$$X = (D + L_{S_0})/L_{S_0} \quad \text{and} \quad Y = (C + L_{S_0})/L_{S_0}.$$

Since S_0 is periodic, by Lemma 2.5, we have

$$G/L_{S_0} = X \oplus Y$$

and X is aperiodic in G/L_{S_0} . We also know that X contains L_{S_0} and X generates G/L_{S_0} by Lemma 2.7. This factorization enables us to make use of Lemma 2.6 and previously established theorems to obtain the pair (S_0, C) via (X, Y) .

Similar to Table 5.10, the columns of Table 5.13 represent all possible values for $(|S_0|, |C|, |L_{S_0}|)$; L_{S_0} , X , Y , S_0 , and C ; which previous theorem or lemma is used to obtain (X, Y) ; and which row in Table 5.11 the result corresponds to. $\square$

Table 5.13: The case when $G = \mathbb{Z}_p \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and S_0 is periodic

(S_0 , C , L_{S_0})	$L_{S_0}, X, Y, S_0, \text{ and } C$	Reference	Result
$(2p, 4, 2)$	$L_{S_0} = \langle \mathbf{e}_2 \rangle$		
	$X = \{i\mathbf{e}_1 + \alpha_i\mathbf{e}_3 + \beta_i\mathbf{e}_4 + L_{S_0} \mid i \in [p]\}$		
	$Y = \{i\mathbf{e}_3 + j\mathbf{e}_4 + L_{S_0} \mid i, j \in [2]\}$	3.6 (row 1)	row 2
	$S_0 = \{i\mathbf{e}_1 + j\mathbf{e}_2 + \alpha_i\mathbf{e}_3 + \beta_i\mathbf{e}_4 \mid i \in [p], j \in [2]\}$		
	$C = \{\alpha_{ij}\mathbf{e}_2 + i\mathbf{e}_3 + j\mathbf{e}_4 \mid i, j \in [2]\}$		
$(2p, 4, p)$	Not possible since no X exists	5.5	
$(4, 2p, 2)$	Not possible since no X exists	3.6	
$(4p, 2, 2)$	$L_{S_0} = \langle \mathbf{e}_2 \rangle$		
	$X = \{i\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_3 + j\mathbf{e}_4 + L_{S_0} \mid i \in [p], j \in [2]\}$		
	$Y = \{i\mathbf{e}_3 + L_{S_0} \mid i \in [2]\}$	3.6 (row 5)	row 7
	$S_0 = \{i\mathbf{e}_1 + k\mathbf{e}_2 + \alpha_{ij}\mathbf{e}_3 + j\mathbf{e}_4 \mid i \in [p], j, k \in [2]\}$		
	$C = \{\alpha_i\mathbf{e}_2 + i\mathbf{e}_3 \mid i \in [2]\}$		
$(4p, 2, p)$	$L_{S_0} = \langle \mathbf{e}_1 \rangle$		
	$X = \{\alpha_{ij}\mathbf{e}_2 + i\mathbf{e}_3 + j\mathbf{e}_4 + L_{S_0} \mid i, j \in [2]\}$		
	$Y = \{i\mathbf{e}_2 + L_{S_0} \mid i \in [2]\}$	5.5	row 8
	$S_0 = \{k\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + i\mathbf{e}_3 + j\mathbf{e}_4 \mid i, j \in [2], k \in [p]\}$		
	$C = \{\alpha_i\mathbf{e}_1 + i\mathbf{e}_2 \mid i \in [2]\}$		
$(4p, 2, 2p)$	Not possible since no X exists	5.2	
$(4p, 2, 4)$	$L_{S_0} = \langle \mathbf{e}_2, \mathbf{e}_3 \rangle$		
	$X = \{i\mathbf{e}_1 + \alpha_i\mathbf{e}_4 + L_{S_0} \mid i \in [p]\}$		
	$Y = \{i\mathbf{e}_4 + L_{S_0} \mid i \in [2]\}$	3.5 (row 1)	row 9
	$S_0 = \{i\mathbf{e}_1 + j\mathbf{e}_2 + k\mathbf{e}_3 + \alpha_i\mathbf{e}_4 \mid i \in [p], j, k \in [2]\}$		
	$C = \{\alpha_i\mathbf{e}_2 + \beta_i\mathbf{e}_3 + i\mathbf{e}_4 \mid i \in [2]\}$		
$(8, p, 2)$	$L_{S_0} = \langle \mathbf{e}_2 \rangle$		
	$X = \{\alpha_{ij}\mathbf{e}_1 + i\mathbf{e}_3 + j\mathbf{e}_4 + L_{S_0} \mid i, j \in [2]\}$		
	$Y = \{i\mathbf{e}_1 + L_{S_0} \mid i \in [p]\}$	3.6 (row 7)	row 11
	$S_0 = \{\alpha_{ij}\mathbf{e}_1 + k\mathbf{e}_2 + i\mathbf{e}_3 + j\mathbf{e}_4 \mid i, j, k \in [2]\}$		
	$C = \{i\mathbf{e}_1 + \alpha_i\mathbf{e}_2 \mid i \in [p]\}$		
$(8, p, 4)$	$L_{S_0} = \langle \mathbf{e}_2, \mathbf{e}_3 \rangle$		
	$X = \{\alpha_i\mathbf{e}_1 + i\mathbf{e}_4 + L_{S_0} \mid i \in [2]\}$		
	$Y = \{i\mathbf{e}_1 + L_{S_0} \mid i \in [p]\}$	3.5 (row 2)	row 12
	$S_0 = \{\alpha_i\mathbf{e}_1 + j\mathbf{e}_2 + k\mathbf{e}_3 + i\mathbf{e}_4 \mid i, j, k \in [2]\}$		
	$C = \{i\mathbf{e}_1 + \alpha_i\mathbf{e}_2 + \beta_i\mathbf{e}_3 \mid i \in [p]\}$		

Remark 5.11. In Theorem 5.10, the generators $\mathbf{e}_2, \mathbf{e}_3$ and $\mathbf{e}_4$ may be replaced by any basis of $H = \{0\} \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, as in (C2).

Lemma 5.12. *Let $G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and let S be a proper subset of $G \setminus \{0\}$ that generates G . Let $0 \in C \subseteq G$. Then C is a perfect code in $\text{Cay}(G, S)$ if and only if (S_0, C) is one of the pairs in Table 5.14.*

Table 5.14: Perfect codes in Cayley graphs of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

Row	S_0	C
1	$\{k\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + i\mathbf{e}_3 + j\mathbf{e}_4 \mid i, j, k \in [2]\}$	$\{\alpha_i\mathbf{e}_1 + i\mathbf{e}_2 \mid i \in [2]\}$
2	$\{\alpha_{ijk}\mathbf{e}_1 + i\mathbf{e}_2 + j\mathbf{e}_3 + k\mathbf{e}_4 \mid i, j, k \in [2]\}$	$\{i\mathbf{e}_1 \mid i \in [2]\}$

Proof. Suppose $\text{Cay}(G, S)$ admits a perfect code C . Then $G = S_0 \oplus C$ and so $|S_0| \cdot |C| = |G| = 16$. Since S_0 generates G , S_0 contains at least four non-zero elements, so $|S_0| \geq 5$. Also, $\text{Cay}(G, S)$ is not a complete graph. Hence $|C| \geq 2$ and therefore $(|S_0|, |C|) = (8, 2)$.

If S_0 is periodic, then $|L_{S_0}| = 2$ or 4 . Suppose $|L_{S_0}| = 4$. Then $L_{S_0} = \langle \mathbf{e}_1, \mathbf{e}_2 \rangle$. Let $S_0 = D \oplus L_{S_0}$ for some subset D . From the factorization of G/L_{S_0} given in Lemma 2.5, we have $G/L_{S_0} = ((D+L_{S_0})/L_{S_0}) \oplus ((C+L_{S_0})/L_{S_0})$. Note that $(D+L_{S_0})/L_{S_0}$ is a set of order 2 containing $0+L_{S_0}$ and generates $G/L_{S_0} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ by Lemma 2.7. Since $\mathbb{Z}_2 \times \mathbb{Z}_2$ is not a cyclic group, $(D+L_{S_0})/L_{S_0}$ cannot generate G/L_{S_0} , a contradiction. Therefore, $|L_{S_0}| = 2$, and hence $L_{S_0} = \langle \mathbf{e}_1 \rangle$. Let $S_0 = D \oplus L_{S_0}$ for some subset D . From the factorization of G/L_{S_0} given in Lemma 2.5, we have $G/L_{S_0} = ((D+L_{S_0})/L_{S_0}) \oplus ((C+L_{S_0})/L_{S_0})$. Note that $(D+L_{S_0})/L_{S_0}$ is an aperiodic set of order 4 and $G/L_{S_0} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. From the case of $|S_0| = 4$ and S_0 is aperiodic in Lemma 5.5, we have $(D+L_{S_0})/L_{S_0} = \{\alpha_{ij}\mathbf{e}_2 + i\mathbf{e}_3 + j\mathbf{e}_4 + L_{S_0} \mid i, j \in [2]\}$ and $(C+L_{S_0})/L_{S_0} = \{i\mathbf{e}_2 + L_{S_0} \mid i \in [2]\}$. Therefore, $S_0 = D \oplus L_{S_0} = \{k\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + i\mathbf{e}_3 + j\mathbf{e}_4 \mid i, j, k \in [2]\}$ and $C = \{\alpha_i\mathbf{e}_1 + i\mathbf{e}_2 \mid i \in [2]\}$. This gives row 1 in Table 5.14.

If $|S_0| = 8$ and S_0 is aperiodic, then $|C| = 2$ and C is periodic, and hence $C = L_C = \langle \mathbf{e}_1 \rangle$. From the factorization of G/L_C given in Lemma 2.5, $(S_0 + L_C)/L_C = G/L_C = \{i\mathbf{e}_2 + j\mathbf{e}_3 + k\mathbf{e}_4 + L_C \mid i, j, k \in [2]\}$, and hence $S_0 = \{\alpha_{ijk}\mathbf{e}_1 + i\mathbf{e}_2 + j\mathbf{e}_3 + k\mathbf{e}_4 \mid i, j, k \in [2]\}$, $\alpha_{ijk} \in \mathbb{Z}$. This gives row 2 in Table 5.14. $\square$

Remark 5.13. In Lemma 5.12, the generators $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ and $\mathbf{e}_4$ may be replaced by any basis of $H = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, as in (C2).

Theorem 5.14. *Let $G = \mathbb{Z}_{p^3} \times \mathbb{Z}_2 \times \mathbb{Z}_2$ for an odd prime p and let S be a proper subset of $G \setminus \{0\}$ that generates G . Let $0 \in C \subseteq G$. Then C is a perfect code in $\text{Cay}(G, S)$ if and only if (S_0, C) is one of the pairs in Table 5.15.*

Table 5.15: Perfect codes in Cayley graphs of $\mathbb{Z}_{p^3} \times \mathbb{Z}_2 \times \mathbb{Z}_2$

Row	S_0	C
1	$\{(i + \alpha_i p)\mathbf{e}_1 + \alpha_i \mathbf{e}_2 + \beta_i \mathbf{e}_3 \mid i \in [p]\}$	$\{i p \mathbf{e}_1 + j \mathbf{e}_2 + k \mathbf{e}_3 \mid i \in [p^2], j, k \in [2]\}$
2	$\{(i + \alpha_i p + j p^2)\mathbf{e}_1 + \alpha_i \mathbf{e}_2 + \beta_i \mathbf{e}_3 \mid i, j \in [p]\}$	$\{(i p + \alpha_{ijk} p^2)\mathbf{e}_1 + j \mathbf{e}_2 + k \mathbf{e}_3 \mid i \in [p], j, k \in [2]\}$
3	$\{(i + \alpha_i p + j p^2)\mathbf{e}_1 + \alpha_{ij} \mathbf{e}_2 + \alpha_i \mathbf{e}_3 \mid i, j \in [p]\}$	$\{(i p + \alpha_{ij} p^2)\mathbf{e}_1 + k \mathbf{e}_2 + j \mathbf{e}_3 \mid i \in [p], j, k \in [2]\}$
4	$\{(i + j p + \alpha_{ij} p^2)\mathbf{e}_1 + \alpha_i \mathbf{e}_2 + \beta_i \mathbf{e}_3 \mid i, j \in [p]\}$	$\{(\alpha_{ij} p + k p^2)\mathbf{e}_1 + i \mathbf{e}_2 + j \mathbf{e}_3 \mid i, j \in [2], k \in [p]\}$
5	$\{(i + \alpha_i p + j p^2)\mathbf{e}_1 + \alpha_{ij} \mathbf{e}_2 + \beta_{ij} \mathbf{e}_3 \mid i, j \in [p]\}$	$\{(i p + \alpha_i p^2)\mathbf{e}_1 + j \mathbf{e}_2 + k \mathbf{e}_3 \mid i \in [p], j, k \in [2]\}$
6	$\{(i + j p + \alpha_{ij} p^2)\mathbf{e}_1 + \alpha_{ij} \mathbf{e}_2 + \alpha_i \mathbf{e}_3 \mid i, j \in [p]\}$	$\{(\alpha_i p + j p^2)\mathbf{e}_1 + k \mathbf{e}_2 + i \mathbf{e}_3 \mid i, k \in [2], j \in [p]\}$
7	$\{(i + \alpha_i p^2)\mathbf{e}_1 + \beta_i \mathbf{e}_2 + \gamma_i \mathbf{e}_3 \mid i \in [p^2]\}$	$\{i p^2 \mathbf{e}_1 + j \mathbf{e}_2 + k \mathbf{e}_3 \mid i \in [p], j, k \in [2]\}$
8	$\{(i + \alpha_i p)\mathbf{e}_1 + j \mathbf{e}_2 + \alpha_i \mathbf{e}_3 \mid i \in [p], j \in [2]\}$	$\{i p \mathbf{e}_1 + \alpha_{ij} \mathbf{e}_2 + j \mathbf{e}_3 \mid i \in [p^2], j \in [2]\}$
9	$\{(i + \alpha_i p)\mathbf{e}_1 + \alpha_{ij} \mathbf{e}_2 + j \mathbf{e}_3 \mid i \in [p], j \in [2]\}$	$\{i p \mathbf{e}_1 + j \mathbf{e}_2 + \alpha_i \mathbf{e}_3 \mid i \in [p^2], j \in [2]\}$
10	$\{(\alpha_i + j p^2)\mathbf{e}_1 + \alpha_{ij} \mathbf{e}_2 + i \mathbf{e}_3 \mid i \in [2], j \in [p]\}$	$\{(i + \alpha_i p^2)\mathbf{e}_1 + j \mathbf{e}_2 \mid i \in [p^2], j \in [2]\}$
11	$\{(i + \alpha_i p + \alpha_{ij} p^2)\mathbf{e}_1 + j \mathbf{e}_2 + \alpha_i \mathbf{e}_3 \mid i \in [p], j \in [2]\}$	$\{(i p + k p^2)\mathbf{e}_1 + \alpha_{ij} \mathbf{e}_2 + j \mathbf{e}_3 \mid i, k \in [p], j \in [2]\}$
12	$\{(i + \alpha_i p + \alpha_{ij} p^2)\mathbf{e}_1 + \alpha_{ij} \mathbf{e}_2 + j \mathbf{e}_3 \mid i \in [p], j \in [2]\}$	$\{(i p + j p^2)\mathbf{e}_1 + k \mathbf{e}_2 + \alpha_i \mathbf{e}_3 \mid i, j \in [p], k \in [2]\}$
13	$\{(\alpha_i + j p + \alpha_{ij} p^2)\mathbf{e}_1 + \alpha_{ij} \mathbf{e}_2 + i \mathbf{e}_3 \mid i \in [2], j \in [p]\}$	$\{(i + \alpha_i p + j p^2)\mathbf{e}_1 + k \mathbf{e}_2 \mid i, j \in [p], k \in [2]\}$
14	$\{(i + \alpha_{ij} p)\mathbf{e}_1 + j \mathbf{e}_2 + \alpha_i \mathbf{e}_3 \mid i \in [p], j \in [2]\}$	$\{j p \mathbf{e}_1 + \alpha_i \mathbf{e}_2 + i \mathbf{e}_3 \mid i \in [2], j \in [p^2]\}$
15	$\{(i + \alpha_{ij} p)\mathbf{e}_1 + \alpha_{ij} \mathbf{e}_2 + j \mathbf{e}_3 \mid i \in [p], j \in [2]\}$	$\{i p \mathbf{e}_1 + j \mathbf{e}_2 \mid i \in [p^2], j \in [2]\}$
16	$\{\alpha_i \mathbf{e}_1 + j \mathbf{e}_2 + i \mathbf{e}_3 \mid i, j \in [2]\}$	$\{i \mathbf{e}_1 + \alpha_i \mathbf{e}_2 \mid i \in [p^3]\}$
17	$\{(\alpha_i + \alpha_{ij} p^2)\mathbf{e}_1 + j \mathbf{e}_2 + i \mathbf{e}_3 \mid i, j \in [2]\}$	$\{(i + j p^2)\mathbf{e}_1 + \alpha_i \mathbf{e}_2 \mid i \in [p^2], j \in [p]\}$
18	$\{(\alpha_i + \alpha_{ij} p)\mathbf{e}_1 + j \mathbf{e}_2 + i \mathbf{e}_3 \mid i, j \in [2]\}$	$\{(i + j p)\mathbf{e}_1 + \alpha_i \mathbf{e}_2 \mid i \in [p], j \in [p^2]\}$

19	$\{\alpha_{ij}\mathbf{e}_1 + i\mathbf{e}_2 + j\mathbf{e}_3 \mid i, j \in [2]\}$	$\{i\mathbf{e}_1 \mid i \in [p^3]\}$
20	$\{(i + jp + kp^2)\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + \alpha_i\mathbf{e}_3 \mid i, j, k \in [p]\}$	$\{\alpha_i p\mathbf{e}_1 + j\mathbf{e}_2 + i\mathbf{e}_3 \mid i, j \in [2]\}$
21	$\{(i + jp^2)\mathbf{e}_1 + \alpha_i\mathbf{e}_2 + \beta_i\mathbf{e}_3 \mid i \in [p^2], j \in [p]\}$	$\{\alpha_{ij}p^2\mathbf{e}_1 + i\mathbf{e}_2 + j\mathbf{e}_3 \mid i, j \in [2]\}$
22	$\{(i + jp)\mathbf{e}_1 + \alpha_i\mathbf{e}_2 + \beta_i\mathbf{e}_3 \mid i \in [p], j \in [p^2]\}$	$\{\alpha_{ij}p\mathbf{e}_1 + i\mathbf{e}_2 + j\mathbf{e}_3 \mid i, j \in [2]\}$
23	$\{(i + jp^2)\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + \alpha_i\mathbf{e}_3 \mid i \in [p^2], j \in [p]\}$	$\{\alpha_i p^2\mathbf{e}_1 + j\mathbf{e}_2 + i\mathbf{e}_3 \mid i, j \in [2]\}$
24	$\{(i + jp)\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + \alpha_i\mathbf{e}_3 \mid i \in [p], j \in [p^2]\}$	$\{\alpha_i p\mathbf{e}_1 + j\mathbf{e}_2 + i\mathbf{e}_3 \mid i, j \in [2]\}$
25	$\{i\mathbf{e}_1 + \alpha_i\mathbf{e}_2 + \beta_i\mathbf{e}_3 \mid i \in [p^3]\}$	$\{i\mathbf{e}_2 + j\mathbf{e}_3 \mid i, j \in [2]\}$
26	$\{(i + \alpha_i p + jp^2)\mathbf{e}_1 + k\mathbf{e}_2 + \alpha_{ij}\mathbf{e}_3 \mid i, j \in [p], k \in [2]\}$	$\{(ip + \alpha_i p^2)\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + j\mathbf{e}_3 \mid i \in [p], j \in [2]\}$
27	$\{(i + jp + \alpha_{ij}p^2)\mathbf{e}_1 + k\mathbf{e}_2 + \alpha_i\mathbf{e}_3 \mid i, j \in [p], k \in [2]\}$	$\{(\alpha_i p + jp^2)\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + i\mathbf{e}_3 \mid i \in [2], j \in [p]\}$
28	$\{(i + \alpha_i p^2)\mathbf{e}_1 + j\mathbf{e}_2 + \alpha_i\mathbf{e}_3 \mid i \in [p^2], j \in [2]\}$	$\{ip^2\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + j\mathbf{e}_3 \mid i \in [p], j \in [2]\}$
29	$\{(i + \alpha_{ij}p + kp^2)\mathbf{e}_1 + j\mathbf{e}_2 + \alpha_i\mathbf{e}_3 \mid i, k \in [p], j \in [2]\}$	$\{(jp + \alpha_{ij}p^2)\mathbf{e}_1 + \alpha_i\mathbf{e}_2 + i\mathbf{e}_3 \mid i \in [2], j \in [p]\}$
30	$\{(i + \alpha_i p + kp^2)\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + j\mathbf{e}_3 \mid i, k \in [p], j \in [2]\}$	$\{(ip + \alpha_{ij}p^2)\mathbf{e}_1 + j\mathbf{e}_2 + \alpha_i\mathbf{e}_3 \mid i \in [p], j \in [2]\}$
31	$\{(\alpha_i + jp + kp^2)\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + i\mathbf{e}_3 \mid i \in [2], j, k \in [p]\}$	$\{(i + \alpha_i p + \alpha_{ij}p^2)\mathbf{e}_1 + j\mathbf{e}_2 \mid i \in [p], j \in [2]\}$
32	$\{(i + \alpha_{ij}p + kp^2)\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + j\mathbf{e}_3 \mid i, k \in [p], j \in [2]\}$	$\{(ip + \alpha_{ij}p^2)\mathbf{e}_1 + j\mathbf{e}_2 \mid i \in [p], j \in [2]\}$
33	$\{(i + \alpha_i p + jp^2)\mathbf{e}_1 + k\mathbf{e}_2 + \alpha_i\mathbf{e}_3 \mid i, j \in [p], k \in [2]\}$	$\{(ip + \alpha_{ij}p^2)\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + j\mathbf{e}_3 \mid i \in [p], j \in [2]\}$
34	$\{(i + \alpha_i p^2)\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + j\mathbf{e}_3 \mid i \in [p^2], j \in [2]\}$	$\{ip^2\mathbf{e}_1 + j\mathbf{e}_2 + \alpha_i\mathbf{e}_3 \mid i \in [p], j \in [2]\}$
35	$\{(i + \alpha_{ik}p + jp^2)\mathbf{e}_1 + \alpha_{ijk}\mathbf{e}_2 + k\mathbf{e}_3 \mid i, j \in [p], k \in [2]\}$	$\{(ip + \alpha_i p^2)\mathbf{e}_1 + j\mathbf{e}_2 \mid i \in [p], j \in [2]\}$
36	$\{(i + \alpha_i p + jp^2)\mathbf{e}_1 + \alpha_{ijk}\mathbf{e}_2 + k\mathbf{e}_3 \mid i, j \in [p], k \in [2]\}$	$\{(ip + \alpha_i p^2)\mathbf{e}_1 + j\mathbf{e}_2 + \alpha_i\mathbf{e}_3 \mid i \in [p], j \in [2]\}$
37	$\{(\alpha_i + jp)\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + i\mathbf{e}_3 \mid i \in [2], j \in [p^2]\}$	$\{(i + \alpha_i p)\mathbf{e}_1 + j\mathbf{e}_2 \mid i \in [p], j \in [2]\}$
38	$\{(i + \alpha_{ij}p^2)\mathbf{e}_1 + j\mathbf{e}_2 + \alpha_i\mathbf{e}_3 \mid i \in [p^2], j \in [2]\}$	$\{jp^2\mathbf{e}_1 + \alpha_i\mathbf{e}_2 + i\mathbf{e}_3 \mid i \in [2], j \in [p]\}$
39	$\{(i + kp + \alpha_{ijk}p^2)\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + j\mathbf{e}_3 \mid i, k \in [p], j \in [2]\}$	$\{(\alpha_i p + jp^2)\mathbf{e}_1 + i\mathbf{e}_2 \mid i \in [2], j \in [p]\}$
40	$\{(i + jp + \alpha_{ijk}p^2)\mathbf{e}_1 + k\mathbf{e}_2 + \alpha_i\mathbf{e}_3 \mid i, j \in [p], k \in [2]\}$	$\{(\alpha_i p + jp^2)\mathbf{e}_1 + \alpha_i\mathbf{e}_2 + i\mathbf{e}_3 \mid i \in [2], j \in [p]\}$
41	$\{(i + \alpha_{ij}p^2)\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + j\mathbf{e}_3 \mid i \in [p^2], j \in [2]\}$	$\{ip^2\mathbf{e}_1 + j\mathbf{e}_2 \mid i \in [p], j \in [2]\}$
42	$\{(i + \alpha_i p + \alpha_{ij}p^2)\mathbf{e}_1 + k\mathbf{e}_2 + j\mathbf{e}_3 \mid i \in [p], j, k \in [2]\}$	$\{(ip + jp^2)\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + \alpha_i\mathbf{e}_3 \mid i, j \in [p]\}$
43	$\{(\alpha_i + jp + \alpha_{ij}p^2)\mathbf{e}_1 + k\mathbf{e}_2 + i\mathbf{e}_3 \mid i, k \in [2], j \in [p]\}$	$\{(i + \alpha_i p + jp^2)\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 \mid i, j \in [p]\}$
44	$\{(i + \alpha_{ij}p)\mathbf{e}_1 + k\mathbf{e}_2 + j\mathbf{e}_3 \mid i \in [p], j, k \in [2]\}$	$\{ipe_1 + \alpha_i\mathbf{e}_2 \mid i \in [p^2]\}$
45	$\{(\alpha_i + \alpha_{ij}p + kp^2)\mathbf{e}_1 + j\mathbf{e}_2 + i\mathbf{e}_3 \mid i, j \in [2], k \in [p]\}$	$\{(i + jp + \alpha_{ij}p^2)\mathbf{e}_1 + \alpha_i\mathbf{e}_2 \mid i, j \in [p]\}$
46	$\{(\alpha_{ij} + kp^2)\mathbf{e}_1 + i\mathbf{e}_2 + j\mathbf{e}_3 \mid i, j \in [2], k \in [p]\}$	$\{(i + \alpha_i p^2)\mathbf{e}_1 \mid i \in [p^2]\}$
47	$\{(i + \alpha_i p)\mathbf{e}_1 + j\mathbf{e}_2 + k\mathbf{e}_3 \mid i \in [p], j, k \in [2]\}$	$\{ipe_1 + \alpha_i\mathbf{e}_2 + \beta_i\mathbf{e}_3 \mid i \in [p^2]\}$
48	$\{(\alpha_i + jp^2)\mathbf{e}_1 + k\mathbf{e}_2 + i\mathbf{e}_3 \mid i, k \in [2], j \in [p]\}$	$\{(i + \alpha_i p^2)\mathbf{e}_1 + \alpha_i\mathbf{e}_2 \mid i \in [p^2]\}$
49	$\{(i + \alpha_{ij}p + \alpha_{ijk}p^2)\mathbf{e}_1 + k\mathbf{e}_2 + j\mathbf{e}_3 \mid i \in [p], j, k \in [2]\}$	$\{(ip + jp^2)\mathbf{e}_1 + \alpha_i\mathbf{e}_2 \mid i, j \in [p]\}$
50	$\{(\alpha_{ij} + kp + \alpha_{ijk}p^2)\mathbf{e}_1 + i\mathbf{e}_2 + j\mathbf{e}_3 \mid i, j \in [2], k \in [p]\}$	$\{(i + \alpha_i p + jp^2)\mathbf{e}_1 \mid i, j \in [p]\}$
51	$\{(\alpha_i + jp + \alpha_{ijk}p^2)\mathbf{e}_1 + k\mathbf{e}_2 + i\mathbf{e}_3 \mid i, k \in [2], j \in [p]\}$	$\{(i + \alpha_i p + jp^2)\mathbf{e}_1 + \alpha_i\mathbf{e}_2 \mid i, j \in [p]\}$
52	$\{(i + \alpha_i p + \alpha_{ijk}p^2)\mathbf{e}_1 + j\mathbf{e}_2 + k\mathbf{e}_3 \mid i \in [p], j, k \in [2]\}$	$\{(ip + jp^2)\mathbf{e}_1 + \alpha_i\mathbf{e}_2 + \beta_i\mathbf{e}_3 \mid i, j \in [p]\}$
53	$\{(i + \alpha_{ijk}p)\mathbf{e}_1 + j\mathbf{e}_2 + k\mathbf{e}_3 \mid i \in [p], j, k \in [2]\}$	$\{ipe_1 \mid i \in [p^2]\}$
54	$\{i\mathbf{e}_1 + j\mathbf{e}_2 + \alpha_i\mathbf{e}_3 \mid i \in [p^3], j \in [2]\}$	$\{\alpha_i\mathbf{e}_2 + i\mathbf{e}_3 \mid i \in [2]\}$
55	$\{(i + kp^2)\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + j\mathbf{e}_3 \mid i \in [p^2], j \in [2], k \in [p]\}$	$\{\alpha_i p^2\mathbf{e}_1 + i\mathbf{e}_2 \mid i \in [2]\}$
56	$\{(i + jp^2)\mathbf{e}_1 + k\mathbf{e}_2 + \alpha_i\mathbf{e}_3 \mid i \in [p^2], j \in [p], k \in [2]\}$	$\{\alpha_i p^2\mathbf{e}_1 + \alpha_i\mathbf{e}_2 + i\mathbf{e}_3 \mid i \in [2]\}$
57	$\{(i + kp)\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + j\mathbf{e}_3 \mid i \in [p], j \in [2], k \in [p^2]\}$	$\{\alpha_i p\mathbf{e}_1 + i\mathbf{e}_2 \mid i \in [2]\}$
58	$\{(i + jp)\mathbf{e}_1 + k\mathbf{e}_2 + \alpha_i\mathbf{e}_3 \mid i \in [p], j \in [p^2], k \in [2]\}$	$\{\alpha_i p\mathbf{e}_1 + \alpha_i\mathbf{e}_2 + i\mathbf{e}_3 \mid i \in [2]\}$
59	$\{i\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + j\mathbf{e}_3 \mid i \in [p^3], j \in [2]\}$	$\{i\mathbf{e}_2 \mid i \in [2]\}$
60	$\{(i + \alpha_{ij}p^2)\mathbf{e}_1 + k\mathbf{e}_2 + j\mathbf{e}_3 \mid i \in [p^2], j, k \in [2]\}$	$\{ip^2\mathbf{e}_1 + \alpha_i\mathbf{e}_2 \mid i \in [p]\}$
61	$\{(i + \alpha_{ijk}p + lp^2)\mathbf{e}_1 + j\mathbf{e}_2 + k\mathbf{e}_3 \mid i, l \in [p], j, k \in [2]\}$	$\{(ip + \alpha_i p^2)\mathbf{e}_1 \mid i \in [p]\}$
62	$\{(i + \alpha_i p^2)\mathbf{e}_1 + j\mathbf{e}_2 + k\mathbf{e}_3 \mid i \in [p^2], j, k \in [2]\}$	$\{ip^2\mathbf{e}_1 + \alpha_i\mathbf{e}_2 + \beta_i\mathbf{e}_3 \mid i \in [p]\}$
63	$\{(i + \alpha_{ij}p + kp^2)\mathbf{e}_1 + l\mathbf{e}_2 + j\mathbf{e}_3 \mid i, k \in [p], j, l \in [2]\}$	$\{(ip + \alpha_i p^2)\mathbf{e}_1 + \alpha_i\mathbf{e}_2 \mid i \in [p]\}$
64	$\{(\alpha_{ij} + kp)\mathbf{e}_1 + i\mathbf{e}_2 + j\mathbf{e}_3 \mid i, j \in [2], k \in [p^2]\}$	$\{(i + \alpha_i p)\mathbf{e}_1 \mid i \in [p]\}$
65	$\{(i + \alpha_i p + jp^2)\mathbf{e}_1 + k\mathbf{e}_2 + l\mathbf{e}_3 \mid i, j \in [p], k, l \in [2]\}$	$\{(ip + \alpha_i p^2)\mathbf{e}_1 + \alpha_i\mathbf{e}_2 + \beta_i\mathbf{e}_3 \mid i \in [p]\}$
66	$\{(\alpha_i + jp)\mathbf{e}_1 + k\mathbf{e}_2 + i\mathbf{e}_3 \mid i, k \in [2], j \in [p^2]\}$	$\{(i + \alpha_i p)\mathbf{e}_1 + \alpha_i\mathbf{e}_2 \mid i \in [p]\}$
67	$\{(i + \alpha_{ijk}p^2)\mathbf{e}_1 + j\mathbf{e}_2 + k\mathbf{e}_3 \mid i \in [p^2], j, k \in [2]\}$	$\{ip^2\mathbf{e}_1 \mid i \in [p]\}$

Proof. Suppose C is a perfect code in $\text{Cay}(G, S)$. Then $G = S_0 \oplus C$ and so $|S_0| \cdot |C| = |G| = 4p^3$. Since G is not a cyclic group, $|S_0| > 2$ by Lemma 2.8. Therefore $2 < |S_0| < 4p^3$.

Case 1. S_0 is aperiodic.

Since G is Hajós and $G = S_0 \oplus C$, in this case C must be periodic, that is, L_C is nontrivial. Write $C = L_C \oplus D$ for some subset D of G whose existence is ensured by

Lemma 2.3. Then $|L_C| \cdot |D| = |C|$ and $2 \leq |L_C| \leq |C|$. If $|S_0|$ is a prime, then $|L_C| = |C|$ by Lemma 2.8 part (c). If $|S_0|$ is not a prime, then $|L_C|$ can be any divisor of $|C|$. Set

$$A = (S_0 + L_C)/L_C \quad \text{and} \quad B = (D + L_C)/L_C.$$

Since C is periodic, by Lemma 2.5, we have

$$G/L_C = A \oplus B$$

and B is aperiodic in G/L_C . Since G/L_C is isomorphic to a subgroup of G and subgroups of Hajós group are also Hajós groups, A is periodic in G/L_C . We also know that A contains L_C and A generates G/L_C by Lemma 2.7. Furthermore, $|A| = |S_0|$ and $|B| = |D|$. Note that if $|L_C| = |C|$, then $C = L_C$ as $0 \in C$. This means we can take $D = \{0\}$ and hence $B = L_C/L_C$ and $G/L_C = A$. We use C to get A and then use Lemma 2.6 to obtain S_0 . If $|L_C| < |C|$, then this factorization enables us to make use of previously established theorems to obtain the pair (S_0, C) via (A, B) .

Similar to Table 5.9, the columns of Table 5.16 represent all possible values for $(|S_0|, |C|, |L_C|)$; C , A , and S_0 when $|L_C| = |C|$ or L_C , A , B , S_0 , and C when $|L_C| < |C|$; which previous theorem is used to obtain (A, B) when $|L_C| < |C|$; and which row in Table 5.15 the result corresponds to.

Table 5.16: The case when $G = \mathbb{Z}_{p^3} \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and S_0 is aperiodic

(S_0 , C , L_C)	$L_C, A, B, S_0, \text{ and } C$	Reference	Result
$(p, 4p^2, 4p^2)$	$C = \langle p\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_3 \rangle$ $A = \{i\mathbf{e}_1 + L_C \mid i \in [p]\}$ $S_0 = \{(i + \alpha_i p)\mathbf{e}_1 + \alpha_i \mathbf{e}_2 + \beta_i \mathbf{e}_3 \mid i \in [p]\}$		row 1
$(p^2, 4p, 2)$	$L_C = \langle \mathbf{e}_2 \rangle$ $A = \{(i + \alpha_i p + jp^2)\mathbf{e}_1 + \alpha_i \mathbf{e}_3 + L_C \mid i, j \in [p]\}$ $B = \{(ip + \alpha_{ij} p^2)\mathbf{e}_1 + j\mathbf{e}_3 + L_C \mid i \in [p], j \in [2]\}$ $S_0 = \{(i + \alpha_i p + jp^2)\mathbf{e}_1 + \alpha_{ij} \mathbf{e}_2 + \alpha_i \mathbf{e}_3 \mid i, j \in [p]\}$ $C = \{(ip + \alpha_{ij} p^2)\mathbf{e}_1 + k\mathbf{e}_2 + j\mathbf{e}_3 \mid i \in [p], j, k \in [2]\}$	5.7 (row 3)	row 3
$(p^2, 4p, p)$	$L_C = \langle p^2 \mathbf{e}_1 \rangle$ $A = \{(i + jp)\mathbf{e}_1 + \alpha_i \mathbf{e}_2 + \beta_i \mathbf{e}_3 + L_C \mid i, j \in [p]\}$ $B = \{(\alpha_{ij} p, i, j) + L_C \mid i, j \in [2]\}$ $S_0 = \{(i + jp + \alpha_{ij} p^2)\mathbf{e}_1 + \alpha_i \mathbf{e}_2 + \beta_i \mathbf{e}_3 \mid i, j \in [p]\}$ $C = \{(\alpha_{ij} p + kp^2)\mathbf{e}_1 + i\mathbf{e}_2 + j\mathbf{e}_3 \mid i, j \in [2], k \in [p]\}$	5.8 (row 2)	row 4
$(p^2, 4p, 4)$	$L_C = \langle \mathbf{e}_2, \mathbf{e}_3 \rangle$ $A = \{(i + \alpha_i p + jp^2)\mathbf{e}_1 + L_C \mid i, j \in [p]\}$ $B = \{(ip + \alpha_i p^2)\mathbf{e}_1 + L_C \mid i \in [p]\}$ $S_0 = \{(i + \alpha_i p + jp^2)\mathbf{e}_1 + \alpha_{ij} \mathbf{e}_2 + \beta_{ij} \mathbf{e}_3 \mid i, j \in [p]\}$ $C = \{(ip + \alpha_i p^2)\mathbf{e}_1 + j\mathbf{e}_2 + k\mathbf{e}_3 \mid i \in [p], j, k \in [2]\}$	5.3 (row 2)	row 5
$(p^2, 4p, 2p)$	$L_C = \langle p^2 \mathbf{e}_1 + \mathbf{e}_2 \rangle$ $A = \{(i + jp)\mathbf{e}_1 + \alpha_i \mathbf{e}_3 + L_C \mid i, j \in [p]\}$ $B = \{\alpha_i p \mathbf{e}_1 + i\mathbf{e}_3 + L_C \mid i \in [2]\}$ $S_0 = \{(i + jp + \alpha_{ij} p^2)\mathbf{e}_1 + \alpha_{ij} \mathbf{e}_2 + \alpha_i \mathbf{e}_3 \mid i, j \in [p]\}$ $C = \{(\alpha_i p + jp^2)\mathbf{e}_1 + k\mathbf{e}_2 + i\mathbf{e}_3 \mid i, k \in [2], j \in [p]\}$	5.4 (row 3)	row 6

$(p^2, 4p, 4p)$	$C = \langle p^2 \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \rangle$ $A = \{i\mathbf{e}_1 + L_C \mid i \in [p^2]\}$ $S_0 = \{(i + \alpha_i p^2)\mathbf{e}_1 + \beta_i \mathbf{e}_2 + \gamma_i \mathbf{e}_3 \mid i \in [p^2]\}$		row 7
$(2p, 2p^2, 2)$	$L_C = \langle \mathbf{e}_2 \rangle$ $A = \{(i + \alpha_i p)\mathbf{e}_1 + j\mathbf{e}_3 + L_C \mid i \in [p], j \in [2]\}$ $B = \{ip\mathbf{e}_1 + \alpha_i \mathbf{e}_3 + L_C \mid i \in [p^2]\}$ $S_0 = \{(i + \alpha_i p)\mathbf{e}_1 + \alpha_{ij} \mathbf{e}_2 + j\mathbf{e}_3 \mid i \in [p], j \in [2]\}$ $C = \{ip\mathbf{e}_1 + j\mathbf{e}_2 + \alpha_i \mathbf{e}_3 \mid i \in [p^2], j \in [2]\}$	5.7 (row 7)	row 9
$(2p, 2p^2, 2)$	$L_C = \langle \mathbf{e}_2 \rangle$ $A = \{(\alpha_i + jp^2)\mathbf{e}_1 + i\mathbf{e}_3 + L_C \mid i \in [2], j \in [p]\}$ $B = \{(i + \alpha_i p^2)\mathbf{e}_1 + L_C \mid i \in [p^2]\}$ $S_0 = \{(\alpha_i + jp^2)\mathbf{e}_1 + \alpha_{ij} \mathbf{e}_2 + i\mathbf{e}_3 \mid i \in [2], j \in [p]\}$ $C = \{(i + \alpha_i p^2)\mathbf{e}_1 + j\mathbf{e}_2 \mid i \in [p^2], j \in [2]\}$	5.7 (row 8)	row 10
$(2p, 2p^2, p)$	$L_C = \langle p^2 \mathbf{e}_1 \rangle$ $A = \{(i + \alpha_i p)\mathbf{e}_1 + j\mathbf{e}_2 + \alpha_i \mathbf{e}_3 + L_C \mid i \in [p], j \in [2]\}$ $B = \{ip\mathbf{e}_1 + \alpha_{ij} \mathbf{e}_2 + j\mathbf{e}_3 + L_C \mid i \in [p], j \in [2]\}$ $S_0 = \{(i + \alpha_i p + \alpha_{ij} p^2)\mathbf{e}_1 + j\mathbf{e}_2 + \alpha_i \mathbf{e}_3 \mid i \in [p], j \in [2]\}$ $C = \{(ip + kp^2)\mathbf{e}_1 + \alpha_{ij} \mathbf{e}_2 + j\mathbf{e}_3 \mid i, k \in [p], j \in [2]\}$	5.8 (row 5)	row 11
$(2p, 2p^2, 2p)$	$L_C = \langle p^2 \mathbf{e}_1 + \mathbf{e}_2 \rangle$ $A = \{(i + \alpha_i p)\mathbf{e}_1 + j\mathbf{e}_3 + L_C \mid i \in [p], j \in [2]\}$ $B = \{ip\mathbf{e}_1 + \alpha_i \mathbf{e}_3 + L_C \mid i \in [p]\}$ $S_0 = \{(i + \alpha_i p + \alpha_{ij} p^2)\mathbf{e}_1 + \alpha_{ij} \mathbf{e}_2 + j\mathbf{e}_3 \mid i \in [p], j \in [2]\}$ $C = \{(ip + jp^2)\mathbf{e}_1 + k\mathbf{e}_2 + \alpha_i \mathbf{e}_3 \mid i, j \in [p], k \in [2]\}$	5.4 (row 5)	row 12
$(2p, 2p^2, 2p)$	$L_C = \langle p^2 \mathbf{e}_1 + \mathbf{e}_2 \rangle$ $A = \{(\alpha_i + jp)\mathbf{e}_1 + i\mathbf{e}_3 + L_C \mid i \in [2], j \in [p]\}$ $B = \{(i + \alpha_i p)\mathbf{e}_1 + L_C \mid i \in [p]\}$ $S_0 = \{(\alpha_i + jp + \alpha_{ij} p^2)\mathbf{e}_1 + \alpha_{ij} \mathbf{e}_2 + i\mathbf{e}_3 \mid i \in [2], j \in [p]\}$ $C = \{(i + \alpha_i p + jp^2)\mathbf{e}_1 + k\mathbf{e}_2 \mid i, j \in [p], k \in [2]\}$	5.4 (row 6)	row 13
$(2p, 2p^2, p^2)$	$L_C = \langle p\mathbf{e}_1 \rangle$ $A = \{i\mathbf{e}_1 + j\mathbf{e}_2 + \alpha_i \mathbf{e}_3 + L_C \mid i \in [p], j \in [2]\}$ $B = \{\alpha_i \mathbf{e}_2 + i\mathbf{e}_3 + L_C \mid i \in [2]\}$ $S_0 = \{(i + \alpha_{ij} p)\mathbf{e}_1 + j\mathbf{e}_2 + \alpha_i \mathbf{e}_3 \mid i \in [p], j \in [2]\}$ $C = \{jp\mathbf{e}_1 + \alpha_i \mathbf{e}_2 + i\mathbf{e}_3 \mid i \in [2], j \in [p^2]\}$	3.6 (row 4)	row 14
$(2p, 2p^2, 2p^2)$	$C = \langle p\mathbf{e}_1 + \mathbf{e}_2 \rangle$ $A = \{i\mathbf{e}_1 + j\mathbf{e}_3 + L_C \mid i \in [p], j \in [2]\}$ $S_0 = \{(i + \alpha_{ij} p)\mathbf{e}_1 + \alpha_{ij} \mathbf{e}_2 + j\mathbf{e}_3 \mid i \in [p], j \in [2]\}$		row 15
$(4, p^3, p)$	$L_C = \langle p^2 \mathbf{e}_1 \rangle$ $A = \{\alpha_i \mathbf{e}_1 + j\mathbf{e}_2 + i\mathbf{e}_3 + L_C \mid i, j \in [2]\}$ $B = \{i\mathbf{e}_1 + \alpha_i \mathbf{e}_2 + L_C \mid i \in [p^2]\}$ $S_0 = \{(\alpha_i + \alpha_{ij} p^2)\mathbf{e}_1 + j\mathbf{e}_2 + i\mathbf{e}_3 \mid i, j \in [2]\}$ $C = \{(i + jp^2)\mathbf{e}_1 + \alpha_i \mathbf{e}_2 \mid i \in [p^2], j \in [p]\}$	5.8 (row 10)	row 17
$(4, p^3, p^2)$	$L_C = \langle p\mathbf{e}_1 \rangle$ $A = \{\alpha_i \mathbf{e}_1 + j\mathbf{e}_2 + i\mathbf{e}_3 + L_C \mid i, j \in [2]\}$ $B = \{i\mathbf{e}_1 + \alpha_i \mathbf{e}_2 + L_C \mid i \in [p]\}$ $S_0 = \{(\alpha_i + \alpha_{ij} p)\mathbf{e}_1 + j\mathbf{e}_2 + i\mathbf{e}_3 \mid i, j \in [2]\}$ $C = \{(i + jp)\mathbf{e}_1 + \alpha_i \mathbf{e}_2 \mid i \in [p], j \in [p^2]\}$	3.6 (row 6)	row 18
$(4, p^3, p^3)$	$C = \langle \mathbf{e}_1 \rangle$ $A = \{i\mathbf{e}_2 + j\mathbf{e}_3 + L_C \mid i, j \in [2]\}$ $S_0 = \{\alpha_{ij} \mathbf{e}_1 + i\mathbf{e}_2 + j\mathbf{e}_3 \mid i, j \in [2]\}$		row 19
	$L_C = \langle \mathbf{e}_2 \rangle$ $A = \{(i + jp^2)\mathbf{e}_1 + \alpha_i \mathbf{e}_3 + L_C \mid i \in [p^2], j \in [p]\}$		

$(p^3, 4, 2)$	$B = \{\alpha_i p^2 \mathbf{e}_1 + i \mathbf{e}_3 + L_C \mid i \in [2]\}$ $S_0 = \{(i + jp^2) \mathbf{e}_1 + \alpha_{ij} \mathbf{e}_2 + \alpha_i \mathbf{e}_3 \mid i \in [p^2], j \in [p]\}$ $C = \{\alpha_i p^2 \mathbf{e}_1 + j \mathbf{e}_2 + i \mathbf{e}_3 \mid i, j \in [2]\}$	5.7 (row 12)	row 23
$(p^3, 4, 2)$	$L_C = \langle \mathbf{e}_2 \rangle$ $A = \{(i + jp) \mathbf{e}_1 + \alpha_i \mathbf{e}_3 + L_C \mid i \in [p], j \in [p^2]\}$ $B = \{\alpha_i p \mathbf{e}_1 + i \mathbf{e}_3 + L_C \mid i \in [2]\}$ $S_0 = \{(i + jp) \mathbf{e}_1 + \alpha_{ij} \mathbf{e}_2 + \alpha_i \mathbf{e}_3 \mid i \in [p], j \in [p^2]\}$ $C = \{\alpha_i p \mathbf{e}_1 + j \mathbf{e}_2 + i \mathbf{e}_3 \mid i, j \in [2]\}$	5.7 (row 13)	row 24
$(p^3, 4, 4)$	$C = \langle \mathbf{e}_2, \mathbf{e}_3 \rangle$ $A = \{i \mathbf{e}_1 + L_C \mid i \in [p^3]\}$ $S_0 = \{i \mathbf{e}_1 + \alpha_i \mathbf{e}_2 + \beta_i \mathbf{e}_3 \mid i \in [p^3]\}$		row 25
$(2p^2, 2p, 2)$	$L_C = \langle \mathbf{e}_2 \rangle$ $A = \{(i + \alpha_i p^2) \mathbf{e}_1 + j \mathbf{e}_3 + L_C \mid i \in [p^2], j \in [2]\}$ $B = \{ip^2 \mathbf{e}_1 + \alpha_i \mathbf{e}_3 + L_C \mid i \in [p]\}$ $S_0 = \{(i + \alpha_i p^2) \mathbf{e}_1 + \alpha_{ij} \mathbf{e}_2 + j \mathbf{e}_3 \mid i \in [p^2], j \in [2]\}$ $C = \{ip^2 \mathbf{e}_1 + j \mathbf{e}_2 + \alpha_i \mathbf{e}_3 \mid i \in [p], j \in [2]\}$	5.7 (row 15)	row 34
$(2p^2, 2p, 2)$	$L_C = \langle \mathbf{e}_2 \rangle$ $A = \{(i + \alpha_{ik} p + jp^2) \mathbf{e}_1 + k \mathbf{e}_3 + L_C \mid i, j \in [p], k \in [2]\}$ $B = \{(ip + \alpha_i p^2) \mathbf{e}_1 + L_C \mid i \in [p]\}$ $S_0 = \{(i + \alpha_{ik} p + jp^2) \mathbf{e}_1 + \alpha_{ijk} \mathbf{e}_2 + k \mathbf{e}_3 \mid i, j \in [p], k \in [2]\}$ $C = \{(ip + \alpha_i p^2) \mathbf{e}_1 + j \mathbf{e}_2 \mid i \in [p], j \in [2]\}$	5.7 (row 16)	row 35
$(2p^2, 2p, 2)$	$L_C = \langle \mathbf{e}_2 \rangle$ $A = \{(i + \alpha_i p + jp^2) \mathbf{e}_1 + k \mathbf{e}_3 + L_C \mid i, j \in [p], k \in [2]\}$ $B = \{(ip + \alpha_i p^2) \mathbf{e}_1 + \alpha_i \mathbf{e}_3 + L_C \mid i \in [p]\}$ $S_0 = \{(i + \alpha_i p + jp^2) \mathbf{e}_1 + \alpha_{ijk} \mathbf{e}_2 + k \mathbf{e}_3 \mid i, j \in [p], k \in [2]\}$ $C = \{(ip + \alpha_i p^2) \mathbf{e}_1 + j \mathbf{e}_2 + \alpha_i \mathbf{e}_3 \mid i \in [p], j \in [2]\}$	5.7 (row 17)	row 36
$(2p^2, 2p, 2)$	$L_C = \langle \mathbf{e}_2 \rangle$ $A = \{(\alpha_i + jp) \mathbf{e}_1 + i \mathbf{e}_3 + L_C \mid i \in [2], j \in [p^2]\}$ $B = \{(i + \alpha_i p) \mathbf{e}_1 + L_C \mid i \in [p]\}$ $S_0 = \{(\alpha_i + jp) \mathbf{e}_1 + \alpha_{ij} \mathbf{e}_2 + i \mathbf{e}_3 \mid i \in [2], j \in [p^2]\}$ $C = \{(i + \alpha_i p) \mathbf{e}_1 + j \mathbf{e}_2 \mid i \in [p], j \in [2]\}$	5.7 (row 18)	row 37
$(2p^2, 2p, p)$	$L_C = \langle p^2 \mathbf{e}_1 \rangle$ $A = \{i \mathbf{e}_1 + j \mathbf{e}_2 + \alpha_i \mathbf{e}_3 + L_C \mid i \in [p^2], j \in [2]\}$ $B = \{\alpha_i \mathbf{e}_2 + i \mathbf{e}_3 + L_C \mid i \in [2]\}$ $S_0 = \{(i + \alpha_{ij} p^2) \mathbf{e}_1 + j \mathbf{e}_2 + \alpha_i \mathbf{e}_3 \mid i \in [p^2], j \in [2]\}$ $C = \{jp^2 \mathbf{e}_1 + \alpha_i \mathbf{e}_2 + i \mathbf{e}_3 \mid i \in [2], j \in [p]\}$	5.8 (row 13)	row 38
$(2p^2, 2p, p)$	$L_C = \langle p^2 \mathbf{e}_1 \rangle$ $A = \{(i + kp) \mathbf{e}_1 + \alpha_{ij} \mathbf{e}_2 + j \mathbf{e}_3 + L_C \mid i, k \in [p], j \in [2]\}$ $B = \{\alpha_i p \mathbf{e}_1 + i \mathbf{e}_2 + L_C \mid i \in [2]\}$ $S_0 = \{(i + kp + \alpha_{ijk} p^2) \mathbf{e}_1 + \alpha_{ij} \mathbf{e}_2 + j \mathbf{e}_3 \mid i, k \in [p], j \in [2]\}$ $C = \{(\alpha_i p + jp^2) \mathbf{e}_1 + i \mathbf{e}_2 \mid i \in [2], j \in [p]\}$	5.8 (row 14)	row 39
$(2p^2, 2p, p)$	$L_C = \langle p^2 \mathbf{e}_1 \rangle$ $A = \{(i + jp) \mathbf{e}_1 + k \mathbf{e}_2 + \alpha_i \mathbf{e}_3 + L_C \mid i, j \in [p], k \in [2]\}$ $B = \{\alpha_i p \mathbf{e}_1 + \alpha_i \mathbf{e}_2 + i \mathbf{e}_3 + L_C \mid i \in [2]\}$ $S_0 = \{(i + jp + \alpha_{ijk} p^2) \mathbf{e}_1 + k \mathbf{e}_2 + \alpha_i \mathbf{e}_3 \mid i, j \in [p], k \in [2]\}$ $C = \{(\alpha_i p + jp^2) \mathbf{e}_1 + \alpha_i \mathbf{e}_2 + i \mathbf{e}_3 \mid i \in [2], j \in [p]\}$	5.8 (row 15)	row 40
$(2p^2, 2p, 2p)$	$C = \langle p^2 \mathbf{e}_1 + \mathbf{e}_2 \rangle$ $A = \{i \mathbf{e}_1 + j \mathbf{e}_3 + L_C \mid i \in [p^2], j \in [2]\}$ $S_0 = \{(i + \alpha_{ij} p^2) \mathbf{e}_1 + \alpha_{ij} \mathbf{e}_2 + j \mathbf{e}_3 \mid i \in [p^2], j \in [2]\}$		row 41
	$L_C = \langle p^2 \mathbf{e}_1 \rangle$ $A = \{(i + \alpha_{ij} p) \mathbf{e}_1 + k \mathbf{e}_2 + j \mathbf{e}_3 + L_C \mid i \in [p], j, k \in [2]\}$		

$(4p, p^2, p)$	$B = \{ip\mathbf{e}_1 + \alpha_i\mathbf{e}_2 + L_C \mid i \in [p]\}$ $S_0 = \{(i + \alpha_{ij}p + \alpha_{ijk}p^2)\mathbf{e}_1 + k\mathbf{e}_2 + j\mathbf{e}_3 \mid i \in [p], j, k \in [2]\}$ $C = \{(ip + jp^2)\mathbf{e}_1 + \alpha_i\mathbf{e}_2 \mid i, j \in [p]\}$	5.8 (row 17)	row 49
$(4p, p^2, p)$	$L_C = \langle p^2\mathbf{e}_1 \rangle$ $A = \{(\alpha_{ij} + kp)\mathbf{e}_1 + i\mathbf{e}_2 + j\mathbf{e}_3 + L_C \mid i, j \in [2], k \in [p]\}$ $B = \{(i + \alpha_i p)\mathbf{e}_1 + L_C \mid i \in [p]\}$ $S_0 = \{(\alpha_{ij} + kp + \alpha_{ijk}p^2)\mathbf{e}_1 + i\mathbf{e}_2 + j\mathbf{e}_3 \mid i, j \in [2], k \in [p]\}$ $C = \{(i + \alpha_i p + jp^2)\mathbf{e}_1 \mid i, j \in [p]\}$	5.8 (row 18)	row 50
$(4p, p^2, p)$	$L_C = \langle p^2\mathbf{e}_1 \rangle$ $A = \{(\alpha_i + jp)\mathbf{e}_1 + k\mathbf{e}_2 + i\mathbf{e}_3 + L_C \mid i, k \in [2], j \in [p]\}$ $B = \{(i + \alpha_i p)\mathbf{e}_1 + \alpha_i\mathbf{e}_2 + L_C \mid i \in [p]\}$ $S_0 = \{(\alpha_i + jp + \alpha_{ijk}p^2)\mathbf{e}_1 + k\mathbf{e}_2 + i\mathbf{e}_3 \mid i, k \in [2], j \in [p]\}$ $C = \{(i + \alpha_i p + jp^2)\mathbf{e}_1 + \alpha_i\mathbf{e}_2 \mid i, j \in [p]\}$	5.8 (row 19)	row 51
$(4p, p^2, p)$	$L_C = \langle p^2\mathbf{e}_1 \rangle$ $A = \{(i + \alpha_i p, j, k) + L_C \mid i \in [p], j, k \in [2]\}$ $B = \{ip\mathbf{e}_1 + \alpha_i\mathbf{e}_2 + \beta_i\mathbf{e}_3 + L_C \mid i \in [p]\}$ $S_0 = \{(i + \alpha_i p + \alpha_{ijk}p^2)\mathbf{e}_1 + j\mathbf{e}_2 + k\mathbf{e}_3 \mid i \in [p], j, k \in [2]\}$ $C = \{(ip + jp^2)\mathbf{e}_1 + \alpha_i\mathbf{e}_2 + \beta_i\mathbf{e}_3 \mid i, j \in [p]\}$	5.8 (row 20)	row 52
$(4p, p^2, p^2)$	$C = \langle p\mathbf{e}_1 \rangle$ $A = \{i\mathbf{e}_1 + j\mathbf{e}_2 + k\mathbf{e}_3 + L_C \mid i \in [p], j, k \in [2]\}$ $S_0 = \{(i + \alpha_{ijk}p)\mathbf{e}_1 + j\mathbf{e}_2 + k\mathbf{e}_3 \mid i \in [p], j, k \in [2]\}$		row 53
$(2p^3, 2, 2)$	$C = \langle \mathbf{e}_2 \rangle$ $A = \{i\mathbf{e}_1 + j\mathbf{e}_3 + L_C \mid i \in [p^3], j \in [2]\}$ $S_0 = \{i\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + j\mathbf{e}_3 \mid i \in [p^3], j \in [2]\}$		row 59
$(4p^2, p, p)$	$C = \langle p^2\mathbf{e}_1 \rangle$ $A = \{i\mathbf{e}_1 + j\mathbf{e}_2 + k\mathbf{e}_3 + L_C \mid i \in [p^2], j, k \in [2]\}$ $S_0 = \{(i + \alpha_{ijk}p^2)\mathbf{e}_1 + j\mathbf{e}_2 + k\mathbf{e}_3 \mid i \in [p^2], j, k \in [2]\}$		row 67

Case 2. S_0 is periodic.

In this case we have $|L_{S_0}| \geq 2$ and $S_0 = D \oplus L_{S_0}$ for some subset D of G by Lemma 2.3. So $|D| \cdot |L_{S_0}| = |S_0|$ and $1 < |L_{S_0}| < |S_0|$ by Lemma 2.8 part (b). Set

$$X = (D + L_{S_0})/L_{S_0} \quad \text{and} \quad Y = (C + L_{S_0})/L_{S_0}.$$

Since S_0 is periodic, by Lemma 2.5, we have

$$G/L_{S_0} = X \oplus Y$$

and X is aperiodic in G/L_{S_0} . We also know that X contains L_{S_0} and X generates G/L_{S_0} by Lemma 2.7. This factorization enables us to make use of Lemma 2.6 and previously established theorems to obtain the pair (S_0, C) via (X, Y) .

Similar to Table 5.10, the columns of Table 5.17 represent all possible values for $(|S_0|, |C|, |L_{S_0}|)$; L_{S_0} , X , Y , S_0 , and C ; which previous theorem or lemma is used to obtain (X, Y) ; and which row in Table 5.15 the result corresponds to. $\square$

Table 5.17: The case when $G = \mathbb{Z}_{p^3} \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and S_0 is periodic

(S_0 , C , L_{S_0})	L_{S_0}, X, Y, S_0 , and C	Reference	Result
$(p^2, 4p, p)$	$L_{S_0} = \langle p^2 \mathbf{e}_1 \rangle$ $X = \{(i + \alpha_i p) \mathbf{e}_1 + \alpha_i \mathbf{e}_2 + \beta_i \mathbf{e}_3 + L_{S_0} \mid i \in [p]\}$ $Y = \{i p \mathbf{e}_1 + j \mathbf{e}_2 + k \mathbf{e}_3 + L_{S_0} \mid i \in [p], j, k \in [2]\}$ $S_0 = \{(i + \alpha_i p + j p^2) \mathbf{e}_1 + \alpha_i \mathbf{e}_2 + \beta_i \mathbf{e}_3 \mid i, j \in [p]\}$ $C = \{(i p + \alpha_{ijk} p^2) \mathbf{e}_1 + j \mathbf{e}_2 + k \mathbf{e}_3 \mid i \in [p], j, k \in [2]\}$	5.8 (row 1)	row 2
$(2p, 2p^2, 2)$	$L_{S_0} = \langle \mathbf{e}_2 \rangle$ $X = \{(i + \alpha_i p) \mathbf{e}_1 + \alpha_i \mathbf{e}_3 + L_{S_0} \mid i \in [p]\}$ $Y = \{i p \mathbf{e}_1 + j \mathbf{e}_3 + L_{S_0} \mid i \in [p^2], j \in [2]\}$ $S_0 = \{(i + \alpha_i p) \mathbf{e}_1 + j \mathbf{e}_2 + \alpha_i \mathbf{e}_3 \mid i \in [p], j \in [2]\}$ $C = \{i p \mathbf{e}_1 + \alpha_{ij} \mathbf{e}_2 + j \mathbf{e}_3 \mid i \in [p^2], j \in [2]\}$	5.7 (row 1)	row 8
$(2p, 2p^2, p)$	Not possible since no X exists	5.8	
$(4, p^3, 2)$	$L_{S_0} = \langle \mathbf{e}_2 \rangle$ $X = \{\alpha_i \mathbf{e}_1 + i \mathbf{e}_3 + L_{S_0} \mid i \in [2]\}$ $Y = \{i \mathbf{e}_1 + L_{S_0} \mid i \in [p^3]\}$ $S_0 = \{\alpha_i \mathbf{e}_1 + j \mathbf{e}_2 + i \mathbf{e}_3 \mid i, j \in [2]\}$ $C = \{i \mathbf{e}_1 + \alpha_i \mathbf{e}_2 \mid i \in [p^3]\}$	5.7 (row 2)	row 16
$(p^3, 4, p)$	$L_{S_0} = \langle p^2 \mathbf{e}_1 \rangle$ $X = \{(i + j p) \mathbf{e}_1 + \alpha_{ij} \mathbf{e}_2 + \alpha_i \mathbf{e}_3 + L_{S_0} \mid i, j \in [p]\}$ $Y = \{\alpha_i p \mathbf{e}_1 + j \mathbf{e}_2 + i \mathbf{e}_3 + L_{S_0} \mid i, j \in [2]\}$ $S_0 = \{(i + j p + k p^2) \mathbf{e}_1 + \alpha_{ij} \mathbf{e}_2 + \alpha_i \mathbf{e}_3 \mid i, j, k \in [p]\}$ $C = \{\alpha_i p \mathbf{e}_1 + j \mathbf{e}_2 + i \mathbf{e}_3 \mid i, j \in [2]\}$	5.8 (row 3)	row 20
$(p^3, 4, p)$	$L_{S_0} = \langle p^2 \mathbf{e}_1 \rangle$ $X = \{i \mathbf{e}_1 + \alpha_i \mathbf{e}_2 + \beta_i \mathbf{e}_3 + L_{S_0} \mid i \in [p^2]\}$ $Y = \{i \mathbf{e}_2 + j \mathbf{e}_3 + L_{S_0} \mid i, j \in [2]\}$ $S_0 = \{(i + j p^2) \mathbf{e}_1 + \alpha_i \mathbf{e}_2 + \beta_i \mathbf{e}_3 \mid i \in [p^2], j \in [p]\}$ $C = \{\alpha_{ij} p^2 \mathbf{e}_1 + i \mathbf{e}_2 + j \mathbf{e}_3 \mid i, j \in [2]\}$	5.8 (row 4)	row 21
$(p^3, 4, p^2)$	$L_{S_0} = \langle p \mathbf{e}_1 \rangle$ $X = \{i \mathbf{e}_1 + \alpha_i \mathbf{e}_2 + \beta_i \mathbf{e}_3 + L_{S_0} \mid i \in [p]\}$ $Y = \{i \mathbf{e}_2 + j \mathbf{e}_3 + L_{S_0} \mid i, j \in [2]\}$ $S_0 = \{(i + j p) \mathbf{e}_1 + \alpha_i \mathbf{e}_2 + \beta_i \mathbf{e}_3 \mid i \in [p], j \in [p^2]\}$ $C = \{\alpha_{ij} p \mathbf{e}_1 + i \mathbf{e}_2 + j \mathbf{e}_3 \mid i, j \in [2]\}$	3.6 (row 1)	row 22
$(2p^2, 2p, 2)$	$L_{S_0} = \langle \mathbf{e}_2 \rangle$ $X = \{(i + \alpha_i p + j p^2) \mathbf{e}_1 + \alpha_{ij} \mathbf{e}_3 + L_{S_0} \mid i, j \in [p]\}$ $Y = \{(i p + \alpha_i p^2) \mathbf{e}_1 + j \mathbf{e}_3 + L_{S_0} \mid i \in [p], j \in [2]\}$ $S_0 = \{(i + \alpha_i p + j p^2) \mathbf{e}_1 + k \mathbf{e}_2 + \alpha_{ij} \mathbf{e}_3 \mid i, j \in [p], k \in [2]\}$ $C = \{(i p + \alpha_i p^2) \mathbf{e}_1 + \alpha_{ij} \mathbf{e}_2 + j \mathbf{e}_3 \mid i \in [p], j \in [2]\}$	5.7 (row 4)	row 26
$(2p^2, 2p, 2)$	$L_{S_0} = \langle \mathbf{e}_2 \rangle$ $X = \{(i + j p + \alpha_{ij} p^2) \mathbf{e}_1 + \alpha_i \mathbf{e}_3 + L_{S_0} \mid i, j \in [p]\}$ $Y = \{(\alpha_i p + j p^2) \mathbf{e}_1 + i \mathbf{e}_3 + L_{S_0} \mid i \in [2], j \in [p]\}$ $S_0 = \{(i + j p + \alpha_{ij} p^2) \mathbf{e}_1 + k \mathbf{e}_2 + \alpha_i \mathbf{e}_3 \mid i, j \in [p], k \in [2]\}$ $C = \{(\alpha_i p + j p^2) \mathbf{e}_1 + \alpha_{ij} \mathbf{e}_2 + i \mathbf{e}_3 \mid i \in [2], j \in [p]\}$	5.7 (row 5)	row 27
$(2p^2, 2p, 2)$	$L_{S_0} = \langle \mathbf{e}_2 \rangle$ $X = \{(i + \alpha_i p^2) \mathbf{e}_1 + \alpha_i \mathbf{e}_3 + L_{S_0} \mid i \in [p^2]\}$ $Y = \{i p^2 \mathbf{e}_1 + j \mathbf{e}_3 + L_{S_0} \mid i \in [p], j \in [2]\}$ $S_0 = \{(i + \alpha_i p^2) \mathbf{e}_1 + j \mathbf{e}_2 + \alpha_i \mathbf{e}_3 \mid i \in [p^2], j \in [2]\}$ $C = \{i p^2 \mathbf{e}_1 + \alpha_{ij} \mathbf{e}_2 + j \mathbf{e}_3 \mid i \in [p], j \in [2]\}$	5.7 (row 6)	row 28
$(2p^2, 2p, p)$	$L_{S_0} = \langle p^2 \mathbf{e}_1 \rangle$ $X = \{(i + \alpha_{ij} p) \mathbf{e}_1 + j \mathbf{e}_2 + \alpha_i \mathbf{e}_3 + L_{S_0} \mid i \in [p], j \in [2]\}$ $Y = \{j p \mathbf{e}_1 + \alpha_i \mathbf{e}_2 + i \mathbf{e}_3 + L_{S_0} \mid i \in [2], j \in [p]\}$	5.8 (row 6)	row 29

	$S_0 = \{(i + \alpha_{ij}p + kp^2)\mathbf{e}_1 + j\mathbf{e}_2 + \alpha_i\mathbf{e}_3 \mid i, k \in [p], j \in [2]\}$ $C = \{(jp + \alpha_{ij}p^2)\mathbf{e}_1 + \alpha_i\mathbf{e}_2 + i\mathbf{e}_3 \mid i \in [2], j \in [p]\}$		
$(2p^2, 2p, p)$	$L_{S_0} = \langle p^2\mathbf{e}_1 \rangle$ $X = \{(i + \alpha_i p)\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + j\mathbf{e}_3 + L_{S_0} \mid i \in [p], j \in [2]\}$ $Y = \{ip\mathbf{e}_1 + j\mathbf{e}_2 + \alpha_i\mathbf{e}_3 + L_{S_0} \mid i \in [p], j \in [2]\}$ $S_0 = \{(i + \alpha_i p + kp^2)\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + j\mathbf{e}_3 \mid i, k \in [p], j \in [2]\}$ $C = \{(ip + \alpha_{ij}p^2)\mathbf{e}_1 + j\mathbf{e}_2 + \alpha_i\mathbf{e}_3 \mid i \in [p], j \in [2]\}$	5.8 (row 7)	row 30
$(2p^2, 2p, p)$	$L_{S_0} = \langle p^2\mathbf{e}_1 \rangle$ $X = \{(\alpha_i + jp)\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + i\mathbf{e}_3 + L_{S_0} \mid i \in [2], j \in [p]\}$ $Y = \{(i + \alpha_i p)\mathbf{e}_1 + j\mathbf{e}_2 + L_{S_0} \mid i \in [p], j \in [2]\}$ $S_0 = \{(\alpha_i + jp + kp^2)\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + i\mathbf{e}_3 \mid i \in [2], j, k \in [p]\}$ $C = \{(i + \alpha_i p + \alpha_{ij}p^2)\mathbf{e}_1 + j\mathbf{e}_2 \mid i \in [p], j \in [2]\}$	5.8 (row 8)	row 31
$(2p^2, 2p, p)$	$L_{S_0} = \langle p^2\mathbf{e}_1 \rangle$ $X = \{(i + \alpha_{ij}p)\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + j\mathbf{e}_3 + L_{S_0} \mid i \in [p], j \in [2]\}$ $Y = \{ip\mathbf{e}_1 + j\mathbf{e}_2 + L_{S_0} \mid i \in [p], j \in [2]\}$ $S_0 = \{(i + \alpha_{ij}p + kp^2)\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + j\mathbf{e}_3 \mid i, k \in [p], j \in [2]\}$ $C = \{(ip + \alpha_{ij}p^2)\mathbf{e}_1 + j\mathbf{e}_2 \mid i \in [p], j \in [2]\}$	5.8 (row 9)	row 32
$(2p^2, 2p, 2p)$	$L_{S_0} = \langle p^2\mathbf{e}_1 + \mathbf{e}_2 \rangle$ $X = \{(i + \alpha_i p)\mathbf{e}_1 + \alpha_i\mathbf{e}_3 + L_{S_0} \mid i \in [p]\}$ $Y = \{ip\mathbf{e}_1 + j\mathbf{e}_3 + L_{S_0} \mid i \in [p], j \in [2]\}$ $S_0 = \{(i + \alpha_i p + jp^2)\mathbf{e}_1 + k\mathbf{e}_2 + \alpha_i\mathbf{e}_3 \mid i, j \in [p], k \in [2]\}$ $C = \{(ip + \alpha_{ij}p^2)\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + j\mathbf{e}_3 \mid i \in [p], j \in [2]\}$	5.4 (row 1)	row 33
$(2p^2, 2p, p^2)$	Not possible since no X exists	3.6	
$(4p, p^2, 2)$	$L_{S_0} = \langle \mathbf{e}_2 \rangle$ $X = \{(i + \alpha_i p + \alpha_{ij}p^2)\mathbf{e}_1 + j\mathbf{e}_3 + L_{S_0} \mid i \in [p], j \in [2]\}$ $Y = \{(ip + jp^2)\mathbf{e}_1 + \alpha_i\mathbf{e}_3 + L_{S_0} \mid i, j \in [p]\}$ $S_0 = \{(i + \alpha_i p + \alpha_{ij}p^2)\mathbf{e}_1 + k\mathbf{e}_2 + j\mathbf{e}_3 \mid i \in [p], j, k \in [2]\}$ $C = \{(ip + jp^2)\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + \alpha_i\mathbf{e}_3 \mid i, j \in [p]\}$	5.7 (row 9)	row 42
$(4p, p^2, 2)$	$L_{S_0} = \langle \mathbf{e}_2 \rangle$ $X = \{(\alpha_i + jp + \alpha_{ij}p^2)\mathbf{e}_1 + i\mathbf{e}_3 + L_{S_0} \mid i \in [2], j \in [p]\}$ $Y = \{(i + \alpha_i p + jp^2)\mathbf{e}_1 + L_{S_0} \mid i, j \in [p]\}$ $S_0 = \{(\alpha_i + jp + \alpha_{ij}p^2)\mathbf{e}_1 + k\mathbf{e}_2 + i\mathbf{e}_3 \mid i, k \in [2], j \in [p]\}$ $C = \{(i + \alpha_i p + jp^2)\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 \mid i, j \in [p]\}$	5.7 (row 10)	row 43
$(4p, p^2, 2)$	$L_{S_0} = \langle \mathbf{e}_2 \rangle$ $X = \{(i + \alpha_{ij}p)\mathbf{e}_1 + j\mathbf{e}_3 + L_{S_0} \mid i \in [p], j \in [2]\}$ $Y = \{ip\mathbf{e}_1 + L_{S_0} \mid i \in [p^2]\}$ $S_0 = \{(i + \alpha_{ij}p)\mathbf{e}_1 + k\mathbf{e}_2 + j\mathbf{e}_3 \mid i \in [p], j, k \in [2]\}$ $C = \{ip\mathbf{e}_1 + \alpha_i\mathbf{e}_2 \mid i \in [p^2]\}$	5.7 (row 11)	row 44
$(4p, p^2, p)$	$L_{S_0} = \langle p^2\mathbf{e}_1 \rangle$ $X = \{(\alpha_i + \alpha_{ij}p)\mathbf{e}_1 + j\mathbf{e}_2 + i\mathbf{e}_3 + L_{S_0} \mid i, j \in [2]\}$ $Y = \{(i + jp)\mathbf{e}_1 + \alpha_i\mathbf{e}_2 + L_{S_0} \mid i, j \in [p]\}$ $S_0 = \{(\alpha_i + \alpha_{ij}p + kp^2)\mathbf{e}_1 + j\mathbf{e}_2 + i\mathbf{e}_3 \mid i, j \in [2], k \in [p]\}$ $C = \{(i + jp + \alpha_{ij}p^2)\mathbf{e}_1 + \alpha_i\mathbf{e}_2 \mid i, j \in [p]\}$	5.8 (row 11)	row 45
$(4p, p^2, p)$	$L_{S_0} = \langle p^2\mathbf{e}_1 \rangle$ $X = \{\alpha_{ij}\mathbf{e}_1 + i\mathbf{e}_2 + j\mathbf{e}_3 + L_{S_0} \mid i, j \in [2]\}$ $Y = \{i\mathbf{e}_1 + L_{S_0} \mid i \in [p^2]\}$ $S_0 = \{(\alpha_{ij} + kp^2)\mathbf{e}_1 + i\mathbf{e}_2 + j\mathbf{e}_3 \mid i, j \in [2], k \in [p]\}$ $C = \{(i + \alpha_i p^2)\mathbf{e}_1 \mid i \in [p^2]\}$	5.8 (row 12)	row 46
$(4p, p^2, 4)$	$L_{S_0} = \langle \mathbf{e}_2, \mathbf{e}_3 \rangle$ $X = \{(i + \alpha_i p)\mathbf{e}_1 + L_{S_0} \mid i \in [p]\}$ $Y = \{ip\mathbf{e}_1 + L_{S_0} \mid i \in [p^2]\}$	5.3 (row 1)	row 47

	$S_0 = \{(i + \alpha_i p)\mathbf{e}_1 + j\mathbf{e}_2 + k\mathbf{e}_3 \mid i \in [p], j, k \in [2]\}$ $C = \{ip\mathbf{e}_1 + \alpha_i\mathbf{e}_2 + \beta_i\mathbf{e}_3 \mid i \in [p^2]\}$		
$(4p, p^2, 2p)$	$L_{S_0} = \langle p^2\mathbf{e}_1 + \mathbf{e}_2 \rangle$ $X = \{\alpha_i\mathbf{e}_1 + i\mathbf{e}_3 + L_{S_0} \mid i \in [2]\}$ $Y = \{i\mathbf{e}_1 + L_{S_0} \mid i \in [p^2]\}$ $S_0 = \{(\alpha_i + jp^2)\mathbf{e}_1 + k\mathbf{e}_2 + i\mathbf{e}_3 \mid i, k \in [2], j \in [p]\}$ $C = \{(i + \alpha_i p^2)\mathbf{e}_1 + \alpha_i\mathbf{e}_2 \mid i \in [p^2]\}$	5.4 (row 2)	row 48
$(2p^3, 2, 2)$	$L_{S_0} = \langle \mathbf{e}_2 \rangle$ $X = \{i\mathbf{e}_1 + \alpha_i\mathbf{e}_3 + L_{S_0} \mid i \in [p^3]\}$ $Y = \{i\mathbf{e}_3 + L_{S_0} \mid i \in [2]\}$ $S_0 = \{i\mathbf{e}_1 + j\mathbf{e}_2 + \alpha_i\mathbf{e}_3 \mid i \in [p^3], j \in [2]\}$ $C = \{\alpha_i\mathbf{e}_2 + i\mathbf{e}_3 \mid i \in [2]\}$	5.7 (row 14)	row 54
$(2p^3, 2, p)$	$L_{S_0} = \langle p^2\mathbf{e}_1 \rangle$ $X = \{i\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + j\mathbf{e}_3 + L_{S_0} \mid i \in [p^2], j \in [2]\}$ $Y = \{i\mathbf{e}_2 + L_{S_0} \mid i \in [2]\}$ $S_0 = \{(i + kp^2)\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + j\mathbf{e}_3 \mid i \in [p^2], j \in [2], k \in [p]\}$ $C = \{\alpha_i p^2\mathbf{e}_1 + i\mathbf{e}_2 \mid i \in [2]\}$	5.8 (row 16)	row 55
$(2p^3, 2, 2p)$	$L_{S_0} = \langle p^2\mathbf{e}_1 + \mathbf{e}_2 \rangle$ $X = \{i\mathbf{e}_1 + \alpha_i\mathbf{e}_3 + L_{S_0} \mid i \in [p^2]\}$ $Y = \{i\mathbf{e}_3 + L_{S_0} \mid i \in [2]\}$ $S_0 = \{(i + jp^2)\mathbf{e}_1 + k\mathbf{e}_2 + \alpha_i\mathbf{e}_3 \mid i \in [p^2], j \in [p], k \in [2]\}$ $C = \{\alpha_i p^2\mathbf{e}_1 + \alpha_i\mathbf{e}_2 + i\mathbf{e}_3 \mid i \in [2]\}$	5.4 (row 4)	row 56
$(2p^3, 2, p^2)$	$L_{S_0} = \langle p\mathbf{e}_1 \rangle$ $X = \{i\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + j\mathbf{e}_3 + L_{S_0} \mid i \in [p], j \in [2]\}$ $Y = \{i\mathbf{e}_2 + L_{S_0} \mid i \in [2]\}$ $S_0 = \{(i + kp)\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + j\mathbf{e}_3 \mid i \in [p], j \in [2], k \in [p^2]\}$ $C = \{\alpha_i p\mathbf{e}_1 + i\mathbf{e}_2 \mid i \in [2]\}$	3.6 (row 5)	row 57
$(2p^3, 2, 2p^2)$	$L_{S_0} = \langle p\mathbf{e}_1 + \mathbf{e}_2 \rangle$ $X = \{i\mathbf{e}_1 + \alpha_i\mathbf{e}_3 + L_{S_0} \mid i \in [p]\}$ $Y = \{i\mathbf{e}_3 + L_{S_0} \mid i \in [2]\}$ $S_0 = \{(i + jp)\mathbf{e}_1 + k\mathbf{e}_2 + \alpha_i\mathbf{e}_3 \mid i \in [p], j \in [p^2], k \in [2]\}$ $C = \{\alpha_i p\mathbf{e}_1 + \alpha_i\mathbf{e}_2 + i\mathbf{e}_3 \mid i \in [2]\}$	3.5 (row 1)	row 58
$(2p^3, 2, p^3)$	Not possible since no X exists	5.2	
$(4p^2, p, 2)$	$L_{S_0} = \langle \mathbf{e}_2 \rangle$ $X = \{(i + \alpha_{ij}p^2)\mathbf{e}_1 + j\mathbf{e}_3 + L_{S_0} \mid i \in [p^2], j \in [2]\}$ $Y = \{ip^2\mathbf{e}_1 + L_{S_0} \mid i \in [p]\}$ $S_0 = \{(i + \alpha_{ij}p^2)\mathbf{e}_1 + k\mathbf{e}_2 + j\mathbf{e}_3 \mid i \in [p^2], j, k \in [2]\}$ $C = \{ip^2\mathbf{e}_1 + \alpha_i\mathbf{e}_2 \mid i \in [p]\}$	5.7 (row 19)	row 60
$(4p^2, p, p)$	$L_{S_0} = \langle p^2\mathbf{e}_1 \rangle$ $X = \{(i + \alpha_{ijk}p)\mathbf{e}_1 + j\mathbf{e}_2 + k\mathbf{e}_3 + L_{S_0} \mid i \in [p], j, k \in [2]\}$ $Y = \{ip\mathbf{e}_1 + L_{S_0} \mid i \in [p]\}$ $S_0 = \{(i + \alpha_{ijk}p + lp^2)\mathbf{e}_1 + j\mathbf{e}_2 + k\mathbf{e}_3 \mid i, l \in [p], j, k \in [2]\}$ $C = \{(ip + \alpha_i p^2)\mathbf{e}_1 \mid i \in [p]\}$	5.8 (row 21)	row 61
$(4p^2, p, 4)$	$L_{S_0} = \langle \mathbf{e}_2, \mathbf{e}_3 \rangle$ $X = \{(i + \alpha_i p^2)\mathbf{e}_1 + L_{S_0} \mid i \in [p^2]\}$ $Y = \{ip^2\mathbf{e}_1 + L_{S_0} \mid i \in [p]\}$ $S_0 = \{(i + \alpha_i p^2)\mathbf{e}_1 + j\mathbf{e}_2 + k\mathbf{e}_3 \mid i \in [p^2], j, k \in [2]\}$ $C = \{ip^2\mathbf{e}_1 + \alpha_i\mathbf{e}_2 + \beta_i\mathbf{e}_3 \mid i \in [p]\}$	5.3 (row 3)	row 62
$(4p^2, p, 2p)$	$L_{S_0} = \langle p^2\mathbf{e}_1 + \mathbf{e}_2 \rangle$ $X = \{(i + \alpha_{ij}p)\mathbf{e}_1 + j\mathbf{e}_3 + L_{S_0} \mid i \in [p], j \in [2]\}$ $Y = \{ip\mathbf{e}_1 + L_{S_0} \mid i \in [p]\}$	5.4 (row 7)	row 63

	$S_0 = \{(i + \alpha_{ij}p + kp^2)\mathbf{e}_1 + l\mathbf{e}_2 + j\mathbf{e}_3 \mid i, k \in [p], j, l \in [2]\}$ $C = \{(ip + \alpha_i p^2)\mathbf{e}_1 + \alpha_i \mathbf{e}_2 \mid i \in [p]\}$		
$(4p^2, p, p^2)$	$L_{S_0} = \langle p\mathbf{e}_1 \rangle$ $X = \{\alpha_{ij}\mathbf{e}_1 + i\mathbf{e}_2 + j\mathbf{e}_3 + L_{S_0} \mid i, j \in [2]\}$ $Y = \{i\mathbf{e}_1 + L_{S_0} \mid i \in [p]\}$ $S_0 = \{(\alpha_{ij} + kp)\mathbf{e}_1 + i\mathbf{e}_2 + j\mathbf{e}_3 \mid i, j \in [2], k \in [p^2]\}$ $C = \{(i + \alpha_i p)\mathbf{e}_1 \mid i \in [p]\}$	3.6 (row 7)	row 64
$(4p^2, p, 4p)$	$L_{S_0} = \{p^2\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_3\}$ $X = \{(i + \alpha_i p)\mathbf{e}_1 + L_{S_0} \mid i \in [p]\}$ $Y = \{ip\mathbf{e}_1 + L_{S_0} \mid i \in [p]\}$ $S_0 = \{(i + \alpha_i p + jp^2)\mathbf{e}_1 + k\mathbf{e}_2 + l\mathbf{e}_3 \mid i, j \in [p], k, l \in [2]\}$ $C = \{(ip + \alpha_i p^2)\mathbf{e}_1 + \alpha_i \mathbf{e}_2 + \beta_i \mathbf{e}_3 \mid i \in [p]\}$	5.1	row 65
$(4p^2, p, 2p^2)$	$L_{S_0} = \langle p\mathbf{e}_1 + \mathbf{e}_2 \rangle$ $X = \{\alpha_i \mathbf{e}_1 + i\mathbf{e}_3 + L_{S_0} \mid i \in [2]\}$ $Y = \{i\mathbf{e}_1 + L_{S_0} \mid i \in [p]\}$ $S_0 = \{(\alpha_i + jp)\mathbf{e}_1 + k\mathbf{e}_2 + i\mathbf{e}_3 \mid i, k \in [2], j \in [p^2]\}$ $C = \{(i + \alpha_i p)\mathbf{e}_1 + \alpha_i \mathbf{e}_2 \mid i \in [p]\}$	3.5 (row 2)	row 66

Remark 5.15. In Theorem 5.14, the generators $\mathbf{e}_2$ and $\mathbf{e}_3$ may be replaced by any basis of $H = \{0\} \times \mathbb{Z}_2 \times \mathbb{Z}_2$, as in (C2).

Theorem 5.16. *Let $G = \mathbb{Z}_{p^2} \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ for an odd prime p and let S be a proper subset of $G \setminus \{0\}$ that generates G . Let $0 \in C \subseteq G$. Then C is a perfect code in $\text{Cay}(G, S)$ if and only if (S_0, C) is one of the pairs in Table 5.18.*

Table 5.18: Perfect codes in Cayley graphs of $\mathbb{Z}_{p^2} \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

Row	S_0	C
1	$\{(i + \alpha_i p)\mathbf{e}_1 + \alpha_i \mathbf{e}_2 + \beta_i \mathbf{e}_3 + \gamma_i \mathbf{e}_4 \mid i \in [p]\}$	$\{ipe_1 + je_2 + ke_3 + le_4 \mid i \in [p], j, k, l \in [2]\}$
2	$\{(i + jp)\mathbf{e}_1 + \alpha_i \mathbf{e}_2 + \beta_i \mathbf{e}_3 + \gamma_i \mathbf{e}_4 \mid i, j \in [p]\}$	$\{\alpha_{ijk}pe_1 + ie_2 + je_3 + ke_4 \mid i, j, k \in [2]\}$
3	$\{(i + jp)\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + \alpha_i \mathbf{e}_3 + \beta_i \mathbf{e}_4 \mid i, j \in [p]\}$	$\{\alpha_{ij}pe_1 + ke_2 + ie_3 + je_4 \mid i, j, k \in [2]\}$
4	$\{(i + jp)\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + \beta_{ij}\mathbf{e}_3 + \alpha_i \mathbf{e}_4 \mid i, j \in [p]\}$	$\{\alpha_i pe_1 + je_2 + ke_3 + ie_4 \mid i, j, k \in [2]\}$
5	$\{ie_1 + \alpha_i \mathbf{e}_2 + \beta_i \mathbf{e}_3 + \gamma_i \mathbf{e}_4 \mid i \in [p^2]\}$	$\{ie_2 + je_3 + ke_4 \mid i, j, k \in [2]\}$
6	$\{(i + \alpha_i p)\mathbf{e}_1 + je_2 + \alpha_i \mathbf{e}_3 + \beta_i \mathbf{e}_4 \mid i \in [p], j \in [2]\}$	$\{ipe_1 + \alpha_{ijk}e_2 + je_3 + ke_4 \mid i \in [p], j, k \in [2]\}$
7	$\{(i + \alpha_i p)\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + je_3 + \alpha_i \mathbf{e}_4 \mid i \in [p], j \in [2]\}$	$\{ipe_1 + ke_2 + \alpha_{ij}e_3 + je_4 \mid i \in [p], j, k \in [2]\}$
8	$\{(i + \alpha_{ij}p)\mathbf{e}_1 + je_2 + \alpha_i \mathbf{e}_3 + \beta_i \mathbf{e}_4 \mid i \in [p], j \in [2]\}$	$\{kpe_1 + \alpha_{ij}e_2 + ie_3 + je_4 \mid i, j \in [2], k \in [p]\}$
9	$\{(i + \alpha_i p)\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + \beta_{ij}\mathbf{e}_3 + je_4 \mid i \in [p], j \in [2]\}$	$\{ipe_1 + je_2 + ke_3 + \alpha_i \mathbf{e}_4 \mid i \in [p], j, k \in [2]\}$
10	$\{(\alpha_i + jp)\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + \beta_{ij}\mathbf{e}_3 + ie_4 \mid i \in [2], j \in [p]\}$	$\{(i + \alpha_i p)\mathbf{e}_1 + je_2 + ke_3 \mid i \in [p], j, k \in [2]\}$
11	$\{(i + \alpha_{ij}p)\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + je_3 + \alpha_i \mathbf{e}_4 \mid i \in [p], j \in [2]\}$	$\{jpe_1 + ke_2 + \alpha_i \mathbf{e}_3 + ie_4 \mid j \in [p], i, k \in [2]\}$
12	$\{(i + \alpha_{ij}p)\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + \beta_{ij}\mathbf{e}_3 + je_4 \mid i \in [p], j \in [2]\}$	$\{ipe_1 + je_2 + ke_3 \mid i \in [p], j, k \in [2]\}$
13	$\{\alpha_i \mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + je_3 + ie_4 \mid i, j \in [2]\}$	$\{ie_1 + je_2 + \alpha_i \mathbf{e}_3 \mid i \in [p^2], j \in [2]\}$
14	$\{(\alpha_i + \alpha_{ij}p)\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + je_3 + ie_4 \mid i, j \in [2]\}$	$\{(i + jp)\mathbf{e}_1 + ke_2 + \alpha_i \mathbf{e}_3 \mid i, j \in [p], k \in [2]\}$
15	$\{\alpha_{ij}\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + ie_3 + je_4 \mid i, j \in [2]\}$	$\{ie_1 + je_2 \mid i \in [p^2], j \in [2]\}$
16	$\{(i + jp)\mathbf{e}_1 + ke_2 + \alpha_{ij}\mathbf{e}_3 + \alpha_i \mathbf{e}_4 \mid i, j \in [p], k \in [2]\}$	$\{\alpha_i pe_1 + \alpha_{ij}e_2 + je_3 + ie_4 \mid i, j \in [2]\}$
17	$\{ie_1 + je_2 + \alpha_i \mathbf{e}_3 + \beta_i \mathbf{e}_4 \mid i \in [p^2], j \in [2]\}$	$\{\alpha_{ij}e_2 + ie_3 + je_4 \mid i, j \in [2]\}$
18	$\{(i + kp)\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + je_3 + \alpha_i \mathbf{e}_4 \mid i, k \in [p], j \in [2]\}$	$\{\alpha_{ij}pe_1 + je_2 + \alpha_i \mathbf{e}_3 + ie_4 \mid i, j \in [2]\}$
19	$\{(i + kp)\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + \beta_{ij}\mathbf{e}_3 + je_4 \mid i, k \in [p], j \in [2]\}$	$\{\alpha_{ij}pe_1 + ie_2 + je_3 \mid i, j \in [2]\}$
20	$\{(i + jp)\mathbf{e}_1 + ke_2 + \alpha_i \mathbf{e}_3 + \beta_i \mathbf{e}_4 \mid i, j \in [p], k \in [2]\}$	$\{\alpha_{ij}pe_1 + \alpha_{ij}e_2 + ie_3 + je_4 \mid i, j \in [2]\}$
21	$\{ie_1 + \alpha_{ij}\mathbf{e}_2 + je_3 + \alpha_i \mathbf{e}_4 \mid i \in [p^2], j \in [2]\}$	$\{je_2 + \alpha_i \mathbf{e}_3 + ie_4 \mid i, j \in [2]\}$
22	$\{(i + kp)\mathbf{e}_1 + \alpha_{ijk}\mathbf{e}_2 + \alpha_{ij}\mathbf{e}_3 + je_4 \mid i, k \in [p], j \in [2]\}$	$\{\alpha_i pe_1 + je_2 + ie_3 \mid i, j \in [2]\}$
23	$\{(i + jp)\mathbf{e}_1 + \alpha_{ijk}\mathbf{e}_2 + ke_3 + \alpha_i \mathbf{e}_4 \mid i, j \in [p], k \in [2]\}$	$\{\alpha_i pe_1 + je_2 + \alpha_i \mathbf{e}_3 + ie_4 \mid i, j \in [2]\}$
24	$\{ie_1 + \alpha_{ij}\mathbf{e}_2 + \beta_{ij}\mathbf{e}_3 + je_4 \mid i \in [p^2], j \in [2]\}$	$\{ie_2 + je_3 \mid i, j \in [2]\}$

25	$\{(i + \alpha_{ij}p)\mathbf{e}_1 + k\mathbf{e}_2 + j\mathbf{e}_3 + \alpha_i\mathbf{e}_4 \mid i \in [p], j, k \in [2]\}$	$\{jpe_1 + \alpha_{ij}e_2 + \alpha_ie_3 + ie_4 \mid i \in [2], j \in [p]\}$
26	$\{(i + \alpha_i p)\mathbf{e}_1 + k\mathbf{e}_2 + \alpha_{ij}\mathbf{e}_3 + j\mathbf{e}_4 \mid i \in [p], j, k \in [2]\}$	$\{ipe_1 + \alpha_{ij}e_2 + j\mathbf{e}_3 + \alpha_i\mathbf{e}_4 \mid i \in [p], j \in [2]\}$
27	$\{(\alpha_i + jp)\mathbf{e}_1 + k\mathbf{e}_2 + \alpha_{ij}\mathbf{e}_3 + ie_4 \mid i, k \in [2], j \in [p]\}$	$\{(i + \alpha_i p)\mathbf{e}_1 + \alpha_{ij}e_2 + j\mathbf{e}_3 \mid i \in [p], j \in [2]\}$
28	$\{(i + \alpha_{ij}p)\mathbf{e}_1 + k\mathbf{e}_2 + \alpha_{ij}\mathbf{e}_3 + j\mathbf{e}_4 \mid i \in [p], j, k \in [2]\}$	$\{ipe_1 + \alpha_{ij}e_2 + j\mathbf{e}_3 \mid i \in [p], j \in [2]\}$
29	$\{(\alpha_i + kp)\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + j\mathbf{e}_3 + ie_4 \mid i, j \in [2], k \in [p]\}$	$\{(i + \alpha_{ij}p)\mathbf{e}_1 + j\mathbf{e}_2 + \alpha_i\mathbf{e}_3 \mid i \in [p], j \in [2]\}$
30	$\{(\alpha_{ij} + kp)\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + ie_3 + j\mathbf{e}_4 \mid i, j \in [2], k \in [p]\}$	$\{(i + \alpha_{ij}p)\mathbf{e}_1 + j\mathbf{e}_2 \mid i \in [p], j \in [2]\}$
31	$\{(i + \alpha_i p)\mathbf{e}_1 + j\mathbf{e}_2 + k\mathbf{e}_3 + \alpha_i\mathbf{e}_4 \mid i \in [p], j, k \in [2]\}$	$\{ipe_1 + \alpha_{ij}e_2 + \beta_{ij}e_3 + j\mathbf{e}_4 \mid i \in [p], j \in [2]\}$
32	$\{(i + \alpha_{ij}p)\mathbf{e}_1 + \alpha_{ijk}\mathbf{e}_2 + k\mathbf{e}_3 + j\mathbf{e}_4 \mid i \in [p], j, k \in [2]\}$	$\{ipe_1 + j\mathbf{e}_2 + \alpha_i\mathbf{e}_3 \mid i \in [p], j \in [2]\}$
33	$\{(\alpha_{ij} + kp)\mathbf{e}_1 + \alpha_{ijk}\mathbf{e}_2 + ie_3 + j\mathbf{e}_4 \mid i, j \in [2], k \in [p]\}$	$\{(i + \alpha_i p)\mathbf{e}_1 + j\mathbf{e}_2 \mid i \in [p], j \in [2]\}$
34	$\{(\alpha_i + jp)\mathbf{e}_1 + \alpha_{ijk}\mathbf{e}_2 + k\mathbf{e}_3 + ie_4 \mid i, k \in [2], j \in [p]\}$	$\{(i + \alpha_i p)\mathbf{e}_1 + j\mathbf{e}_2 + \alpha_i\mathbf{e}_3 \mid i \in [p], j \in [2]\}$
35	$\{(i + \alpha_i p)\mathbf{e}_1 + \alpha_{ijk}\mathbf{e}_2 + j\mathbf{e}_3 + k\mathbf{e}_4 \mid i \in [p], j, k \in [2]\}$	$\{ipe_1 + j\mathbf{e}_2 + \alpha_i\mathbf{e}_3 + \beta_i\mathbf{e}_4 \mid i \in [p], j \in [2]\}$
36	$\{(i + \alpha_{ijk}p)\mathbf{e}_1 + k\mathbf{e}_2 + \alpha_{ij}\mathbf{e}_3 + j\mathbf{e}_4 \mid i \in [p], j, k \in [2]\}$	$\{jpe_1 + \alpha_i\mathbf{e}_2 + ie_3 \mid i \in [2], j \in [p]\}$
37	$\{(k + \alpha_{ijk}p)\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + ie_3 + j\mathbf{e}_4 \mid i, j \in [2], k \in [p]\}$	$\{(\alpha_i + jp)\mathbf{e}_1 + ie_2 \mid i \in [2], j \in [p]\}$
38	$\{(i + \alpha_{ijk}p)\mathbf{e}_1 + j\mathbf{e}_2 + k\mathbf{e}_3 + \alpha_i\mathbf{e}_4 \mid i \in [p], j, k \in [2]\}$	$\{jpe_1 + \alpha_i\mathbf{e}_2 + \beta_i\mathbf{e}_3 + ie_4 \mid i \in [2], j \in [p]\}$
39	$\{(i + \alpha_{ijk}p)\mathbf{e}_1 + \alpha_{ijk}\mathbf{e}_2 + j\mathbf{e}_3 + k\mathbf{e}_4 \mid i \in [p], j, k \in [2]\}$	$\{ipe_1 + j\mathbf{e}_2 \mid i \in [p], j \in [2]\}$
40	$\{(\alpha_i + \alpha_{ij}p)\mathbf{e}_1 + k\mathbf{e}_2 + j\mathbf{e}_3 + ie_4 \mid i, j, k \in [2]\}$	$\{(i + jp)\mathbf{e}_1 + \alpha_{ij}e_2 + \alpha_i\mathbf{e}_3 \mid i, j \in [p]\}$
41	$\{\alpha_{ij}\mathbf{e}_1 + k\mathbf{e}_2 + ie_3 + j\mathbf{e}_4 \mid i, j, k \in [2]\}$	$\{ie_1 + \alpha_i\mathbf{e}_2 \mid i \in [p^2]\}$
42	$\{\alpha_i\mathbf{e}_1 + j\mathbf{e}_2 + k\mathbf{e}_3 + ie_4 \mid i, j, k \in [2]\}$	$\{ie_1 + \alpha_i\mathbf{e}_2 + \beta_i\mathbf{e}_3 \mid i \in [p^2]\}$
43	$\{(\alpha_{ij} + \alpha_{ijk}p)\mathbf{e}_1 + k\mathbf{e}_2 + ie_3 + j\mathbf{e}_4 \mid i, j, k \in [2]\}$	$\{(i + jp)\mathbf{e}_1 + \alpha_i\mathbf{e}_2 \mid i, j \in [p]\}$
44	$\{(\alpha_i + \alpha_{ijk}p)\mathbf{e}_1 + j\mathbf{e}_2 + k\mathbf{e}_3 + ie_4 \mid i, j, k \in [2]\}$	$\{(i + jp)\mathbf{e}_1 + \alpha_i\mathbf{e}_2 + \beta_i\mathbf{e}_3 \mid i, j \in [p]\}$
45	$\{(\alpha_{ijk})\mathbf{e}_1 + ie_2 + j\mathbf{e}_3 + k\mathbf{e}_4 \mid i, j, k \in [2]\}$	$\{ie_1 \mid i \in [p^2]\}$
46	$\{ie_1 + k\mathbf{e}_2 + \alpha_{ij}\mathbf{e}_3 + j\mathbf{e}_4 \mid i \in [p^2], j, k \in [2]\}$	$\{\alpha_i\mathbf{e}_2 + ie_3 \mid i \in [2]\}$
47	$\{(i + lp)\mathbf{e}_1 + \alpha_{ijk}\mathbf{e}_2 + j\mathbf{e}_3 + k\mathbf{e}_4 \mid i, l \in [p], j, k \in [2]\}$	$\{\alpha_i p\mathbf{e}_1 + ie_2 \mid i \in [2]\}$
48	$\{ie_1 + j\mathbf{e}_2 + k\mathbf{e}_3 + \alpha_i\mathbf{e}_4 \mid i \in [p^2], j, k \in [2]\}$	$\{\alpha_i\mathbf{e}_2 + \beta_i\mathbf{e}_3 + ie_4 \mid i \in [2]\}$
49	$\{(i + kp)\mathbf{e}_1 + le_2 + \alpha_{ij}\mathbf{e}_3 + j\mathbf{e}_4 \mid i, k \in [p], j, l \in [2]\}$	$\{\alpha_i p\mathbf{e}_1 + \alpha_i\mathbf{e}_2 + ie_3 \mid i \in [2]\}$
50	$\{k\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + ie_3 + j\mathbf{e}_4 \mid i, j \in [2], k \in [p^2]\}$	$\{\alpha_i\mathbf{e}_1 + ie_2 \mid i \in [2]\}$
51	$\{(i + jp)\mathbf{e}_1 + k\mathbf{e}_2 + le_3 + \alpha_i\mathbf{e}_4 \mid i, j \in [p], k, l \in [2]\}$	$\{\alpha_i p\mathbf{e}_1 + \alpha_i\mathbf{e}_2 + \beta_i\mathbf{e}_3 + ie_4 \mid i \in [2]\}$
52	$\{ie_1 + \alpha_{ijk}\mathbf{e}_2 + j\mathbf{e}_3 + k\mathbf{e}_4 \mid i \in [p^2], j, k \in [2]\}$	$\{ie_2 \mid i \in [2]\}$
53	$\{(i + \alpha_{ijk}p)\mathbf{e}_1 + le_2 + j\mathbf{e}_3 + k\mathbf{e}_4 \mid i \in [p], j, k, l \in [2]\}$	$\{ipe_1 + \alpha_i\mathbf{e}_2 \mid i \in [p]\}$
54	$\{(\alpha_{ijk} + lp)\mathbf{e}_1 + ie_2 + j\mathbf{e}_3 + k\mathbf{e}_4 \mid i, j, k \in [2], l \in [p]\}$	$\{(i + \alpha_i p)\mathbf{e}_1 \mid i \in [p]\}$
55	$\{(i + \alpha_{ij}p)\mathbf{e}_1 + k\mathbf{e}_2 + le_3 + j\mathbf{e}_4 \mid i \in [p], j, k, l \in [2]\}$	$\{ipe_1 + \alpha_i\mathbf{e}_2 + \beta_i\mathbf{e}_3 \mid i \in [p]\}$
56	$\{(\alpha_{ij} + kp)\mathbf{e}_1 + le_2 + ie_3 + j\mathbf{e}_4 \mid i, j, l \in [2], k \in [p]\}$	$\{(i + \alpha_i p)\mathbf{e}_1 + \alpha_i\mathbf{e}_2 \mid i \in [p]\}$
57	$\{(i + \alpha_i p)\mathbf{e}_1 + j\mathbf{e}_2 + k\mathbf{e}_3 + le_4 \mid i \in [p], j, k, l \in [2]\}$	$\{ipe_1 + \alpha_i\mathbf{e}_2 + \beta_i\mathbf{e}_3 + \gamma_i\mathbf{e}_4 \mid i \in [p]\}$
58	$\{(\alpha_i + jp)\mathbf{e}_1 + k\mathbf{e}_2 + le_3 + ie_4 \mid i, k, l \in [2], j \in [p]\}$	$\{(i + \alpha_i p)\mathbf{e}_1 + \alpha_i\mathbf{e}_2 + \beta_i\mathbf{e}_3 \mid i \in [p]\}$
59	$\{(i + \alpha_{ijk}lp)\mathbf{e}_1 + j\mathbf{e}_2 + k\mathbf{e}_3 + le_4 \mid i \in [p], j, k, l \in [2]\}$	$\{ipe_1 \mid i \in [p]\}$

Proof. Suppose C is a perfect code in $\text{Cay}(G, S)$. Then $G = S_0 \oplus C$ and so $|S_0| \cdot |C| = |G| = 8p^2$. Since G is not a cyclic group, $|S_0| > 2$ by Lemma 2.8. Therefore, $2 < |S_0| < 8p^2$.

Case 1. S_0 is aperiodic.

Since G is Hajós and $G = S_0 \oplus C$, in this case C must be periodic, that is, L_C is nontrivial. Write $C = L_C \oplus D$ for some subset D of G whose existence is ensured by Lemma 2.3. Then $|L_C| \cdot |D| = |C|$ and $2 \leq |L_C| \leq |C|$. If $|S_0|$ is a prime, then $|L_C| = |C|$ by Lemma 2.8 part (c). If $|S_0|$ is not a prime, then $|L_C|$ could be any divisor of $|C|$. Set

$$A = (S_0 + L_C)/L_C \quad \text{and} \quad B = (D + L_C)/L_C.$$

Since C is periodic, by Lemma 2.5, we have

$$G/L_C = A \oplus B$$

and B is aperiodic in G/L_C . Since G/L_C is isomorphic to a subgroup of G and subgroups of Hajós group are also Hajós groups, A is periodic in G/L_C . We also know that A contains

L_C and A generates G/L_C by Lemma 2.7. Furthermore, $|A| = |S_0|$ and $|B| = |D|$. Note that if $|L_C| = |C|$, then $C = L_C$ as $0 \in C$. This means we can take $D = \{0\}$ and hence $B = L_C/L_C$ and $G/L_C = A$. We use C to get A and then use Lemma 2.6 to obtain S_0 . If $|L_C| < |C|$, then this factorization enables us to make use of previously established theorems to obtain the pair (S_0, C) via (A, B) .

Table 5.19 represent all possible values for $(|S_0|, |C|, |L_C|)$; C , A , and S_0 when $|L_C| = |C|$ or L_C , A , B , S_0 , and C when $|L_C| < |C|$; which previous theorem or lemma is used to obtain (A, B) when $|L_C| < |C|$; and which row in Table 5.18 the result corresponds to.

Table 5.19: The case when $G = \mathbb{Z}_{p^2} \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and S_0 is aperiodic

(S_0 , C , L_C)	L_C, A, B, S_0 , and C	Reference	Result
$(p, 8p, 8p)$	$C = \langle p\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4 \rangle$ $A = \{i\mathbf{e}_1 + L_C \mid i \in [p]\}$ $S_0 = \{(i + \alpha_i p)\mathbf{e}_1 + \alpha_i \mathbf{e}_2 + \beta_i \mathbf{e}_3 + \gamma_i \mathbf{e}_4 \mid i \in [p]\}$		row 1
$(p^2, 8, 2)$	$L_C = \langle \mathbf{e}_2 \rangle$ $A = \{(i + jp)\mathbf{e}_1 + \alpha_i \mathbf{e}_3 + \beta_i \mathbf{e}_4 + L_C \mid i, j \in [p]\}$ $B = \{\alpha_{ij} p\mathbf{e}_1 + i\mathbf{e}_3 + j\mathbf{e}_4 + L_C \mid i, j \in [2]\}$ $S_0 = \{(i + jp)\mathbf{e}_1 + \alpha_{ij} \mathbf{e}_2 + \alpha_i \mathbf{e}_3 + \beta_i \mathbf{e}_4 \mid i, j \in [p]\}$ $C = \{\alpha_{ij} p\mathbf{e}_1 + k\mathbf{e}_2 + i\mathbf{e}_3 + j\mathbf{e}_4 \mid i, j, k \in [2]\}$	5.8 (row 2)	row 3
$(p^2, 8, 4)$	$L_C = \langle \mathbf{e}_2, \mathbf{e}_3 \rangle$ $A = \{(i + jp)\mathbf{e}_1 + \alpha_i \mathbf{e}_4 + L_C \mid i, j \in [p]\}$ $B = \{\alpha_i p\mathbf{e}_1 + i\mathbf{e}_4 + L_C \mid i \in [2]\}$ $S_0 = \{(i + jp)\mathbf{e}_1 + \alpha_{ij} \mathbf{e}_2 + \beta_{ij} \mathbf{e}_3 + \alpha_i \mathbf{e}_4 \mid i, j \in [p]\}$ $C = \{\alpha_i p\mathbf{e}_1 + j\mathbf{e}_2 + k\mathbf{e}_3 + i\mathbf{e}_4 \mid i, j, k \in [2]\}$	5.4 (row 3)	row 4
$(p^2, 8, 8)$	$C = \langle \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4 \rangle$ $A = \{i\mathbf{e}_1 + L_C \mid i \in [p^2]\}$ $S_0 = \{i\mathbf{e}_1 + \alpha_i \mathbf{e}_2 + \beta_i \mathbf{e}_3 + \gamma_i \mathbf{e}_4 \mid i \in [p^2]\}$		row 5
$(2p, 4p, 2)$	$L_C = \langle \mathbf{e}_2 \rangle$ $A = \{(i + \alpha_i p)\mathbf{e}_1 + j\mathbf{e}_3 + \alpha_i \mathbf{e}_4 + L_C \mid i \in [p], j \in [2]\}$ $B = \{i p\mathbf{e}_1 + \alpha_{ij} \mathbf{e}_3 + j\mathbf{e}_4 + L_C \mid i \in [p], j \in [2]\}$ $S_0 = \{(i + \alpha_i p)\mathbf{e}_1 + \alpha_{ij} \mathbf{e}_2 + j\mathbf{e}_3 + \alpha_i \mathbf{e}_4 \mid i \in [p], j \in [2]\}$ $C = \{i p\mathbf{e}_1 + k\mathbf{e}_2 + \alpha_{ij} \mathbf{e}_3 + j\mathbf{e}_4 \mid i \in [p], j, k \in [2]\}$	5.8 (row 5)	row 7
$(2p, 4p, p)$	$L_C = \langle p\mathbf{e}_1 \rangle$ $A = \{i\mathbf{e}_1 + j\mathbf{e}_2 + \alpha_i \mathbf{e}_3 + \beta_i \mathbf{e}_4 + L_C \mid i \in [p], j \in [2]\}$ $B = \{\alpha_{ij} \mathbf{e}_2 + i\mathbf{e}_3 + j\mathbf{e}_4 + L_C \mid i, j \in [2]\}$ $S_0 = \{(i + \alpha_{ij} p)\mathbf{e}_1 + j\mathbf{e}_2 + \alpha_i \mathbf{e}_3 + \beta_i \mathbf{e}_4 \mid i \in [p], j \in [2]\}$ $C = \{k p\mathbf{e}_1 + \alpha_{ij} \mathbf{e}_2 + i\mathbf{e}_3 + j\mathbf{e}_4 \mid i, j \in [2], k \in [p]\}$	5.10 (row 2)	row 8
$(2p, 4p, 4)$	$L_C = \langle \mathbf{e}_2, \mathbf{e}_3 \rangle$ $A = \{(i + \alpha_i p)\mathbf{e}_1 + j\mathbf{e}_4 + L_C \mid i \in [p], j \in [2]\}$ $B = \{i p\mathbf{e}_1 + \alpha_i \mathbf{e}_4 + L_C \mid i \in [p]\}$ $S_0 = \{(i + \alpha_i p)\mathbf{e}_1 + \alpha_{ij} \mathbf{e}_2 + \beta_{ij} \mathbf{e}_3 + j\mathbf{e}_4 \mid i \in [p], j \in [2]\}$ $C = \{i p\mathbf{e}_1 + j\mathbf{e}_2 + k\mathbf{e}_3 + \alpha_i \mathbf{e}_4 \mid i \in [p], j, k \in [2]\}$	5.4 (row 5)	row 9
$(2p, 4p, 4)$	$L_C = \langle \mathbf{e}_2, \mathbf{e}_3 \rangle$ $A = \{(\alpha_i + jp)\mathbf{e}_1 + i\mathbf{e}_4 + L_C \mid i \in [2], j \in [p]\}$ $B = \{(i + \alpha_i p)\mathbf{e}_1 + L_C \mid i \in [p]\}$ $S_0 = \{(\alpha_i + jp)\mathbf{e}_1 + \alpha_{ij} \mathbf{e}_2 + \beta_{ij} \mathbf{e}_3 + i\mathbf{e}_4 \mid i \in [2], j \in [p]\}$ $C = \{(i + \alpha_i p)\mathbf{e}_1 + j\mathbf{e}_2 + k\mathbf{e}_3 \mid i \in [p], j, k \in [2]\}$	5.4 (row 6)	row 10
	$L_C = \langle p\mathbf{e}_1 + \mathbf{e}_2 \rangle$ $A = \{i\mathbf{e}_1 + j\mathbf{e}_3 + \alpha_i \mathbf{e}_4 + L_C \mid i \in [p], j \in [2]\}$		

$(2p, 4p, 2p)$	$B = \{\alpha_i \mathbf{e}_3 + i\mathbf{e}_4 + L_C \mid i \in [2]\}$ $S_0 = \{(i + \alpha_{ij}p)\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + j\mathbf{e}_3 + \alpha_i\mathbf{e}_4 \mid i \in [p], j \in [2]\}$ $C = \{jp\mathbf{e}_1 + k\mathbf{e}_2 + \alpha_i\mathbf{e}_3 + i\mathbf{e}_4 \mid j \in [p], i, k \in [2]\}$	3.6 (row 4)	row 11
$(2p, 4p, 4p)$	$C = \langle p\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_3 \rangle$ $A = \{i\mathbf{e}_1 + j\mathbf{e}_4 + L_C \mid i \in [p], j \in [2]\}$ $S_0 = \{(i + \alpha_{ij}p)\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + \beta_{ij}\mathbf{e}_3 + j\mathbf{e}_4 \mid i \in [p], j \in [2]\}$		row 12
$(4, 2p^2, 2)$	$L_C = \langle \mathbf{e}_2 \rangle$ $A = \{\alpha_i \mathbf{e}_1 + j\mathbf{e}_3 + i\mathbf{e}_4 + L_C \mid i, j \in [2]\}$ $B = \{i\mathbf{e}_1 + \alpha_i \mathbf{e}_3 + L_C \mid i \in [p^2]\}$ $S_0 = \{\alpha_i \mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + j\mathbf{e}_3 + i\mathbf{e}_4 \mid i, j \in [2]\}$ $C = \{i\mathbf{e}_1 + j\mathbf{e}_2 + \alpha_i \mathbf{e}_3 \mid i \in [p^2], j \in [2]\}$	5.8 (row 10)	row 13
$(4, 2p^2, p)$	Not possible since no A exists	5.10	
$(4, 2p^2, 2p)$	$L_C = \langle p\mathbf{e}_1 + \mathbf{e}_2 \rangle$ $A = \{\alpha_i \mathbf{e}_1 + j\mathbf{e}_3 + i\mathbf{e}_4 + L_C \mid i, j \in [2]\}$ $B = \{i\mathbf{e}_1 + \alpha_i \mathbf{e}_3 + L_C \mid i \in [p]\}$ $S_0 = \{(\alpha_i + \alpha_{ij}p)\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + j\mathbf{e}_3 + i\mathbf{e}_4 \mid i, j \in [2]\}$ $C = \{(i + jp)\mathbf{e}_1 + k\mathbf{e}_2 + \alpha_i \mathbf{e}_3 \mid i, j \in [p], k \in [2]\}$	3.6 (row 6)	row 14
$(4, 2p^2, p^2)$	Not possible since no A exists	5.5	
$(4, 2p^2, 2p^2)$	$C = \langle \mathbf{e}_1 + \mathbf{e}_2 \rangle$ $A = \{i\mathbf{e}_3 + j\mathbf{e}_4 + L_C \mid i, j \in [2]\}$ $S_0 = \{\alpha_{ij}\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + i\mathbf{e}_3 + j\mathbf{e}_4 \mid i, j \in [2]\}$		row 15
$(2p^2, 4, 2)$	$L_C = \langle \mathbf{e}_2 \rangle$ $A = \{i\mathbf{e}_1 + j\mathbf{e}_3 + \alpha_i \mathbf{e}_4 + L_C \mid i \in [p^2], j \in [2]\}$ $B = \{\alpha_i \mathbf{e}_3 + i\mathbf{e}_4 + L_C \mid i \in [2]\}$ $S_0 = \{i\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + j\mathbf{e}_3 + \alpha_i \mathbf{e}_4 \mid i \in [p^2], j \in [2]\}$ $C = \{j\mathbf{e}_2 + \alpha_i \mathbf{e}_3 + i\mathbf{e}_4 \mid i, j \in [2]\}$	5.8 (row 13)	row 21
$(2p^2, 4, 2)$	$L_C = \langle \mathbf{e}_2 \rangle$ $A = \{(i + kp)\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_3 + j\mathbf{e}_4 + L_C \mid i, k \in [p], j \in [2]\}$ $B = \{\alpha_i p\mathbf{e}_1 + i\mathbf{e}_3 + L_C \mid i \in [2]\}$ $S_0 = \{(i + kp)\mathbf{e}_1 + \alpha_{ijk}\mathbf{e}_2 + \alpha_{ij}\mathbf{e}_3 + j\mathbf{e}_4 \mid i, k \in [p], j \in [2]\}$ $C = \{\alpha_i p\mathbf{e}_1 + j\mathbf{e}_2 + i\mathbf{e}_3 \mid i, j \in [2]\}$	5.8 (row 14)	row 22
$(2p^2, 4, 2)$	$L_C = \langle \mathbf{e}_2 \rangle$ $A = \{(i + jp)\mathbf{e}_1 + k\mathbf{e}_3 + \alpha_i \mathbf{e}_4 + L_C \mid i, j \in [p], k \in [2]\}$ $B = \{\alpha_i p\mathbf{e}_1 + \alpha_i \mathbf{e}_3 + i\mathbf{e}_4 + L_C \mid i \in [2]\}$ $S_0 = \{(i + jp)\mathbf{e}_1 + \alpha_{ijk}\mathbf{e}_2 + k\mathbf{e}_3 + \alpha_i \mathbf{e}_4 \mid i, j \in [p], k \in [2]\}$ $C = \{\alpha_i p\mathbf{e}_1 + j\mathbf{e}_2 + \alpha_i \mathbf{e}_3 + i\mathbf{e}_4 \mid i, j \in [2]\}$	5.8 (row 15)	row 23
$(2p^2, 4, 4)$	$C = \langle \mathbf{e}_2, \mathbf{e}_3 \rangle$ $A = \{i\mathbf{e}_1 + j\mathbf{e}_4 + L_C \mid i \in [p^2], j \in [2]\}$ $S_0 = \{i\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + \beta_{ij}\mathbf{e}_3 + j\mathbf{e}_4 \mid i \in [p^2], j \in [2]\}$		row 24
$(4p, 2p, 2)$	$L_C = \langle \mathbf{e}_2 \rangle$ $A = \{(i + \alpha_{ij}p)\mathbf{e}_1 + k\mathbf{e}_3 + j\mathbf{e}_4 + L_C \mid i \in [p], j, k \in [2]\}$ $B = \{ip\mathbf{e}_1 + \alpha_i \mathbf{e}_3 + L_C \mid i \in [p]\}$ $S_0 = \{(i + \alpha_{ij}p)\mathbf{e}_1 + \alpha_{ijk}\mathbf{e}_2 + k\mathbf{e}_3 + j\mathbf{e}_4 \mid i \in [p], j, k \in [2]\}$ $C = \{ip\mathbf{e}_1 + j\mathbf{e}_2 + \alpha_i \mathbf{e}_3 \mid i \in [p], j \in [2]\}$	5.8 (row 17)	row 32
$(4p, 2p, 2)$	$L_C = \langle \mathbf{e}_2 \rangle$ $A = \{(\alpha_{ij} + kp)\mathbf{e}_1 + i\mathbf{e}_3 + j\mathbf{e}_4 + L_C \mid i, j \in [2], k \in [p]\}$ $B = \{(i + \alpha_i p)\mathbf{e}_1 + L_C \mid i \in [p]\}$ $S_0 = \{(\alpha_{ij} + kp)\mathbf{e}_1 + \alpha_{ijk}\mathbf{e}_2 + i\mathbf{e}_3 + j\mathbf{e}_4 \mid i, j \in [2], k \in [p]\}$ $C = \{(i + \alpha_i p)\mathbf{e}_1 + j\mathbf{e}_2 \mid i \in [p], j \in [2]\}$	5.8 (row 18)	row 33
	$L_C = \langle \mathbf{e}_2 \rangle$ $A = \{(\alpha_i + jp)\mathbf{e}_1 + k\mathbf{e}_3 + i\mathbf{e}_4 + L_C \mid i, k \in [2], j \in [p]\}$		

$(4p, 2p, 2)$	$B = \{(i + \alpha_i p)\mathbf{e}_1 + \alpha_i \mathbf{e}_3 + L_C \mid i \in [p]\}$ $S_0 = \{(\alpha_i + jp)\mathbf{e}_1 + \alpha_{ijk}\mathbf{e}_2 + k\mathbf{e}_3 + i\mathbf{e}_4 \mid i, k \in [2], j \in [p]\}$ $C = \{(i + \alpha_i p)\mathbf{e}_1 + j\mathbf{e}_2 + \alpha_i \mathbf{e}_3 \mid i \in [p], j \in [2]\}$	5.8 (row 19)	row 34
$(4p, 2p, 2)$	$L_C = \langle \mathbf{e}_2 \rangle$ $A = \{(i + \alpha_i p)\mathbf{e}_1 + j\mathbf{e}_3 + k\mathbf{e}_4 + L_C \mid i \in [p], j, k \in [2]\}$ $B = \{i p \mathbf{e}_1 + \alpha_i \mathbf{e}_3 + \beta_i \mathbf{e}_4 + L_C \mid i \in [p]\}$ $S_0 = \{(i + \alpha_i p)\mathbf{e}_1 + \alpha_{ijk}\mathbf{e}_2 + j\mathbf{e}_3 + k\mathbf{e}_4 \mid i \in [p], j, k \in [2]\}$ $C = \{i p \mathbf{e}_1 + j\mathbf{e}_2 + \alpha_i \mathbf{e}_3 + \beta_i \mathbf{e}_4 \mid i \in [p], j \in [2]\}$	5.8 (row 20)	row 35
$(4p, 2p, p)$	$L_C = \langle p\mathbf{e}_1 \rangle$ $A = \{i\mathbf{e}_1 + k\mathbf{e}_2 + \alpha_{ij}\mathbf{e}_3 + j\mathbf{e}_4 + L_C \mid i \in [p], j, k \in [2]\}$ $B = \{\alpha_i \mathbf{e}_2 + i\mathbf{e}_3 + L_C \mid i \in [2]\}$ $S_0 = \{(i + \alpha_{ijk}p)\mathbf{e}_1 + k\mathbf{e}_2 + \alpha_{ij}\mathbf{e}_3 + j\mathbf{e}_4 \mid i \in [p], j, k \in [2]\}$ $C = \{j p \mathbf{e}_1 + \alpha_i \mathbf{e}_2 + i\mathbf{e}_3 \mid i \in [2], j \in [p]\}$	5.10 (row 7)	row 36
$(4p, 2p, p)$	$L_C = \langle p\mathbf{e}_1 \rangle$ $A = \{k\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + i\mathbf{e}_3 + j\mathbf{e}_4 + L_C \mid i, j \in [2], k \in [p]\}$ $B = \{\alpha_i \mathbf{e}_1 + i\mathbf{e}_2 + L_C \mid i \in [2]\}$ $S_0 = \{(k + \alpha_{ijk}p)\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + i\mathbf{e}_3 + j\mathbf{e}_4 \mid i, j \in [2], k \in [p]\}$ $C = \{(\alpha_i + jp)\mathbf{e}_1 + i\mathbf{e}_2 \mid i \in [2], j \in [p]\}$	5.10 (row 8)	row 37
$(4p, 2p, p)$	$L_C = \langle p\mathbf{e}_1 \rangle$ $A = \{i\mathbf{e}_1 + j\mathbf{e}_2 + k\mathbf{e}_3 + \alpha_i \mathbf{e}_4 + L_C \mid i \in [p], j, k \in [2]\}$ $B = \{\alpha_i \mathbf{e}_2 + \beta_i \mathbf{e}_3 + i\mathbf{e}_4 + L_C \mid i \in [2]\}$ $S_0 = \{(i + \alpha_{ijk}p)\mathbf{e}_1 + j\mathbf{e}_2 + k\mathbf{e}_3 + \alpha_i \mathbf{e}_4 \mid i \in [p], j, k \in [2]\}$ $C = \{j p \mathbf{e}_1 + \alpha_i \mathbf{e}_2 + \beta_i \mathbf{e}_3 + i\mathbf{e}_4 \mid i \in [2], j \in [p]\}$	5.10 (row 9)	row 38
$(4p, 2p, 2p)$	$C = \langle p\mathbf{e}_1 + \mathbf{e}_2 \rangle$ $A = \{i\mathbf{e}_1 + j\mathbf{e}_3 + k\mathbf{e}_4 + L_C \mid i \in [p], j, k \in [2]\}$ $S_0 = \{(i + \alpha_{ijk}p)\mathbf{e}_1 + \alpha_{ijk}\mathbf{e}_2 + j\mathbf{e}_3 + k\mathbf{e}_4 \mid i \in [p], j, k \in [2]\}$		row 39
$(8, p^2, p)$	$L_C = \langle p\mathbf{e}_1 \rangle$ $A = \{\alpha_{ij}\mathbf{e}_1 + k\mathbf{e}_2 + i\mathbf{e}_3 + j\mathbf{e}_4 + L_C \mid i, j, k \in [2]\}$ $B = \{i\mathbf{e}_1 + \alpha_i \mathbf{e}_2 + L_C \mid i \in [p]\}$ $S_0 = \{(\alpha_{ij} + \alpha_{ijk}p)\mathbf{e}_1 + k\mathbf{e}_2 + i\mathbf{e}_3 + j\mathbf{e}_4 \mid i, j, k \in [2]\}$ $C = \{(i + jp)\mathbf{e}_1 + \alpha_i \mathbf{e}_2 \mid i, j \in [p]\}$	5.10 (row 11)	row 43
$(8, p^2, p)$	$L_C = \langle p\mathbf{e}_1 \rangle$ $A = \{\alpha_i \mathbf{e}_1 + j\mathbf{e}_2 + k\mathbf{e}_3 + i\mathbf{e}_4 + L_C \mid i, j, k \in [2]\}$ $B = \{i\mathbf{e}_1 + \alpha_i \mathbf{e}_2 + \beta_i \mathbf{e}_3 + L_C \mid i \in [p]\}$ $S_0 = \{(\alpha_i + \alpha_{ijk}p)\mathbf{e}_1 + j\mathbf{e}_2 + k\mathbf{e}_3 + i\mathbf{e}_4 \mid i, j, k \in [2]\}$ $C = \{(i + jp)\mathbf{e}_1 + \alpha_i \mathbf{e}_2 + \beta_i \mathbf{e}_3 \mid i, j \in [p]\}$	5.10 (row 12)	row 44
$(8, p^2, p^2)$	$C = \langle \mathbf{e}_1 \rangle$ $A = \{i\mathbf{e}_2 + j\mathbf{e}_3 + k\mathbf{e}_4 + L_C \mid i, j, k \in [2]\}$ $S_0 = \{(\alpha_{ijk})\mathbf{e}_1 + i\mathbf{e}_2 + j\mathbf{e}_3 + k\mathbf{e}_4 \mid i, j, k \in [2]\}$		row 45
$(4p^2, 2, 2)$	$C = \langle \mathbf{e}_2 \rangle$ $A = \{i\mathbf{e}_1 + j\mathbf{e}_3 + k\mathbf{e}_4 + L_C \mid i \in [p^2], j, k \in [2]\}$ $S_0 = \{i\mathbf{e}_1 + \alpha_{ijk}\mathbf{e}_2 + j\mathbf{e}_3 + k\mathbf{e}_4 \mid i \in [p^2], j, k \in [2]\}$		row 52
$(8p, p, p)$	$C = \langle p\mathbf{e}_1 \rangle$ $A = \{i\mathbf{e}_1 + j\mathbf{e}_2 + k\mathbf{e}_3 + l\mathbf{e}_4 + L_C \mid i \in [p], j, k, l \in [2]\}$ $S_0 = \{(i + \alpha_{ijkl}p)\mathbf{e}_1 + j\mathbf{e}_2 + k\mathbf{e}_3 + l\mathbf{e}_4 \mid i \in [p], j, k, l \in [2]\}$		row 59

Case 2. S_0 is periodic.

In this case we have $|L_{S_0}| \geq 2$ and $S_0 = D \oplus L_{S_0}$ for some subset D of G by Lemma 2.3. So $|D| \cdot |L_{S_0}| = |S_0|$ and $1 < |L_{S_0}| < |S_0|$ by Lemma 2.8 part (b). Set

$$X = (D + L_{S_0})/L_{S_0} \quad \text{and} \quad Y = (C + L_{S_0})/L_{S_0}.$$

Since S_0 is periodic, by Lemma 2.5, we have

$$G/L_{S_0} = X \oplus Y$$

and X is aperiodic in G/L_{S_0} . We also know that X contains L_{S_0} and X generates G/L_{S_0} by Lemma 2.7. This factorization enables us to make use of Lemma 2.6 and previously established theorems to obtain the pair (S_0, C) via (X, Y) .

Table 5.20 represent all possible values for $(|S_0|, |C|, |L_{S_0}|)$; L_{S_0} , X , Y , S_0 , and C ; which previous theorem or lemma is used to obtain (X, Y) ; and which row in Table 5.18 the result corresponds to. $\square$

Table 5.20: The case when $G = \mathbb{Z}_{p^2} \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and S_0 is periodic

(S_0 , C , L_{S_0})	$L_{S_0}, X, Y, S_0, \text{ and } C$	Reference	Result
$(p^2, 8, p)$	$L_{S_0} = \langle p\mathbf{e}_1 \rangle$ $X = \{i\mathbf{e}_1 + \alpha_i\mathbf{e}_2 + \beta_i\mathbf{e}_3 + \gamma_i\mathbf{e}_4 + L_{S_0} \mid i \in [p]\}$ $Y = \{i\mathbf{e}_2 + j\mathbf{e}_3 + k\mathbf{e}_4 + L_{S_0} \mid i, j, k \in [2]\}$ $S_0 = \{(i + jp)\mathbf{e}_1 + \alpha_i\mathbf{e}_2 + \beta_i\mathbf{e}_3 + \gamma_i\mathbf{e}_4 \mid i, j \in [p]\}$ $C = \{\alpha_{ijk}p\mathbf{e}_1 + i\mathbf{e}_2 + j\mathbf{e}_3 + k\mathbf{e}_4 \mid i, j, k \in [2]\}$	5.10 (row 1)	row 2
$(2p, 4p, 2)$	$L_{S_0} = \langle \mathbf{e}_2 \rangle$ $X = \{(i + \alpha_i p)\mathbf{e}_1 + \alpha_i\mathbf{e}_3 + \beta_i\mathbf{e}_4 + L_{S_0} \mid i \in [p]\}$ $Y = \{ipe_1 + j\mathbf{e}_3 + k\mathbf{e}_4 + L_{S_0} \mid i \in [p], j, k \in [2]\}$ $S_0 = \{(i + \alpha_i p)\mathbf{e}_1 + j\mathbf{e}_2 + \alpha_i\mathbf{e}_3 + \beta_i\mathbf{e}_4 \mid i \in [p], j \in [2]\}$ $C = \{ipe_1 + \alpha_{ijk}\mathbf{e}_2 + j\mathbf{e}_3 + k\mathbf{e}_4 \mid i \in [p], j, k \in [2]\}$	5.8 (row 1)	row 6
$(2p, 4p, p)$	Not possible since no X exists	5.10	
$(4, 2p^2, 2)$	Not possible since no X exists	5.8	
$(2p^2, 4, 2)$	$L_{S_0} = \langle \mathbf{e}_2 \rangle$ $X = \{(i + jp)\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_3 + \alpha_i\mathbf{e}_4 + L_{S_0} \mid i, j \in [p]\}$ $Y = \{\alpha_i p\mathbf{e}_1 + j\mathbf{e}_3 + i\mathbf{e}_4 + L_{S_0} \mid i, j \in [2]\}$ $S_0 = \{(i + jp)\mathbf{e}_1 + k\mathbf{e}_2 + \alpha_{ij}\mathbf{e}_3 + \alpha_i\mathbf{e}_4 \mid i, j \in [p], k \in [2]\}$ $C = \{\alpha_i p\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + j\mathbf{e}_3 + i\mathbf{e}_4 \mid i, j \in [2]\}$	5.8 (row 3)	row 16
$(2p^2, 4, 2)$	$L_{S_0} = \langle \mathbf{e}_2 \rangle$ $X = \{i\mathbf{e}_1 + \alpha_i\mathbf{e}_3 + \beta_i\mathbf{e}_4 + L_{S_0} \mid i \in [p^2]\}$ $Y = \{i\mathbf{e}_3 + j\mathbf{e}_4 + L_{S_0} \mid i, j \in [2]\}$ $S_0 = \{i\mathbf{e}_1 + j\mathbf{e}_2 + \alpha_i\mathbf{e}_3 + \beta_i\mathbf{e}_4 \mid i \in [p^2], j \in [2]\}$ $C = \{\alpha_{ij}\mathbf{e}_2 + i\mathbf{e}_3 + j\mathbf{e}_4 \mid i, j \in [2]\}$	5.8 (row 4)	row 17
$(2p^2, 4, p)$	$L_{S_0} = \langle p\mathbf{e}_1 \rangle$ $X = \{i\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + j\mathbf{e}_3 + \alpha_i\mathbf{e}_4 + L_{S_0} \mid i \in [p], j \in [2]\}$ $Y = \{j\mathbf{e}_2 + \alpha_i\mathbf{e}_3 + i\mathbf{e}_4 \mid i, j \in [2]\}$ $S_0 = \{(i + kp)\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + j\mathbf{e}_3 + \alpha_i\mathbf{e}_4 \mid i, k \in [p], j \in [2]\}$ $C = \{\alpha_{ij}p\mathbf{e}_1 + j\mathbf{e}_2 + \alpha_i\mathbf{e}_3 + i\mathbf{e}_4 \mid i, j \in [2]\}$	5.10 (row 3)	row 18
$(2p^2, 4, p)$	$L_{S_0} = \langle p\mathbf{e}_1 \rangle$ $X = \{i\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + \beta_{ij}\mathbf{e}_3 + j\mathbf{e}_4 + L_{S_0} \mid i \in [p], j \in [2]\}$ $Y = \{i\mathbf{e}_2 + j\mathbf{e}_3 + L_{S_0} \mid i, j \in [2]\}$ $S_0 = \{(i + kp)\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + \beta_{ij}\mathbf{e}_3 + j\mathbf{e}_4 \mid i, k \in [p], j \in [2]\}$ $C = \{\alpha_{ij}p\mathbf{e}_1 + i\mathbf{e}_2 + j\mathbf{e}_3 \mid i, j \in [2]\}$	5.10 (row 4)	row 19
$(2p^2, 4, 2p)$	$L_{S_0} = \langle p\mathbf{e}_1 + \mathbf{e}_2 \rangle$ $X = \{i\mathbf{e}_1 + \alpha_i\mathbf{e}_3 + \beta_i\mathbf{e}_4 + L_{S_0} \mid i \in [p]\}$ $Y = \{i\mathbf{e}_3 + j\mathbf{e}_4 + L_{S_0} \mid i, j \in [2]\}$ $S_0 = \{(i + jp)\mathbf{e}_1 + k\mathbf{e}_2 + \alpha_i\mathbf{e}_3 + \beta_i\mathbf{e}_4 \mid i, j \in [p], k \in [2]\}$	3.6 (row 1)	row 20

	$C = \{\alpha_{ij}p\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + i\mathbf{e}_3 + j\mathbf{e}_4 \mid i, j \in [2]\}$		
$(2p^2, 4, p^2)$	Not possible since no X exists	5.5	
	$L_{S_0} = \langle \mathbf{e}_2 \rangle$ $X = \{(i + \alpha_{ij}p)\mathbf{e}_1 + j\mathbf{e}_3 + \alpha_i\mathbf{e}_4 + L_{S_0} \mid i \in [p], j \in [2]\}$ $Y = \{jp\mathbf{e}_1 + \alpha_i\mathbf{e}_3 + i\mathbf{e}_4 + L_{S_0} \mid i \in [2], j \in [p]\}$ $S_0 = \{(i + \alpha_{ij}p)\mathbf{e}_1 + k\mathbf{e}_2 + j\mathbf{e}_3 + \alpha_i\mathbf{e}_4 \mid i \in [p], j, k \in [2]\}$ $C = \{jp\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + \alpha_i\mathbf{e}_3 + i\mathbf{e}_4 \mid i \in [2], j \in [p]\}$	5.8 (row 6)	row 25
	$L_{S_0} = \langle \mathbf{e}_2 \rangle$ $X = \{(i + \alpha_i p)\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_3 + j\mathbf{e}_4 + L_{S_0} \mid i \in [p], j \in [2]\}$ $Y = \{ip\mathbf{e}_1 + j\mathbf{e}_3 + \alpha_i\mathbf{e}_4 + L_{S_0} \mid i \in [p], j \in [2]\}$ $S_0 = \{(i + \alpha_i p)\mathbf{e}_1 + k\mathbf{e}_2 + \alpha_{ij}\mathbf{e}_3 + j\mathbf{e}_4 \mid i \in [p], j, k \in [2]\}$ $C = \{ip\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + j\mathbf{e}_3 + \alpha_i\mathbf{e}_4 \mid i \in [p], j \in [2]\}$	5.8 (row 7)	row 26
	$L_{S_0} = \langle \mathbf{e}_2 \rangle$ $X = \{(\alpha_i + jp)\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_3 + i\mathbf{e}_4 + L_{S_0} \mid i \in [2], j \in [p]\}$ $Y = \{(i + \alpha_i p)\mathbf{e}_1 + j\mathbf{e}_3 + L_{S_0} \mid i \in [p], j \in [2]\}$ $S_0 = \{(\alpha_i + jp)\mathbf{e}_1 + k\mathbf{e}_2 + \alpha_{ij}\mathbf{e}_3 + i\mathbf{e}_4 \mid i, k \in [2], j \in [p]\}$ $C = \{(i + \alpha_i p)\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + j\mathbf{e}_3 \mid i \in [p], j \in [2]\}$	5.8 (row 8)	row 27
	$L_{S_0} = \langle \mathbf{e}_2 \rangle$ $X = \{(i + \alpha_{ij}p)\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_3 + j\mathbf{e}_4 + L_{S_0} \mid i \in [p], j \in [2]\}$ $Y = \{ip\mathbf{e}_1 + j\mathbf{e}_3 + L_{S_0} \mid i \in [p], j \in [2]\}$ $S_0 = \{(i + \alpha_{ij}p)\mathbf{e}_1 + k\mathbf{e}_2 + \alpha_{ij}\mathbf{e}_3 + j\mathbf{e}_4 \mid i \in [p], j, k \in [2]\}$ $C = \{ip\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + j\mathbf{e}_3 \mid i \in [p], j \in [2]\}$	5.8 (row 9)	row 28
	$L_{S_0} = \langle p\mathbf{e}_1 \rangle$ $X = \{\alpha_i\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + j\mathbf{e}_3 + i\mathbf{e}_4 + L_{S_0} \mid i, j \in [2]\}$ $Y = \{i\mathbf{e}_1 + j\mathbf{e}_2 + \alpha_i\mathbf{e}_3 + L_{S_0} \mid i \in [p], j \in [2]\}$ $S_0 = \{(\alpha_i + kp)\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + j\mathbf{e}_3 + i\mathbf{e}_4 \mid i, j \in [2], k \in [p]\}$ $C = \{(i + \alpha_{ij}p)\mathbf{e}_1 + j\mathbf{e}_2 + \alpha_i\mathbf{e}_3 \mid i \in [p], j \in [2]\}$	5.10 (row 5)	row 29
	$L_{S_0} = \langle p\mathbf{e}_1 \rangle$ $X = \{\alpha_{ij}\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + i\mathbf{e}_3 + j\mathbf{e}_4 + L_{S_0} \mid i, j \in [2]\}$ $Y = \{i\mathbf{e}_1 + j\mathbf{e}_2 + L_{S_0} \mid i \in [p], j \in [2]\}$ $S_0 = \{(\alpha_{ij} + kp)\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + i\mathbf{e}_3 + j\mathbf{e}_4 \mid i, j \in [2], k \in [p]\}$ $C = \{(i + \alpha_{ij}p)\mathbf{e}_1 + j\mathbf{e}_2 \mid i \in [p], j \in [2]\}$	5.10 (row 6)	row 30
	$L_{S_0} = \langle \mathbf{e}_2, \mathbf{e}_3 \rangle$ $X = \{(i + \alpha_i p)\mathbf{e}_1 + \alpha_i\mathbf{e}_4 + L_{S_0} \mid i \in [p]\}$ $Y = \{ip\mathbf{e}_1 + j\mathbf{e}_4 + L_{S_0} \mid i \in [p], j \in [2]\}$ $S_0 = \{(i + \alpha_i p)\mathbf{e}_1 + j\mathbf{e}_2 + k\mathbf{e}_3 + \alpha_i\mathbf{e}_4 \mid i \in [p], j, k \in [2]\}$ $C = \{ip\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + \beta_{ij}\mathbf{e}_3 + j\mathbf{e}_4 \mid i \in [p], j \in [2]\}$	5.4 (row 1)	row 31
$(4p, 2p, 2p)$	Not possible since no X exists	3.6	
	$L_{S_0} = \langle \mathbf{e}_2 \rangle$ $X = \{(\alpha_i + \alpha_{ij}p)\mathbf{e}_1 + j\mathbf{e}_3 + i\mathbf{e}_4 + L_{S_0} \mid i, j \in [2]\}$ $Y = \{(i + jp)\mathbf{e}_1 + \alpha_i\mathbf{e}_3 + L_{S_0} \mid i, j \in [p]\}$ $S_0 = \{(\alpha_i + \alpha_{ij}p)\mathbf{e}_1 + k\mathbf{e}_2 + j\mathbf{e}_3 + i\mathbf{e}_4 \mid i, j, k \in [2]\}$ $C = \{(i + jp)\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + \alpha_i\mathbf{e}_3 \mid i, j \in [p]\}$	5.8 (row 11)	row 40
	$L_{S_0} = \langle \mathbf{e}_2 \rangle$ $X = \{\alpha_{ij}\mathbf{e}_1 + i\mathbf{e}_3 + j\mathbf{e}_4 + L_{S_0} \mid i, j \in [2]\}$ $Y = \{i\mathbf{e}_1 + L_{S_0} \mid i \in [p^2]\}$ $S_0 = \{\alpha_{ij}\mathbf{e}_1 + k\mathbf{e}_2 + i\mathbf{e}_3 + j\mathbf{e}_4 \mid i, j, k \in [2]\}$ $C = \{i\mathbf{e}_1 + \alpha_i\mathbf{e}_2 \mid i \in [p^2]\}$	5.8 (row 12)	row 41
	$L_{S_0} = \langle \mathbf{e}_2, \mathbf{e}_3 \rangle$ $X = \{\alpha_i\mathbf{e}_1 + i\mathbf{e}_4 + L_{S_0} \mid i \in [2]\}$ $Y = \{i\mathbf{e}_1 + L_{S_0} \mid i \in [p^2]\}$	5.4 (row 2)	row 42

	$S_0 = \{\alpha_i \mathbf{e}_1 + j\mathbf{e}_2 + k\mathbf{e}_3 + i\mathbf{e}_4 \mid i, j, k \in [2]\}$ $C = \{i\mathbf{e}_1 + \alpha_i \mathbf{e}_2 + \beta_i \mathbf{e}_3 \mid i \in [p^2]\}$		
$(4p^2, 2, 2)$	$L_{S_0} = \langle \mathbf{e}_2 \rangle$ $X = \{i\mathbf{e}_1 + \alpha_{ij} \mathbf{e}_3 + j\mathbf{e}_4 + L_{S_0} \mid i \in [p^2], j \in [2]\}$ $Y = \{i\mathbf{e}_3 + L_{S_0} \mid i \in [2]\}$ $S_0 = \{i\mathbf{e}_1 + k\mathbf{e}_2 + \alpha_{ij} \mathbf{e}_3 + j\mathbf{e}_4 \mid i \in [p^2], j, k \in [2]\}$ $C = \{\alpha_i \mathbf{e}_2 + i\mathbf{e}_3 \mid i \in [2]\}$	5.8 (row 16)	row 46
$(4p^2, 2, p)$	$L_{S_0} = \langle p\mathbf{e}_1 \rangle$ $X = \{i\mathbf{e}_1 + \alpha_{ijk} \mathbf{e}_2 + j\mathbf{e}_3 + k\mathbf{e}_4 + L_{S_0} \mid i \in [p], j, k \in [2]\}$ $Y = \{i\mathbf{e}_2 + L_{S_0} \mid i \in [2]\}$ $S_0 = \{(i + lp)\mathbf{e}_1 + \alpha_{ijk} \mathbf{e}_2 + j\mathbf{e}_3 + k\mathbf{e}_4 \mid i, l \in [p], j, k \in [2]\}$ $C = \{\alpha_i p\mathbf{e}_1 + i\mathbf{e}_2 \mid i \in [2]\}$	5.10 (row 10)	row 47
$(4p^2, 2, 4)$	$L_{S_0} = \langle \mathbf{e}_2, \mathbf{e}_3 \rangle$ $X = \{i\mathbf{e}_1 + \alpha_i \mathbf{e}_4 + L_{S_0} \mid i \in [p^2]\}$ $Y = \{i\mathbf{e}_4 + L_{S_0} \mid i \in [2]\}$ $S_0 = \{i\mathbf{e}_1 + j\mathbf{e}_2 + k\mathbf{e}_3 + \alpha_i \mathbf{e}_4 \mid i \in [p^2], j, k \in [2]\}$ $C = \{\alpha_i \mathbf{e}_2 + \beta_i \mathbf{e}_3 + i\mathbf{e}_4 \mid i \in [2]\}$	5.4 (row 4)	row 48
$(4p^2, 2, 2p)$	$L_{S_0} = \langle p\mathbf{e}_1 + \mathbf{e}_2 \rangle$ $X = \{i\mathbf{e}_1 + \alpha_{ij} \mathbf{e}_3 + j\mathbf{e}_4 + L_{S_0} \mid i \in [p], j \in [2]\}$ $Y = \{i\mathbf{e}_3 + L_{S_0} \mid i \in [2]\}$ $S_0 = \{(i + kp)\mathbf{e}_1 + l\mathbf{e}_2 + \alpha_{ij} \mathbf{e}_3 + j\mathbf{e}_4 \mid i, k \in [p], j, l \in [2]\}$ $C = \{\alpha_i p\mathbf{e}_1 + \alpha_i \mathbf{e}_2 + i\mathbf{e}_3 \mid i \in [2]\}$	3.6 (row 5)	row 49
$(4p^2, 2, p^2)$	$L_{S_0} = \langle \mathbf{e}_1 \rangle$ $X = \{\alpha_{ij} \mathbf{e}_2 + i\mathbf{e}_3 + j\mathbf{e}_4 + L_{S_0} \mid i, j \in [2]\}$ $Y = \{i\mathbf{e}_2 + L_{S_0} \mid i \in [2]\}$ $S_0 = \{k\mathbf{e}_1 + \alpha_{ij} \mathbf{e}_2 + i\mathbf{e}_3 + j\mathbf{e}_4 \mid i, j \in [2], k \in [p^2]\}$ $C = \{\alpha_i \mathbf{e}_1 + i\mathbf{e}_2 \mid i \in [2]\}$	5.5	row 50
$(4p^2, 2, 4p)$	$L_{S_0} = \langle p\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_3 \rangle$ $X = \{i\mathbf{e}_1 + \alpha_i \mathbf{e}_4 + L_{S_0} \mid i \in [p]\}$ $Y = \{i\mathbf{e}_4 + L_{S_0} \mid i \in [2]\}$ $S_0 = \{(i + jp)\mathbf{e}_1 + k\mathbf{e}_2 + l\mathbf{e}_3 + \alpha_i \mathbf{e}_4 \mid i, j \in [p], k, l \in [2]\}$ $C = \{\alpha_i p\mathbf{e}_1 + \alpha_i \mathbf{e}_2 + \beta_i \mathbf{e}_3 + i\mathbf{e}_4 \mid i \in [2]\}$	3.5 (row 1)	row 51
$(4p^2, 2, 2p^2)$	Not possible since no X exists	5.2	
$(8p, p, 2)$	$L_{S_0} = \langle \mathbf{e}_2 \rangle$ $X = \{(i + \alpha_{ijk}p)\mathbf{e}_1 + j\mathbf{e}_3 + k\mathbf{e}_4 + L_{S_0} \mid i \in [p], j, k \in [2]\}$ $Y = \{ip\mathbf{e}_1 + L_{S_0} \mid i \in [p]\}$ $S_0 = \{(i + \alpha_{ijk}p)\mathbf{e}_1 + l\mathbf{e}_2 + j\mathbf{e}_3 + k\mathbf{e}_4 \mid i \in [p], j, k, l \in [2]\}$ $C = \{ip\mathbf{e}_1 + \alpha_i \mathbf{e}_2 \mid i \in [p]\}$	5.8 (row 21)	row 53
$(8p, p, p)$	$L_{S_0} = \langle p\mathbf{e}_1 \rangle$ $X = \{(\alpha_{ijk})\mathbf{e}_1 + i\mathbf{e}_2 + j\mathbf{e}_3 + k\mathbf{e}_4 + L_{S_0} \mid i, j, k \in [2]\}$ $Y = \{i\mathbf{e}_1 + L_{S_0} \mid i \in [p]\}$ $S_0 = \{(\alpha_{ijk} + lp)\mathbf{e}_1 + i\mathbf{e}_2 + j\mathbf{e}_3 + k\mathbf{e}_4 \mid i, j, k \in [2], l \in [p]\}$ $C = \{(i + \alpha_i p)\mathbf{e}_1 \mid i \in [p]\}$	5.10 (row 13)	row 54
$(8p, p, 4)$	$L_{S_0} = \langle \mathbf{e}_2, \mathbf{e}_3 \rangle$ $X = \{(i + \alpha_{ij}p)\mathbf{e}_1 + j\mathbf{e}_4 + L_{S_0} \mid i \in [p], j \in [2]\}$ $Y = \{ip\mathbf{e}_1 + L_{S_0} \mid i \in [p]\}$ $S_0 = \{(i + \alpha_{ij}p)\mathbf{e}_1 + k\mathbf{e}_2 + l\mathbf{e}_3 + j\mathbf{e}_4 \mid i \in [p], j, k, l \in [2]\}$ $C = \{ip\mathbf{e}_1 + \alpha_i \mathbf{e}_2 + \beta_i \mathbf{e}_3 \mid i \in [p]\}$	5.4 (row 7)	row 55
$(8p, p, 2p)$	$L_{S_0} = \langle p\mathbf{e}_1 + \mathbf{e}_2 \rangle$ $X = \{\alpha_{ij} \mathbf{e}_1 + i\mathbf{e}_3 + j\mathbf{e}_4 + L_{S_0} \mid i, j \in [2]\}$ $Y = \{i\mathbf{e}_1 + L_{S_0} \mid i \in [p]\}$	3.6 (row 7)	row 56

	$S_0 = \{(\alpha_{ij} + kp)\mathbf{e}_1 + l\mathbf{e}_2 + i\mathbf{e}_3 + j\mathbf{e}_4 \mid i, j, l \in [2], k \in [p]\}$ $C = \{(i + \alpha_i p)\mathbf{e}_1 + \alpha_i \mathbf{e}_2 \mid i \in [p]\}$		
$(8p, p, 8)$	$L_{S_0} = \langle \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4 \rangle$ $X = \{(i + \alpha_i p)\mathbf{e}_1 + L_{S_0} \mid i \in [p]\}$ $Y = \{i p \mathbf{e}_1 + L_{S_0} \mid i \in [p]\}$ $S_0 = \{(i + \alpha_i p)\mathbf{e}_1 + j\mathbf{e}_2 + k\mathbf{e}_3 + l\mathbf{e}_4 \mid i \in [p], j, k, l \in [2]\}$ $C = \{i p \mathbf{e}_1 + \alpha_i \mathbf{e}_2 + \beta_i \mathbf{e}_3 + \gamma_i \mathbf{e}_4 \mid i \in [p]\}$	5.1	row 57
$(8p, p, 4p)$	$L_{S_0} = \langle p\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_3 \rangle$ $X = \{\alpha_i \mathbf{e}_1 + i\mathbf{e}_4 + L_{S_0} \mid i \in [2]\}$ $Y = \{i\mathbf{e}_1 + L_{S_0} \mid i \in [p]\}$ $S_0 = \{(\alpha_i + jp)\mathbf{e}_1 + k\mathbf{e}_2 + l\mathbf{e}_3 + i\mathbf{e}_4 \mid i, k, l \in [2], j \in [p]\}$ $C = \{(i + \alpha_i p)\mathbf{e}_1 + \alpha_i \mathbf{e}_2 + \beta_i \mathbf{e}_3 \mid i \in [p]\}$	3.5 (row 2)	row 58

Remark 5.17. In Theorem 5.16, the generators $\mathbf{e}_2$, $\mathbf{e}_3$ and $\mathbf{e}_4$ may be replaced by any basis of $H = \{0\} \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, as in (C2).

Theorem 5.18. Let $G = \mathbb{Z}_p \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ for an odd prime p and let S be a proper subset of $G \setminus \{0\}$ that generates G . Let $0 \in C \subseteq G$. Then C is a perfect code in $\text{Cay}(G, S)$ if and only if (S_0, C) is one of the pairs in Table 5.21.

Table 5.21: Perfect codes in Cayley graphs of $\mathbb{Z}_p \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

Row	S_0	C
1	$\{i\mathbf{e}_1 + \alpha_i \mathbf{e}_2 + \beta_i \mathbf{e}_3 + \gamma_i \mathbf{e}_4 + \delta_i \mathbf{e}_5 \mid i \in [p]\}$	$\{i\mathbf{e}_2 + j\mathbf{e}_3 + k\mathbf{e}_4 + l\mathbf{e}_5 \mid i, j, k, l \in [2]\}$
2	$\{i\mathbf{e}_1 + j\mathbf{e}_2 + \alpha_i \mathbf{e}_3 + \beta_i \mathbf{e}_4 + \gamma_i \mathbf{e}_5 \mid i \in [p], j \in [2]\}$	$\{\alpha_{ijk} \mathbf{e}_2 + i\mathbf{e}_3 + j\mathbf{e}_4 + k\mathbf{e}_5 \mid i, j, k \in [2]\}$
3	$\{i\mathbf{e}_1 + \alpha_{ij} \mathbf{e}_2 + j\mathbf{e}_3 + \alpha_i \mathbf{e}_4 + \beta_i \mathbf{e}_5 \mid i \in [p], j \in [2]\}$	$\{k\mathbf{e}_2 + \alpha_{ij} \mathbf{e}_3 + i\mathbf{e}_4 + j\mathbf{e}_5 \mid i, j, k \in [2]\}$
4	$\{i\mathbf{e}_1 + \alpha_{ij} \mathbf{e}_2 + \beta_{ij} \mathbf{e}_3 + j\mathbf{e}_4 + \alpha_i \mathbf{e}_5 \mid i \in [p], j \in [2]\}$	$\{j\mathbf{e}_2 + k\mathbf{e}_3 + \alpha_i \mathbf{e}_4 + i\mathbf{e}_5 \mid i, j, k \in [2]\}$
5	$\{i\mathbf{e}_1 + \alpha_{ij} \mathbf{e}_2 + \beta_{ij} \mathbf{e}_3 + \gamma_{ij} \mathbf{e}_4 + j\mathbf{e}_5 \mid i \in [p], j \in [2]\}$	$\{i\mathbf{e}_2 + j\mathbf{e}_3 + k\mathbf{e}_4 \mid i, j, k \in [2]\}$
6	$\{i\mathbf{e}_1 + k\mathbf{e}_2 + \alpha_{ij} \mathbf{e}_3 + j\mathbf{e}_4 + \alpha_i \mathbf{e}_5 \mid i \in [p], j, k \in [2]\}$	$\{\alpha_{ij} \mathbf{e}_2 + j\mathbf{e}_3 + \alpha_i \mathbf{e}_4 + i\mathbf{e}_5 \mid i, j \in [2]\}$
7	$\{i\mathbf{e}_1 + k\mathbf{e}_2 + \alpha_{ij} \mathbf{e}_3 + \beta_{ij} \mathbf{e}_4 + j\mathbf{e}_5 \mid i \in [p], j, k \in [2]\}$	$\{\alpha_{ij} \mathbf{e}_2 + i\mathbf{e}_3 + j\mathbf{e}_4 \mid i, j \in [2]\}$
8	$\{i\mathbf{e}_1 + j\mathbf{e}_2 + k\mathbf{e}_3 + \alpha_i \mathbf{e}_4 + \beta_i \mathbf{e}_5 \mid i \in [p], j, k \in [2]\}$	$\{\alpha_{ij} \mathbf{e}_2 + \beta_{ij} \mathbf{e}_3 + i\mathbf{e}_4 + j\mathbf{e}_5 \mid i, j \in [2]\}$
9	$\{i\mathbf{e}_1 + \alpha_{ijk} \mathbf{e}_2 + k\mathbf{e}_3 + \alpha_{ij} \mathbf{e}_4 + j\mathbf{e}_5 \mid i \in [p], j, k \in [2]\}$	$\{j\mathbf{e}_2 + \alpha_i \mathbf{e}_3 + i\mathbf{e}_4 \mid i, j \in [2]\}$
10	$\{k\mathbf{e}_1 + \alpha_{ijk} \mathbf{e}_2 + \alpha_{ij} \mathbf{e}_3 + i\mathbf{e}_4 + j\mathbf{e}_5 \mid i, j \in [2], k \in [p]\}$	$\{\alpha_i \mathbf{e}_1 + j\mathbf{e}_2 + i\mathbf{e}_3 \mid i, j \in [2]\}$
11	$\{i\mathbf{e}_1 + \alpha_{ijk} \mathbf{e}_2 + j\mathbf{e}_3 + k\mathbf{e}_4 + \alpha_i \mathbf{e}_5 \mid i \in [p], j, k \in [2]\}$	$\{j\mathbf{e}_2 + \alpha_i \mathbf{e}_3 + \beta_i \mathbf{e}_4 + i\mathbf{e}_5 \mid i, j \in [2]\}$
12	$\{i\mathbf{e}_1 + \alpha_{ijk} \mathbf{e}_2 + \beta_{ijk} \mathbf{e}_3 + j\mathbf{e}_4 + k\mathbf{e}_5 \mid i \in [p], j, k \in [2]\}$	$\{i\mathbf{e}_2 + j\mathbf{e}_3 \mid i, j \in [2]\}$
13	$\{\alpha_i \mathbf{e}_1 + k\mathbf{e}_2 + \alpha_{ij} \mathbf{e}_3 + j\mathbf{e}_4 + i\mathbf{e}_5 \mid i, j, k \in [2]\}$	$\{i\mathbf{e}_1 + \alpha_{ij} \mathbf{e}_2 + j\mathbf{e}_3 + \alpha_i \mathbf{e}_4 \mid i \in [p], j \in [2]\}$
14	$\{\alpha_{ij} \mathbf{e}_1 + k\mathbf{e}_2 + \alpha_{ij} \mathbf{e}_3 + i\mathbf{e}_4 + j\mathbf{e}_5 \mid i, j, k \in [2]\}$	$\{i\mathbf{e}_1 + \alpha_{ij} \mathbf{e}_2 + j\mathbf{e}_3 \mid i \in [p], j \in [2]\}$
15	$\{\alpha_{ij} \mathbf{e}_1 + \alpha_{ijk} \mathbf{e}_2 + k\mathbf{e}_3 + i\mathbf{e}_4 + j\mathbf{e}_5 \mid i, j, k \in [2]\}$	$\{i\mathbf{e}_1 + j\mathbf{e}_2 + \alpha_i \mathbf{e}_3 \mid i \in [p], j \in [2]\}$
16	$\{\alpha_i \mathbf{e}_1 + \alpha_{ijk} \mathbf{e}_2 + j\mathbf{e}_3 + k\mathbf{e}_4 + i\mathbf{e}_5 \mid i, j, k \in [2]\}$	$\{i\mathbf{e}_1 + j\mathbf{e}_2 + \alpha_i \mathbf{e}_3 + \beta_i \mathbf{e}_4 \mid i \in [p], j \in [2]\}$
17	$\{\alpha_{ijk} \mathbf{e}_1 + k\mathbf{e}_2 + \alpha_{ij} \mathbf{e}_3 + i\mathbf{e}_4 + j\mathbf{e}_5 \mid i, j, k \in [2]\}$	$\{j\mathbf{e}_1 + \alpha_i \mathbf{e}_2 + i\mathbf{e}_3 \mid i \in [2], j \in [p]\}$
18	$\{\alpha_{ijk} \mathbf{e}_1 + \alpha_{ijk} \mathbf{e}_2 + i\mathbf{e}_3 + j\mathbf{e}_4 + k\mathbf{e}_5 \mid i, j, k \in [2]\}$	$\{i\mathbf{e}_1 + j\mathbf{e}_2 \mid i \in [p], j \in [2]\}$
19	$\{i\mathbf{e}_1 + l\mathbf{e}_2 + \alpha_{ijk} \mathbf{e}_3 + j\mathbf{e}_4 + k\mathbf{e}_5 \mid i \in [p], j, k, l \in [2]\}$	$\{\alpha_i \mathbf{e}_2 + i\mathbf{e}_3 \mid i \in [2]\}$
20	$\{l\mathbf{e}_1 + \alpha_{ijk} \mathbf{e}_2 + i\mathbf{e}_3 + j\mathbf{e}_4 + k\mathbf{e}_5 \mid i, j, k \in [2], l \in [p]\}$	$\{\alpha_i \mathbf{e}_1 + i\mathbf{e}_2 \mid i \in [2]\}$
21	$\{i\mathbf{e}_1 + k\mathbf{e}_2 + l\mathbf{e}_3 + \alpha_{ij} \mathbf{e}_4 + j\mathbf{e}_5 \mid i \in [p], j, k, l \in [2]\}$	$\{\alpha_i \mathbf{e}_2 + \beta_i \mathbf{e}_3 + i\mathbf{e}_4 \mid i \in [2]\}$
22	$\{k\mathbf{e}_1 + l\mathbf{e}_2 + \alpha_{ij} \mathbf{e}_3 + i\mathbf{e}_4 + j\mathbf{e}_5 \mid i, j, l \in [2], k \in [p]\}$	$\{\alpha_i \mathbf{e}_1 + \alpha_i \mathbf{e}_2 + i\mathbf{e}_3 \mid i \in [2]\}$
23	$\{i\mathbf{e}_1 + j\mathbf{e}_2 + k\mathbf{e}_3 + l\mathbf{e}_4 + \alpha_i \mathbf{e}_5 \mid i \in [p], j, k, l \in [2]\}$	$\{\alpha_i \mathbf{e}_2 + \beta_i \mathbf{e}_3 + \gamma_i \mathbf{e}_4 + i\mathbf{e}_5 \mid i \in [2]\}$
24	$\{i\mathbf{e}_1 + \alpha_{ijk} \mathbf{e}_2 + j\mathbf{e}_3 + k\mathbf{e}_4 + l\mathbf{e}_5 \mid i \in [p], j, k, l \in [2]\}$	$\{i\mathbf{e}_2 \mid i \in [2]\}$
25	$\{\alpha_{ijk} \mathbf{e}_1 + l\mathbf{e}_2 + i\mathbf{e}_3 + j\mathbf{e}_4 + k\mathbf{e}_5 \mid i, j, k, l \in [2]\}$	$\{i\mathbf{e}_1 + \alpha_i \mathbf{e}_2 \mid i \in [p]\}$
26	$\{\alpha_{ij} \mathbf{e}_1 + k\mathbf{e}_2 + l\mathbf{e}_3 + i\mathbf{e}_4 + j\mathbf{e}_5 \mid i, j, k, l \in [2]\}$	$\{i\mathbf{e}_1 + \alpha_i \mathbf{e}_2 + \beta_i \mathbf{e}_3 \mid i \in [p]\}$
27	$\{\alpha_i \mathbf{e}_1 + j\mathbf{e}_2 + k\mathbf{e}_3 + l\mathbf{e}_4 + i\mathbf{e}_5 \mid i, j, k, l \in [2]\}$	$\{i\mathbf{e}_1 + \alpha_i \mathbf{e}_2 + \beta_i \mathbf{e}_3 + \gamma_i \mathbf{e}_4 \mid i \in [p]\}$
28	$\{\alpha_{ijkl} \mathbf{e}_1 + i\mathbf{e}_2 + j\mathbf{e}_3 + k\mathbf{e}_4 + l\mathbf{e}_5 \mid i, j, k, l \in [2]\}$	$\{i\mathbf{e}_1 \mid i \in [p]\}$

Proof. Suppose C is a perfect code in $\text{Cay}(G, S)$. Then $G = S_0 \oplus C$ and so $|S_0| \cdot |C| = |G| = 16p$. Since S_0 generates G , S_0 contains at least four non-zero elements, so $|S_0| \geq 5$. Therefore, $4 < |S_0| < 16p$.

Case 1. S_0 is aperiodic.

Since G is Hajós and $G = S_0 \oplus C$, in this case C must be periodic, that is, L_C is nontrivial. Write $C = L_C \oplus D$ for some subset D of G whose existence is ensured by Lemma 2.3. Then $|L_C| \cdot |D| = |C|$ and $2 \leq |L_C| \leq |C|$. If $|S_0|$ is a prime, then $|L_C| = |C|$ by Lemma 2.8 part (c). If $|S_0|$ is not a prime, then $|L_C|$ can be any divisor of $|C|$. Set

$$A = (S_0 + L_C)/L_C \quad \text{and} \quad B = (D + L_C)/L_C.$$

Since C is periodic, by Lemma 2.5, we have

$$G/L_C = A \oplus B$$

and B is aperiodic in G/L_C . Since G/L_C is isomorphic to a subgroup of G and subgroups of Hajós group are also Hajós groups, A is periodic in G/L_C . We also know that A contains L_C and A generates G/L_C by Lemma 2.7. Furthermore, $|A| = |S_0|$ and $|B| = |D|$. Note that if $|L_C| = |C|$, then $C = L_C$ as $0 \in C$. This means we can take $D = \{0\}$ and hence $B = L_C/L_C$ and $G/L_C = A$. We use C to get A and then use Lemma 2.6 to obtain S_0 . If $|L_C| < |C|$, then this factorization enables us to make use of previously established theorems to obtain the pair (S_0, C) via (A, B) .

Table 5.22 represent all possible values for $(|S_0|, |C|, |L_C|)$; C , A , and S_0 when $|L_C| = |C|$ or L_C , A , B , S_0 , and C when $|L_C| < |C|$; which previous theorem or lemma is used to obtain (A, B) when $|L_C| < |C|$; and which row in Table 5.21 the result corresponds to.

Table 5.22: The case when $G = \mathbb{Z}_p \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and S_0 is aperiodic

(S_0 , C , L_C)	$L_C, A, B, S_0, \text{ and } C$	Reference	Result
$(p, 16, 16)$	$C = \langle \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5 \rangle$ $A = \{i\mathbf{e}_1 + L_C \mid i \in [p]\}$ $S_0 = \{i\mathbf{e}_1 + \alpha_i\mathbf{e}_2 + \beta_i\mathbf{e}_3 + \gamma_i\mathbf{e}_4 + \delta_i\mathbf{e}_5 \mid i \in [p]\}$		row 1
$(2p, 8, 2)$	$L_C = \langle \mathbf{e}_2 \rangle$ $A = \{i\mathbf{e}_1 + j\mathbf{e}_3 + \alpha_i\mathbf{e}_4 + \beta_i\mathbf{e}_5 + L_C \mid i \in [p], j \in [2]\}$ $B = \{\alpha_{ij}\mathbf{e}_3 + i\mathbf{e}_4 + j\mathbf{e}_5 + L_C \mid i, j \in [2]\}$ $S_0 = \{i\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + j\mathbf{e}_3 + \alpha_i\mathbf{e}_4 + \beta_i\mathbf{e}_5 \mid i \in [p], j \in [2]\}$ $C = \{k\mathbf{e}_2 + \alpha_{ij}\mathbf{e}_3 + i\mathbf{e}_4 + j\mathbf{e}_5 \mid i, j, k \in [2]\}$	5.10 (row 2)	row 3
$(2p, 8, 4)$	$L_C = \langle \mathbf{e}_2, \mathbf{e}_3 \rangle$ $A = \{i\mathbf{e}_1 + j\mathbf{e}_4 + \alpha_i\mathbf{e}_5 + L_C \mid i \in [p], j \in [2]\}$ $B = \{\alpha_i\mathbf{e}_4 + i\mathbf{e}_5 + L_C \mid i \in [2]\}$ $S_0 = \{i\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + \beta_{ij}\mathbf{e}_3 + j\mathbf{e}_4 + \alpha_i\mathbf{e}_5 \mid i \in [p], j \in [2]\}$ $C = \{j\mathbf{e}_2 + k\mathbf{e}_3 + \alpha_i\mathbf{e}_4 + i\mathbf{e}_5 \mid i, j, k \in [2]\}$	3.6 (row 4)	row 4
$(2p, 8, 8)$	$C = \langle \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4 \rangle$ $A = \{i\mathbf{e}_1 + j\mathbf{e}_5 + L_C \mid i \in [p], j \in [2]\}$ $S_0 = \{i\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + \beta_{ij}\mathbf{e}_3 + \gamma_{ij}\mathbf{e}_4 + j\mathbf{e}_5 \mid i \in [p], j \in [2]\}$		row 5
	$L_C = \langle \mathbf{e}_2 \rangle$ $A = \{i\mathbf{e}_1 + k\mathbf{e}_3 + \alpha_{ij}\mathbf{e}_4 + j\mathbf{e}_5 + L_C \mid i \in [p], j, k \in [2]\}$		

$(4p, 4, 2)$	$B = \{\alpha_i \mathbf{e}_3 + i\mathbf{e}_4 + L_C \mid i \in [2]\}$ $S_0 = \{i\mathbf{e}_1 + \alpha_{ijk}\mathbf{e}_2 + k\mathbf{e}_3 + \alpha_{ij}\mathbf{e}_4 + j\mathbf{e}_5 \mid i \in [p], j, k \in [2]\}$ $C = \{j\mathbf{e}_2 + \alpha_i\mathbf{e}_3 + i\mathbf{e}_4 \mid i, j \in [2]\}$	5.10 (row 7)	row 9
$(4p, 4, 2)$	$L_C = \langle \mathbf{e}_2 \rangle$ $A = \{k\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_3 + i\mathbf{e}_4 + j\mathbf{e}_5 + L_C \mid i, j \in [2], k \in [p]\}$ $B = \{\alpha_i \mathbf{e}_1 + i\mathbf{e}_3 + L_C \mid i \in [2]\}$ $S_0 = \{k\mathbf{e}_1 + \alpha_{ijk}\mathbf{e}_2 + \alpha_{ij}\mathbf{e}_3 + i\mathbf{e}_4 + j\mathbf{e}_5 \mid i, j \in [2], k \in [p]\}$ $C = \{\alpha_i \mathbf{e}_1 + j\mathbf{e}_2 + i\mathbf{e}_3 \mid i, j \in [2]\}$	5.10 (row 8)	row 10
$(4p, 4, 2)$	$L_C = \langle \mathbf{e}_2 \rangle$ $A = \{i\mathbf{e}_1 + j\mathbf{e}_3 + k\mathbf{e}_4 + \alpha_i\mathbf{e}_5 + L_C \mid i \in [p], j, k \in [2]\}$ $B = \{\alpha_i \mathbf{e}_3 + \beta_i \mathbf{e}_4 + i\mathbf{e}_5 + L_C \mid i \in [2]\}$ $S_0 = \{i\mathbf{e}_1 + \alpha_{ijk}\mathbf{e}_2 + j\mathbf{e}_3 + k\mathbf{e}_4 + \alpha_i\mathbf{e}_5 \mid i \in [p], j, k \in [2]\}$ $C = \{j\mathbf{e}_2 + \alpha_i\mathbf{e}_3 + \beta_i \mathbf{e}_4 + i\mathbf{e}_5 \mid i, j \in [2]\}$	5.10 (row 9)	row 11
$(4p, 4, 4)$	$C = \langle \mathbf{e}_2, \mathbf{e}_3 \rangle$ $A = \{i\mathbf{e}_1 + j\mathbf{e}_4 + k\mathbf{e}_5 + L_C \mid i \in [p], j, k \in [2]\}$ $S_0 = \{i\mathbf{e}_1 + \alpha_{ijk}\mathbf{e}_2 + \beta_{ijk}\mathbf{e}_3 + j\mathbf{e}_4 + k\mathbf{e}_5 \mid i \in [p], j, k \in [2]\}$		row 12
$(8, 2p, 2)$	$L_C = \langle \mathbf{e}_2 \rangle$ $A = \{\alpha_{ij}\mathbf{e}_1 + k\mathbf{e}_3 + i\mathbf{e}_4 + j\mathbf{e}_5 + L_C \mid i, j, k \in [2]\}$ $B = \{i\mathbf{e}_1 + \alpha_i\mathbf{e}_3 + L_C \mid i \in [p]\}$ $S_0 = \{\alpha_{ij}\mathbf{e}_1 + \alpha_{ijk}\mathbf{e}_2 + k\mathbf{e}_3 + i\mathbf{e}_4 + j\mathbf{e}_5 \mid i, j, k \in [2]\}$ $C = \{i\mathbf{e}_1 + j\mathbf{e}_2 + \alpha_i\mathbf{e}_3 \mid i \in [p], j \in [2]\}$	5.10 (row 11)	row 15
$(8, 2p, 2)$	$L_C = \langle \mathbf{e}_2 \rangle$ $A = \{\alpha_i \mathbf{e}_1 + j\mathbf{e}_3 + k\mathbf{e}_4 + i\mathbf{e}_5 + L_C \mid i, j, k \in [2]\}$ $B = \{i\mathbf{e}_1 + \alpha_i\mathbf{e}_3 + \beta_i \mathbf{e}_4 + L_C \mid i \in [p]\}$ $S_0 = \{\alpha_i \mathbf{e}_1 + \alpha_{ijk}\mathbf{e}_2 + j\mathbf{e}_3 + k\mathbf{e}_4 + i\mathbf{e}_5 \mid i, j, k \in [2]\}$ $C = \{i\mathbf{e}_1 + j\mathbf{e}_2 + \alpha_i\mathbf{e}_3 + \beta_i \mathbf{e}_4 \mid i \in [p], j \in [2]\}$	5.10 (row 12)	row 16
$(8, 2p, p)$	$L_C = \langle \mathbf{e}_1 \rangle$ $A = \{k\mathbf{e}_2 + \alpha_{ij}\mathbf{e}_3 + i\mathbf{e}_4 + j\mathbf{e}_5 + L_C \mid i, j, k \in [2]\}$ $B = \{\alpha_i \mathbf{e}_2 + i\mathbf{e}_3 + L_C \mid i \in [2]\}$ $S_0 = \{\alpha_{ijk}\mathbf{e}_1 + k\mathbf{e}_2 + \alpha_{ij}\mathbf{e}_3 + i\mathbf{e}_4 + j\mathbf{e}_5 \mid i, j, k \in [2]\}$ $C = \{j\mathbf{e}_1 + \alpha_i\mathbf{e}_2 + i\mathbf{e}_3 \mid i \in [2], j \in [p]\}$	5.12 (row 1)	row 17
$(8, 2p, 2p)$	$C = \langle \mathbf{e}_1 + \mathbf{e}_2 \rangle$ $A = \{i\mathbf{e}_3 + j\mathbf{e}_4 + k\mathbf{e}_5 + L_C \mid i, j, k \in [2]\}$ $S_0 = \{\alpha_{ijk}\mathbf{e}_1 + \alpha_{ijk}\mathbf{e}_2 + i\mathbf{e}_3 + j\mathbf{e}_4 + k\mathbf{e}_5 \mid i, j, k \in [2]\}$		row 18
$(8p, 2, 2)$	$C = \langle \mathbf{e}_2 \rangle$ $A = \{i\mathbf{e}_1 + j\mathbf{e}_3 + k\mathbf{e}_4 + l\mathbf{e}_5 + L_C \mid i \in [p], j, k, l \in [2]\}$ $S_0 = \{i\mathbf{e}_1 + \alpha_{ijkl}\mathbf{e}_2 + j\mathbf{e}_3 + k\mathbf{e}_4 + l\mathbf{e}_5 \mid i \in [p], j, k, l \in [2]\}$		row 24
$(16, p, p)$	$C = \langle \mathbf{e}_1 \rangle$ $A = \{i\mathbf{e}_2 + j\mathbf{e}_3 + k\mathbf{e}_4 + l\mathbf{e}_5 + L_C \mid i, j, k, l \in [2]\}$ $S_0 = \{\alpha_{ijkl}\mathbf{e}_1 + i\mathbf{e}_2 + j\mathbf{e}_3 + k\mathbf{e}_4 + l\mathbf{e}_5 \mid i, j, k, l \in [2]\}$		row 28

Case 2. S_0 is periodic.

In this case we have $|L_{S_0}| \geq 2$ and $S_0 = D \oplus L_{S_0}$ for some subset D of G by Lemma 2.3. So $|D| \cdot |L_{S_0}| = |S_0|$ and $1 < |L_{S_0}| < |S_0|$ by Lemma 2.8 part (b). Set

$$X = (D + L_{S_0})/L_{S_0} \quad \text{and} \quad Y = (C + L_{S_0})/L_{S_0}.$$

Since S_0 is periodic, by Lemma 2.5, we have

$$G/L_{S_0} = X \oplus Y$$

and X is aperiodic in G/L_{S_0} . We also know that X contains L_{S_0} and X generates G/L_{S_0} by Lemma 2.7. This factorization enables us to make use of Lemma 2.6 and previously established theorems to obtain the pair (S_0, C) via (X, Y) .

Table 5.23 represent all possible values for $(|S_0|, |C|, |L_{S_0}|)$; L_{S_0} , X , Y , S_0 , and C ; which previous theorem or lemma is used to obtain (X, Y) ; and which row in Table 5.21 the result corresponds to. $\square$

Table 5.23: The case when $G = \mathbb{Z}_p \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and S_0 is periodic

(S_0 , C , L_{S_0})	$L_{S_0}, X, Y, S_0, \text{ and } C$	Reference	Result
$(2p, 8, 2)$	$L_{S_0} = \langle \mathbf{e}_2 \rangle$		
	$X = \{i\mathbf{e}_1 + \alpha_i\mathbf{e}_3 + \beta_i\mathbf{e}_4 + \gamma_i\mathbf{e}_5 + L_{S_0} \mid i \in [p]\}$		
	$Y = \{i\mathbf{e}_3 + j\mathbf{e}_4 + k\mathbf{e}_5 + L_{S_0} \mid i, j, k \in [2]\}$	5.10 (row 1)	row 2
	$S_0 = \{i\mathbf{e}_1 + j\mathbf{e}_2 + \alpha_i\mathbf{e}_3 + \beta_i\mathbf{e}_4 + \gamma_i\mathbf{e}_5 \mid i \in [p], j \in [2]\}$		
	$C = \{\alpha_{ijk}\mathbf{e}_2 + i\mathbf{e}_3 + j\mathbf{e}_4 + k\mathbf{e}_5 \mid i, j, k \in [2]\}$		
$(2p, 8, p)$	Not possible since no X exists	5.12	
$(4p, 4, 2)$	$L_{S_0} = \langle \mathbf{e}_2 \rangle$		
	$X = \{i\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_3 + j\mathbf{e}_4 + \alpha_i\mathbf{e}_5 + L_{S_0} \mid i \in [p], j \in [2]\}$		
	$Y = \{j\mathbf{e}_3 + \alpha_i\mathbf{e}_4 + i\mathbf{e}_5 + L_{S_0} \mid i, j \in [2]\}$	5.10 (row 3)	row 6
	$S_0 = \{i\mathbf{e}_1 + k\mathbf{e}_2 + \alpha_{ij}\mathbf{e}_3 + j\mathbf{e}_4 + \alpha_i\mathbf{e}_5 \mid i \in [p], j, k \in [2]\}$		
	$C = \{\alpha_{ij}\mathbf{e}_2 + j\mathbf{e}_3 + \alpha_i\mathbf{e}_4 + i\mathbf{e}_5 \mid i, j \in [2]\}$		
$(4p, 4, 2)$	$L_{S_0} = \langle \mathbf{e}_2 \rangle$		
	$X = \{i\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_3 + \beta_{ij}\mathbf{e}_4 + j\mathbf{e}_5 + L_{S_0} \mid i \in [p], j \in [2]\}$		
	$Y = \{i\mathbf{e}_3 + j\mathbf{e}_4 + L_{S_0} \mid i, j \in [2]\}$	5.10 (row 4)	row 7
	$S_0 = \{i\mathbf{e}_1 + k\mathbf{e}_2 + \alpha_{ij}\mathbf{e}_3 + \beta_{ij}\mathbf{e}_4 + j\mathbf{e}_5 \mid i \in [p], j, k \in [2]\}$		
	$C = \{\alpha_{ij}\mathbf{e}_2 + i\mathbf{e}_3 + j\mathbf{e}_4 \mid i, j \in [2]\}$		
$(4p, 4, p)$	Not possible since no X exists	5.12	
$(4p, 4, 4)$	$L_{S_0} = \langle \mathbf{e}_2, \mathbf{e}_3 \rangle$		
	$X = \{i\mathbf{e}_1 + \alpha_i\mathbf{e}_4 + \beta_i\mathbf{e}_5 + L_{S_0} \mid i \in [p]\}$		
	$Y = \{i\mathbf{e}_4 + j\mathbf{e}_5 + L_{S_0} \mid i, j \in [2]\}$	3.6 (row 1)	row 8
	$S_0 = \{i\mathbf{e}_1 + j\mathbf{e}_2 + k\mathbf{e}_3 + \alpha_i\mathbf{e}_4 + \beta_i\mathbf{e}_5 \mid i \in [p], j, k \in [2]\}$		
	$C = \{\alpha_{ij}\mathbf{e}_2 + \beta_{ij}\mathbf{e}_3 + i\mathbf{e}_4 + j\mathbf{e}_5 \mid i, j \in [2]\}$		
$(4p, 4, 2p)$	Not possible since no X exists	5.5	
$(8, 2p, 2)$	$L_{S_0} = \langle \mathbf{e}_2 \rangle$		
	$X = \{\alpha_i\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_3 + j\mathbf{e}_4 + i\mathbf{e}_5 + L_{S_0} \mid i, j \in [2]\}$		
	$Y = \{i\mathbf{e}_1 + j\mathbf{e}_3 + \alpha_i\mathbf{e}_4 + L_{S_0} \mid i \in [p], j \in [2]\}$	5.10 (row 5)	row 13
	$S_0 = \{\alpha_i\mathbf{e}_1 + k\mathbf{e}_2 + \alpha_{ij}\mathbf{e}_3 + j\mathbf{e}_4 + i\mathbf{e}_5 \mid i, j, k \in [2]\}$		
	$C = \{i\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + j\mathbf{e}_3 + \alpha_i\mathbf{e}_4 \mid i \in [p], j \in [2]\}$		
$(8, 2p, 2)$	$L_{S_0} = \langle \mathbf{e}_2 \rangle$		
	$X = \{\alpha_{ij}\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_3 + i\mathbf{e}_4 + j\mathbf{e}_5 + L_{S_0} \mid i, j \in [2]\}$		
	$Y = \{i\mathbf{e}_1 + j\mathbf{e}_3 + L_{S_0} \mid i \in [p], j \in [2]\}$	5.10 (row 6)	row 14
	$S_0 = \{\alpha_{ij}\mathbf{e}_1 + k\mathbf{e}_2 + \alpha_{ij}\mathbf{e}_3 + i\mathbf{e}_4 + j\mathbf{e}_5 \mid i, j, k \in [2]\}$		
	$C = \{i\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + j\mathbf{e}_3 \mid i \in [p], j \in [2]\}$		
$(8, 2p, 4)$	Not possible since no X exists	3.6	
$(8p, 2, 2)$	$L_{S_0} = \langle \mathbf{e}_2 \rangle$		
	$X = \{i\mathbf{e}_1 + \alpha_{ijk}\mathbf{e}_3 + j\mathbf{e}_4 + k\mathbf{e}_5 + L_{S_0} \mid i \in [p], j, k \in [2]\}$		
	$Y = \{i\mathbf{e}_3 + L_{S_0} \mid i \in [2]\}$	5.10 (row 10)	row 19
	$S_0 = \{i\mathbf{e}_1 + l\mathbf{e}_2 + \alpha_{ijk}\mathbf{e}_3 + j\mathbf{e}_4 + k\mathbf{e}_5 \mid i \in [p], j, k, l \in [2]\}$		
	$C = \{\alpha_i\mathbf{e}_2 + i\mathbf{e}_3 \mid i \in [2]\}$		
	$L_{S_0} = \langle \mathbf{e}_1 \rangle$		
	$X = \{\alpha_{ijk}\mathbf{e}_2 + i\mathbf{e}_3 + j\mathbf{e}_4 + k\mathbf{e}_5 + L_{S_0} \mid i, j, k \in [2]\}$		

$(8p, 2, p)$	$Y = \{i\mathbf{e}_2 + L_{S_0} \mid i \in [2]\}$ $S_0 = \{l\mathbf{e}_1 + \alpha_{ijk}\mathbf{e}_2 + i\mathbf{e}_3 + j\mathbf{e}_4 + k\mathbf{e}_5 \mid i, j, k \in [2], l \in [p]\}$ $C = \{\alpha_i\mathbf{e}_1 + i\mathbf{e}_2 \mid i \in [2]\}$	5.12 (row 2)	row 20
$(8p, 2, 4)$	$L_{S_0} = \langle \mathbf{e}_2, \mathbf{e}_3 \rangle$ $X = \{i\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_4 + j\mathbf{e}_5 + L_{S_0} \mid i \in [p], j \in [2]\}$ $Y = \{i\mathbf{e}_4 + L_{S_0} \mid i \in [2]\}$ $S_0 = \{i\mathbf{e}_1 + k\mathbf{e}_2 + l\mathbf{e}_3 + \alpha_{ij}\mathbf{e}_4 + j\mathbf{e}_5 \mid i \in [p], j, k, l \in [2]\}$ $C = \{\alpha_i\mathbf{e}_2 + \beta_i\mathbf{e}_3 + i\mathbf{e}_4 \mid i \in [2]\}$	3.6 (row 5)	row 21
$(8p, 2, 2p)$	$L_{S_0} = \langle \mathbf{e}_1 + \mathbf{e}_2 \rangle$ $X = \{\alpha_{ij}\mathbf{e}_3 + i\mathbf{e}_4 + j\mathbf{e}_5 + L_{S_0} \mid i, j \in [2]\}$ $Y = \{i\mathbf{e}_3 + L_{S_0} \mid i \in [2]\}$ $S_0 = \{k\mathbf{e}_1 + l\mathbf{e}_2 + \alpha_{ij}\mathbf{e}_3 + i\mathbf{e}_4 + j\mathbf{e}_5 \mid i, j, l \in [2], k \in [p]\}$ $C = \{\alpha_i\mathbf{e}_1 + \alpha_i\mathbf{e}_2 + i\mathbf{e}_3 \mid i \in [2]\}$	5.5	row 22
$(8p, 2, 8)$	$L_{S_0} = \langle \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4 \rangle$ $X = \{i\mathbf{e}_1 + \alpha_i\mathbf{e}_5 + L_{S_0} \mid i \in [p]\}$ $Y = \{i\mathbf{e}_5 + L_{S_0} \mid i \in [2]\}$ $S_0 = \{i\mathbf{e}_1 + j\mathbf{e}_2 + k\mathbf{e}_3 + l\mathbf{e}_4 + \alpha_i\mathbf{e}_5 \mid i \in [p], j, k, l \in [2]\}$ $C = \{\alpha_i\mathbf{e}_2 + \beta_i\mathbf{e}_3 + \gamma_i\mathbf{e}_4 + i\mathbf{e}_5 \mid i \in [2]\}$	3.5 (row 1)	row 23
$(8p, 2, 4p)$	Not possible since no X exists	5.2	
$(16, p, 2)$	$L_{S_0} = \langle \mathbf{e}_2 \rangle$ $X = \{\alpha_{ijk}\mathbf{e}_1 + i\mathbf{e}_3 + j\mathbf{e}_4 + k\mathbf{e}_5 + L_{S_0} \mid i, j, k \in [2]\}$ $Y = \{i\mathbf{e}_1 + L_{S_0} \mid i \in [p]\}$ $S_0 = \{\alpha_{ijk}\mathbf{e}_1 + l\mathbf{e}_2 + i\mathbf{e}_3 + j\mathbf{e}_4 + k\mathbf{e}_5 \mid i, j, k, l \in [2]\}$ $C = \{i\mathbf{e}_1 + \alpha_i\mathbf{e}_2 \mid i \in [p]\}$	5.10 (row 13)	row 25
$(16, p, 4)$	$L_{S_0} = \langle \mathbf{e}_2, \mathbf{e}_3 \rangle$ $X = \{\alpha_{ij}\mathbf{e}_1 + i\mathbf{e}_4 + j\mathbf{e}_5 + L_{S_0} \mid i, j \in [2]\}$ $Y = \{i\mathbf{e}_1 + L_{S_0} \mid i \in [p]\}$ $S_0 = \{\alpha_{ij}\mathbf{e}_1 + k\mathbf{e}_2 + l\mathbf{e}_3 + i\mathbf{e}_4 + j\mathbf{e}_5 \mid i, j, k, l \in [2]\}$ $C = \{i\mathbf{e}_1 + \alpha_i\mathbf{e}_2 + \beta_i\mathbf{e}_3 \mid i \in [p]\}$	3.6 (row 7)	row 26
$(16, p, 8)$	$L_{S_0} = \langle \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4 \rangle$ $X = \{\alpha_i\mathbf{e}_1 + i\mathbf{e}_5 + L_{S_0} \mid i \in [2]\}$ $Y = \{i\mathbf{e}_1 + L_{S_0} \mid i \in [p]\}$ $S_0 = \{\alpha_i\mathbf{e}_1 + j\mathbf{e}_2 + k\mathbf{e}_3 + l\mathbf{e}_4 + i\mathbf{e}_5 \mid i, j, k, l \in [2]\}$ $C = \{i\mathbf{e}_1 + \alpha_i\mathbf{e}_2 + \beta_i\mathbf{e}_3 + \gamma_i\mathbf{e}_4 \mid i \in [p]\}$	3.5 (row 2)	row 27

Remark 5.19. In Theorem 5.18, the generators $\mathbf{e}_2$, $\mathbf{e}_3$, $\mathbf{e}_4$ and $\mathbf{e}_5$ may be replaced by any basis of $H = \{0\} \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, as in (C2).

6 Remaining Hajós groups

In this section, we characterize perfect codes in Cayley graphs of the remaining Hajós groups. These groups are subgroups of $\mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_p \times \mathbb{Z}_4 \times \mathbb{Z}_2$, $\mathbb{Z}_9 \times \mathbb{Z}_3$, $\mathbb{Z}_{p^2} \times \mathbb{Z}_{q^2}$, and $\mathbb{Z}_{p^2} \times \mathbb{Z}_q \times \mathbb{Z}_r$.

6.1 Subgroups of $\mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_2 \times \mathbb{Z}_2$

Theorem 6.1. *Let $G = \mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r$ for distinct primes p , q and r , and let S be a proper subset of $G \setminus \{0\}$ that generates G . Let $0 \in C \subseteq G$. Then C is a perfect code in $\text{Cay}(G, S)$ if and only if (S_0, C) is one of the pairs in Table 6.1.*

Table 6.1: Perfect codes in Cayley graphs of $\mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r$

Row	S_0	C
1	$\{(i, \alpha_i, \beta_i) \mid i \in [p]\}$	$\{(0, i, j) \mid i \in [q], j \in [r]\}$
2	$\{(\alpha_i, i, \beta_i) \mid i \in [q]\}$	$\{(i, 0, j) \mid i \in [p], j \in [r]\}$
3	$\{(\alpha_i, \beta_i, i) \mid i \in [r]\}$	$\{(i, j, 0) \mid i \in [p], j \in [q]\}$
4	$\{(j, i, \alpha_i) \mid i \in [q], j \in [p]\}$	$\{(\alpha_i, 0, i) \mid i \in [r]\}$
5	$\{(i, j, \alpha_i) \mid i \in [p], j \in [q]\}$	$\{(0, \alpha_i, i) \mid i \in [r]\}$
6	$\{(j, \alpha_i, i) \mid i \in [r], j \in [p]\}$	$\{(\alpha_i, i, 0) \mid i \in [q]\}$
7	$\{(i, \alpha_i, j) \mid i \in [p], j \in [r]\}$	$\{(0, i, \alpha_i) \mid i \in [q]\}$
8	$\{(\alpha_i, j, i) \mid i \in [r], j \in [q]\}$	$\{(i, \alpha_i, 0) \mid i \in [p]\}$
9	$\{(\alpha_i, i, j) \mid i \in [q], j \in [r]\}$	$\{(i, 0, \alpha_i) \mid i \in [p]\}$
10	$\{(i, j, \alpha_{ij}) \mid i \in [p], j \in [q]\}$	$\{(0, 0, i) \mid i \in [r]\}$
11	$\{(i, \alpha_{ij}, j) \mid i \in [p], j \in [r]\}$	$\{(0, i, 0) \mid i \in [q]\}$
12	$\{(\alpha_{ij}, i, j) \mid i \in [q], j \in [r]\}$	$\{(i, 0, 0) \mid i \in [p]\}$

Proof. Suppose C is a perfect code in $\text{Cay}(G, S)$.

If $|S_0| = p$, then from Lemma 2.8, S_0 is aperiodic and C is a subgroup of order qr , so $C = L_C = \{0\} \times \mathbb{Z}_q \times \mathbb{Z}_r$. From the factorization of G/L_C given in Lemma 2.5, $(S_0 + L_C)/L_C = G/L_C = \{(i, 0, 0) + L_C \mid i \in [p]\}$, and hence $S_0 = \{(i, \alpha_i, \beta_i) \mid i \in [p]\}$. This gives row 1 in Table 6.1. The same argument can be applied when $|S_0| = q$ and $|S_0| = r$ to get rows 2 and 3 in Table 6.1.

If $|S_0| = pq$ and S_0 periodic, then $|L_{S_0}| = p$ or q . Let $S_0 = D \oplus L_{S_0}$ for some subset D . From the factorization of G/L_{S_0} given in Lemma 2.5, we have $G/L_{S_0} = (D + L_{S_0})/L_{S_0} \oplus (C + L_{S_0})/L_{S_0}$. If $|L_{S_0}| = p$, then $L_{S_0} = \langle (1, 0, 0) \rangle$. Note that $(D + L_{S_0})/L_{S_0}$ is an aperiodic set of order q and $G/L_{S_0} \cong \mathbb{Z}_q \times \mathbb{Z}_r$. From Theorem 3.5 (row 2), we get $(D + L_{S_0})/L_{S_0} = \{(0, i, \alpha_i) + L_{S_0} \mid i \in [q]\}$ and $(C + L_{S_0})/L_{S_0} = \{(0, 0, i) + L_{S_0} \mid i \in [r]\}$. Therefore, $S_0 = D \oplus L_{S_0} = \{(j, i, \alpha_i) \mid i \in [q], j \in [p]\}$ and $C = \{(\alpha_i, 0, i) \mid i \in [r]\}$. This is row 4 in Table 6.1. By using the same arguments, we can get the case of $|L_{S_0}| = q$. This is row 5 in Table 6.1. We can get rows 6 to 9 by changing $|S_0|$ to pr and qr , and also set $|L_{S_0}|$ to a prime divisor of $|S_0|$.

If $|S_0| = pq$ and S_0 aperiodic, then $|C| = r$ and C is periodic, which means $C = L_C = \langle (0, 0, 1) \rangle$. From the factorization of G/L_C , $(S_0 + L_C)/L_C = G/L_C = \{(i, j, 0) + L_C \mid i \in [p], j \in [q]\}$, and hence $S_0 = \{(i, j, \alpha_{ij}) \mid i \in [p], j \in [q]\}$, $\alpha_{ij} \in \mathbb{Z}$. This is row 10 in Table 6.1. Rows 11 and 12 are the cases $|S_0|$ equal to pr and qr with the same arguments. $\square$

Theorem 6.2. Let $G = \mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_2 \times \mathbb{Z}_2$ for distinct odd primes p and q , and let S be a proper subset of $G \setminus \{0\}$ that generates G . Let $0 \in C \subseteq G$. Then C is a perfect code in $\text{Cay}(G, S)$ if and only if (S_0, C) is one of the pairs in Table 6.2.

 Table 6.2: Perfect codes in Cayley graphs of $\mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_2 \times \mathbb{Z}_2$

Row	S_0	C
1	$\{i\mathbf{e}_1 + \alpha_i\mathbf{e}_2 + \beta_i\mathbf{e}_3 + \gamma_i\mathbf{e}_4 \mid i \in [p]\}$	$\{i\mathbf{e}_2 + j\mathbf{e}_3 + k\mathbf{e}_4 \mid i \in [q], j, k \in [2]\}$
2	$\{\alpha_i\mathbf{e}_1 + i\mathbf{e}_2 + \beta_i\mathbf{e}_3 + \gamma_i\mathbf{e}_4 \mid i \in [q]\}$	$\{i\mathbf{e}_1 + j\mathbf{e}_3 + k\mathbf{e}_4 \mid i \in [p], j, k \in [2]\}$

3	$\{j\mathbf{e}_1 + i\mathbf{e}_2 + \alpha_i\mathbf{e}_3 + \beta_i\mathbf{e}_4 \mid i \in [q], j \in [p]\}$	$\{\alpha_{ij}\mathbf{e}_1 + i\mathbf{e}_3 + j\mathbf{e}_4 \mid i, j \in [2]\}$
4	$\{i\mathbf{e}_1 + j\mathbf{e}_2 + \alpha_i\mathbf{e}_3 + \beta_i\mathbf{e}_4 \mid i \in [p], j \in [q]\}$	$\{\alpha_{ij}\mathbf{e}_2 + i\mathbf{e}_3 + j\mathbf{e}_4 \mid i, j \in [2]\}$
5	$\{j\mathbf{e}_1 + i\mathbf{e}_2 + \alpha_{ij}\mathbf{e}_3 + \alpha_i\mathbf{e}_4 \mid i \in [q], j \in [p]\}$	$\{\alpha_i\mathbf{e}_1 + j\mathbf{e}_3 + i\mathbf{e}_4 \mid i, j \in [2]\}$
6	$\{i\mathbf{e}_1 + j\mathbf{e}_2 + \alpha_{ij}\mathbf{e}_3 + \alpha_i\mathbf{e}_4 \mid i \in [p], j \in [q]\}$	$\{\alpha_i\mathbf{e}_2 + j\mathbf{e}_3 + i\mathbf{e}_4 \mid i, j \in [2]\}$
7	$\{i\mathbf{e}_1 + j\mathbf{e}_2 + \alpha_{ij}\mathbf{e}_3 + \beta_{ij}\mathbf{e}_4 \mid i \in [p], j \in [q]\}$	$\{i\mathbf{e}_3 + j\mathbf{e}_4 \mid i, j \in [2]\}$
8	$\{i\mathbf{e}_1 + \alpha_i\mathbf{e}_2 + j\mathbf{e}_3 + \beta_i\mathbf{e}_4 \mid i \in [p], j \in [2]\}$	$\{i\mathbf{e}_2 + \alpha_{ij}\mathbf{e}_3 + j\mathbf{e}_4 \mid i \in [q], j \in [2]\}$
9	$\{j\mathbf{e}_1 + \alpha_i\mathbf{e}_2 + \alpha_{ij}\mathbf{e}_3 + i\mathbf{e}_4 \mid i \in [2], j \in [p]\}$	$\{\alpha_i\mathbf{e}_1 + i\mathbf{e}_2 + j\mathbf{e}_3 \mid i \in [q], j \in [2]\}$
10	$\{i\mathbf{e}_1 + \alpha_i\mathbf{e}_2 + \alpha_{ij}\mathbf{e}_3 + j\mathbf{e}_4 \mid i \in [p], j \in [2]\}$	$\{i\mathbf{e}_2 + j\mathbf{e}_3 + \alpha_i\mathbf{e}_4 \mid i \in [q], j \in [2]\}$
11	$\{i\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + j\mathbf{e}_3 + \alpha_i\mathbf{e}_4 \mid i \in [p], j \in [2]\}$	$\{j\mathbf{e}_2 + \alpha_i\mathbf{e}_3 + i\mathbf{e}_4 \mid i \in [2], j \in [q]\}$
12	$\{i\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + \alpha_{ij}\mathbf{e}_3 + j\mathbf{e}_4 \mid i \in [p], j \in [2]\}$	$\{i\mathbf{e}_2 + j\mathbf{e}_3 \mid i \in [q], j \in [2]\}$
13	$\{\alpha_i\mathbf{e}_1 + i\mathbf{e}_2 + j\mathbf{e}_3 + \beta_i\mathbf{e}_4 \mid i \in [q], j \in [2]\}$	$\{i\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_3 + j\mathbf{e}_4 \mid i \in [p], j \in [2]\}$
14	$\{\alpha_i\mathbf{e}_1 + j\mathbf{e}_2 + \alpha_{ij}\mathbf{e}_3 + i\mathbf{e}_4 \mid i \in [2], j \in [q]\}$	$\{i\mathbf{e}_1 + \alpha_i\mathbf{e}_2 + j\mathbf{e}_3 \mid i \in [p], j \in [2]\}$
15	$\{\alpha_i\mathbf{e}_1 + i\mathbf{e}_2 + \alpha_{ij}\mathbf{e}_3 + j\mathbf{e}_4 \mid i \in [q], j \in [2]\}$	$\{i\mathbf{e}_1 + j\mathbf{e}_3 + \alpha_i\mathbf{e}_4 \mid i \in [p], j \in [2]\}$
16	$\{\alpha_{ij}\mathbf{e}_1 + i\mathbf{e}_2 + j\mathbf{e}_3 + \alpha_i\mathbf{e}_4 \mid i \in [q], j \in [2]\}$	$\{j\mathbf{e}_1 + \alpha_i\mathbf{e}_3 + i\mathbf{e}_4 \mid i \in [2], j \in [p]\}$
17	$\{\alpha_{ij}\mathbf{e}_1 + i\mathbf{e}_2 + \alpha_{ij}\mathbf{e}_3 + j\mathbf{e}_4 \mid i \in [q], j \in [2]\}$	$\{i\mathbf{e}_1 + j\mathbf{e}_3 \mid i \in [p], j \in [2]\}$
18	$\{\alpha_i\mathbf{e}_1 + \beta_i\mathbf{e}_2 + j\mathbf{e}_3 + i\mathbf{e}_4 \mid i, j \in [2]\}$	$\{i\mathbf{e}_1 + j\mathbf{e}_2 + \alpha_{ij}\mathbf{e}_3 \mid i \in [p], j \in [q]\}$
19	$\{\alpha_{ij}\mathbf{e}_1 + \alpha_i\mathbf{e}_2 + j\mathbf{e}_3 + i\mathbf{e}_4 \mid i, j \in [2]\}$	$\{j\mathbf{e}_1 + i\mathbf{e}_2 + \alpha_i\mathbf{e}_3 \mid i \in [q], j \in [p]\}$
20	$\{\alpha_i\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + j\mathbf{e}_3 + i\mathbf{e}_4 \mid i, j \in [2]\}$	$\{i\mathbf{e}_1 + j\mathbf{e}_2 + \alpha_i\mathbf{e}_3 \mid i \in [p], j \in [q]\}$
21	$\{\alpha_{ij}\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + i\mathbf{e}_3 + j\mathbf{e}_4 \mid i, j \in [2]\}$	$\{i\mathbf{e}_1 + j\mathbf{e}_2 \mid i \in [p], j \in [q]\}$
22	$\{i\mathbf{e}_1 + j\mathbf{e}_2 + k\mathbf{e}_3 + \alpha_{ij}\mathbf{e}_4 \mid i \in [p], j \in [q], k \in [2]\}$	$\{\alpha_i\mathbf{e}_3 + i\mathbf{e}_4 \mid i \in [2]\}$
23	$\{k\mathbf{e}_1 + i\mathbf{e}_2 + \alpha_{ij}\mathbf{e}_3 + j\mathbf{e}_4 \mid i \in [q], j \in [2], k \in [p]\}$	$\{\alpha_i\mathbf{e}_1 + i\mathbf{e}_3 \mid i \in [2]\}$
24	$\{i\mathbf{e}_1 + k\mathbf{e}_2 + \alpha_{ij}\mathbf{e}_3 + j\mathbf{e}_4 \mid i \in [p], j \in [2], k \in [q]\}$	$\{\alpha_i\mathbf{e}_2 + i\mathbf{e}_3 \mid i \in [2]\}$
25	$\{j\mathbf{e}_1 + i\mathbf{e}_2 + k\mathbf{e}_3 + \alpha_i\mathbf{e}_4 \mid i \in [q], j \in [p], k \in [2]\}$	$\{\alpha_i\mathbf{e}_1 + \alpha_i\mathbf{e}_3 + i\mathbf{e}_4 \mid i \in [2]\}$
26	$\{i\mathbf{e}_1 + j\mathbf{e}_2 + k\mathbf{e}_3 + \alpha_i\mathbf{e}_4 \mid i \in [p], j \in [q], k \in [2]\}$	$\{\alpha_i\mathbf{e}_2 + \alpha_i\mathbf{e}_3 + i\mathbf{e}_4 \mid i \in [2]\}$
27	$\{i\mathbf{e}_1 + j\mathbf{e}_2 + \alpha_{ijk}\mathbf{e}_3 + k\mathbf{e}_4 \mid i \in [p], j \in [q], k \in [2]\}$	$\{i\mathbf{e}_3 \mid i \in [2]\}$
28	$\{i\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + k\mathbf{e}_3 + j\mathbf{e}_4 \mid i \in [p], j, k \in [2]\}$	$\{i\mathbf{e}_2 + \alpha_i\mathbf{e}_3 \mid i \in [q]\}$
29	$\{k\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + i\mathbf{e}_3 + j\mathbf{e}_4 \mid i, j \in [2], k \in [p]\}$	$\{\alpha_i\mathbf{e}_1 + i\mathbf{e}_2 \mid i \in [q]\}$
30	$\{i\mathbf{e}_1 + \alpha_i\mathbf{e}_2 + j\mathbf{e}_3 + k\mathbf{e}_4 \mid i \in [p], j, k \in [2]\}$	$\{i\mathbf{e}_2 + \alpha_i\mathbf{e}_3 + \beta_i\mathbf{e}_4 \mid i \in [q]\}$
31	$\{j\mathbf{e}_1 + \alpha_i\mathbf{e}_2 + k\mathbf{e}_3 + i\mathbf{e}_4 \mid i, k \in [2], j \in [p]\}$	$\{\alpha_i\mathbf{e}_1 + i\mathbf{e}_2 + \alpha_i\mathbf{e}_3 \mid i \in [q]\}$
32	$\{i\mathbf{e}_1 + \alpha_{ijk}\mathbf{e}_2 + j\mathbf{e}_3 + k\mathbf{e}_4 \mid i \in [p], j, k \in [2]\}$	$\{i\mathbf{e}_2 \mid i \in [q]\}$
33	$\{\alpha_{ij}\mathbf{e}_1 + i\mathbf{e}_2 + k\mathbf{e}_3 + j\mathbf{e}_4 \mid i \in [q], j, k \in [2]\}$	$\{i\mathbf{e}_1 + \alpha_i\mathbf{e}_3 \mid i \in [p]\}$
34	$\{\alpha_{ij}\mathbf{e}_1 + k\mathbf{e}_2 + i\mathbf{e}_3 + j\mathbf{e}_4 \mid i, j \in [2], k \in [q]\}$	$\{i\mathbf{e}_1 + \alpha_i\mathbf{e}_2 \mid i \in [p]\}$
35	$\{\alpha_i\mathbf{e}_1 + i\mathbf{e}_2 + j\mathbf{e}_3 + k\mathbf{e}_4 \mid i \in [q], j, k \in [2]\}$	$\{i\mathbf{e}_1 + \alpha_i\mathbf{e}_3 + \beta_i\mathbf{e}_4 \mid i \in [p]\}$
36	$\{\alpha_i\mathbf{e}_1 + j\mathbf{e}_2 + k\mathbf{e}_3 + i\mathbf{e}_4 \mid i, k \in [2], j \in [q]\}$	$\{i\mathbf{e}_1 + \alpha_i\mathbf{e}_2 + \alpha_i\mathbf{e}_3 \mid i \in [p]\}$
37	$\{\alpha_{ijk}\mathbf{e}_1 + i\mathbf{e}_2 + j\mathbf{e}_3 + k\mathbf{e}_4 \mid i \in [q], j, k \in [2]\}$	$\{i\mathbf{e}_1 \mid i \in [p]\}$

Proof. Suppose C is a perfect code in $\text{Cay}(G, S)$. Then $G = S_0 \oplus C$ and so $|S_0| \cdot |C| = |G| = 4pq$. Since G is not a cyclic group, $|S_0| > 2$ by Lemma 2.8. Therefore, $2 < |S_0| < 4pq$.

Case 1. S_0 is aperiodic.

Since G is Hajós and $G = S_0 \oplus C$, in this case C must be periodic, that is, L_C is nontrivial. Write $C = L_C \oplus D$ for some subset D of G whose existence is ensured by Lemma 2.3. Then $|L_C| \cdot |D| = |C|$ and $2 \leq |L_C| \leq |C|$. If $|S_0|$ is a prime, then $|L_C| = |C|$ by Lemma 2.8 part (c). If $|S_0|$ is not a prime, then $|L_C|$ can be any divisor of $|C|$. Set

$$A = (S_0 + L_C)/L_C \quad \text{and} \quad B = (D + L_C)/L_C.$$

Since C is periodic, by Lemma 2.5, we have

$$G/L_C = A \oplus B$$

and B is aperiodic in G/L_C . Since G/L_C is isomorphic to a subgroup of G and subgroups of Hajós groups are also Hajós groups, A is periodic in G/L_C . We also know that A contains L_C and A generates G/L_C by Lemma 2.7. Furthermore, $|A| = |S_0|$ and $|B| = |D|$. Note that if $|L_C| = |C|$, then $C = L_C$ as $0 \in C$. This means we can take $D = \{0\}$ and hence $B = L_C/L_C$ and $G/L_C = A$. We use C to get A and then use Lemma 2.6 to obtain S_0 . If $|L_C| < |C|$, then this factorization enables us to make use of previously established theorems to obtain the pair (S_0, C) via (A, B) .

All possible values for $(|S_0|, |C|, |L_C|)$ are given in the first column of Table 6.3. In the second column, we list C , A , and S_0 when $|L_C| = |C|$ and we list L_C , A , B , S_0 , and C when $|L_C| < |C|$. The third column shows which previous theorem is used to obtain (A, B) when $|L_C| < |C|$. The last column indicates which row in Table 6.2 the result corresponds to. The result in the cases when $|S_0| = q$, $2q$, and $4q$ can be derived by swapping $\mathbf{e}_1$ and $\mathbf{e}_2$ from the cases when $|S_0| = p$, $2p$, and $4p$, respectively. The resulting forms of S_0 and C are in rows 2, 14 to 17, and 37 in Table 6.2.

Table 6.3: The case when $G = \mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and S_0 is aperiodic

(S_0 , C , L_C)	$L_C, A, B, S_0, \text{ and } C$	Reference	Result
$(p, 4q, 4q)$	$C = \langle \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4 \rangle$ $A = \{i\mathbf{e}_1 + L_C \mid i \in [p]\}$ $S_0 = \{i\mathbf{e}_1 + \alpha_i\mathbf{e}_2 + \beta_i\mathbf{e}_3 + \gamma_i\mathbf{e}_4 \mid i \in [p]\}$		row 1
$(pq, 4, 2)$	$L_C = \langle \mathbf{e}_3 \rangle$ $A = \{j\mathbf{e}_1 + i\mathbf{e}_2 + \alpha_i\mathbf{e}_4 + L_C \mid i \in [q], j \in [p]\}$ $B = \{\alpha_i\mathbf{e}_1 + i\mathbf{e}_4 + L_C \mid i \in [2]\}$ $S_0 = \{j\mathbf{e}_1 + i\mathbf{e}_2 + \alpha_{ij}\mathbf{e}_3 + \alpha_i\mathbf{e}_4 \mid i \in [q], j \in [p]\}$ $C = \{\alpha_i\mathbf{e}_1 + j\mathbf{e}_3 + i\mathbf{e}_4 \mid i, j \in [2]\}$	6.1 (row 4)	row 5
$(pq, 4, 2)$	$L_C = \langle \mathbf{e}_3 \rangle$ $A = \{i\mathbf{e}_1 + j\mathbf{e}_2 + \alpha_i\mathbf{e}_4 + L_C \mid i \in [p], j \in [q]\}$ $B = \{\alpha_i\mathbf{e}_2 + i\mathbf{e}_4 + L_C \mid i \in [2]\}$ $S_0 = \{i\mathbf{e}_1 + j\mathbf{e}_2 + \alpha_{ij}\mathbf{e}_3 + \alpha_i\mathbf{e}_4 \mid i \in [p], j \in [q]\}$ $C = \{\alpha_i\mathbf{e}_2 + j\mathbf{e}_3 + i\mathbf{e}_4 \mid i, j \in [2]\}$	6.1 (row 5)	row 6
$(pq, 4, 4)$	$C = \langle \mathbf{e}_3, \mathbf{e}_4 \rangle$ $A = \{i\mathbf{e}_1 + j\mathbf{e}_2 + L_C \mid i \in [p], j \in [q]\}$ $S_0 = \{i\mathbf{e}_1 + j\mathbf{e}_2 + \alpha_{ij}\mathbf{e}_3 + \beta_{ij}\mathbf{e}_4 \mid i \in [p], j \in [q]\}$		row 7
$(2p, 2q, 2)$	$L_C = \langle \mathbf{e}_3 \rangle$ $A = \{j\mathbf{e}_1 + \alpha_i\mathbf{e}_2 + i\mathbf{e}_4 + L_C \mid i \in [2], j \in [p]\}$ $B = \{\alpha_i\mathbf{e}_1 + i\mathbf{e}_2 + L_C \mid i \in [q]\}$ $S_0 = \{j\mathbf{e}_1 + \alpha_i\mathbf{e}_2 + \alpha_{ij}\mathbf{e}_3 + i\mathbf{e}_4 \mid i \in [2], j \in [p]\}$ $C = \{\alpha_i\mathbf{e}_1 + i\mathbf{e}_2 + j\mathbf{e}_3 \mid i \in [q], j \in [2]\}$	6.1 (row 6)	row 9
$(2p, 2q, 2)$	$L_C = \langle \mathbf{e}_3 \rangle$ $A = \{i\mathbf{e}_1 + \alpha_i\mathbf{e}_2 + j\mathbf{e}_4 + L_C \mid i \in [p], j \in [2]\}$ $B = \{i\mathbf{e}_2 + \alpha_i\mathbf{e}_4 + L_C \mid i \in [q]\}$ $S_0 = \{i\mathbf{e}_1 + \alpha_i\mathbf{e}_2 + \alpha_{ij}\mathbf{e}_3 + j\mathbf{e}_4 \mid i \in [p], j \in [2]\}$ $C = \{i\mathbf{e}_2 + j\mathbf{e}_3 + \alpha_i\mathbf{e}_4 \mid i \in [q], j \in [2]\}$	6.1 (row 7)	row 10
$(2p, 2q, q)$	$L_C = \langle \mathbf{e}_2 \rangle$ $A = \{i\mathbf{e}_1 + j\mathbf{e}_3 + \alpha_i\mathbf{e}_4 + L_C \mid i \in [p], j \in [2]\}$ $B = \{\alpha_i\mathbf{e}_3 + i\mathbf{e}_4 + L_C \mid i \in [2]\}$ $S_0 = \{i\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + j\mathbf{e}_3 + \alpha_i\mathbf{e}_4 \mid i \in [p], j \in [2]\}$ $C = \{j\mathbf{e}_2 + \alpha_i\mathbf{e}_3 + i\mathbf{e}_4 \mid i \in [2], j \in [q]\}$	3.6 (row 4)	row 11

$(2p, 2q, 2q)$	$C = \langle \mathbf{e}_2 + \mathbf{e}_3 \rangle$ $A = \{i\mathbf{e}_1 + j\mathbf{e}_4 + L_C \mid i \in [p], j \in [2]\}$ $S_0 = \{i\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + \alpha_{ij}\mathbf{e}_3 + j\mathbf{e}_4 \mid i \in [p], j \in [2]\}$		row 12
$(4, pq, p)$	$L_C = \langle \mathbf{e}_1 \rangle$ $A = \{\alpha_i\mathbf{e}_2 + j\mathbf{e}_3 + i\mathbf{e}_4 + L_C \mid i, j \in [2]\}$ $B = \{i\mathbf{e}_2 + \alpha_i\mathbf{e}_3 + L_C \mid i \in [q]\}$ $S_0 = \{\alpha_{ij}\mathbf{e}_1 + \alpha_i\mathbf{e}_2 + j\mathbf{e}_3 + i\mathbf{e}_4 \mid i, j \in [2]\}$ $C = \{j\mathbf{e}_1 + i\mathbf{e}_2 + \alpha_i\mathbf{e}_3 \mid i \in [q], j \in [p]\}$	3.6 (row 6)	row 19
$(4, pq, q)$	$L_C = \langle \mathbf{e}_2 \rangle$ $A = \{\alpha_i\mathbf{e}_1 + j\mathbf{e}_3 + i\mathbf{e}_4 + L_C \mid i, j \in [2]\}$ $B = \{i\mathbf{e}_1 + \alpha_i\mathbf{e}_3 + L_C \mid i \in [p]\}$ $S_0 = \{\alpha_i\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + j\mathbf{e}_3 + i\mathbf{e}_4 \mid i, j \in [2]\}$ $C = \{i\mathbf{e}_1 + j\mathbf{e}_2 + \alpha_i\mathbf{e}_3 \mid i \in [p], j \in [q]\}$	3.6 (row 6)	row 20
$(4, pq, pq)$	$C = \langle \mathbf{e}_1 + \mathbf{e}_2 \rangle$ $A = \{i\mathbf{e}_3 + j\mathbf{e}_4 + L_C \mid i, j \in [2]\}$ $S_0 = \{\alpha_{ij}\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + i\mathbf{e}_3 + j\mathbf{e}_4 \mid i, j \in [2]\}$		row 21
$(2pq, 2, 2)$	$C = \langle \mathbf{e}_3 \rangle$ $A = \{i\mathbf{e}_1 + j\mathbf{e}_2 + k\mathbf{e}_4 + L_C \mid i \in [p], j \in [q], k \in [2]\}$ $S_0 = \{i\mathbf{e}_1 + j\mathbf{e}_2 + \alpha_{ijk}\mathbf{e}_3 + k\mathbf{e}_4 \mid i \in [p], j \in [q], k \in [2]\}$		row 27
$(4p, q, q)$	$C = \langle \mathbf{e}_2 \rangle$ $A = \{i\mathbf{e}_1 + j\mathbf{e}_3 + k\mathbf{e}_4 + L_C \mid i \in [p], j, k \in [2]\}$ $S_0 = \{i\mathbf{e}_1 + \alpha_{ijk}\mathbf{e}_2 + j\mathbf{e}_3 + k\mathbf{e}_4 \mid i \in [p], j, k \in [2]\}$		row 32

Case 2. S_0 is periodic.

In this case we have $|L_{S_0}| \geq 2$ and $S_0 = D \oplus L_{S_0}$ for some subset D of G by Lemma 2.3. So $|D| \cdot |L_{S_0}| = |S_0|$ and $1 < |L_{S_0}| < |S_0|$ by Lemma 2.8 part (b). Set

$$X = (D + L_{S_0})/L_{S_0} \quad \text{and} \quad Y = (C + L_{S_0})/L_{S_0}.$$

Since S_0 is periodic, by Lemma 2.5, we have

$$G/L_{S_0} = X \oplus Y$$

and X is aperiodic in G/L_{S_0} . We also know that X contains L_{S_0} and X generates G/L_{S_0} by Lemma 2.7. This factorization enables us to make use of Lemma 2.6 and previously established theorems to obtain the pair (S_0, C) via (X, Y) .

All possible values for $(|S_0|, |C|, |L_{S_0}|)$ are given in the first column of Table 6.4. In the second column we list L_{S_0} , X , Y , S_0 , and C . The third column shows which previous theorem or lemma is used to obtain (X, Y) . The last column indicates which row in Table 6.2 the result corresponds to. The cases for $|S_0| = 2q$ and $4q$ are derived from the cases of $|S_0| = 2p$ and $4p$, respectively. The resulting forms of S_0 and C are in rows 13, and 33 to 36 in Table 6.2. $\square$

Table 6.4: The case when $G = \mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and S_0 is periodic

(S_0 , C , L_{S_0})	L_{S_0} , X , Y , S_0 , and C	Reference	Result
	$L_{S_0} = \langle \mathbf{e}_1 \rangle$		

$(pq, 4, p)$	$X = \{i\mathbf{e}_2 + \alpha_i\mathbf{e}_3 + \beta_i\mathbf{e}_4 + L_{S_0} \mid i \in [q]\}$ $Y = \{i\mathbf{e}_3 + j\mathbf{e}_4 + L_{S_0} \mid i, j \in [2]\}$ $S_0 = \{j\mathbf{e}_1 + i\mathbf{e}_2 + \alpha_i\mathbf{e}_3 + \beta_i\mathbf{e}_4 \mid i \in [q], j \in [p]\}$ $C = \{\alpha_{ij}\mathbf{e}_1 + i\mathbf{e}_3 + j\mathbf{e}_4 \mid i, j \in [2]\}$	3.6 (row 1)	row 3
$(pq, 4, q)$	$L_{S_0} = \langle \mathbf{e}_2 \rangle$ $X = \{i\mathbf{e}_1 + \alpha_i\mathbf{e}_3 + \beta_i\mathbf{e}_4 + L_{S_0} \mid i \in [p]\}$ $Y = \{i\mathbf{e}_3 + j\mathbf{e}_4 + L_{S_0} \mid i, j \in [2]\}$ $S_0 = \{i\mathbf{e}_1 + j\mathbf{e}_2 + \alpha_i\mathbf{e}_3 + \beta_i\mathbf{e}_4 \mid i \in [p], j \in [q]\}$ $C = \{\alpha_{ij}\mathbf{e}_2 + i\mathbf{e}_3 + j\mathbf{e}_4 \mid i, j \in [2]\}$	3.6 (row 1)	row 4
$(2p, 2q, 2)$	$L_{S_0} = \langle \mathbf{e}_3 \rangle$ $X = \{i\mathbf{e}_1 + \alpha_i\mathbf{e}_2 + \beta_i\mathbf{e}_4 + L_{S_0} \mid i \in [p]\}$ $Y = \{i\mathbf{e}_2 + j\mathbf{e}_4 + L_{S_0} \mid i \in [q], j \in [2]\}$ $S_0 = \{i\mathbf{e}_1 + \alpha_i\mathbf{e}_2 + j\mathbf{e}_3 + \beta_i\mathbf{e}_4 \mid i \in [p], j \in [2]\}$ $C = \{i\mathbf{e}_2 + \alpha_{ij}\mathbf{e}_3 + j\mathbf{e}_4 \mid i \in [q], j \in [2]\}$	6.1 (row 1)	row 8
$(2p, 2q, p)$	Not possible since no X exists	3.6	
$(4, pq, 2)$	$L_{S_0} = \langle \mathbf{e}_3 \rangle$ $X = \{\alpha_i\mathbf{e}_1 + \beta_i\mathbf{e}_2 + i\mathbf{e}_4 + L_{S_0} \mid i \in [2]\}$ $Y = \{i\mathbf{e}_1 + j\mathbf{e}_2 + L_{S_0} \mid i \in [p], j \in [q]\}$ $S_0 = \{\alpha_i\mathbf{e}_1 + \beta_i\mathbf{e}_2 + j\mathbf{e}_3 + i\mathbf{e}_4 \mid i, j \in [2]\}$ $C = \{i\mathbf{e}_1 + j\mathbf{e}_2 + \alpha_{ij}\mathbf{e}_3 \mid i \in [p], j \in [q]\}$	6.1 (row 3)	row 18
$(2pq, 2, 2)$	$L_{S_0} = \langle \mathbf{e}_3 \rangle$ $X = \{i\mathbf{e}_1 + j\mathbf{e}_2 + \alpha_{ij}\mathbf{e}_4 + L_{S_0} \mid i \in [p], j \in [q]\}$ $Y = \{i\mathbf{e}_4 + L_{S_0} \mid i \in [2]\}$ $S_0 = \{i\mathbf{e}_1 + j\mathbf{e}_2 + k\mathbf{e}_3 + \alpha_{ij}\mathbf{e}_4 \mid i \in [p], j \in [q], k \in [2]\}$ $C = \{\alpha_i\mathbf{e}_3 + i\mathbf{e}_4 \mid i \in [2]\}$	6.1 (row 10)	row 22
$(2pq, 2, p)$	$L_{S_0} = \langle \mathbf{e}_1 \rangle$ $X = \{i\mathbf{e}_2 + \alpha_{ij}\mathbf{e}_3 + j\mathbf{e}_4 + L_{S_0} \mid i \in [q], j \in [2]\}$ $Y = \{i\mathbf{e}_3 + L_{S_0} \mid i \in [2]\}$ $S_0 = \{k\mathbf{e}_1 + i\mathbf{e}_2 + \alpha_{ij}\mathbf{e}_3 + j\mathbf{e}_4 \mid i \in [q], j \in [2], k \in [p]\}$ $C = \{\alpha_i\mathbf{e}_1 + i\mathbf{e}_3 \mid i \in [2]\}$	3.6 (row 5)	row 23
$(2pq, 2, q)$	$L_{S_0} = \langle \mathbf{e}_2 \rangle$ $X = \{i\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_3 + j\mathbf{e}_4 + L_{S_0} \mid i \in [p], j \in [2]\}$ $Y = \{i\mathbf{e}_3 + L_{S_0} \mid i \in [2]\}$ $S_0 = \{i\mathbf{e}_1 + k\mathbf{e}_2 + \alpha_{ij}\mathbf{e}_3 + j\mathbf{e}_4 \mid i \in [p], j \in [2], k \in [q]\}$ $C = \{\alpha_i\mathbf{e}_2 + i\mathbf{e}_3 \mid i \in [2]\}$	3.6 (row 5)	row 24
$(2pq, 2, 2p)$	$L_{S_0} = \langle \mathbf{e}_1 + \mathbf{e}_3 \rangle$ $X = \{i\mathbf{e}_2 + \alpha_i\mathbf{e}_4 + L_{S_0} \mid i \in [q]\}$ $Y = \{i\mathbf{e}_4 + L_{S_0} \mid i \in [2]\}$ $S_0 = \{j\mathbf{e}_1 + i\mathbf{e}_2 + k\mathbf{e}_3 + \alpha_i\mathbf{e}_4 \mid i \in [q], j \in [p], k \in [2]\}$ $C = \{\alpha_i\mathbf{e}_1 + \alpha_i\mathbf{e}_3 + i\mathbf{e}_4 \mid i \in [2]\}$	3.5 (row 1)	row 25
$(2pq, 2, 2q)$	$L_{S_0} = \langle \mathbf{e}_2 + \mathbf{e}_3 \rangle$ $X = \{i\mathbf{e}_1 + \alpha_i\mathbf{e}_4 + L_{S_0} \mid i \in [p]\}$ $Y = \{i\mathbf{e}_4 + L_{S_0} \mid i \in [2]\}$ $S_0 = \{i\mathbf{e}_1 + j\mathbf{e}_2 + k\mathbf{e}_3 + \alpha_i\mathbf{e}_4 \mid i \in [p], j \in [q], k \in [2]\}$ $C = \{\alpha_i\mathbf{e}_2 + \alpha_i\mathbf{e}_3 + i\mathbf{e}_4 \mid i \in [2]\}$	3.5 (row 1)	row 26
$(2pq, 2, pq)$	Not possible since no X exists	5.2	
$(4p, q, 2)$	$L_{S_0} = \langle \mathbf{e}_3 \rangle$ $X = \{i\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + j\mathbf{e}_4 + L_{S_0} \mid i \in [p], j \in [2]\}$ $Y = \{i\mathbf{e}_2 + L_{S_0} \mid i \in [q]\}$ $S_0 = \{i\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + k\mathbf{e}_3 + j\mathbf{e}_4 \mid i \in [p], j, k \in [2]\}$ $C = \{i\mathbf{e}_2 + \alpha_i\mathbf{e}_3 \mid i \in [q]\}$	6.1 (row 11)	row 28
	$L_{S_0} = \langle \mathbf{e}_1 \rangle$		

$(4p, q, p)$	$X = \{\alpha_{ij}\mathbf{e}_2 + i\mathbf{e}_3 + j\mathbf{e}_4 + L_{S_0} \mid i, j \in [2]\}$	3.6 (row 7)	row 29
	$Y = \{i\mathbf{e}_2 + L_{S_0} \mid i \in [q]\}$		
	$S_0 = \{k\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + i\mathbf{e}_3 + j\mathbf{e}_4 \mid i, j \in [2], k \in [p]\}$		
	$C = \{\alpha_i\mathbf{e}_1 + i\mathbf{e}_2 \mid i \in [q]\}$		
$(4p, q, 4)$	$L_{S_0} = \langle \mathbf{e}_3, \mathbf{e}_4 \rangle$	3.5 (row 1)	row 30
	$X = \{i\mathbf{e}_1 + \alpha_i\mathbf{e}_2 + L_{S_0} \mid i \in [p]\}$		
	$Y = \{i\mathbf{e}_2 + L_{S_0} \mid i \in [q]\}$		
	$S_0 = \{i\mathbf{e}_1 + \alpha_i\mathbf{e}_2 + j\mathbf{e}_3 + k\mathbf{e}_4 \mid i \in [p], j, k \in [2]\}$		
$(4p, q, 2p)$	$C = \{i\mathbf{e}_2 + \alpha_i\mathbf{e}_3 + \beta_i\mathbf{e}_4 \mid i \in [q]\}$	3.5 (row 2)	row 31
	$L_{S_0} = \langle \mathbf{e}_1 + \mathbf{e}_3 \rangle$		
	$X = \{\alpha_i\mathbf{e}_2 + i\mathbf{e}_4 + L_{S_0} \mid i \in [2]\}$		
	$Y = \{i\mathbf{e}_2 + L_{S_0} \mid i \in [q]\}$		
$(4p, q, 2p)$	$S_0 = \{j\mathbf{e}_1 + \alpha_i\mathbf{e}_2 + k\mathbf{e}_3 + i\mathbf{e}_4 \mid i, k \in [2], j \in [p]\}$	3.5 (row 2)	row 31
	$C = \{\alpha_i\mathbf{e}_1 + i\mathbf{e}_2 + \alpha_i\mathbf{e}_3 \mid i \in [q]\}$		

Remark 6.3. In Theorem 6.2, the generators $\mathbf{e}_3$ and $\mathbf{e}_4$ may be replaced by any basis of $H = \{0\} \times \{0\} \times \mathbb{Z}_2 \times \mathbb{Z}_2$, as in (C2).

6.2 Subgroups of $\mathbb{Z}_p \times \mathbb{Z}_{q^2} \times \mathbb{Z}_q$

Lemma 6.4. *Let $G = \mathbb{Z}_{p^2} \times \mathbb{Z}_p$ for $p = 2$ or 3 , and let S be a proper subset of $G \setminus \{0\}$ that generates G . Let $0 \in C \subseteq G$. Then C is a perfect code in $\text{Cay}(G, S)$ if and only if (S_0, C) is one of the pairs in Table 6.5 for some $a \in [3]$, $b \in [3] \setminus \{a\}$ and $c \in [p] \setminus \{0\}$. In this lemma, repeated occurrences of each individual structural parameter a and c in the same row have the same value across the two factors.*

Table 6.5: Perfect codes in Cayley graphs of $\mathbb{Z}_{p^2} \times \mathbb{Z}_p$

Row	S_0	C	Condition
1	$\{(\alpha_i, i + a\alpha_i) \mid i \in [3]\}$	$\{(i, ai) \mid i \in [9]\}$	$p = 3$
2	$\{(i + 3\alpha_i, \beta_i) \mid i \in [3]\}$	$\{(3i, j) \mid i, j \in [3]\}$	$p = 3$
3	$\{(i + 3j, ai + \alpha_i) \mid i, j \in [3]\}$	$\{(3\alpha_i, i) \mid i \in [3]\}$	$p = 3$
4	$\{(\alpha_i + 3j, a\alpha_i + i) \mid i, j \in [3]\}$	$\{(i + 3\alpha_i, ai) \mid i \in [3]\}$	$p = 3$
5	$\{(\alpha_i + i + 3j, a\alpha_i + bi) \mid i, j \in [3]\}$	$\{(i + 3\alpha_i, ai) \mid i \in [3]\}$	$p = 3$
6	$\{(i + (\alpha_i + j)p, cj) \mid i, j \in [p]\}$	$\{((i + \alpha_i)p, c\alpha_i) \mid i \in [p]\}$	
7	$\{(i + p\alpha_i, j) \mid i, j \in [p]\}$	$\{(pi, \alpha_i) \mid i \in [p]\}$	
8	$\{(i + p\alpha_{ij}, j) \mid i, j \in [p]\}$	$\{(pi, 0) \mid i \in [p]\}$	
9	$\{(i + \alpha_i p, c\alpha_i) \mid i \in [p^2]\}$	$\{(pi, ci) \mid i \in [p]\}$	
10	$\{(i, \alpha_i) \mid i \in [p^2]\}$	$\{(0, i) \mid i \in [p]\}$	

Proof. Suppose C is a perfect code in $\text{Cay}(G, S)$. From Lemma 2.8, $|S_0| \geq 3$. If $|S_0| = p$, then $p = 3$. From Lemma 2.8, S_0 is aperiodic and C is a subgroup of order 9, so $C = L_C = \langle (1, a) \rangle$ for some $a \in [3]$, or $\langle (3, 0), (0, 1) \rangle$. From the factorization of G/L_C given in Lemma 2.5, $G/L_C = (S_0 + L_C)/L_C$. If $L_C = \langle (1, a) \rangle$, then $(S_0 + L_C)/L_C = \{(0, i) + L_C \mid i \in [3]\}$, and hence $S_0 = \{(\alpha_i, i + a\alpha_i) \mid i \in [3]\}$, for some α_i 's in $\mathbb{Z}$. This gives row 1 in Table 6.5. If $L_C = \langle (3, 0), (0, 1) \rangle$, then $(S_0 + L_C)/L_C = \{(i, 0) + L_C \mid i \in [3]\}$, and hence

$S_0 = \{(i, 0) + \alpha_i(3, 0) + \beta_i(0, 1) \mid i \in [3]\} = \{(i + 3\alpha_i, \beta_i) \mid i \in [3]\}$, for some α_i 's and β_i 's in $\mathbb{Z}$. This gives row 2 in Table 6.5.

If $|S_0| = p^2$ and S_0 is periodic, then $|L_{S_0}| = p$ and $L_{S_0} = \langle(p, 0)\rangle, \langle(p, c)\rangle$ for $c \in [p] \setminus \{0\}$, or $\langle(0, 1)\rangle$. Let $S_0 = D \oplus L_{S_0}$ for some subset D . From the factorization of G/L_{S_0} given in Lemma 2.5, we have $G/L_{S_0} = (D + L_{S_0})/L_{S_0} \oplus (C + L_{S_0})/L_{S_0}$. If $L_{S_0} = \langle(p, 0)\rangle$, then $G/L_{S_0} \cong \mathbb{Z}_p \times \mathbb{Z}_p$. If $p = 2$, then from Lemma 5.2 we know this case is not possible. So $p = 3$. From Theorem 3.1 and Remark 3.4, we get $(D + L_{S_0})/L_{S_0} = \{\alpha_i \mathbf{v}_1 + i \mathbf{v}_2 + L_{S_0} \mid i \in [3]\}$ and $(C + L_{S_0})/L_{S_0} = \{i \mathbf{v}_1 + L_{S_0} \mid i \in [3]\}$ for some generator $\mathbf{v}_1$ and $\mathbf{v}_2$ of $\mathbb{Z}_3 \times \mathbb{Z}_3$ and $\alpha_i \in \mathbb{Z}$. Now, all 8 elements of $\mathbb{Z}_3 \times \mathbb{Z}_3 \setminus \{(0, 0)\}$ is a candidate for $\mathbf{v}_1$. However, for every candidate $\mathbf{v}_1$, $\langle \mathbf{v}_1 \rangle = \langle 2\mathbf{v}_1 \rangle$, so we only need to consider four candidates: $(0, 1)$, $(1, 0)$, $(1, 1)$, and $(1, 2)$. Similarly, we can choose $\mathbf{v}_2$ to be one of the three remaining candidates not chosen by $\mathbf{v}_1$. We divide this case into smaller cases based on $\mathbf{v}_1$ and $\mathbf{v}_2$. If $\mathbf{v}_1 = (0, 1)$ and $\mathbf{v}_2 = (1, a)$ for some $a \in [3]$. In this case, $S_0 = D \oplus L_{S_0} = \{(i + 3j, ai + \alpha_i) \mid i, j \in [3]\}$ and $C = \{(3\alpha_i, i) \mid i \in [3]\}$. This gives row 3 in Table 6.5. If $\mathbf{v}_1 = (1, a)$ and $\mathbf{v}_2 = (0, 1)$ for some $a \in [3]$. In this case, $S_0 = D \oplus L_{S_0} = \{(\alpha_i + 3j, a\alpha_i + i) \mid i, j \in [3]\}$ and $C = \{(i + 3\alpha_i, ai) \mid i \in [3]\}$. This gives row 4 in Table 6.5. If $\mathbf{v}_1 = (1, a)$ and $\mathbf{v}_2 = (1, b)$ for some $a \in [3]$ and $b \in [3] \setminus \{a\}$. In this case, $S_0 = D \oplus L_{S_0} = \{(\alpha_i + i + 3j, a\alpha_i + bi) \mid i, j \in [3]\}$ and $C = \{(i + 3\alpha_i, ai) \mid i \in [3]\}$. This gives row 5 in Table 6.5.

If $|S_0| = p^2$, S_0 is periodic, and $L_{S_0} = \langle(p, c)\rangle$ for $c \in [p] \setminus \{0\}$ or $\langle(0, 1)\rangle$, then $G/L_{S_0} \cong \mathbb{Z}_{p^2}$. From Theorem 5.1, we get $(D + L_{S_0})/L_{S_0} = \{(i + p\alpha_i, 0) + L_{S_0} \mid i \in [p]\}$ and $(C + L_{S_0})/L_{S_0} = \{(pi, 0) + L_{S_0} \mid i \in [p]\}$. If $L_{S_0} = \langle(p, c)\rangle$ for $c \in [p] \setminus \{0\}$, then $S_0 = D \oplus L_{S_0} = \{(i + (\alpha_i + j)p, cj) \mid i, j \in [p]\}$ and $C = \{((i + \alpha_i)p, c\alpha_i) \mid i \in [p]\}$. This gives row 6 in Table 6.5. If $L_{S_0} = \langle(0, 1)\rangle$, then $S_0 = D \oplus L_{S_0} = \{(i + p\alpha_i, j) \mid i, j \in [p]\}$ and $C = \{(pi, \alpha_i) \mid i \in [p]\}$. This gives row 7 in Table 6.5.

If $|S_0| = p^2$ and S_0 is aperiodic, then $|C| = p$ and periodic, so $C = L_C = \langle(p, 0)\rangle, \langle(p, c)\rangle$ for $c \in [p] \setminus \{0\}$, or $\langle(0, 1)\rangle$. From the factorization of G/L_C given in Lemma 2.5, we have $G/L_C = (S_0 + L_C)/L_C \oplus (D + L_C)/L_C$. If $C = L_C = \langle(p, 0)\rangle$, from the factorization of G/L_C , $(S_0 + L_C)/L_C = G/L_C = \{(i, j) + L_C \mid i, j \in [p]\}$, and hence $S_0 = \{(i + p\alpha_{ij}, j) \mid i, j \in [p]\}$, $\alpha_{ij} \in \mathbb{Z}$. This gives row 8 in Table 6.5. If $C = L_C = \langle(p, c)\rangle$ for $c \in [p] \setminus \{0\}$, from the factorization of G/L_C , $(S_0 + L_C)/L_C = G/L_C = \{(i, 0) + L_C \mid i \in [p^2]\}$, and hence $S_0 = \{(i + \alpha_i p, c\alpha_i) \mid i \in [p^2]\}$, $\alpha_i \in \mathbb{Z}$. This gives row 9 in Table 6.5. If $C = L_C = \langle(0, 1)\rangle$, from the factorization of G/L_C , $(S_0 + L_C)/L_C = G/L_C = \{(i, 0) + L_C \mid i \in [p^2]\}$, and hence $S_0 = \{(i, \alpha_i) \mid i \in [p^2]\}$, $\alpha_i \in \mathbb{Z}$. This gives row 10 in Table 6.5. $\square$

Theorem 6.5. *Let $G = \mathbb{Z}_p \times \mathbb{Z}_4$ for an odd prime p , and let S be a proper subset of $G \setminus \{0\}$ that generates G . Let $0 \in C \subseteq G$. Then C is a perfect code in $\text{Cay}(G, S)$ if and only if (S_0, C) is one of the pairs in Table 6.6.*

Table 6.6: Perfect codes in Cayley graphs of $\mathbb{Z}_p \times \mathbb{Z}_4$

Row	S_0	C
1	$\{(\alpha_i, i + 2\alpha_i) \mid i \in [2]\}$	$\{(i, 2j) \mid i \in [p], j \in [2]\}$
2	$\{(i, \alpha_i) \mid i \in [p]\}$	$\{(0, i) \mid i \in [4]\}$

3	$\{(i, \alpha_i + 2j) \mid i \in [p], j \in [2]\}$	$\{(0, i + 2\alpha_i) \mid i \in [2]\}$
4	$\{(j, i + 2\alpha_i) \mid i \in [2], j \in [p]\}$	$\{(\alpha_i, 2i) \mid i \in [2]\}$
5	$\{(i, j + 2\alpha_{ij}) \mid i \in [p], j \in [2]\}$	$\{(0, 2i) \mid i \in [2]\}$
6	$\{(\alpha_i, i + 2j) \mid i, j \in [2]\}$	$\{(i, 2\alpha_i) \mid i \in [p]\}$
7	$\{(\alpha_i, i) \mid i \in [4]\}$	$\{(i, 0) \mid i \in [p]\}$

Proof. Suppose C is a perfect code in $\text{Cay}(G, S)$.

If $|S_0| = 2$, then from Lemma 2.8, S_0 is aperiodic and C is a subgroup of order $2p$, so $C = L_C = \langle(1, 2)\rangle$. From the factorization of G/L_C given in Lemma 2.5, $(S_0 + L_C)/L_C = G/L_C = \{(0, i) + L_C \mid i \in [2]\}$, and hence $S_0 = \{(\alpha_i, i + 2\alpha_i) \mid i \in [2]\}$. This gives row 1 in Table 6.6.

If $|S_0| = p$, then from Lemma 2.8, S_0 is aperiodic and C is a subgroup of order 4, so $C = L_C = \langle(0, 1)\rangle$. From the factorization of G/L_C given in Lemma 2.5, $(S_0 + L_C)/L_C = G/L_C = \{(i, 0) + L_C \mid i \in [p]\}$, and hence $S_0 = \{(i, \alpha_i) \mid i \in [p]\}$. This gives row 2 in Table 6.6.

If $|S_0| = 2p$ and S_0 is periodic, then $|L_{S_0}| = 2$ or p . Let $S_0 = D \oplus L_{S_0}$ for some subset D . From the factorization of G/L_{S_0} given in Lemma 2.5, we have $G/L_{S_0} = (D + L_{S_0})/L_{S_0} \oplus (C + L_{S_0})/L_{S_0}$. If $|L_{S_0}| = 2$, then $L_{S_0} = \langle(0, 2)\rangle$. Note that $(D + L_{S_0})/L_{S_0}$ is an aperiodic set of order p and $G/L_{S_0} \cong \mathbb{Z}_p \times \mathbb{Z}_2$. From the case of $|S_0| = p$ in Theorem 3.5 (row 1), we get $(D + L_{S_0})/L_{S_0} = \{(i, \alpha_i) + L_{S_0} \mid i \in [p]\}$ and $(C + L_{S_0})/L_{S_0} = \{(0, i) + L_{S_0} \mid i \in [2]\}$. Therefore, $S_0 = D \oplus L_{S_0} = \{(i, \alpha_i + 2j) \mid i \in [p], j \in [2]\}$ and $C = \{(0, i + 2\alpha_i) \mid i \in [2]\}$. This gives row 3 in Table 6.6.

If $|S_0| = 2p$, S_0 is periodic, and $|L_{S_0}| = p$, then $L_{S_0} = \langle(1, 0)\rangle$. Note that $(D + L_{S_0})/L_{S_0}$ is an aperiodic set of cardinality 2 and $G/L_{S_0} \cong \mathbb{Z}_4$. From Theorem 5.1, we get $(D + L_{S_0})/L_{S_0} = \{(0, i + 2\alpha_i) + L_{S_0} \mid i \in [2]\}$ and $(C + L_{S_0})/L_{S_0} = \{(0, 2i) + L_{S_0} \mid i \in [2]\}$. Therefore, $S_0 = D \oplus L_{S_0} = \{(j, i + 2\alpha_i) \mid i \in [2], j \in [p]\}$ and $C = \{(\alpha_i, 2i) \mid i \in [2]\}$. This gives row 4 in Table 6.6.

If $|S_0| = 2p$ and S_0 is aperiodic, then $|C| = 2$ and C is periodic, which means $C = L_C = \langle(0, 2)\rangle$. From the factorization of G/L_C , $(S_0 + L_C)/L_C = G/L_C = \{(i, j) + L_C \mid i \in [p], j \in [2]\}$, and hence $S_0 = \{(i, j + 2\alpha_{ij}) \mid i \in [p], j \in [2]\}$. This gives row 5 in Table 6.6.

If $|S_0| = 4$ and S_0 is periodic, then $|L_{S_0}| = 2$ and hence $L_{S_0} = \langle(0, 2)\rangle$. Let $S_0 = D \oplus L_{S_0}$ for some subset D . From the factorization of G/L_{S_0} given in Lemma 2.5, we have $G/L_{S_0} = (D + L_{S_0})/L_{S_0} \oplus (C + L_{S_0})/L_{S_0}$. Note that $(D + L_{S_0})/L_{S_0}$ is an aperiodic set of cardinality 2 and $G/L_{S_0} \cong \mathbb{Z}_p \times \mathbb{Z}_2$. From the case of $|S_0| = 2$ in Theorem 3.5 (row 2), we get $(D + L_{S_0})/L_{S_0} = \{(\alpha_i, i) + L_{S_0} \mid i \in [2]\}$ and $(C + L_{S_0})/L_{S_0} = \{(i, 0) + L_{S_0} \mid i \in [p]\}$. Therefore, $S_0 = D \oplus L_{S_0} = \{(\alpha_i, i + 2j) \mid i, j \in [2]\}$ and $C = \{(i, 2\alpha_i) \mid i \in [p]\}$. This gives row 6 in Table 6.6.

If $|S_0| = 4$ and S_0 is aperiodic, then $|C| = p$ and periodic, which means $C = L_C = \langle(1, 0)\rangle$. From the factorization of G/L_C , $(S_0 + L_C)/L_C = G/L_C = \{(0, i) + L_C \mid i \in [4]\}$, and hence $S_0 = \{(\alpha_i, i) \mid i \in [4]\}$. This gives row 7 in Table 6.6. $\square$

Theorem 6.6. Let $G = \mathbb{Z}_p \times \mathbb{Z}_4 \times \mathbb{Z}_2$ for an odd prime p , and let S be a proper subset of $G \setminus \{0\}$ that generates G . Let $0 \in C \subseteq G$. Then C is a perfect code in $\text{Cay}(G, S)$ if and

only if (S_0, C) is one of the pairs in Table 6.7.

Table 6.7: Perfect codes in Cayley graphs of $\mathbb{Z}_p \times \mathbb{Z}_4 \times \mathbb{Z}_2$

Row	S_0	C
1	$\{(i, \alpha_i, \beta_i) \mid i \in [p]\}$	$\{(0, i, j) \mid i \in [4], j \in [2]\}$
2	$\{(i, \alpha_i + 2j, \beta_i) \mid i \in [p], j \in [2]\}$	$\{(0, i + 2\alpha_{ij}, j) \mid i, j \in [2]\}$
3	$\{(i, \alpha_i + 2j, j) \mid i \in [p], j \in [2]\}$	$\{(0, i + 2\alpha_i, \alpha_i) \mid i \in [4]\}$
4	$\{(i, \alpha_i, j) \mid i \in [p], j \in [2]\}$	$\{(0, i, \alpha_i) \mid i \in [4]\}$
5	$\{(i, j + 2\alpha_{ij}, \alpha_i) \mid i \in [p], j \in [2]\}$	$\{(0, \alpha_i + 2j, i) \mid i, j \in [2]\}$
6	$\{(i, \alpha_i + 2j + 2\alpha_{ij}, \alpha_{ij}) \mid i \in [p], j \in [2]\}$	$\{(0, i + 2\alpha_i + 2j, j) \mid i, j \in [2]\}$
7	$\{(j, i + 2\alpha_i + 2\alpha_{ij}, \alpha_{ij}) \mid i \in [2], j \in [p]\}$	$\{(\alpha_i, 2i + 2j, j) \mid i, j \in [2]\}$
8	$\{(i, \alpha_i + 2j, \alpha_{ij}) \mid i \in [p], j \in [2]\}$	$\{(0, i + 2\alpha_i, j) \mid i, j \in [2]\}$
9	$\{(j, i + 2\alpha_i, \alpha_{ij}) \mid i \in [2], j \in [p]\}$	$\{(\alpha_i, 2i, j) \mid i, j \in [2]\}$
10	$\{(i, \alpha_{ij}, j) \mid i \in [p], j \in [2]\}$	$\{(0, i, 0) \mid i \in [4]\}$
11	$\{(i, \alpha_{ij}, j + \alpha_{ij}) \mid i \in [p], j \in [2]\}$	$\{(0, i, i) \mid i \in [4]\}$
12	$\{(i, j + 2\alpha_{ij}, j + \beta_{ij}) \mid i \in [p], j \in [2]\}$	$\{(0, 2i, j) \mid i, j \in [2]\}$
13	$\{(\alpha_i, i + 2\alpha_i + 2j, j) \mid i, j \in [2]\}$	$\{(i, 2j + 2\alpha_{ij}, \alpha_{ij}) \mid i \in [p], j \in [2]\}$
14	$\{(\alpha_i, i + 2\alpha_i, j) \mid i, j \in [2]\}$	$\{(i, 2j, \alpha_{ij}) \mid i \in [p], j \in [2]\}$
15	$\{(\alpha_i, j + 2\alpha_{ij}, i) \mid i, j \in [2]\}$	$\{(i, \alpha_i + 2j, 0) \mid i \in [p], j \in [2]\}$
16	$\{(\alpha_i, i + 2j + 2\alpha_{ij}, \alpha_{ij}) \mid i, j \in [2]\}$	$\{(i, 2\alpha_i + 2j, j) \mid i \in [p], j \in [2]\}$
17	$\{(\alpha_i, i + 2j, \alpha_{ij}) \mid i, j \in [2]\}$	$\{(i, 2\alpha_i, j) \mid i \in [p], j \in [2]\}$
18	$\{(\alpha_{ij}, i + 2\alpha_i + 2j, j) \mid i, j \in [2]\}$	$\{(j, 2i + 2\alpha_i, \alpha_i) \mid i \in [2], j \in [p]\}$
19	$\{(\alpha_{ij}, i + 2\alpha_i, j) \mid i, j \in [2]\}$	$\{(j, 2i, \alpha_i) \mid i \in [2], j \in [p]\}$
20	$\{(\alpha_{ij}, i + 2\alpha_{ij}, j) \mid i, j \in [2]\}$	$\{(i, 2j, 0) \mid i \in [p], j \in [2]\}$
21	$\{(\alpha_i, i + 2\alpha_i, \alpha_i) \mid i \in [4]\}$	$\{(i, 2j, j) \mid i \in [p], j \in [2]\}$
22	$\{(\alpha_i, i, \alpha_i) \mid i \in [4]\}$	$\{(i, 0, j) \mid i \in [p], j \in [2]\}$
23	$\{(i, \alpha_{ij} + 2k, j) \mid i \in [p], j, k \in [2]\}$	$\{(0, i + 2\alpha_i, 0) \mid i \in [2]\}$
24	$\{(i, j + 2\alpha_{ij} + 2k, k) \mid i \in [p], j, k \in [2]\}$	$\{(0, 2i + 2\alpha_i, \alpha_i) \mid i \in [2]\}$
25	$\{(i, j + 2\alpha_{ij}, k) \mid i \in [p], j, k \in [2]\}$	$\{(0, 2i, \alpha_i) \mid i \in [2]\}$
26	$\{(k, i + 2\alpha_{ij}, j) \mid i, j \in [2], k \in [p]\}$	$\{(\alpha_i, 2i, 0) \mid i \in [2]\}$
27	$\{(k, i + 2\alpha_{ij}, j + \alpha_{ij}) \mid i, j \in [2], k \in [p]\}$	$\{(\alpha_i, 2i, i) \mid i \in [2]\}$
28	$\{(j, i, \alpha_i) \mid i \in [4], j \in [p]\}$	$\{(\alpha_i, 0, i) \mid i \in [2]\}$
29	$\{(i, j, \alpha_i) \mid i \in [p], j \in [4]\}$	$\{(0, \alpha_i, i) \mid i \in [2]\}$
30	$\{(i, j, \alpha_i + j) \mid i \in [p], j \in [4]\}$	$\{(0, \alpha_i, i + \alpha_i) \mid i \in [2]\}$
31	$\{(i, \alpha_i + 2j, k) \mid i \in [p], j, k \in [2]\}$	$\{(0, i + 2\alpha_i, \beta_i) \mid i \in [2]\}$
32	$\{(j, i + 2\alpha_i + 2k, k) \mid i, k \in [2], j \in [p]\}$	$\{(\alpha_i, 2i + 2\alpha_i, \alpha_i) \mid i \in [2]\}$
33	$\{(j, i + 2\alpha_i, k) \mid i, k \in [2], j \in [p]\}$	$\{(\alpha_i, 2i, \alpha_i) \mid i \in [2]\}$
34	$\{(i, j + 2\alpha_{ijk}, k) \mid i \in [p], j, k \in [2]\}$	$\{(0, 2i, 0) \mid i \in [2]\}$
35	$\{(i, j + 2\alpha_{ij}, \alpha_{ij}) \mid i \in [p], j \in [4]\}$	$\{(0, 2i, i) \mid i \in [2]\}$
36	$\{(i, j, \alpha_{ij}) \mid i \in [p], j \in [4]\}$	$\{(0, 0, i) \mid i \in [2]\}$
37	$\{(\alpha_{ij}, i + 2k, j) \mid i, j, k \in [2]\}$	$\{(i, 2\alpha_i, 0) \mid i \in [p]\}$
38	$\{(\alpha_i, i + 2j, j) \mid i \in [4], j \in [2]\}$	$\{(i, 2\alpha_i, \alpha_i) \mid i \in [p]\}$
39	$\{(\alpha_i, i, j) \mid i \in [4], j \in [2]\}$	$\{(i, 0, \alpha_i) \mid i \in [p]\}$
40	$\{(\alpha_i, j, i) \mid i \in [2], j \in [4]\}$	$\{(i, \alpha_i, 0) \mid i \in [p]\}$
41	$\{(\alpha_i, j, i + j) \mid i \in [2], j \in [4]\}$	$\{(i, \alpha_i, \alpha_i) \mid i \in [p]\}$

42	$\{(\alpha_i, i + 2j, k) \mid i, j, k \in [2]\}$	$\{(i, 2\alpha_i, \beta_i) \mid i \in [p]\}$
43	$\{(\alpha_{ij}, i, j) \mid i \in [4], j \in [2]\}$	$\{(i, 0, 0) \mid i \in [p]\}$

Proof. Suppose C is a perfect code in $\text{Cay}(G, S)$. Then $G = S_0 \oplus C$ and so $|S_0| \cdot |C| = |G| = 8p$. Since G is not a cyclic group, $|S_0| > 2$ by Lemma 2.8. Therefore, $2 < |S_0| < 8p$.

Case 1. S_0 is aperiodic.

Since G is Hajós and $G = S_0 \oplus C$, in this case C must be periodic, that is, L_C is nontrivial. Write $C = L_C \oplus D$ for some subset D of G whose existence is ensured by Lemma 2.3. Then $|L_C| \cdot |D| = |C|$ and $2 \leq |L_C| \leq |C|$. If $|S_0|$ is a prime, then $|L_C| = |C|$ by Lemma 2.8 part (c). If $|S_0|$ is not a prime, then $|L_C|$ could be any divisor of $|C|$. Set

$$A = (S_0 + L_C)/L_C \quad \text{and} \quad B = (D + L_C)/L_C.$$

Since C is periodic, by Lemma 2.5, we have

$$G/L_C = A \oplus B$$

and B is aperiodic in G/L_C . Since G/L_C is isomorphic to a subgroup of G and subgroups of Hajós groups are also Hajós groups, A is periodic in G/L_C . We also know that A contains L_C and A generates G/L_C by Lemma 2.7. Furthermore, $|A| = |S_0|$ and $|B| = |D|$. Note that if $|L_C| = |C|$, then $C = L_C$ as $0 \in C$. This means we can take $D = \{0\}$ and hence $B = L_C/L_C$ and $G/L_C = A$. We use C to get A and then use Lemma 2.6 to obtain S_0 . If $|L_C| < |C|$, then this factorization enables us to make use of previously established theorems to obtain the pair (S_0, C) via (A, B) . Note that for some triples $(|S_0|, |C|, |L_C|)$, there are more than one possible L_C , see Observation 4.5.

The columns of Table 6.8 represent all possible values of $(|S_0|, |C|, |L_C|)$; C , A , and S_0 when $|L_C| = |C|$ or L_C , A , B , S_0 , and C when $|L_C| < |C|$; which previous theorem or lemma is used to obtain (A, B) when $|L_C| < |C|$; and which row in Table 6.7 the result corresponds to.

Table 6.8: The case when $G = \mathbb{Z}_p \times \mathbb{Z}_4 \times \mathbb{Z}_2$ and S_0 is aperiodic

(S_0 , C , L_C)	$L_C, A, B, S_0, \text{ and } C$	Reference	Result
$(p, 8, 8)$	$C = \langle (0, 1, 0), (0, 0, 1) \rangle$ $A = \{(i, 0, 0) + L_C \mid i \in [p]\}$ $S_0 = \{(i, \alpha_i, \beta_i) \mid i \in [p]\}$		row 1
$(2p, 4, 2)$	$L_C = \langle (0, 2, 0) \rangle$ $A = \{i\mathbf{e}_1 + j\mathbf{e}_2 + \alpha_i\mathbf{e}_3 + L_C \mid i \in [p], j \in [2]\}$ $B = \{\alpha_i\mathbf{e}_2 + i\mathbf{e}_3 + L_C \mid i \in [2]\}$ $S_0 = \{(i, j + 2\alpha_{ij}, \alpha_i) \mid i \in [p], j \in [2]\}$ $C = \{(0, 2j + \alpha_i, i) \mid i, j \in [2]\}$	3.6 (row 4)	row 5
$(2p, 4, 2)$	$L_C = \langle (0, 2, 1) \rangle$ $A = \{(i, \alpha_i + 2j, 0) + L_C \mid i \in [p], j \in [2]\}$ $B = \{(0, i + 2\alpha_i, 0) + L_C \mid i \in [2]\}$ $S_0 = \{(i, \alpha_i + 2j + 2\alpha_{ij}, \alpha_{ij}) \mid i \in [p], j \in [2]\}$ $C = \{(0, i + 2\alpha_i + 2j, j) \mid i, j \in [2]\}$	6.5 (row 3)	row 6

	$L_C = \langle (0, 2, 1) \rangle$		
$(2p, 4, 2)$	$A = \{(j, i + 2\alpha_i, 0) + L_C \mid i \in [2], j \in [p]\}$ $B = \{(\alpha_i, 2i, 0) + L_C \mid i \in [2]\}$ $S_0 = \{(j, i + 2\alpha_i + 2\alpha_{ij}, \alpha_{ij}) \mid i \in [2], j \in [p]\}$ $C = \{(\alpha_i, 2i + 2j, j) \mid i, j \in [2]\}$	6.5 (row 4)	row 7
	$L_C = \langle (0, 0, 1) \rangle$		
$(2p, 4, 2)$	$A = \{(i, \alpha_i + 2j, 0) + L_C \mid i \in [p], j \in [2]\}$ $B = \{(0, i + 2\alpha_i, 0) + L_C \mid i \in [2]\}$ $S_0 = \{(i, \alpha_i + 2j, \alpha_{ij}) \mid i \in [p], j \in [2]\}$ $C = \{(0, i + 2\alpha_i, j) \mid i, j \in [2]\}$	6.5 (row 3)	row 8
	$L_C = \langle (0, 0, 1) \rangle$		
$(2p, 4, 2)$	$A = \{(j, i + 2\alpha_i, 0) + L_C \mid i \in [2], j \in [p]\}$ $B = \{(\alpha_i, 2i, 0) + L_C \mid i \in [2]\}$ $S_0 = \{(j, i + 2\alpha_i, \alpha_{ij}) \mid i \in [2], j \in [p]\}$ $C = \{(\alpha_i, 2i, j) \mid i, j \in [2]\}$	6.5 (row 4)	row 9
	$C = \langle (0, 1, 0) \rangle$		
$(2p, 4, 4)$	$A = \{(i, 0, j) + L_C \mid i \in [p], j \in [2]\}$ $S_0 = \{(i, \alpha_{ij}, j) \mid i \in [p], j \in [2]\}$		row 10
	$C = \langle (0, 1, 1) \rangle$		
$(2p, 4, 4)$	$A = \{(i, 0, j) + L_C \mid i \in [p], j \in [2]\}$ $S_0 = \{(i, \alpha_{ij}, j + \alpha_{ij}) \mid i \in [p], j \in [2]\}$		row 11
	$C = \langle (0, 2, 0), (0, 0, 1) \rangle$		
$(2p, 4, 4)$	$A = \{(i, j, 0) + L_C \mid i \in [p], j \in [2]\}$ $S_0 = \{(i, j + 2\alpha_{ij}, j + \beta_{ij}) \mid i \in [p], j \in [2]\}$		row 12
	$L_C = \langle (0, 2, 0) \rangle$		
$(4, 2p, 2)$	$A = \{\alpha_i \mathbf{e}_1 + j \mathbf{e}_2 + i \mathbf{e}_3 + L_C \mid i, j \in [2]\}$ $B = \{i \mathbf{e}_1 + \alpha_i \mathbf{e}_2 + L_C \mid i \in [p]\}$ $S_0 = \{(\alpha_i, j + 2\alpha_{ij}, i) \mid i, j \in [2]\}$ $C = \{(i, \alpha_i + 2j, 0) \mid i \in [p], j \in [2]\}$	3.6 (row 6)	row 15
	$L_C = \langle (0, 2, 1) \rangle$		
$(4, 2p, 2)$	$A = \{(\alpha_i, i + 2j, 0) + L_C \mid i, j \in [2]\}$ $B = \{(i, 2\alpha_i, 0) + L_C \mid i \in [p]\}$ $S_0 = \{(\alpha_i, i + 2j + 2\alpha_{ij}, \alpha_{ij}) \mid i, j \in [2]\}$ $C = \{(i, 2\alpha_i + 2j, j) \mid i \in [p], j \in [2]\}$	6.5 (row 6)	row 16
	$L_C = \langle (0, 0, 1) \rangle$		
$(4, 2p, 2)$	$A = \{(\alpha_i, i + 2j, 0) + L_C \mid i, j \in [2]\}$ $B = \{(i, 2\alpha_i, 0) + L_C \mid i \in [p]\}$ $S_0 = \{(\alpha_i, i + 2j, \alpha_{ij}) \mid i, j \in [2]\}$ $C = \{(i, 2\alpha_i, j) \mid i \in [p], j \in [2]\}$	6.5 (row 6)	row 17
	$L_C = \langle (1, 0, 0) \rangle$		
$(4, 2p, p)$	$A = \{(0, i + 2\alpha_i + 2j, j) + L_C \mid i, j \in [2]\}$ $B = \{(0, 2i + 2\alpha_i, \alpha_i) + L_C \mid i \in [2]\}$ $S_0 = \{(\alpha_{ij}, i + 2\alpha_i + 2j, j) \mid i, j \in [2]\}$ $C = \{(j, 2i + 2\alpha_i, \alpha_i) \mid i \in [2], j \in [p]\}$	6.4 (row 6)	row 18
	$L_C = \langle (1, 0, 0) \rangle$		
$(4, 2p, p)$	$A = \{(0, i + 2\alpha_i, j) + L_C \mid i, j \in [2]\}$ $B = \{(0, 2i, \alpha_i) + L_C \mid i \in [2]\}$ $S_0 = \{(\alpha_{ij}, i + 2\alpha_i, j) \mid i, j \in [2]\}$ $C = \{(j, 2i, \alpha_i) \mid i \in [2], j \in [p]\}$	6.4 (row 7)	row 19
	$C = \langle (1, 2, 0) \rangle$		
$(4, 2p, 2p)$	$A = \{(0, i, j) + L_C \mid i, j \in [2]\}$		row 20

	$S_0 = \{(\alpha_{ij}, i + 2\alpha_{ij}, j) \mid i, j \in [2]\}$		
$(4, 2p, 2p)$	$C = \langle(1, 2, 1)\rangle$ $A = \{(0, i, 0) + L_C \mid i \in [4]\}$ $S_0 = \{(\alpha_i, i + 2\alpha_i, \alpha_i) \mid i \in [4]\}$		row 21
$(4, 2p, 2p)$	$C = \langle(1, 0, 1)\rangle$ $A = \{(0, i, 0) + L_C \mid i \in [4]\}$ $S_0 = \{(\alpha_i, i, \alpha_i) \mid i \in [4]\}$		row 22
$(4p, 2, 2)$	$C = \langle(0, 2, 0)\rangle$ $A = \{(i, j, k) + L_C \mid i \in [p], j, k \in [2]\}$ $S_0 = \{(i, j + 2\alpha_{ijk}, k) \mid i \in [p], j, k \in [2]\}$		row 34
$(4p, 2, 2)$	$C = \langle(0, 2, 1)\rangle$ $A = \{(i, j, 0) + L_C \mid i \in [p], j \in [4]\}$ $S_0 = \{(i, j + 2\alpha_{ij}, \alpha_{ij}) \mid i \in [p], j \in [4]\}$		row 35
$(4p, 2, 2)$	$C = \langle(0, 0, 1)\rangle$ $A = \{(i, j, 0) + L_C \mid i \in [p], j \in [4]\}$ $S_0 = \{(i, j, \alpha_{ij}) \mid i \in [p], j \in [4]\}$		row 36
$(8, p, p)$	$C = \langle(1, 0, 0)\rangle$ $A = \{(0, i, j) + L_C \mid i \in [4], j \in [2]\}$ $S_0 = \{(\alpha_{ij}, i, j) \mid i \in [4], j \in [2]\}$		row 43

Case 2. S_0 is periodic.

In this case we have $|L_{S_0}| \geq 2$ and $S_0 = D \oplus L_{S_0}$ for some subset D of G by Lemma 2.3. So $|D| \cdot |L_{S_0}| = |S_0|$ and $1 < |L_{S_0}| < |S_0|$ by Lemma 2.8 part (b). Set

$$X = (D + L_{S_0})/L_{S_0} \quad \text{and} \quad Y = (C + L_{S_0})/L_{S_0}.$$

Since S_0 is periodic, by Lemma 2.5, we have

$$G/L_{S_0} = X \oplus Y$$

and X is aperiodic in G/L_{S_0} . We also know that X contains L_{S_0} and X generates G/L_{S_0} by Lemma 2.7. This factorization enables us to make use of Lemma 2.6 and previously established theorems to obtain the pair (S_0, C) via (X, Y) . Note that for some triples $(|S_0|, |C|, |L_{S_0}|)$, there are more than one possible L_{S_0} , see Observation 4.5.

The columns of Table 6.9 represent all possible values for $(|S_0|, |C|, |L_{S_0}|)$; L_{S_0} , X , Y , S_0 , and C ; which previous theorem or lemma is used to obtain (X, Y) ; and which row in Table 6.7 the result corresponds to. $\square$

Table 6.9: The case when $G = \mathbb{Z}_p \times \mathbb{Z}_4 \times \mathbb{Z}_2$ and S_0 is periodic

(S_0 , C , L_{S_0})	$L_{S_0}, X, Y, S_0, \text{ and } C$	Reference	Result
$(2p, 4, 2)$	$L_{S_0} = \langle(0, 2, 0)\rangle$	3.6 (row 1)	row 2
	$X = \{(i, \alpha_i, \beta_i) + L_{S_0} \mid i \in [p]\}$		
	$Y = \{(0, i, j) + L_{S_0} \mid i, j \in [2]\}$		
	$S_0 = \{(i, \alpha_i + 2j, \beta_i) \mid i \in [p], j \in [2]\}$ $C = \{(0, i + 2\alpha_{ij}, j) \mid i, j \in [2]\}$		
	$L_{S_0} = \langle(0, 2, 1)\rangle$ $X = \{(i, \alpha_i, 0) + L_{S_0} \mid i \in [p]\}$		

(2p, 4, 2)	$Y = \{(0, i, 0) + L_{S_0} \mid i \in [4]\}$ $S_0 = \{(i, \alpha_i + 2j, j) \mid i \in [p], j \in [2]\}$ $C = \{(0, i + 2\alpha_i, \alpha_i) \mid i \in [4]\}$	6.5 (row 2)	row 3
	$L_{S_0} = \langle (0, 0, 1) \rangle$ $X = \{(i, \alpha_i, 0) + L_{S_0} \mid i \in [p]\}$		
(2p, 4, 2)	$Y = \{(0, i, 0) + L_{S_0} \mid i \in [4]\}$ $S_0 = \{(i, \alpha_i, j) \mid i \in [p], j \in [2]\}$ $C = \{(0, i, \alpha_i) \mid i \in [4]\}$	6.5 (row 2)	row 4
(2p, 4, p)	Not possible since no X exists	6.4	
(4, 2p, 2)	Not possible for $L_{S_0} = \langle (0, 2, 0) \rangle$	3.6	
	$L_{S_0} = \langle (0, 2, 1) \rangle$ $X = \{(\alpha_i, i + 2\alpha_i, 0) + L_{S_0} \mid i \in [2]\}$		
(4, 2p, 2)	$Y = \{(i, 2j, 0) + L_{S_0} \mid i \in [p], j \in [2]\}$ $S_0 = \{(\alpha_i, i + 2\alpha_i + 2j, j) \mid i, j \in [2]\}$ $C = \{(i, 2j + 2\alpha_{ij}, \alpha_{ij}) \mid i \in [p], j \in [2]\}$	6.5 (row 1)	row 13
	$L_{S_0} = \langle (0, 0, 1) \rangle$ $X = \{(\alpha_i, i + 2\alpha_i, 0) + L_{S_0} \mid i \in [2]\}$		
(4, 2p, 2)	$Y = \{(i, 2j, 0) + L_{S_0} \mid i \in [p], j \in [2]\}$ $S_0 = \{(\alpha_i, i + 2\alpha_i, j) \mid i, j \in [2]\}$ $C = \{(i, 2j, \alpha_{ij}) \mid i \in [p], j \in [2]\}$	6.5 (row 1)	row 14
	$L_{S_0} = \langle (0, 2, 0) \rangle$ $X = \{i\mathbf{e}_1 + \alpha_{ij}\mathbf{e}_2 + j\mathbf{e}_3 + L_{S_0} \mid i \in [p], j \in [2]\}$		
(4p, 2, 2)	$Y = \{i\mathbf{e}_2 + L_{S_0} \mid i \in [2]\}$ $S_0 = \{(i, \alpha_{ij} + 2k, j) \mid i \in [p], j, k \in [2]\}$ $C = \{(0, i + 2\alpha_i, 0) \mid i \in [2]\}$	3.6 (row 5)	row 23
	$L_{S_0} = \langle (0, 2, 1) \rangle$ $X = \{(i, j + 2\alpha_{ij}, 0) + L_{S_0} \mid i \in [p], j \in [2]\}$		
(4p, 2, 2)	$Y = \{(0, 2i, 0) + L_{S_0} \mid i \in [2]\}$ $S_0 = \{(i, j + 2\alpha_{ij} + 2k, k) \mid i \in [p], j, k \in [2]\}$ $C = \{(0, 2i + 2\alpha_i, \alpha_i) \mid i \in [2]\}$	6.5 (row 5)	row 24
	$L_{S_0} = \langle (0, 0, 1) \rangle$ $X = \{(i, j + 2\alpha_{ij}, 0) + L_{S_0} \mid i \in [p], j \in [2]\}$		
(4p, 2, 2)	$Y = \{(0, 2i, 0) + L_{S_0} \mid i \in [2]\}$ $S_0 = \{(i, j + 2\alpha_{ij}, k) \mid i \in [p], j, k \in [2]\}$ $C = \{(0, 2i, \alpha_i) \mid i \in [2]\}$	6.5 (row 5)	row 25
	$L_{S_0} = \langle (1, 0, 0) \rangle$ $X = \{(0, i + 2\alpha_{ij}, j) + L_{S_0} \mid i, j \in [2]\}$		
(4p, 2, p)	$Y = \{(0, 2i, 0) + L_{S_0} \mid i \in [2]\}$ $S_0 = \{(k, i + 2\alpha_{ij}, j) \mid i, j \in [2], k \in [p]\}$ $C = \{(\alpha_i, 2i, 0) \mid i \in [2]\}$	6.4 (row 8)	row 26
	$L_{S_0} = \langle (1, 0, 0) \rangle$ $X = \{(0, i + 2\alpha_{ij}, j + \alpha_{ij}) + L_{S_0} \mid i, j \in [2]\}$		
(4p, 2, p)	$Y = \{(0, 2i, i) + L_{S_0} \mid i \in [2]\}$ $S_0 = \{(k, i + 2\alpha_{ij}, j + \alpha_{ij}) \mid i, j \in [2], k \in [p]\}$ $C = \{(\alpha_i, 2i, i) \mid i \in [2]\}$	6.4 (row 9)	row 27
	$L_{S_0} = \langle (1, 0, 0) \rangle$ $X = \{(0, i, \alpha_i) + L_{S_0} \mid i \in [4]\}$		
(4p, 2, p)	$Y = \{(0, 0, i) + L_{S_0} \mid i \in [2]\}$ $S_0 = \{(j, i, \alpha_i) \mid i \in [4], j \in [p]\}$ $C = \{(\alpha_i, 0, i) \mid i \in [2]\}$	6.4 (row 10)	row 28
	$L_{S_0} = \langle (0, 1, 0) \rangle$ $X = \{(i, 0, \alpha_i) + L_{S_0} \mid i \in [p]\}$		

(4p, 2, 4)	$Y = \{(0, 0, i) + L_{S_0} \mid i \in [2]\}$ $S_0 = \{(i, j, \alpha_i) \mid i \in [p], j \in [4]\}$ $C = \{(0, \alpha_i, i) \mid i \in [2]\}$	3.5 (row 1)	row 29
(4p, 2, 4)	$L_{S_0} = \langle (0, 1, 1) \rangle$ $X = \{(i, 0, \alpha_i) + L_{S_0} \mid i \in [p]\}$ $Y = \{(0, 0, i) + L_{S_0} \mid i \in [2]\}$ $S_0 = \{(i, j, \alpha_i + j) \mid i \in [p], j \in [4]\}$ $C = \{(0, \alpha_i, i + \alpha_i) \mid i \in [2]\}$	3.5 (row 1)	row 30
(4p, 2, 4)	$L_{S_0} = \langle (0, 2, 0), (0, 0, 1) \rangle$ $X = \{(i, \alpha_i, 0) + L_{S_0} \mid i \in [p]\}$ $Y = \{(0, i, 0) + L_{S_0} \mid i \in [2]\}$ $S_0 = \{(i, \alpha_i + 2j, k) \mid i \in [p], j, k \in [2]\}$ $C = \{(0, i + 2\alpha_i, \beta_i) \mid i \in [2]\}$	3.5 (row 1)	row 31
(4p, 2, 2p)	Not possible for $L_{S_0} = \langle (1, 2, 0) \rangle$	5.2	
(4p, 2, 2p)	$L_{S_0} = \langle (1, 2, 1) \rangle$ $X = \{(0, i + 2\alpha_i, 0) + L_{S_0} \mid i \in [2]\}$ $Y = \{(0, 2i, 0) + L_{S_0} \mid i \in [2]\}$ $S_0 = \{(j, i + 2\alpha_i + 2k, k) \mid i, k \in [2], j \in [p]\}$ $C = \{(\alpha_i, 2i + 2\alpha_i, \alpha_i) \mid i \in [2]\}$	5.1	row 32
(4p, 2, 2p)	$L_{S_0} = \langle (1, 0, 1) \rangle$ $X = \{(0, i + 2\alpha_i, 0) + L_{S_0} \mid i \in [2]\}$ $Y = \{(0, 2i, 0) + L_{S_0} \mid i \in [2]\}$ $S_0 = \{(j, i + 2\alpha_i, k) \mid i, k \in [2], j \in [p]\}$ $C = \{(\alpha_i, 2i, \alpha_i) \mid i \in [2]\}$	5.1	row 33
(8, p, 2)	$L_{S_0} = \langle (0, 2, 0) \rangle$ $X = \{(\alpha_{ij}, i, j) + L_{S_0} \mid i, j \in [2]\}$ $Y = \{(i, 0, 0) + L_{S_0} \mid i \in [p]\}$ $S_0 = \{(\alpha_{ij}, i + 2k, j) \mid i, j, k \in [2]\}$ $C = \{(i, 2\alpha_i, 0) \mid i \in [p]\}$	3.6 (row 7)	row 37
(8, p, 2)	$L_{S_0} = \langle (0, 2, 1) \rangle$ $X = \{(\alpha_i, i, 0) + L_{S_0} \mid i \in [4]\}$ $Y = \{(i, 0, 0) + L_{S_0} \mid i \in [p]\}$ $S_0 = \{(\alpha_i, i + 2j, j) \mid i \in [4], j \in [2]\}$ $C = \{(i, 2\alpha_i, \alpha_i) \mid i \in [p]\}$	6.5 (row 7)	row 38
(8, p, 2)	$L_{S_0} = \langle (0, 0, 1) \rangle$ $X = \{(\alpha_i, i, 0) + L_{S_0} \mid i \in [4]\}$ $Y = \{(i, 0, 0) + L_{S_0} \mid i \in [p]\}$ $S_0 = \{(\alpha_i, i, j) \mid i \in [4], j \in [2]\}$ $C = \{(i, 0, \alpha_i) \mid i \in [p]\}$	6.5 (row 7)	row 39
(8, p, 4)	$L_{S_0} = \langle (0, 1, 0) \rangle$ $X = \{(\alpha_i, 0, i) + L_{S_0} \mid i \in [2]\}$ $Y = \{(i, 0, 0) + L_{S_0} \mid i \in [p]\}$ $S_0 = \{(\alpha_i, j, i) \mid i \in [2], j \in [4]\}$ $C = \{(i, \alpha_i, 0) \mid i \in [p]\}$	3.5 (row 2)	row 40
(8, p, 4)	$L_{S_0} = \langle (0, 1, 1) \rangle$ $X = \{(\alpha_i, 0, i) + L_{S_0} \mid i \in [2]\}$ $Y = \{(i, 0, 0) + L_{S_0} \mid i \in [p]\}$ $S_0 = \{(\alpha_i, j, i + j) \mid i \in [2], j \in [4]\}$ $C = \{(i, \alpha_i, \alpha_i) \mid i \in [p]\}$	3.5 (row 2)	row 41
(8, p, 4)	$L_{S_0} = \langle (0, 2, 0), (0, 0, 1) \rangle$ $X = \{(\alpha_i, i, 0) + L_{S_0} \mid i \in [2]\}$ $Y = \{(i, 0, 0) + L_{S_0} \mid i \in [p]\}$	3.5 (row 2)	row 42

$S_0 = \{(\alpha_i, i + 2j, k) \mid i, j, k \in [2]\}$		
$C = \{(i, 2\alpha_i, \beta_i) \mid i \in [p]\}$		

6.3 Circulant graphs

As mentioned in the introduction, cyclic Hajós groups are subgroups of $\mathbb{Z}_{p^k} \times \mathbb{Z}_q$, $\mathbb{Z}_{p^2} \times \mathbb{Z}_{q^2}$, $\mathbb{Z}_{p^2} \times \mathbb{Z}_q \times \mathbb{Z}_r$, and $\mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r \times \mathbb{Z}_s$. Recall that circulant graphs having perfect codes with prime size have been classified in [6]. Circulant graphs on Hajós groups having perfect codes with size relatively prime to the degree plus one have been classified in [7].

If a circulant graph on a subgroup of $\mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r \times \mathbb{Z}_s$ has a perfect code, then the size of the perfect code will be relatively prime to the degree plus one. This means that circulant graphs on subgroups of $\mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r \times \mathbb{Z}_s$ have been characterized. Subgroups of $\mathbb{Z}_{p^k} \times \mathbb{Z}_q$ are covered in Theorems 3.5, 4.1, and 4.2. As every proper subgroup of $\mathbb{Z}_{p^2} \times \mathbb{Z}_{q^2}$ or of $\mathbb{Z}_{p^2} \times \mathbb{Z}_q \times \mathbb{Z}_r$ is also a subgroup of $\mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r \times \mathbb{Z}_s$ or $\mathbb{Z}_{p^k} \times \mathbb{Z}_q$, we are left to characterize circulant graphs on $\mathbb{Z}_{p^2} \times \mathbb{Z}_{q^2}$ and $\mathbb{Z}_{p^2} \times \mathbb{Z}_q \times \mathbb{Z}_r$ having a perfect code C where $|C|$ is not relatively prime to $|G|/|C|$.

Theorem 6.7. *Let $G = \mathbb{Z}_{p^2} \times \mathbb{Z}_{q^2}$ for distinct primes p and q , and let S be a proper subset of $G \setminus \{0\}$ that generates G . Let $0 \in C \subseteq G$ and let $|C|$ be not relatively prime to $|S_0|$. Then C is a perfect code in $\text{Cay}(G, S)$ if and only if (S_0, C) is one of the pairs in Table 6.10.*

Table 6.10: Perfect codes in Cayley graphs of $\mathbb{Z}_{p^2} \times \mathbb{Z}_{q^2}$

Row	S_0	C
1	$\{(i + \alpha_i p, \beta_i) \mid i \in [p]\}$	$\{(ip, j) \mid i \in [p], j \in [q^2]\}$
2	$\{(\alpha_i, i + \beta_i q) \mid i \in [q]\}$	$\{(i, jq) \mid i \in [p^2], j \in [q]\}$
3	$\{(i + \alpha_i p, \alpha_i + jq) \mid i \in [p], j \in [q]\}$	$\{(ip, j + \alpha_{ij} q) \mid i \in [p], j \in [q]\}$
4	$\{(\alpha_i + jp, i + \alpha_i q) \mid i \in [q], j \in [p]\}$	$\{(i + \alpha_{ij} p, jq) \mid i \in [p], j \in [q]\}$
5	$\{(j + \alpha_{ij} p, i + \alpha_i q) \mid i \in [q], j \in [p]\}$	$\{(\alpha_i + jp, iq) \mid i \in [q], j \in [p]\}$
6	$\{(i + \alpha_{ij} p, \alpha_i + jq) \mid i \in [p], j \in [q]\}$	$\{(jp, i + \alpha_i q) \mid i \in [q], j \in [p]\}$
7	$\{(i + \alpha_i p, j + \alpha_{ij} q) \mid i \in [p], j \in [q]\}$	$\{(ip, \alpha_i + jq) \mid i \in [p], j \in [q]\}$
8	$\{(\alpha_i + jp, i + \alpha_{ij} q) \mid i \in [q], j \in [p]\}$	$\{(i + \alpha_i p, jq) \mid i \in [p], j \in [q]\}$
9	$\{(i + \alpha_{ij} p, j + \alpha_{ij} q) \mid i \in [p], j \in [q]\}$	$\{(ip, jq) \mid i \in [p], j \in [q]\}$
10	$\{(j + kp, i + \alpha_{ij} q) \mid i \in [q], j, k \in [p]\}$	$\{(p\alpha_i, iq) \mid i \in [q]\}$
11	$\{(i, \alpha_i + jq) \mid i \in [p^2], j \in [q]\}$	$\{(0, i + \alpha_i q) \mid i \in [q]\}$
12	$\{(j, i + \alpha_i q) \mid i \in [q], j \in [p^2]\}$	$\{(\alpha_i, iq) \mid i \in [q]\}$
13	$\{(i + jp, \alpha_i + kq) \mid i, j \in [p], k \in [q]\}$	$\{(p\alpha_i, i + q\alpha_i) \mid i \in [q]\}$
14	$\{(i, j + \alpha_{ij} q) \mid i \in [p^2], j \in [q]\}$	$\{(0, iq) \mid i \in [q]\}$
15	$\{(i + \alpha_{ij} p, j + kq) \mid i \in [p], j, k \in [q]\}$	$\{(ip, \alpha_i q) \mid i \in [p]\}$
16	$\{(\alpha_i + jp, i) \mid i \in [q^2], j \in [p]\}$	$\{(i + \alpha_i p, 0) \mid i \in [p]\}$
17	$\{(i + \alpha_i p, j) \mid i \in [p], j \in [q^2]\}$	$\{(ip, \alpha_i) \mid i \in [p]\}$
18	$\{(\alpha_i + jp, i + kq) \mid i, k \in [q], j \in [p]\}$	$\{(i + p\alpha_i, q\alpha_i) \mid i \in [p]\}$
19	$\{(i + \alpha_{ij} p, j) \mid i \in [p], j \in [q^2]\}$	$\{(ip, 0) \mid i \in [p]\}$

Proof. Suppose C is a perfect code in $\text{Cay}(G, S)$. Then $G = S_0 \oplus C$ and so $|S_0| \cdot |C| = |G| = p^2 q^2$. Since $\text{Cay}(G, S)$ is a connected non-complete graph, $1 < |S_0| < p^2 q^2$.

Case 1. S_0 is aperiodic.

Since G is Hajós and $G = S_0 \oplus C$, in this case C must be periodic, that is, L_C is nontrivial. Write $C = L_C \oplus D$ for some subset D of G whose existence is ensured by Lemma 2.3. Then $|L_C| \cdot |D| = |C|$ and $2 \leq |L_C| \leq |C|$. If $|S_0|$ is a prime, then $|L_C| = |C|$ by Lemma 2.8 part (c). If $|S_0|$ is not a prime, then $|L_C|$ can be any divisor of $|C|$. Set

$$A = (S_0 + L_C)/L_C \quad \text{and} \quad B = (D + L_C)/L_C.$$

Since C is periodic, by Lemma 2.5, we have

$$G/L_C = A \oplus B$$

and B is aperiodic in G/L_C . Since G/L_C is isomorphic to a subgroup of G and subgroups of Hajós group are also Hajós groups, A is periodic in G/L_C . We also know that A contains L_C and A generates G/L_C by Lemma 2.7. Furthermore, $|A| = |S_0|$ and $|B| = |D|$. Note that if $|L_C| = |C|$, then $C = L_C$ as $0 \in C$. This means we can take $D = \{0\}$ and hence $B = L_C/L_C$ and $G/L_C = A$. We use C to get A and then use Lemma 2.6 to obtain S_0 . If $|L_C| < |C|$, then this factorization enables us to make use of previously established theorems to obtain the pair (S_0, C) via (A, B) .

All possible values for $(|S_0|, |C|, |L_C|)$ when $|S_0|$ and $|C|$ are not relatively prime are given in the first column of Table 6.11. In the second column, we list C , A , and S_0 when $|L_C| = |C|$ and we list L_C , A , B , S_0 , and C when $|L_C| < |C|$. The third column shows which previous theorem is used to obtain (A, B) when $|L_C| < |C|$. The last column indicates which row in Table 6.10 the result corresponds to.

Table 6.11: The case when $G = \mathbb{Z}_{p^2} \times \mathbb{Z}_{q^2}$ and S_0 is aperiodic

(S_0 , C , L_C)	L_C, A, B, S_0 , and C	Reference	Result
(p, pq^2, pq^2)	$C = \{(ip, j) \mid i \in [p], j \in [q^2]\}$ $A = \{(i, 0) + L_C \mid i \in [p]\}$ $S_0 = \{(i + \alpha_i p, \beta_i) \mid i \in [p]\}$		row 1
$(q, p^2 q, p^2 q)$	$C = \{(i, jq) \mid i \in [p^2], j \in [q]\}$ $A = \{(0, i) + L_C \mid i \in [q]\}$ $S_0 = \{(\alpha_i, i + \beta_i q) \mid i \in [q]\}$		row 2
(pq, pq, p)	$L_C = \langle p\mathbf{e}_1 \rangle$ $A = \{(j, i + \alpha_i q) + L_C \mid i \in [q], j \in [p]\}$ $B = \{(\alpha_i, iq) + L_C \mid i \in [q]\}$ $S_0 = \{(j + \alpha_{ij} p, i + \alpha_i q) \mid i \in [q], j \in [p]\}$ $C = \{(\alpha_i + jp, iq) \mid i \in [q], j \in [p]\}$	5.4 (row 5)	row 5
(pq, pq, p)	$L_C = \langle p\mathbf{e}_1 \rangle$ $A = \{(i, \alpha_i + jq) + L_C \mid i \in [p], j \in [q]\}$ $B = \{(0, i + \alpha_i q) + L_C \mid i \in [q]\}$ $S_0 = \{(i + \alpha_{ij} p, \alpha_i + jq) \mid i \in [p], j \in [q]\}$	5.4 (row 6)	row 6

	$C = \{(jp, i + \alpha_i q) \mid i \in [q], j \in [p]\}$		
	$L_C = \langle q\mathbf{e}_2 \rangle$		
(pq, pq, q)	$A = \{(i + \alpha_i p, j) + L_C \mid i \in [p], j \in [q]\}$ $B = \{(ip, \alpha_i) + L_C \mid i \in [p]\}$	5.4 (row 5)	row 7
	$S_0 = \{(i + \alpha_i p, j + \alpha_{ij} q) \mid i \in [p], j \in [q]\}$		
	$C = \{(ip, \alpha_i + jq) \mid i \in [p], j \in [q]\}$		
	$L_C = \langle q\mathbf{e}_2 \rangle$		
(pq, pq, q)	$A = \{(\alpha_i + jp, i) + L_C \mid i \in [q], j \in [p]\}$ $B = \{(i + \alpha_i p, 0) + L_C \mid i \in [p]\}$	5.4 (row 6)	row 8
	$S_0 = \{(\alpha_i + jp, i + \alpha_{ij} q) \mid i \in [q], j \in [p]\}$		
	$C = \{(i + \alpha_i p, jq) \mid i \in [p], j \in [q]\}$		
(pq, pq, pq)	$C = \{(ip, jq) \mid i \in [p], j \in [q]\}$ $A = \{(i, j) + L_C \mid i \in [p], j \in [q]\}$		row 9
	$S_0 = \{(i + \alpha_{ij} p, j + \alpha_{ij} q) \mid i \in [p], j \in [q]\}$		
$(p^2 q, q, q)$	$C = \{(0, iq) \mid i \in [q]\}$ $A = \{(i, j) + L_C \mid i \in [p^2], j \in [q]\}$		row 14
	$S_0 = \{(i, j + \alpha_{ij} q) \mid i \in [p^2], j \in [q]\}$		
(pq^2, p, p)	$C = \{(ip, 0) \mid i \in [p]\}$ $A = \{(i, j) + L_C \mid i \in [p], j \in [q^2]\}$		row 19
	$S_0 = \{(i + \alpha_{ij} p, j) \mid i \in [p], j \in [q^2]\}$		

Case 2. S_0 is periodic.

In this case we have $|L_{S_0}| \geq 2$ and $S_0 = D \oplus L_{S_0}$ for some subset D of G by Lemma 2.3. So $|D| \cdot |L_{S_0}| = |S_0|$ and $1 < |L_{S_0}| < |S_0|$ by Lemma 2.8 part (b). Set

$$X = (D + L_{S_0})/L_{S_0} \quad \text{and} \quad Y = (C + L_{S_0})/L_{S_0}.$$

Since S_0 is periodic, by Lemma 2.5, we have

$$G/L_{S_0} = X \oplus Y$$

and X is aperiodic in G/L_{S_0} . We also know that X contains L_{S_0} and X generates G/L_{S_0} by Lemma 2.7. This factorization enables us to make use of Lemma 2.6 and previously established theorems to obtain the pair (S_0, C) via (X, Y) .

All possible values for $(|S_0|, |C|, |L_{S_0}|)$ when $|S_0|$ and $|C|$ are not relatively prime are given in the first column of Table 6.12. In the second column we list L_{S_0} , X , Y , S_0 , and C . The third column shows which previous theorem is used to obtain (X, Y) . The last column indicates which row in Table 6.10 the result corresponds to. $\square$

Table 6.12: The case when $G = \mathbb{Z}_{p^2} \times \mathbb{Z}_{q^2}$ and S_0 is periodic

(S_0 , C , L_{S_0})	$L_{S_0}, X, Y, S_0, \text{ and } C$	Reference	Result
	$L_{S_0} = \langle q\mathbf{e}_2 \rangle$		
	$X = \{(i + \alpha_i p, \alpha_i) + L_{S_0} \mid i \in [p]\}$		
(pq, pq, q)	$Y = \{(ip, j) + L_{S_0} \mid i \in [p], j \in [q]\}$	5.4 (row 1)	row 3

	$S_0 = \{(i + \alpha_i p, \alpha_i + jq) \mid i \in [p], j \in [q]\}$ $C = \{(ip, j + \alpha_{ij} q) \mid i \in [p], j \in [q]\}$		
	$L_{S_0} = \langle p\mathbf{e}_1 \rangle$ $X = \{(\alpha_i, i + \alpha_i q) + L_{S_0} \mid i \in [q]\}$ $Y = \{(i, jq) + L_{S_0} \mid i \in [p], j \in [q]\}$ $S_0 = \{(\alpha_i + jp, i + \alpha_i q) \mid i \in [q], j \in [p]\}$ $C = \{(i + \alpha_{ij} p, jq) \mid i \in [p], j \in [q]\}$	5.4 (row 1)	row 4
(pq, pq, p)			
	$L_{S_0} = \langle p\mathbf{e}_1 \rangle$ $X = \{(j, i + \alpha_{ij} q) + L_{S_0} \mid i \in [q], j \in [p]\}$ $Y = \{(0, iq) + L_{S_0} \mid i \in [q]\}$ $S_0 = \{(j + kp, i + \alpha_{ij} q) \mid i \in [q], j, k \in [p]\}$ $C = \{(p\alpha_i, iq) \mid i \in [q]\}$	5.4 (row 7)	row 10
$(p^2 q, q, p)$			
	$L_{S_0} = \langle q\mathbf{e}_2 \rangle$ $X = \{(i, \alpha_i) + L_{S_0} \mid i \in [p^2]\}$ $Y = \{(0, i) + L_{S_0} \mid i \in [q]\}$ $S_0 = \{(i, \alpha_i + jq) \mid i \in [p^2], j \in [q]\}$ $C = \{(0, i + \alpha_i q) \mid i \in [q]\}$	5.4 (row 4)	row 11
$(p^2 q, q, q)$			
	$L_{S_0} = \langle \mathbf{e}_1 \rangle$ $X = \{(0, i + \alpha_i q) + L_{S_0} \mid i \in [q]\}$ $Y = \{(0, iq) + L_{S_0} \mid i \in [q]\}$ $S_0 = \{(j, i + \alpha_i q) \mid i \in [q], j \in [p^2]\}$ $C = \{(\alpha_i, iq) \mid i \in [q]\}$	5.1	row 12
$(p^2 q, q, p^2)$			
	$L_{S_0} = \langle p\mathbf{e}_1 + q\mathbf{e}_2 \rangle$ $X = \{(i, \alpha_i) + L_{S_0} \mid i \in [p]\}$ $Y = \{(0, i) + L_{S_0} \mid i \in [q]\}$ $S_0 = \{(i + jp, \alpha_i + kq) \mid i, j \in [p], k \in [q]\}$ $C = \{(p\alpha_i, i + q\alpha_i) \mid i \in [q]\}$	3.5 (row 1)	row 13
$(p^2 q, q, pq)$			
	$L_{S_0} = \langle q\mathbf{e}_2 \rangle$ $X = \{(i + \alpha_{ij} p, j) + L_{S_0} \mid i \in [p], j \in [q]\}$ $Y = \{(ip, 0) + L_{S_0} \mid i \in [p]\}$ $S_0 = \{(i + \alpha_{ij} p, j + kq) \mid i \in [p], j, k \in [q]\}$ $C = \{(ip, \alpha_i q) \mid i \in [p]\}$	5.4 (row 7)	row 15
(pq^2, p, q)			
	$L_{S_0} = \langle p\mathbf{e}_1 \rangle$ $X = \{(\alpha_i, i) + L_{S_0} \mid i \in [q^2]\}$ $Y = \{(i, 0) + L_{S_0} \mid i \in [p]\}$ $S_0 = \{(\alpha_i + jp, i) \mid i \in [q^2], j \in [p]\}$ $C = \{(i + \alpha_i p, 0) \mid i \in [p]\}$	5.4 (row 4)	row 16
(pq^2, p, p)			
	$L_{S_0} = \langle \mathbf{e}_2 \rangle$ $X = \{(i + \alpha_i p, 0) + L_{S_0} \mid i \in [p]\}$ $Y = \{(ip, 0) + L_{S_0} \mid i \in [p]\}$ $S_0 = \{(i + \alpha_i p, j) \mid i \in [p], j \in [q^2]\}$ $C = \{(ip, \alpha_i) \mid i \in [p]\}$	5.1	row 17
(pq^2, p, q^2)			
	$L_{S_0} = \langle p\mathbf{e}_1 + q\mathbf{e}_2 \rangle$ $X = \{(\alpha_i, i) + L_{S_0} \mid i \in [q]\}$ $Y = \{(i, 0) + L_{S_0} \mid i \in [p]\}$ $S_0 = \{(\alpha_i + jp, i + kq) \mid i, k \in [q], j \in [p]\}$	3.5 (row 2)	row 18
(pq^2, p, pq)			

	$C = \{(i + p\alpha_i, q\alpha_i) \mid i \in [p]\}$		
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Theorem 6.8. Let $G = \mathbb{Z}_{p^2} \times \mathbb{Z}_q \times \mathbb{Z}_r$ for distinct primes p, q and r , and let S be a proper subset of $G \setminus \{0\}$ that generates G . Let $0 \in C \subseteq G$ and let $|C|$ be not relatively prime to $|S_0|$. Then C is a perfect code in $\text{Cay}(G, S)$ if and only if (S_0, C) is one of the pairs in Table 6.13.

Table 6.13: Perfect codes in Cayley graphs of $\mathbb{Z}_{p^2} \times \mathbb{Z}_q \times \mathbb{Z}_r$

Row	S_0	C
1	$\{(i + \alpha_i p, \alpha_i, \alpha_i) \mid i \in [p]\}$	$\{(ip, j, k) \mid i \in [p], j \in [q], k \in [r]\}$
2	$\{(\alpha_i + jp, i, \beta_i) \mid i \in [q], j \in [p]\}$	$\{(i + \alpha_{ij} p, 0, j) \mid i \in [p], j \in [r]\}$
3	$\{(i + \alpha_i p, j, \alpha_i) \mid i \in [p], j \in [q]\}$	$\{(ip, \alpha_{ij}, j) \mid i \in [p], j \in [r]\}$
4	$\{(j + \alpha_{ij} p, i, \alpha_i) \mid i \in [q], j \in [p]\}$	$\{(\alpha_i + jp, 0, i) \mid i \in [r], j \in [p]\}$
5	$\{(i + \alpha_{ij} p, j, \alpha_i) \mid i \in [p], j \in [q]\}$	$\{(jp, \alpha_i, i) \mid i \in [r], j \in [p]\}$
6	$\{(i + \alpha_i p, j, \alpha_{ij}) \mid i \in [p], j \in [q]\}$	$\{(ip, \alpha_i, j) \mid i \in [p], j \in [r]\}$
7	$\{(\alpha_i + jp, i, \alpha_{ij}) \mid i \in [q], j \in [p]\}$	$\{(i + \alpha_i p, 0, j) \mid i \in [p], j \in [r]\}$
8	$\{(i + \alpha_{ij} p, j, \beta_{ij}) \mid i \in [p], j \in [q]\}$	$\{(ip, 0, j) \mid i \in [p], j \in [r]\}$
9	$\{(\alpha_i + jp, \beta_i, i) \mid i \in [r], j \in [p]\}$	$\{(i + \alpha_{ij} p, j, 0) \mid i \in [p], j \in [q]\}$
10	$\{(i + \alpha_i p, \alpha_i, j) \mid i \in [p], j \in [r]\}$	$\{(ip, j, \alpha_{ij}) \mid i \in [p], j \in [q]\}$
11	$\{(j + \alpha_{ij} p, \alpha_i, i) \mid i \in [r], j \in [p]\}$	$\{(\alpha_i + jp, i, 0) \mid i \in [q], j \in [p]\}$
12	$\{(i + \alpha_{ij} p, \alpha_i, j) \mid i \in [p], j \in [r]\}$	$\{(jp, i, \alpha_i) \mid i \in [q], j \in [p]\}$
13	$\{(i + \alpha_i p, \alpha_{ij}, j) \mid i \in [p], j \in [r]\}$	$\{(ip, j, \alpha_i) \mid i \in [p], j \in [q]\}$
14	$\{(\alpha_i + jp, \alpha_{ij}, i) \mid i \in [r], j \in [p]\}$	$\{(i + \alpha_i p, j, 0) \mid i \in [p], j \in [q]\}$
15	$\{(i + \alpha_{ij} p, \beta_{ij}, j) \mid i \in [p], j \in [r]\}$	$\{(ip, j, 0) \mid i \in [p], j \in [q]\}$
16	$\{(\alpha_{ij} + kp, i, j) \mid i \in [q], j \in [r], k \in [p]\}$	$\{(i + \alpha_i p, 0, 0) \mid i \in [p]\}$
17	$\{(i + \alpha_{ij} p, k, j) \mid i \in [p], j \in [r], k \in [q]\}$	$\{(ip, \alpha_i, 0) \mid i \in [p]\}$
18	$\{(i + \alpha_{ij} p, j, k) \mid i \in [p], j \in [q], k \in [r]\}$	$\{(ip, 0, \alpha_i) \mid i \in [p]\}$
19	$\{(\alpha_i + pj, k, i) \mid i \in [r], j \in [p], k \in [q]\}$	$\{(i + \alpha_i p, \alpha_i, 0) \mid i \in [p]\}$
20	$\{(\alpha_i + jp, i, k) \mid i \in [q], j \in [p], k \in [r]\}$	$\{(i + \alpha_i p, 0, \alpha_i) \mid i \in [p]\}$
21	$\{(i + \alpha_i p, j, k) \mid i \in [p], j \in [q], k \in [r]\}$	$\{(ip, \alpha_i, \alpha_i) \mid i \in [p]\}$
22	$\{(i + \alpha_{ijk} p, j, k) \mid i \in [p], j \in [q], k \in [r]\}$	$\{(ip, 0, 0) \mid i \in [p]\}$

Proof. Suppose C is a perfect code in $\text{Cay}(G, S)$. Then $G = S_0 \oplus C$ and so $|S_0| \cdot |C| = |G| = p^2 qr$. Since $\text{Cay}(G, S)$ is a connected non-complete graph, $1 < |S_0| < p^2 qr$.

Case 1. S_0 is aperiodic.

Since G is Hajós and $G = S_0 \oplus C$, in this case C must be periodic, that is, L_C is nontrivial. Write $C = L_C \oplus D$ for some subset D of G whose existence is ensured by Lemma 2.3. Then $|L_C| \cdot |D| = |C|$ and $2 \leq |L_C| \leq |C|$. If $|S_0|$ is a prime, then $|L_C| = |C|$ by Lemma 2.8 part (c). If $|S_0|$ is not a prime, then $|L_C|$ can be any divisor of $|C|$. Set

$$A = (S_0 + L_C)/L_C \quad \text{and} \quad B = (D + L_C)/L_C.$$

Since C is periodic, by Lemma 2.5, we have

$$G/L_C = A \oplus B$$

and B is aperiodic in G/L_C . Since G/L_C is isomorphic to a subgroup of G and subgroups of Hajós group are also Hajós groups, A is periodic in G/L_C . We also know that A contains L_C and A generates G/L_C by Lemma 2.7. Furthermore, $|A| = |S_0|$ and $|B| = |D|$. Note that if $|L_C| = |C|$, then $C = L_C$ as $0 \in C$. This means we can take $D = \{0\}$ and hence $B = L_C/L_C$ and $G/L_C = A$. We use C to get A and then use Lemma 2.6 to obtain S_0 . If $|L_C| < |C|$, then this factorization enables us to make use of previously established theorems to obtain the pair (S_0, C) via (A, B) .

As in Table 6.11, the first column of Table 6.14 is $(|S_0|, |C|, |L_C|)$ when $|S_0|$ and $|C|$ are not relatively prime. The remaining columns contain C , A , and S_0 when $|L_C| = |C|$ or L_C , A , B , S_0 , and C when $|L_C| < |C|$; which previous theorem is used to obtain (A, B) when $|L_C| < |C|$; and which row in Table 6.13 the result corresponds to.

Table 6.14: The case when $G = \mathbb{Z}_{p^2} \times \mathbb{Z}_q \times \mathbb{Z}_r$ and S_0 is aperiodic

(S_0 , C , L_C)	L_C, A, B, S_0 , and C	Reference	Result
(p, pqr, pqr)	$C = \{(ip, j, k) \mid i \in [p], j \in [q], k \in [r]\}$ $A = \{(i, 0, 0) + L_C \mid i \in [p]\}$ $S_0 = \{(i + \alpha_i p, \alpha_i, \alpha_i) \mid i \in [p]\}$		row 1
(pq, pr, p)	$L_C = \langle p\mathbf{e}_1 \rangle$ $A = \{(j, i, \alpha_i) + L_C \mid i \in [q], j \in [p]\}$ $B = \{(\alpha_i, 0, i) + L_C \mid i \in [r]\}$ $S_0 = \{(j + \alpha_{ij} p, i, \alpha_i) \mid i \in [q], j \in [p]\}$ $C = \{(\alpha_i + jp, 0, i) \mid i \in [r], j \in [p]\}$	6.1 (row 4)	row 4
(pq, pr, p)	$L_C = \langle p\mathbf{e}_1 \rangle$ $A = \{(i, j, \alpha_i) + L_C \mid i \in [p], j \in [q]\}$ $B = \{(0, \alpha_i, i) + L_C \mid i \in [r]\}$ $S_0 = \{(i + \alpha_{ij} p, j, \alpha_i) \mid i \in [p], j \in [q]\}$ $C = \{(jp, \alpha_i, i) \mid i \in [r], j \in [p]\}$	6.1 (row 5)	row 5
(pq, pr, r)	$L_C = \langle \mathbf{e}_3 \rangle$ $A = \{(i + \alpha_i p, j, 0) + L_C \mid i \in [p], j \in [q]\}$ $B = \{(ip, \alpha_i, 0) + L_C \mid i \in [p]\}$ $S_0 = \{(i + \alpha_i p, j, \alpha_{ij}) \mid i \in [p], j \in [q]\}$ $C = \{(ip, \alpha_i, j) \mid i \in [p], j \in [r]\}$	5.4 (row 5)	row 6
(pq, pr, r)	$L_C = \langle \mathbf{e}_3 \rangle$ $A = \{(\alpha_i + jp, i, 0) + L_C \mid i \in [q], j \in [p]\}$ $B = \{(i + \alpha_i p, 0, 0) + L_C \mid i \in [p]\}$ $S_0 = \{(\alpha_i + jp, i, \alpha_{ij}) \mid i \in [q], j \in [p]\}$ $C = \{(i + \alpha_i p, 0, j) \mid i \in [p], j \in [r]\}$	5.4 (row 6)	row 7
(pq, pr, pr)	$C = \{(ip, 0, j) \mid i \in [p], j \in [r]\}$ $A = \{(i, j, 0) + L_C \mid i \in [p], j \in [q]\}$ $S_0 = \{(i + \alpha_{ij} p, j, \beta_{ij}) \mid i \in [p], j \in [q]\}$		row 8
(pr, pq, p)	$L_C = \langle p\mathbf{e}_1 \rangle$ $A = \{(j, \alpha_i, i) + L_C \mid i \in [r], j \in [p]\}$ $B = \{(\alpha_i, i, 0) + L_C \mid i \in [q]\}$ $S_0 = \{(j + \alpha_{ij} p, \alpha_i, i) \mid i \in [r], j \in [p]\}$ $C = \{(\alpha_i + jp, i, 0) \mid i \in [q], j \in [p]\}$	6.1 (row 6)	row 11
(pr, pq, p)	$L_C = \langle p\mathbf{e}_1 \rangle$ $A = \{(i, \alpha_i, j) + L_C \mid i \in [p], j \in [r]\}$ $B = \{(0, i, \alpha_i) + L_C \mid i \in [q]\}$ $S_0 = \{(i + \alpha_{ij} p, \alpha_i, j) \mid i \in [p], j \in [r]\}$	6.1 (row 7)	row 12

	$C = \{(jp, i, \alpha_i) \mid i \in [q], j \in [p]\}$		
	$L_C = \langle \mathbf{e}_2 \rangle$		
(pr, pq, q)	$A = \{(i + \alpha_i p, 0, j) + L_C \mid i \in [p], j \in [r]\}$ $B = \{(ip, 0, \alpha_i) + L_C \mid i \in [p]\}$ $S_0 = \{(i + \alpha_i p, \alpha_{ij}, j) \mid i \in [p], j \in [r]\}$ $C = \{(ip, j, \alpha_i) \mid i \in [p], j \in [q]\}$	5.4 (row 5)	row 13
	$L_C = \langle \mathbf{e}_2 \rangle$		
(pr, pq, q)	$A = \{(\alpha_i + jp, 0, i) + L_C \mid i \in [r], j \in [p]\}$ $B = \{(i + \alpha_i p, 0, 0) + L_C \mid i \in [p]\}$ $S_0 = \{(\alpha_i + jp, \alpha_{ij}, i) \mid i \in [r], j \in [p]\}$ $C = \{(i + \alpha_i p, j, 0) \mid i \in [p], j \in [q]\}$	5.4 (row 6)	row 14
(pr, pq, pq)	$C = \{(ip, j, 0) \mid i \in [p], j \in [q]\}$ $A = \{(i, 0, j) + L_C \mid i \in [p], j \in [r]\}$ $S_0 = \{(i + \alpha_{ij} p, \beta_{ij}, j) \mid i \in [p], j \in [r]\}$		row 15
(pqr, p, p)	$C = \{(ip, 0, 0) \mid i \in [p]\}$ $A = \{(i, j, k) + L_C \mid i \in [p], j \in [q], k \in [r]\}$ $S_0 = \{(i + \alpha_{ijk} p, j, k) \mid i \in [p], j \in [q], k \in [r]\}$		row 22

Case 2. S_0 is periodic.

In this case we have $|L_{S_0}| \geq 2$ and $S_0 = D \oplus L_{S_0}$ for some subset D of G by Lemma 2.3. So $|D| \cdot |L_{S_0}| = |S_0|$ and $1 < |L_{S_0}| < |S_0|$ by Lemma 2.8 part (b). Set

$$X = (D + L_{S_0})/L_{S_0} \quad \text{and} \quad Y = (C + L_{S_0})/L_{S_0}.$$

Since S_0 is periodic, by Lemma 2.5, we have

$$G/L_{S_0} = X \oplus Y$$

and X is aperiodic in G/L_{S_0} . We also know that X contains L_{S_0} and X generates G/L_{S_0} by Lemma 2.7. This factorization enables us to make use of Lemma 2.6 and previously established theorems to obtain the pair (S_0, C) via (X, Y) .

As in Table 6.12, the first column of Table 6.15 contains all possible values for $(|S_0|, |C|, |L_{S_0}|)$ when $|S_0|$ and $|C|$ are not relatively prime. The next three columns are L_{S_0} , X , Y , S_0 , and C ; which previous theorem is used to obtain (X, Y) ; and which row in Table 6.13 the result corresponds to. $\square$

Table 6.15: The case when $G = \mathbb{Z}_{p^2} \times \mathbb{Z}_q \times \mathbb{Z}_r$ and S_0 is periodic

(S_0 , C , L_{S_0})	$L_{S_0}, X, Y, S_0, \text{ and } C$	Reference	Result
	$L_{S_0} = \langle p\mathbf{e}_1 \rangle$		
(pq, pr, p)	$X = \{(\alpha_i, i, \beta_i) + L_{S_0} \mid i \in [q]\}$ $Y = \{(i, 0, j) + L_{S_0} \mid i \in [p], j \in [r]\}$ $S_0 = \{(\alpha_i + jp, i, \beta_i) \mid i \in [q], j \in [p]\}$ $C = \{(i + \alpha_{ij} p, 0, j) \mid i \in [p], j \in [r]\}$	6.1 (row 2)	row 2
	$L_{S_0} = \langle \mathbf{e}_2 \rangle$		
(pq, pr, q)	$X = \{(i + \alpha_i p, 0, \alpha_i) + L_{S_0} \mid i \in [p]\}$ $Y = \{(ip, 0, j) + L_{S_0} \mid i \in [p], j \in [r]\}$ $S_0 = \{(i + \alpha_i p, j, \alpha_i) \mid i \in [p], j \in [q]\}$	5.4 (row 1)	row 3

	$C = \{(ip, \alpha_{ij}, j) \mid i \in [p], j \in [r]\}$		
(pr, pq, p)	$L_{S_0} = \langle p\mathbf{e}_1 \rangle$ $X = \{(\alpha_i, \beta_i, i) + L_{S_0} \mid i \in [r]\}$ $Y = \{(i, j, 0) + L_{S_0} \mid i \in [p], j \in [q]\}$ $S_0 = \{(\alpha_i + jp, \beta_i, i) \mid i \in [r], j \in [p]\}$ $C = \{(i + \alpha_{ij}p, j, 0) \mid i \in [p], j \in [q]\}$	6.1 (row 3)	row 9
(pr, pq, r)	$L_{S_0} = \langle \mathbf{e}_3 \rangle$ $X = \{(i + \alpha_{ij}p, \alpha_i, 0) + L_{S_0} \mid i \in [p]\}$ $Y = \{(ip, j, 0) + L_{S_0} \mid i \in [p], j \in [q]\}$ $S_0 = \{(i + \alpha_{ij}p, \alpha_i, j) \mid i \in [p], j \in [r]\}$ $C = \{(ip, j, \alpha_{ij}) \mid i \in [p], j \in [q]\}$	5.4 (row 1)	row 10
(pqr, p, p)	$L_{S_0} = \langle p\mathbf{e}_1 \rangle$ $X = \{(\alpha_{ij}, i, j) + L_{S_0} \mid i \in [q], j \in [r]\}$ $Y = \{(i, 0, 0) + L_{S_0} \mid i \in [p]\}$ $S_0 = \{(\alpha_{ij} + kp, i, j) \mid i \in [q], j \in [r], k \in [p]\}$ $C = \{(i + \alpha_{ij}p, 0, 0) \mid i \in [p]\}$	6.1 (row 12)	row 16
(pqr, p, q)	$L_{S_0} = \langle \mathbf{e}_2 \rangle$ $X = \{(i + \alpha_{ij}p, 0, j) + L_{S_0} \mid i \in [p], j \in [r]\}$ $Y = \{(ip, 0, 0) + L_{S_0} \mid i \in [p]\}$ $S_0 = \{(i + \alpha_{ij}p, k, j) \mid i \in [p], j \in [r], k \in [q]\}$ $C = \{(ip, \alpha_i, 0) \mid i \in [p]\}$	5.4 (row 7)	row 17
(pqr, p, r)	$L_{S_0} = \langle \mathbf{e}_3 \rangle$ $X = \{(i + \alpha_{ij}p, j, 0) + L_{S_0} \mid i \in [p], j \in [q]\}$ $Y = \{(ip, 0, 0) + L_{S_0} \mid i \in [p]\}$ $S_0 = \{(i + \alpha_{ij}p, j, k) \mid i \in [p], j \in [q], k \in [r]\}$ $C = \{(ip, 0, \alpha_i) \mid i \in [p]\}$	5.4 (row 7)	row 18
(pqr, p, pq)	$L_{S_0} = \langle p\mathbf{e}_1 + \mathbf{e}_2 \rangle$ $X = \{(\alpha_i, 0, i) + L_{S_0} \mid i \in [r]\}$ $Y = \{(i, 0, 0) + L_{S_0} \mid i \in [p]\}$ $S_0 = \{(\alpha_i + pj, k, i) \mid i \in [r], j \in [p], k \in [q]\}$ $C = \{(i + \alpha_{ij}p, \alpha_i, 0) \mid i \in [p]\}$	3.5 (row 2)	row 19
(pqr, p, pr)	$L_{S_0} = \langle p\mathbf{e}_1 + \mathbf{e}_3 \rangle$ $X = \{(\alpha_i, i, 0) + L_{S_0} \mid i \in [q]\}$ $Y = \{(i, 0, 0) + L_{S_0} \mid i \in [p]\}$ $S_0 = \{(\alpha_i + jp, i, k) \mid i \in [q], j \in [p], k \in [r]\}$ $C = \{(i + \alpha_{ij}p, 0, \alpha_i) \mid i \in [p]\}$	3.5 (row 2)	row 20
(pqr, p, qr)	$L_{S_0} = \langle \mathbf{e}_2 + \mathbf{e}_3 \rangle$ $X = \{(i + \alpha_{ij}p, 0, 0) + L_{S_0} \mid i \in [p]\}$ $Y = \{(ip, 0, 0) + L_{S_0} \mid i \in [p]\}$ $S_0 = \{(i + \alpha_{ij}p, j, k) \mid i \in [p], j \in [q], k \in [r]\}$ $C = \{(ip, \alpha_i, \alpha_i) \mid i \in [p]\}$	5.1	row 21

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