

Hypergraph Turán problem of the generalized triangle with bounded matching number

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Abstract

Let $\mathcal{H}$ be a 3-graph on n vertices. The matching number $\nu(\mathcal{H})$ is defined as the maximum number of disjoint edges in $\mathcal{H}$. The generalized triangle F_5 is a 3-graph on the vertex set $\{a, b, c, d, e\}$ with the edge set $\{abc, abd, cde\}$. In this paper, we determine the maximum number of edges in an F_5 -free 3-graph $\mathcal{H}$ satisfying $\nu(\mathcal{H}) \leq s$ when $n \geq 30(s+1)$. More precisely, the maximum is $\binom{n-1}{2}$ for $s = 1, 2$ and is $s \lfloor (n-s)^2/4 \rfloor$ for $s \geq 3$. We also determine the unique extremal graph for each case. For the proof, we establish a 2-colored version of Mantel's theorem, which may be of independent interest.

Mathematics Subject Classifications: 05C65, 05C35

1 Introduction

For $r \geq 2$, an r -graph $\mathcal{H} = (V(\mathcal{H}), E(\mathcal{H}))$ consists of a vertex set $V(\mathcal{H})$ and an edge set $E(\mathcal{H})$, where $E(\mathcal{H})$ is a collection of r -element subsets of $V(\mathcal{H})$. For $r \geq 3$, we often identify $\mathcal{H}$ with $E(\mathcal{H})$. Given a family $\mathcal{F}$ of r -uniform hypergraphs, an r -graph $\mathcal{H}$ is called $\mathcal{F}$ -free if it contains no member of $\mathcal{F}$ as a subgraph.

The Turán number $\text{ex}_r(n, \mathcal{F})$ is defined as the maximum number of edges in an $\mathcal{F}$ -free r -graph on n vertices. Let $T_r(n, \ell)$ denote the *generalized Turán graph*, which is the complete ℓ -partite r -graph on n vertices with ℓ partition classes, with each of size either $\lfloor \frac{n}{\ell} \rfloor$ or $\lceil \frac{n}{\ell} \rceil$. Denote by $t_r(n, \ell)$ the number of edges in $T_r(n, \ell)$. The classical Turán theorem [16] states that for $\ell \geq 2$, $\text{ex}_2(n, K_{\ell+1})$ is uniquely achieved by $T_2(n, \ell)$.

A *matching* is a collection of vertex-pairwise disjoint edges, and its size is the number of its edges. For an r -graph $\mathcal{H}$, the *matching number* $\nu(\mathcal{H})$ is defined as the maximum size of a matching in $\mathcal{H}$. Let M_{s+1}^r denote a matching of size $s+1$ with each edge being an r -set. The study of Turán problems under a matching-number constraint begins with

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the theorem of Erdős and Gallai [3], who proved in 1959 that for $n \geq 2s + 1$,

$$\text{ex}_2(n, M_{s+1}^2) = \max \left\{ \binom{2s+1}{2}, \binom{s}{2} + s(n-s) \right\}.$$

Using the shifting technique, Akiyama and Frankl [1] gave a short proof of the above result. An almost identical proof yields a rainbow extension that will be used later in this paper.

Recently, Alon and Frankl [2] determined the maximum number of edges in a graph with bounded clique number and bounded matching number. They proved that for $n \geq 2s + 1$ and $\ell \geq 2$,

$$\text{ex}_2(n, \{K_{\ell+1}, M_{s+1}^2\}) = \max\{t_2(2s+1, \ell), g(n, \ell, s)\},$$

where $g(n, \ell, s)$ denotes the number of edges in the complete ℓ -partite graph $G(n, \ell, s)$ on n vertices, which has one part of size $n - s$ and the other $\ell - 1$ parts inducing a copy of $T_2(s, \ell - 1)$. The same result for the range $n > 3s$ was independently obtained in [9].

A *weakly independent set* of an r -graph $\mathcal{H}$ is a subset of $V(\mathcal{H})$ that does not contain any edge of $\mathcal{H}$. A proper k -coloring of $\mathcal{H}$ is a mapping from $V(\mathcal{H})$ to a set of k colors such that no edge of $\mathcal{H}$ is monochromatic. The *chromatic number* $\chi(\mathcal{H})$ is defined as the minimum number of colors needed for a proper coloring of $\mathcal{H}$.

More recently, Gerbner, Tompkins and Zhou [10] considered analogous Turán problems for hypergraphs with bounded matching number. In particular, they proved that if $\chi(F) > 2$ and n is sufficiently large, then

$$\text{ex}_r(n, \{F, M_{s+1}^r\}) = \text{ex}_r(s, \mathcal{F}) + \sum_{1 \leq i \leq t} \binom{s}{i} \binom{n-s}{r-i},$$

where $t = \min\{s, r-1\}$ and $\mathcal{F}$ denotes the family of r -graphs obtained from F by deleting a weakly independent set.

The generalized triangle F_5 is a 3-graph on the vertex set $\{a, b, c, d, e\}$ with the edge set $\{abc, abd, cde\}$. It is easy to check that $\chi(F_5) = 2$. Although F_5 is 2-colorable, it has positive Turán density. Frankl and Füredi [8] proved that, for $n \geq 3000$,

$$\text{ex}_3(n, F_5) = \left\lfloor \frac{n}{3} \right\rfloor \left\lfloor \frac{n+1}{3} \right\rfloor \left\lfloor \frac{n+2}{3} \right\rfloor.$$

Later, the condition $n \geq 3000$ was improved to $n \geq 33$ by Keevash and Mubayi [13].

Let $\mathcal{H}_3(n, s)$ denote the complete 3-partite 3-graph on n vertices with vertex partition $X \cup Y \cup Z$, where $|X| = s$, $|Y| = \lfloor (n-s)/2 \rfloor$, and $|Z| = \lceil (n-s)/2 \rceil$. Thus every edge has one vertex in each part and

$$|\mathcal{H}_3(n, s)| = s \left\lfloor \frac{(n-s)^2}{4} \right\rfloor.$$

It is easy to check that $\mathcal{H}_3(n, s)$ is F_5 -free and has matching number at most s . If two edges share two vertices, then their two remaining vertices lie in the same part, so no third edge

can contain both of them. Moreover, every edge meets X , and hence $\nu(\mathcal{H}_3(n, s)) \leq s$. For $x \in [n]$, define the *full star of center x* as the 3-graph $\mathcal{S}_x = \{E \in \binom{[n]}{3} : x \in E\}$. Clearly, $\mathcal{S}_x$ is $\{F_5, M_2^3\}$ -free with $|\mathcal{S}_x| = \binom{n-1}{2}$.

In this paper, we show that $\mathcal{H}_3(n, s)$ attains the maximum number of edges among all n -vertex F_5 -free 3-graphs with matching number s for $n \geq 30(s+1)$ and $s \geq 3$.

Theorem 1. For $n \geq 30(s+1)$,

$$\text{ex}_3(n, \{F_5, M_{s+1}^3\}) = \begin{cases} \binom{n-1}{2}, & \text{if } s = 1, 2, \\ s \left\lfloor \frac{(n-s)^2}{4} \right\rfloor, & \text{if } s \geq 3. \end{cases}$$

Moreover, for $s = 1, 2$ the full star is the unique $\{F_5, M_{s+1}^3\}$ -free 3-graph attaining the maximum size and for $s \geq 3$, $\mathcal{H}_3(n, s)$ is the unique $\{F_5, M_{s+1}^3\}$ -free 3-graph attaining the maximum size up to isomorphism.

Let $G_1, G_2, \dots, G_s$ be simple graphs on a common vertex set V . We write $G_1 + \dots + G_s$ for the *edge-colored multigraph* on V whose edges are the colored copies (xy, i) with $xy \in E(G_i)$, where the copy (xy, i) has color i . Thus, if an edge xy belongs to several of the graphs G_i , then $G_1 + \dots + G_p$ contains several copies of xy , one in each of the corresponding colors. An edge-colored copy of a graph H is called *rainbow* if its selected edge copies have pairwise distinct colors. We say that the edge-colored multigraph $G_1 + \dots + G_p$ is *rainbow H -free* if it contains no rainbow copy of H .

Frankl [5] proved the following rainbow version of Mantel's theorem. If $p \geq 3$, $G_1, \dots, G_p \subset \binom{[n]}{2}$, and the edge-colored multigraph $G_1 + \dots + G_p$ is rainbow triangle-free, then

$$\sum_{i=1}^p |E(G_i)| \leq \max \left\{ 2 \binom{n}{2}, p \left\lfloor \frac{n^2}{4} \right\rfloor \right\}.$$

Related multicolor Turán problems have been studied systematically in [14, 15, 12, 7].

A triangle in $G_1 + \dots + G_p$ is called *2-colored* if its three selected edge copies use exactly two colors. We say that the edge-colored multigraph $G_1 + \dots + G_p$ is *2-colored triangle-free* if it contains no such triangle.

For example, if $xy, xz \in E(G_1)$ and $yz \in E(G_2)$, then the three colored edge copies $(xy, 1), (xz, 1), (yz, 2)$ form a 2-colored triangle in $G_1 + \dots + G_p$. On the other hand, a monochromatic triangle contained entirely in G_1 is not 2-colored. If an edge belongs to several graphs G_i , then each of its colored copies is available when selecting a colored triangle.

As a key ingredient in the proof of Theorem 1, we prove the following 2-colored version of Mantel's theorem.

Theorem 2. Let $p \geq 2$, and let $G_1, \dots, G_p \subset \binom{[n]}{2}$. If the edge-colored multigraph $G_1 + \dots + G_p$ is 2-colored triangle-free, then

$$\sum_{i=1}^p |E(G_i)| \leq p \left\lfloor \frac{n^2}{4} \right\rfloor.$$

Equality holds if and only if there is a common bipartition of $[n]$ into parts of sizes $\lfloor n/2 \rfloor$ and $\lceil n/2 \rceil$ such that

$$G_1 = \cdots = G_p = K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}.$$

Note that both Frankl's rainbow version of Mantel's theorem and Theorem 2 imply Mantel's Theorem by setting $G_1 = G_2 = \cdots = G_p$.

The relevance of Theorem 2 to the proof of Theorem 1 is as follows. Given an F_5 -free hypergraph, we select a small set of vertices and consider their link graphs on a common complementary vertex set. A 2-colored triangle in these link graphs would lead to a copy of F_5 . Thus, Theorem 2 provides a sharp upper bound on the sum of the corresponding vertex degrees. The other main ingredient in the proof of Theorem 1 is a degree-decomposition Lemma, which converts the matching-number constraint into the existence of a small set of high-degree vertices. Combining these two ingredients with case analysis and detailed structural analysis, we prove Theorem 1.

The paper is organized as follows. In Section 2, we first introduce some notation and auxiliary lemmas, then prove Theorem 2, and finally establish the degree-decomposition lemma. In Section 3, we prove Theorem 1 for $s \leq 2$. In Section 4, we prove it for $s \geq 3$. In Section 5, we conclude with two open problems.

2 Preliminaries

2.1 The 2-colored Mantel's theorem

Lemma 3. *Let $p \geq 2$. Let $G_1, \dots, G_p \subset \binom{[n]}{2}$ and $G_1 + \cdots + G_p$ be 2-colored triangle-free. If $xy \in E(G_1)$ and $N_{G_1}(x) \cap N_{G_1}(y) \neq \emptyset$, then for $2 \leq i \leq p$,*

$$\deg_{G_i}(x) + \deg_{G_i}(y) + \deg_{G_1}(x) + \deg_{G_1}(y) \leq 2n - 2. \quad (1)$$

Proof. Let $z \in N_{G_1}(x) \cap N_{G_1}(y)$. Then xyz forms a triangle in G_1 . Since $G_1 + \cdots + G_p$ is 2-colored triangle-free, $xy \notin E(G_i)$. If $w \in N_{G_1}(x) \cap N_{G_i}(y)$, then $xy, xw \in E(G_1)$ and $yw \in E(G_i)$. That is, xyw forms a 2-colored triangle, a contradiction. Thus $N_{G_1}(x) \cap N_{G_i}(y) = \emptyset$. Similarly, we have $N_{G_i}(x) \cap N_{G_1}(y) = \emptyset$. Note that $x \notin N_{G_1}(x) \cup N_{G_i}(y)$ and $y \notin N_{G_i}(x) \cup N_{G_1}(y)$. It follows that

$$\deg_{G_1}(x) + \deg_{G_i}(y) \leq n - 1 \text{ and } \deg_{G_i}(x) + \deg_{G_1}(y) \leq n - 1.$$

Thus (1) follows. □

Lemma 4. *For every graph G ,*

$$\sum_{v \in V(G)} \deg_G(v)^2 \geq \frac{4|E(G)|^2}{|V(G)|}. \quad (2)$$

Proof. By the Cauchy–Schwarz inequality,

$$\sum_{v \in V(G)} \deg_G(v)^2 \geq \frac{\left(\sum_{v \in V(G)} \deg_G(v)\right)^2}{|V(G)|} = \frac{4|E(G)|^2}{|V(G)|}. \quad \square$$

Proof of Theorem 2. We prove the theorem by induction on n . For $n = 1$ and $n = 2$, the theorem holds trivially. Let $n = 3$. If some G_i is a triangle, then every G_j with $j \neq i$ is empty, because any edge of G_j together with the other two edges of G_i would form a 2-colored triangle. Hence the total number of edges is $3 < 2p$. Otherwise, every G_i has at most two edges, so the total is at most $2p = p\lfloor 3^2/4 \rfloor$. Equality forces every G_i to have two edges. If two of these graphs omitted different edges of the triangle, then the two edges of one graph together with the missing edge supplied by the other would form a 2-colored triangle. Thus all G_i are the same copy of $K_{1,2}$, as required.

Now assume that the theorem holds for $1, 2, \dots, n-1$ and we prove it for n . We distinguish two cases.

Case 1. $\deg_{G_i}(x) + \deg_{G_i}(y) \leq n$ for all $xy \in E(G_i)$ and $i \in [p]$.

Note that by (2),

$$\sum_{xy \in E(G_i)} (\deg_{G_i}(x) + \deg_{G_i}(y)) = \sum_{x \in V(G_i)} \deg_{G_i}^2(x) \geq \frac{4|E(G_i)|^2}{n}. \quad (3)$$

It follows that

$$\frac{4|E(G_i)|^2}{n} \leq \sum_{xy \in E(G_i)} (\deg_{G_i}(x) + \deg_{G_i}(y)) \leq n \cdot |E(G_i)|.$$

Thus $|E(G_i)| \leq \lfloor \frac{n^2}{4} \rfloor$ for each $i \in [p]$ and $\sum_{i=1}^p |E(G_i)| \leq p \cdot \lfloor \frac{n^2}{4} \rfloor$ follows.

It remains to characterize the equality case. Suppose that

$$|E(G_i)| = \left\lfloor \frac{n^2}{4} \right\rfloor \quad \text{for every } i \in [p].$$

We first consider the case $n = 2k$. For every $i \in [p]$, the assumption of Case 1 gives

$$\sum_{v \in [n]} \deg_{G_i}(v)^2 = \sum_{xy \in E(G_i)} (\deg_{G_i}(x) + \deg_{G_i}(y)) \leq n|E(G_i)| = 2k^3.$$

On the other hand, by the Cauchy–Schwarz inequality,

$$\sum_{v \in [n]} \deg_{G_i}(v)^2 \geq \frac{(\sum_{v \in [n]} \deg_{G_i}(v))^2}{n} = \frac{(2|E(G_i)|)^2}{n} = 2k^3.$$

Hence equality holds in the Cauchy–Schwarz inequality, and therefore every G_i is k -regular.

We claim that every G_i is triangle-free. Suppose, to the contrary, that xyz is a triangle in G_i . For any $j \neq i$, applying (1) to the edge $xy \in E(G_i)$ and its common neighbor z gives

$$\deg_{G_i}(x) + \deg_{G_i}(y) + \deg_{G_j}(x) + \deg_{G_j}(y) \leq 2n - 2.$$

Since both G_i and G_j are k -regular, the left-hand side is equal to $4k = 2n$, a contradiction. Thus every G_i is triangle-free. By the equality case of Mantel's theorem, $G_i = K_{k,k}$ for every $i \in [p]$.

We next consider the case $n = 2k + 1$. In this case,

$$|E(G_i)| = k(k+1) \quad \text{and} \quad \sum_{v \in [n]} \deg_{G_i}(v) = 2k(k+1).$$

Since all degrees are integers, the convexity of the function $x \mapsto x^2$ implies that

$$\sum_{v \in [n]} \deg_{G_i}(v)^2 \geq k(k+1)^2 + (k+1)k^2 = (2k+1)k(k+1) = n|E(G_i)|.$$

Together with

$$\sum_{v \in [n]} \deg_{G_i}(v)^2 = \sum_{xy \in E(G_i)} (\deg_{G_i}(x) + \deg_{G_i}(y)) \leq n|E(G_i)|,$$

this shows that equality holds throughout. Consequently, exactly k vertices of G_i have degree $k+1$, and the remaining $k+1$ vertices have degree k . Moreover,

$$\deg_{G_i}(x) + \deg_{G_i}(y) = n \quad \text{for every } xy \in E(G_i).$$

It follows that every edge of G_i joins a vertex of degree $k+1$ to a vertex of degree k . Since there are $k(k+1)$ such possible edges and $|E(G_i)| = k(k+1)$, we obtain $G_i = K_{k,k+1}$ for every $i \in [p]$.

Finally, the balanced bipartitions of the graphs $G_1, \dots, G_p$ must coincide. Indeed, suppose that the bipartitions of G_i and G_j are different, even after interchanging their two parts. Then some part of the bipartition of G_i contains two vertices y, z lying in different parts of the bipartition of G_j . Choosing a vertex x in the other part of the bipartition of G_i , we have $xy, xz \in E(G_i)$ and $yz \in E(G_j)$. Thus xyz is a 2-colored triangle, a contradiction. Therefore all the graphs have the same balanced bipartition, and hence $G_1 = \dots = G_p = K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$.

Case 2. There exist $i \in [p]$ and $xy \in E(G_i)$ such that $\deg_{G_i}(x) + \deg_{G_i}(y) > n$.

Assume that $\deg_{G_i}(x) + \deg_{G_i}(y) = n + t$, where $t \geq 1$. The two vertices have a common neighbor in G_i , so by (1), for every $\ell \neq i$,

$$xy \notin E(G_\ell) \quad \text{and} \quad \deg_{G_\ell}(x) + \deg_{G_\ell}(y) \leq 2n - 2 - n - t = n - t - 2.$$

It follows that for $p \geq 2$

$$\sum_{\ell=1}^p (\deg_{G_\ell}(x) + \deg_{G_\ell}(y)) \leq n + t + (p-1)(n - t - 2) \leq p(n-1).$$

After deleting x and y , the resulting edge-colored multigraph is still 2-colored triangle-free. Since xy occurs only in G_i , the induction hypothesis gives

$$\begin{aligned}\sum_{\ell=1}^p |E(G_\ell)| &= \sum_{\ell=1}^p |E(G_\ell - \{x, y\})| + \sum_{\ell=1}^p (\deg_{G_\ell}(x) + \deg_{G_\ell}(y)) - 1 \\ &\leq p \left\lfloor \frac{(n-2)^2}{4} \right\rfloor + p(n-1) - 1 = p \left\lfloor \frac{n^2}{4} \right\rfloor - 1.\end{aligned}\quad \square$$

Corollary 5. *Let $p \geq 2$. Let $G_1, \dots, G_p \subset \binom{[n]}{2}$ and $G_1 + \dots + G_p$ be 2-colored triangle-free. Then*

$$\prod_{i=1}^p |E(G_i)| \leq \left\lfloor \frac{n^2}{4} \right\rfloor^p.$$

Moreover, the equality holds if and only if $G_1 = \dots = G_p = K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$.

Proof. By applying the AM–GM inequality to Theorem 2, we obtain

$$\prod_{i=1}^p |E(G_i)| \leq \left(\frac{\sum_{i=1}^p |E(G_i)|}{p} \right)^p \leq \left\lfloor \frac{n^2}{4} \right\rfloor^p.$$

The equality characterization follows directly from that of Theorem 2. $\square$

Remark. Let us mention that in [5] Frankl conjectured that if $G_1, G_2, G_3 \subset \binom{[n]}{2}$ and $G_1 + G_2 + G_3$ is rainbow triangle-free, then

$$|E(G_1)| |E(G_2)| |E(G_3)| \leq \left\lfloor \frac{n^2}{4} \right\rfloor^3.$$

However, it was disproved by Frankl, Győri, He, Lv, Salia, Tompkins, Varga and Zhu [7].

2.2 The degree-decomposition lemma

Let us recall some definitions and notations. Let $\mathcal{H}$ be a 3-graph and let $u \in V(\mathcal{H})$. The *link graph* $L_{\mathcal{H}}(u)$ is defined as

$$L_{\mathcal{H}}(u) = \left\{ vw \in \binom{V(\mathcal{H})}{2} : uvw \in \mathcal{H} \right\}.$$

The *degree* $\deg_{\mathcal{H}}(u)$ of u is defined as the number of edges in $L_{\mathcal{H}}(u)$. For $W \subset V(\mathcal{H})$, we also use

$$L_{\mathcal{H}}(u, W) = \{vw \in L_{\mathcal{H}}(u) : \{v, w\} \subset W\}$$

and $\deg_{\mathcal{H}}(u, W) = |L_{\mathcal{H}}(u, W)|$. Let $N_{\mathcal{H}}(v)$ denote the set of vertices u such that uv is covered by an edge of $\mathcal{H}$. For $\{u, v\} \subset V(\mathcal{H})$, the *neighborhood* $N_{\mathcal{H}}(u, v)$ is defined as

$$N_{\mathcal{H}}(u, v) = \{w \in V(\mathcal{H}) : uvw \in \mathcal{H}\}$$

and $\deg_{\mathcal{H}}(u, v) = |N_{\mathcal{H}}(u, v)|$. When the context is clear, we often omit the subscript $\mathcal{H}$.

For $S \subseteq V(\mathcal{H})$, let $\mathcal{H}[S]$ denote the subgraph of $\mathcal{H}$ induced by S and let $\mathcal{H} - S$ denote the subgraph $\mathcal{H}[V(\mathcal{H}) \setminus S]$.

The following result is essential for the proof of Lemma 7.

Lemma 6 (Akiyama–Frankl [1]). *Let $G_1, \dots, G_s \subset \binom{[n]}{2}$ with $n \geq 2s + 1$. If $|E(G_i)| > \max\{\binom{2s-1}{2}, \binom{s-1}{2} + (s-1)(n-s+1)\}$ for all $i = 1, 2, \dots, s$, then there exist pairwise disjoint edges $e_1 \in E(G_1), \dots, e_s \in E(G_s)$.*

For any $X \subset [n]$, we use $\bar{X}$ to denote $[n] \setminus X$. The following lemma plays a crucial role in our proof, it converts the matching-number constraint into a restriction on the high-degree vertices of $\mathcal{H}$, and consequently yields an upper bound on $|\mathcal{H}|$ in terms of the size of the set of high-degree vertices.

Lemma 7. *Let $\mathcal{H} \subset \binom{[n]}{3}$ be a 3-graph with $\nu(\mathcal{H}) \leq s$, $s \geq 2$ and $n \geq 3(s+1)$. Then there exist $0 \leq i \leq s$ and $A \subset S \subset [n]$ with $|A| = s - i$ and $|S| = s$ such that*

$$\nu(\mathcal{H}[\bar{A}]) \leq i, \quad \deg(x) \leq (s+2i)n \quad \text{for all } x \in \bar{A},$$

and

$$|\mathcal{H}| \leq \sum_{v \in A} \deg(v, \bar{S}) + \sum_{\{u,v\} \in \binom{A}{2}} \deg(u, v) + isn + 6i^2n.$$

Proof. Without loss of generality, assume that $\deg(1) \geq \deg(2) \geq \dots \geq \deg(n)$.

Claim 8. *There exists $0 \leq i \leq s$ such that $\deg(s+1-i) \leq (s+2i)n$.*

Proof of the claim. Suppose for contradiction that $\deg(s+1-i) > (s+2i)n$ holds for all $i = 0, 1, 2, \dots, s$. Then one can find a matching of size $s+1$ by the following procedure. In the first step, by $\deg(s+1) > sn$ we can choose e_{s+1} with $s+1 \in e_{s+1}$ such that $e_{s+1} \cap [s+1] = \{s+1\}$. In the second step by $\deg(s) > (s+2)n$ there is e_s with $s \in e_s$ such that $e_s \cap ([s+1] \cup e_{s+1}) = \{s\}$. In the $(j+1)$ th step with $j = 0, 1, 2, \dots, s$, by $\deg(s+1-j) > (s+2j)n$ there is e_{s+1-j} with $s+1-j \in e_{s+1-j}$ such that $e_{s+1-j} \cap ([s+1] \cup e_{s+1} \cup \dots \cup e_{s+2-j}) = \{s+1-j\}$. Eventually we will obtain a matching $e_{s+1}, e_s, \dots, e_1$, contradicting $\nu(\mathcal{H}) \leq s$. $\diamond$

Choose i maximal subject to Claim 8, and set $A = [s-i]$. Then every vertex of $\bar{A}$ has degree at most $(s+2i)n$. Moreover, maximality gives

$$\deg(r) > (3s+2-2r)n \quad \text{for every } r \in [s-i]. \quad (4)$$

Let $\nu(\mathcal{H}[\bar{A}]) = k$ and let $e_1, e_2, \dots, e_k$ be a maximal matching in $\mathcal{H}[\bar{A}]$.

Claim 9. $k \leq i$.

Proof of the claim. Suppose that $k \geq i + 1$, and set $r_0 = s + 1 - k$. Since $r_0 \leq s - i$, we may greedily extend the matching $e_1, e_2, \dots, e_k$ by adding edges that contain the vertices $r_0, r_0 - 1, \dots, 1$ in succession. When choosing an edge containing r , the forbidden vertices consist of the $3k$ vertices of $e_1, e_2, \dots, e_k$, the other $r_0 - 1 = s - k$ vertices of $[r_0]$, and at most two new vertices from each of the $r_0 - r$ edges already added. Hence the total number of forbidden vertices is at most

$$3k + (s - k) + 2(r_0 - r) = 3s + 2 - 2r.$$

By (4), there are more than $(3s + 2 - 2r)n$ edges containing r . Thus, one can extend $e_1, e_2, \dots, e_k$ to a matching of size $s + 1$ in $\mathcal{H}$, contradicting $\nu(\mathcal{H}) \leq s$. $\diamond$

Let $e_1 \cup e_2 \cup \dots \cup e_k = \{u_1, u_2, \dots, u_{3k}\} = X$ and assume

$$\deg_{\mathcal{H}[\bar{A}]}(u_1) \geq \deg_{\mathcal{H}[\bar{A}]}(u_2) \geq \dots \geq \deg_{\mathcal{H}[\bar{A}]}(u_{3k}).$$

Since $e_1, e_2, \dots, e_k$ form a maximal matching in $\mathcal{H}[\bar{A}]$, we infer that $e \cap X \neq \emptyset$ for all $e \in \mathcal{H}[\bar{A}]$.

If $3k \leq i$, choose any i -element set $B \subseteq \bar{A}$ containing X . Such a set exists because $|\bar{A}| = n - s + i \geq i$. Put $S = A \cup B$. Since every edge of $\mathcal{H}[\bar{A}]$ meets $X \subseteq B$, we have $\mathcal{H}[\bar{S}] = \emptyset$.

If $3k \geq i + 1$, we first prove that

$$\deg_{\mathcal{H}[\bar{A}]}(u_{i+1}) \leq 2in.$$

Suppose to the contrary that $\deg_{\mathcal{H}[\bar{A}]}(u_{i+1}) > 2in$, then let $W = \bar{A} \setminus \{u_1, \dots, u_{i+1}\}$ and $|L_{\mathcal{H}}(u_j, W)| > in$ holds for each $j = 1, 2, \dots, i + 1$. By Lemma 6, there is an $(i + 1)$ -matching in $\mathcal{H}[\bar{A}]$, contradicting Claim 9. Thus,

$$\deg_{\mathcal{H}[\bar{A}]}(u_{3k}) \leq \dots \leq \deg_{\mathcal{H}[\bar{A}]}(u_{i+2}) \leq \deg_{\mathcal{H}[\bar{A}]}(u_{i+1}) \leq 2in.$$

Now let $B = \{u_1, u_2, \dots, u_i\}$ and $S = A \cup B$. Then every edge of $\mathcal{H}[\bar{S}]$ meets $X \setminus B = \{u_{i+1}, \dots, u_{3k}\}$, and hence

$$|\mathcal{H}[\bar{S}]| \leq \sum_{j=i+1}^{3k} \deg_{\mathcal{H}[\bar{A}]}(u_j) \leq (3k - i) 2in \leq 4i^2n,$$

where the last inequality follows from $k \leq i$.

Recall that $\deg(u) \leq (s + 2i)n$ for $u \in B$. Thus,

$$|\mathcal{H}| \leq \sum_{u \in B} \deg(u) + |\mathcal{H}[\bar{B}]| \leq i(s + 2i)n + |\mathcal{H}[\bar{B}]|$$

and

$$\begin{aligned} |\mathcal{H}[\bar{B}]| &\leq \sum_{v \in A} \deg(v, \bar{S}) + \sum_{\{u, v\} \in \binom{A}{2}} \deg(u, v) + |\mathcal{H}[\bar{S}]| \\ &\leq \sum_{v \in A} \deg(v, \bar{S}) + \sum_{\{u, v\} \in \binom{A}{2}} \deg(u, v) + 4i^2n. \end{aligned}$$

Thus,

$$|\mathcal{H}| \leq \sum_{v \in A} \deg(v, \bar{S}) + \sum_{\{u,v\} \in \binom{A}{2}} \deg(u, v) + isn + 6i^2n. \quad \square$$

2.3 A result used in Section 4

For a graph G on vertex set V , the *square graph* of G , denoted by G^2 , is the graph on the same vertex set V in which two distinct vertices $x, y \in V$ are adjacent if and only if they have a common neighbor in G , that is,

$$xy \in E(G^2) \iff \text{there exists } z \in V \text{ such that } xz, yz \in E(G).$$

We also need the following result to prove the Theorem 1 for $s \geq 3$.

Theorem 10. *Let $0 < \alpha, \beta < \frac{1}{4}$ be reals satisfying $\beta + 2\alpha^{\frac{3}{4}} < \frac{1}{4}$ and let $p \geq 2$, $n > \frac{4}{1-4\beta}$. Suppose that $G_1, \dots, G_p \subset \binom{[n]}{2}$ and $G_1 + \dots + G_p$ is 2-colored triangle-free. If $E(G_i) \cap E(G_j) = \emptyset$ and $|E(G_i)|, |E(G_j)| > \frac{n^2}{4} - \alpha n^2$ for some $1 \leq i < j \leq p$, then*

$$\sum_{i=1}^p |E(G_i)| \leq p \cdot \frac{n^2}{4} - (p-2)\beta n^2. \quad (5)$$

Proof. By symmetry assume $E(G_1) \cap E(G_2) = \emptyset$ and $|E(G_1)|, |E(G_2)| > \frac{n^2}{4} - \alpha n^2$. If $p = 2$, then, since $E(G_1) \cap E(G_2) = \emptyset$, we have $|E(G_1)| + |E(G_2)| \leq \binom{n}{2} < n^2/2$, which is exactly (5). Hence we may assume $p \geq 3$. If $|E(G_k)| \leq \frac{n^2}{4} - \beta n^2$ for all $k \in \{3, 4, \dots, p\}$, then (5) follows. Thus by symmetry we may assume $|E(G_3)| > \frac{n^2}{4} - \beta n^2$.

Claim 11. *The maximum degree $\Delta(G_3) < 2\sqrt{\alpha}n$.*

Proof of the claim. Suppose that there exists $v \in [n]$ with $\deg_{G_3}(v) \geq 2\sqrt{\alpha}n$. Let $X = N_{G_3}(v)$. Since $G_1 + \dots + G_p$ is 2-colored triangle-free, X is an independent set of G_k for all $k \neq 3$. Thus,

$$|E(G_1)| + |E(G_2)| \leq \binom{n}{2} - \binom{|X|}{2} \leq \binom{n}{2} - \binom{2\sqrt{\alpha}n}{2}.$$

Because $\alpha < 1/4$, the right-hand side is strictly smaller than $\frac{n^2}{2} - 2\alpha n^2$, contradicting the fact that $|E(G_i)| > \frac{n^2}{4} - \alpha n^2$, $i = 1, 2$. $\diamond$

Consider the square graph G_3^2 . The number of pairs $(z, \{x, y\})$ such that $xz, yz \in E(G_3)$ is $\sum_{i=1}^n \binom{\deg_{G_3}(i)}{2}$. Every edge of G_3^2 is counted at most $\Delta(G_3) < 2\sqrt{\alpha}n$ times.

Thus,

$$\begin{aligned}
|E(G_3^2)| &> \frac{1}{2\sqrt{\alpha n}} \cdot \sum_{i=1}^n \binom{\deg_{G_3}(i)}{2} = \frac{1}{4\sqrt{\alpha n}} \cdot \sum_{i=1}^n (\deg_{G_3}^2(i) - \deg_{G_3}(i)) \\
&\stackrel{(2)}{\geq} \frac{1}{4\sqrt{\alpha n}} \cdot \left(\frac{4|E(G_3)|^2}{n} - 2|E(G_3)| \right) \quad (6) \\
&= \frac{|E(G_3)|}{\sqrt{\alpha n^2}} \left(|E(G_3)| - \frac{n}{2} \right).
\end{aligned}$$

Note that $n > \frac{4}{1-4\beta}$ implies $|E(G_3)| > \frac{n^2}{4} - \beta n^2 > n$. It follows that $|E(G_3)| - \frac{n}{2} > \frac{|E(G_3)|}{2}$. Then by $|E(G_3)| > \frac{n^2}{4} - \beta n^2 > 2\alpha^{\frac{3}{4}}n^2$,

$$|E(G_3^2)| \geq \frac{|E(G_3)|^2}{2\sqrt{\alpha n^2}} > \frac{(2\alpha^{\frac{3}{4}}n^2)^2}{2\sqrt{\alpha n^2}} = 2\alpha n^2.$$

Since $G_1 + G_2 + G_3$ is 2-colored triangle-free, we infer that $(E(G_1) \cup E(G_2)) \cap E(G_3^2) = \emptyset$. Thus,

$$|E(G_1)| + |E(G_2)| \leq \binom{n}{2} - |E(G_3^2)| \leq \binom{n}{2} - 2\alpha n^2.$$

contradicting $|E(G_i)| > \frac{n^2}{4} - \alpha n^2$, $i = 1, 2$. Therefore, $|E(G_k)| \leq \frac{n^2}{4} - \beta n^2$ for all $k = 3, 4, \dots, p$ and $\sum_{i=1}^p |E(G_i)| \leq p \cdot \frac{n^2}{4} - (p-2)\beta n^2$. $\square$

3 Proof of Theorem 1 for $s \leq 2$

A family $\mathcal{F} \subseteq \binom{[n]}{k}$ is t -intersecting if $|A \cap B| \geq t$ for every $A, B \in \mathcal{F}$. We need the following two classical results from extremal set theory.

Theorem 12 (The Exact Erdős-Ko-Rado Theorem [4, 6, 17]). *Let $\mathcal{F} \subset \binom{[n]}{k}$ be a t -intersecting family. If $n > (t+1)(k-t+1)$ then $|\mathcal{F}| \leq \binom{n-t}{k-t}$. Moreover, equality holds if and only if $\mathcal{F}$ is a full t -star.*

Theorem 13 (The Hilton-Milner Theorem [11]). *Let $n \geq 2k$. If $\mathcal{F} \subset \binom{[n]}{k}$ is an intersecting family that is not a star, then*

$$|\mathcal{F}| \leq \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1.$$

We now prove the $s \leq 2$ case of Theorem 1.

Proposition 14. *For $n \geq 60$ and $s = 1, 2$,*

$$\text{ex}_3(n, \{F_5, M_{s+1}^3\}) = \binom{n-1}{2}$$

and the full star is the unique $\{F_5, M_{s+1}^3\}$ -free 3-graph attaining the maximum size.

Proof. When $s = 1$, the condition $\nu(\mathcal{H}) \leq 1$ means that $\mathcal{H}$ is an intersecting family. Hence the result follows from Theorem 12 with $t = 1$ and $k = 3$. Thus we are left with the $s = 2$ case.

Let $\mathcal{H} \subset \binom{[n]}{3}$ be an F_5 -free 3-graph with $\nu(\mathcal{H}) \leq 2$. If $\nu(\mathcal{H}) = 1$, then Theorem 12 gives $|\mathcal{H}| \leq \binom{n-1}{2}$, with equality only for a full star. Hence assume $\nu(\mathcal{H}) = 2$.

By Lemma 7, there exist $0 \leq i \leq 2$ and $A \subset S \subset [n]$ with $|A| = 2 - i$ and $|S| = 2$ such that

$$\nu(\mathcal{H}[\bar{A}]) \leq i, \quad \deg(x) \leq 2(i+1)n \text{ for all } x \in \bar{A},$$

and

$$|\mathcal{H}| \leq \sum_{v \in A} \deg(v, \bar{S}) + \sum_{\{u,v\} \in \binom{A}{2}} \deg(u, v) + 2in + 6i^2n.$$

If $i = 2$, then $A = \emptyset$ and for $n \geq 60$,

$$|\mathcal{H}| \leq 28n < \binom{n-1}{2}.$$

Thus we may assume $i = 1$ or $i = 0$. We distinguish two cases.

Case 1. $i = 1$.

Assume $A = \{1\}$. Let

$$\tilde{\mathcal{H}} = \mathcal{H} - \{1\} = \mathcal{H}[V(\mathcal{H}) \setminus \{1\}], \quad G = L_{\mathcal{H}}(1) = \left\{ vw \in \binom{V(\mathcal{H})}{2} : 1vw \in \mathcal{H} \right\}.$$

The additional conclusion $\nu(\mathcal{H}[\bar{A}]) \leq i$ in Lemma 7 shows that $\tilde{\mathcal{H}}$ is intersecting. It is nonempty because otherwise every edge of $\mathcal{H}$ would contain 1, contradicting $\nu(\mathcal{H}) = 2$.

Claim 15. *For each $x_1x_2x_3 \in \tilde{\mathcal{H}}$ and $v \in [2, n] \setminus \{x_1, x_2, x_3\}$, there is at most one edge in G between v and $\{x_1, x_2, x_3\}$.*

Proof of the claim. If $vx_1, vx_2 \in E(G)$, then $x_1x_2x_3, 1vx_1, 1vx_2$ form an F_5 , a contradiction. $\diamond$

Recall that $\nu(\mathcal{H}) = 2$ implies $\tilde{\mathcal{H}} \neq \emptyset$. By Claim 15 we have

$$|E(G)| \leq \binom{n-1}{2} - 2(n-4).$$

If $|\tilde{\mathcal{H}}| < 2(n-4)$, then

$$|\mathcal{H}| = |E(G)| + |\tilde{\mathcal{H}}| < \binom{n-1}{2}.$$

Thus we may assume $|\tilde{\mathcal{H}}| \geq 2(n-4)$.

If $\tilde{\mathcal{H}}$ is 2-intersecting, then by Theorem 12

$$|\tilde{\mathcal{H}}| \leq \binom{\binom{n-1}{2} - 2}{3-2} = n-3 < 2(n-4),$$

a contradiction. Thus there exist $x_1x_2x_3, x_3x_4x_5 \in \tilde{\mathcal{H}}$. Then by Claim 15, for each $x \in [2, n] \setminus \{x_1, x_2, x_3, x_4, x_5\}$ there are at most 2 edges in G between x and $\{x_1, x_2, x_3, x_4, x_5\}$. Moreover, there are at most 2 edges in G between $\{x_1, x_2, x_3\}$ and $\{x_4, x_5\}$. Thus,

$$|E(G)| \leq \binom{n-1}{2} - 3(n-6) - 4 = \binom{n-1}{2} - 3n + 14. \quad (7)$$

Subcase 1.1. $\tilde{\mathcal{H}}$ is a star.

By symmetry we may assume $\tilde{\mathcal{H}}$ is a star of center 2. Then $|\tilde{\mathcal{H}}| \leq \deg(2) \leq 4n$.

Claim 16. *If there are $E_1, E_2, \dots, E_t \in \tilde{\mathcal{H}}$ such that $E_i \cap E_j = \{2\}$ for all $1 \leq i < j \leq t$, then*

$$|E(G)| \leq \binom{n-1}{2} - (t+1)n + t^2 + 3t + 4. \quad (8)$$

Proof of the claim. Let $X = E_1 \cup E_2 \cup \dots \cup E_t$. By Claim 15, for each $x \in [2, n] \setminus X$ there are at most t edges in G between x and X . Moreover for $i = t, t-1, \dots, 2$, each $x \in E_i \setminus \{2\}$ there are at most $i-1$ edges in G between x and $E_1 \cup \dots \cup E_{i-1}$. Then

$$\begin{aligned} |E(G)| &\leq \binom{n-1}{2} - (t+1)(n-2t-2) - 2(t + (t-1) + \dots + 2) \\ &= \binom{n-1}{2} - (t+1)n + t^2 + 3t + 4. \end{aligned}$$

◇

Let $\tilde{\mathcal{H}}(2) := \{E \setminus \{2\} : E \in \tilde{\mathcal{H}}\}$. If $\nu(\tilde{\mathcal{H}}(2)) \geq 4$, then one can find E_1, E_2, E_3, E_4 such that $E_i \cap E_j = \{2\}$ for all $1 \leq i < j \leq 4$. By (8) and $n \geq 60$,

$$|\mathcal{H}| = |E(G)| + |\tilde{\mathcal{H}}| \leq \binom{n-1}{2} - 5n + 32 + 4n < \binom{n-1}{2}.$$

Thus we may assume $\nu(\tilde{\mathcal{H}}(2)) = t \in \{1, 2, 3\}$. By the Erdős–Gallai theorem [3], we have $|\tilde{\mathcal{H}}| = |\tilde{\mathcal{H}}(2)| \leq t(n-2)$. By (8),

$$\begin{aligned} |\mathcal{H}| = |E(G)| + |\tilde{\mathcal{H}}| &\leq \binom{n-1}{2} - (t+1)n + t^2 + 3t + 4 + t(n-2) \\ &\leq \binom{n-1}{2} - n + 16 < \binom{n-1}{2}. \end{aligned}$$

Subcase 1.2. $\tilde{\mathcal{H}}$ is not a star.

By Theorem 13 we have

$$|\tilde{\mathcal{H}}| \leq \binom{n-2}{2} - \binom{n-3-2}{2} + 1 = 3n - 11. \quad (9)$$

Let

$$\Delta_2(\tilde{\mathcal{H}}) = \max_{\{u,v\} \subset [2,n]} |\{E \in \tilde{\mathcal{H}} : \{u,v\} \subset E\}|.$$

Claim 17. We may assume that $\Delta_2(\tilde{\mathcal{H}}) < n/3$.

Proof of the claim. Let $\{u, v\} \subseteq [2, n]$ such that $|\{E \in \tilde{\mathcal{H}}: \{u, v\} \subset E\}|$ is maximal and let $X = \{w \in [2, n]: uvw \in \mathcal{H}\}$. Since $\mathcal{H}$ is F_5 -free, X has to be an independent set of G . If $|X| \geq n/3$, then for $n \geq 60$,

$$|E(G)| \leq \binom{n-1}{2} - \binom{|X|}{2} \leq \binom{n-1}{2} - \frac{(n/3)(n/3-1)}{2} < \binom{n-1}{2} - (3n-11).$$

By (9),

$$|\mathcal{H}| = |E(G)| + |\tilde{\mathcal{H}}| < \binom{n-1}{2} - (3n-11) + (3n-11) = \binom{n-1}{2}.$$

Thus we may assume $|X| < n/3$. $\diamond$

Since $\tilde{\mathcal{H}}$ is not 2-intersecting, there exist $x_1x_2x_3, x_3x_4x_5 \in \tilde{\mathcal{H}}$. Since $\tilde{\mathcal{H}}$ is not a star, there exists $y_1y_2y_3 \in \tilde{\mathcal{H}}$ with $x_3 \notin \{y_1, y_2, y_3\}$. Let $E \in \tilde{\mathcal{H}}$. If $x_3 \notin E$, then E must meet both $\{x_1, x_2\}$ and $\{x_4, x_5\}$, so it contains one of the four pairs

$$\{x_1, x_4\}, \{x_1, x_5\}, \{x_2, x_4\}, \{x_2, x_5\}.$$

If $x_3 \in E$, then E must also meet $\{y_1, y_2, y_3\}$, so it contains one of

$$\{x_3, y_1\}, \{x_3, y_2\}, \{x_3, y_3\}.$$

Thus every edge contains one of these seven pairs. By Claim 17,

$$|\tilde{\mathcal{H}}| \leq 7 \cdot \Delta_2(\tilde{\mathcal{H}}) \leq \frac{7n}{3}.$$

Using (7),

$$|\mathcal{H}| = |E(G)| + |\tilde{\mathcal{H}}| \leq \binom{n-1}{2} - 3n + 14 + \frac{7n}{3} < \binom{n-1}{2}.$$

Case 2. $i = 0$.

Assume $S = A = \{1, 2\}$. Since $\nu(\mathcal{H}[\bar{A}]) = 0$, we know $\mathcal{H}[[3, n]]$ is empty. Thus

$$|\mathcal{H}| = \deg(1, [3, n]) + \deg(2, [3, n]) + \deg(1, 2).$$

Since $\mathcal{H}$ is F_5 -free, we infer that $L(1, [3, n]) + L(2, [3, n])$ is 2-colored triangle-free. By Theorem 2,

$$\deg(1, [3, n]) + \deg(2, [3, n]) \leq 2 \left\lfloor \frac{(n-2)^2}{4} \right\rfloor \leq \frac{(n-2)^2}{2}.$$

If $\deg(1, 2) < \frac{n-2}{2}$, then

$$|\mathcal{H}| < \frac{(n-2)^2}{2} + \frac{n-2}{2} = \binom{n-1}{2}.$$

Thus we may assume $\deg(1, 2) \geq \frac{n-2}{2} \geq 3$.

Now by the F_5 -free property, we infer that $L(1, [3, n]) \cap L(2, [3, n]) = \emptyset$. It follows that

$$|\mathcal{H}| \leq \deg(1, [3, n]) + \deg(2, [3, n]) + \deg(1, 2) \leq \binom{n-2}{2} + (n-2) = \binom{n-1}{2}, \quad (10)$$

with equality holding if and only if $\deg(1, 2) = n-2$ and $L(1, [3, n]) \cup L(2, [3, n]) = \binom{[3, n]}{2}$.

Claim 18. *There exist edges $xy \in L(1, [3, n])$ and $xz \in L(2, [3, n])$.*

Proof of the claim. Since $\nu(\mathcal{H}) = 2$, we have $L(1, [3, n]) \neq \emptyset \neq L(2, [3, n])$. Choose $x_1y_1 \in L(1, [3, n])$ and $x_2y_2 \in L(2, [3, n])$. If x_1y_1 and x_2y_2 share a vertex, then the claim follows. Otherwise, suppose that x_1y_1 and x_2y_2 are disjoint. Since $L(1, [3, n]) \cap L(2, [3, n]) = \emptyset$ and $L(1, [3, n]) \cup L(2, [3, n]) = \binom{[3, n]}{2}$, the edge x_1y_2 must belong to exactly one of the two link graphs. Together with x_1y_1 or x_2y_2 , as appropriate, this proves the claim. $\diamond$

By Claim 18, $1yx, 1y2, 2xz$ form an F_5 , a contradiction. Thus we cannot have equality in (10) and equality holds if and only if $\mathcal{H}$ is the full star. $\square$

4 Proof of Theorem 1 for $s \geq 3$

In the following proof, we will repeatedly use the elementary fact that a quadratic polynomial in i with positive coefficient of i^2 is convex, and hence attains its maximum on a closed interval at one of the two endpoints.

Proof of Theorem 1. Let $\mathcal{H} \subset \binom{[n]}{3}$ be an F_5 -free 3-graph with $\nu(\mathcal{H}) \leq s$, where $s \geq 3$ and $n \geq 30(s+1)$. By Lemma 7, there exist $0 \leq i \leq s$ and $A \subset S \subset [n]$ with $|A| = s-i$ and $|S| = s$ such that $\deg(x) \leq (s+2i)n$ for all $x \in \bar{A}$ and

$$|\mathcal{H}| \leq \sum_{v \in A} \deg(v, \bar{S}) + \sum_{\{u,v\} \in \binom{A}{2}} \deg(u, v) + isn + 6i^2n. \quad (11)$$

Without loss of generality, we assume that $A = \{1, 2, \dots, s-i\}$ and $S = \{1, 2, \dots, s\}$.

If $i = s$, then $A = \emptyset$ and for $n \geq 30s$ it follows from (11) that

$$|\mathcal{H}| \leq 7s^2n < s \cdot \frac{(n-s)^2}{4} - s \leq s \left\lfloor \frac{(n-s)^2}{4} \right\rfloor.$$

Now we distinguish three cases.

Case 1. $i = s-1$.

Here $A = \{1\}$. Suppose that $\mathcal{H}[\bar{S}] \neq \emptyset$, and choose an edge $abc \in \mathcal{H}[\bar{S}]$. We claim that for every $v \in \bar{S} \setminus \{a, b, c\}$, at most one of va, vb, vc belongs to $L_{\mathcal{H}}(1)$. Indeed, if $va, vb \in L_{\mathcal{H}}(1)$, then $1va, 1vb, abc$ form a copy of F_5 , a contradiction. Thus

$$\deg(1, \bar{S}) \leq \binom{n-s}{2} - 2(n-s-3).$$

By (11) and $n \geq 30(s+1)$, we have

$$\begin{aligned} |\mathcal{H}| &\leq \binom{n-s}{2} - 2(n-s-3) + (s-1)sn + 6(s-1)^2n \\ &< s \cdot \left(\frac{(n-s)^2}{4} - 1 \right) - \frac{(s-2)(n-s)^2}{4} - s + 2(n-s-3) - (s-1)sn - 6(s-1)^2n. \end{aligned}$$

Note that

$$\begin{aligned} &\frac{(s-2)(n-s)^2}{4} - s + 2(n-s-3) - (s-1)sn - 6(s-1)^2n \\ &= \frac{1}{4} ((s-2)n^2 - (30s^2 - 56s + 16)n + s^3 - 2s^2 - 12s - 24) \\ &\geq \frac{1}{4} (30(s-2)(s+1)n - (30s^2 - 56s + 16)n + s^3 - 2s^2 - 12s - 24) \\ &= \frac{1}{4} ((26s - 76)n + s^3 - 2s^2 - 12s - 24) \\ &\geq \frac{1}{4} (30 \times (26s - 76)(s+1) + s^3 - 2s^2 - 12s - 24) \\ &\geq \frac{1}{4} (s^3 + 778s^2 - 1512s - 2304) \\ &\geq \frac{1}{4} (s^3 + 778 \times 3s - 1512s - 2304) \\ &\geq \frac{1}{4} (s^3 + 822s - 2304) > 0. \end{aligned}$$

Thus,

$$|\mathcal{H}| < s \cdot \left(\frac{(n-s)^2}{4} - 1 \right) \leq s \left\lfloor \frac{(n-s)^2}{4} \right\rfloor.$$

If $\mathcal{H}[\bar{S}] = \emptyset$, noting that $\deg(v) \leq (3s-2)n$ for all $v \in \bar{A}$, then for $n \geq 30s$ we have

$$\begin{aligned} |\mathcal{H}| &\leq \deg(1, \bar{S}) + \sum_{j=2}^s \deg(j) \leq \binom{n-s}{2} + (s-1)(3s-2)n \\ &< s \cdot \frac{(n-s)^2}{4} - s \leq s \left\lfloor \frac{(n-s)^2}{4} \right\rfloor. \end{aligned}$$

Case 2. $\frac{s}{2} \leq i \leq s-2$.

Since $\mathcal{H}$ is F_5 -free, we infer that $L(1, \bar{S}) + L(2, \bar{S}) + \cdots + L(s-i, \bar{S})$ is 2-colored triangle-free. By Theorem 2,

$$\sum_{v \in A} \deg(v, \bar{S}) \leq (s-i) \left\lfloor \frac{(n-s)^2}{4} \right\rfloor.$$

Using (11), we obtain that

$$\begin{aligned}
|\mathcal{H}| &\leq (s-i) \left\lfloor \frac{(n-s)^2}{4} \right\rfloor + \binom{s-i}{2} n + 6i^2 n + isn \\
&= s \left\lfloor \frac{(n-s)^2}{4} \right\rfloor - i \left\lfloor \frac{(n-s)^2}{4} \right\rfloor + \frac{1}{2}(s-i)(s-i-1)n + 6i^2 n + isn \\
&= s \left\lfloor \frac{(n-s)^2}{4} \right\rfloor + \frac{1}{2}s(s-1)n + i \left(\frac{13}{2}in + \frac{n}{2} - \left\lfloor \frac{(n-s)^2}{4} \right\rfloor \right) \\
&\leq s \left\lfloor \frac{(n-s)^2}{4} \right\rfloor + \left(\frac{s(s-1) + 13i^2 + i}{2} n - \frac{i(n-s)^2}{4} + i \right).
\end{aligned}$$

Since the coefficient of i^2 is $\frac{13n}{2} > 0$, the last expression is a convex quadratic in i , and hence it suffices to check $i = \frac{s}{2}$ and $i = s-2$. For $n \geq 30s$, a direct computation shows that

$$\frac{s(s-1) + 13i^2 + i}{2} n - \frac{i(n-s)^2}{4} + i < 0.$$

Thus, $|\mathcal{H}| < s \left\lfloor \frac{(n-s)^2}{4} \right\rfloor$.

Case 3. $0 \leq i < \frac{s}{2}$.

Define

$$T = \left\{ v \in A : \deg(v, \bar{S}) > \frac{(n-s)^2}{5} \right\}.$$

Recall that $A = \{1, 2, \dots, s-i\}$ and $S = \{1, 2, \dots, s\}$. By symmetry we may assume $T = \{1, 2, \dots, t\}$. Then for $j \in \{t+1, \dots, s-i\}$ we have

$$\deg(j, \bar{S}) \leq \frac{(n-s)^2}{5}. \quad (12)$$

When $t \geq 2$, since $L(1, \bar{S}) + L(2, \bar{S}) + \dots + L(t, \bar{S})$ is 2-colored triangle-free, by Theorem 2 we infer that

$$\sum_{v \in T} \deg(v, \bar{S}) \leq t \left\lfloor \frac{(n-s)^2}{4} \right\rfloor. \quad (13)$$

Subcase 3.1. $|T| \leq \frac{s}{2}$.

Note that $s \geq 3$ and $i < \frac{s}{2}$ implies $s-i \geq 2$. Since $L(1, \bar{S}) + L(2, \bar{S})$ is 2-colored triangle-free, by Theorem 2 we have

$$\deg(1, \bar{S}) + \deg(2, \bar{S}) \leq 2 \left\lfloor \frac{(n-s)^2}{4} \right\rfloor. \quad (14)$$

If $|T| \leq 1$, then by (11), (12) and (14) we have

$$\begin{aligned}
|\mathcal{H}| &\leq 2 \left\lfloor \frac{(n-s)^2}{4} \right\rfloor + (s-i-2) \frac{(n-s)^2}{5} + \binom{s-i}{2} n + 6i^2 n + isn \\
&\leq s \left\lfloor \frac{(n-s)^2}{4} \right\rfloor + s - \frac{1}{20}(s-2)(n-s)^2 + \left(-i \frac{(n-s)^2}{5} + \binom{s-i}{2} n + 6i^2 n + isn \right).
\end{aligned}$$

Since the coefficient of i^2 is $\frac{13n}{2} > 0$, the last expression is a convex quadratic in i , and hence it suffices to check $i = 0$ and $i = \frac{s}{2}$. For $n \geq 23s$, a direct computation shows that

$$-i\frac{(n-s)^2}{5} + \binom{s-i}{2}n + 6i^2n + isn < \frac{1}{20}(s-2)(n-s)^2 - s.$$

Thus $|\mathcal{H}| < s \left\lfloor \frac{(n-s)^2}{4} \right\rfloor$.

Therefore we may assume $2 \leq |T| \leq \frac{s}{2}$. Using (11), (12) and (13), we get

$$\begin{aligned} |\mathcal{H}| &\leq t \left\lfloor \frac{(n-s)^2}{4} \right\rfloor + (s-i-t)\frac{(n-s)^2}{5} + \binom{s-i}{2}n + 6i^2n + isn \\ &\leq t \left\lfloor \frac{(n-s)^2}{4} \right\rfloor + (s-t)\frac{(n-s)^2}{5} - i\frac{(n-s)^2}{5} + \binom{s-i}{2}n + 6i^2n + isn \\ &\leq s \left\lfloor \frac{(n-s)^2}{4} \right\rfloor + s - (s-t)\frac{(n-s)^2}{20} - i\frac{(n-s)^2}{5} + \binom{s-i}{2}n + 6i^2n + isn \\ &\leq s \left\lfloor \frac{(n-s)^2}{4} \right\rfloor + s - \frac{s(n-s)^2}{40} + \left(-i\frac{(n-s)^2}{5} + \binom{s-i}{2}n + 6i^2n + isn \right). \end{aligned}$$

Since the last expression is a convex quadratic in i , it suffices to check $i = 0$ and $i = \frac{s}{2}$. For $n \geq 22s$, a direct computation shows that

$$-i\frac{(n-s)^2}{5} + \binom{s-i}{2}n + 6i^2n + isn < \frac{s(n-s)^2}{40} - s.$$

Hence $|\mathcal{H}| < s \left\lfloor \frac{(n-s)^2}{4} \right\rfloor$.

Subcase 3.2. $|T| > \frac{s}{2}$ and there exist $x, y \in T$ such that $L(x, \bar{S}) \cap L(y, \bar{S}) = \emptyset$.

Since $\mathcal{H}$ is F_5 -free, we infer that $L(1, \bar{S}) + L(2, \bar{S}) + \cdots + L(s-i, \bar{S})$ is 2-colored triangle-free. Moreover,

$$L(x, \bar{S}) \cap L(y, \bar{S}) = \emptyset, \deg(x, \bar{S}) > \frac{(n-s)^2}{5} \text{ and } \deg(y, \bar{S}) > \frac{(n-s)^2}{5}.$$

Applying Theorem 10 with $\alpha = \frac{1}{20}$ and $\beta = \frac{1}{28}$, we obtain that

$$\sum_{v \in A} \deg(v, \bar{S}) \leq (s-i)\frac{(n-s)^2}{4} - \frac{1}{28}(s-i-2)(n-s)^2.$$

Then by (11) we obtain that

$$\begin{aligned} |\mathcal{H}| &\leq (s-i)\frac{(n-s)^2}{4} - \frac{1}{28}(s-i-2)(n-s)^2 + \binom{s-i}{2}n + 6i^2n + isn \\ &\leq s \left\lfloor \frac{(n-s)^2}{4} \right\rfloor + s - \frac{1}{28}(s-2)(n-s)^2 \\ &\quad + \left(-i\frac{(n-s)^2}{4} + \frac{i}{28}(n-s)^2 + \binom{s-i}{2}n + 6i^2n + isn \right). \end{aligned}$$

Since the coefficient of i^2 is $\frac{13n}{2} > 0$, the last expression is a convex quadratic in i , and hence it suffices to check $i = 0$ and $i = \frac{s}{2}$. For $n \geq 30(s+1)$, a direct computation shows that

$$-i\frac{(n-s)^2}{4} + \frac{i}{28}(n-s)^2 + \binom{s-i}{2}n + 6i^2n + isn < \frac{1}{28}(s-2)(n-s)^2 - s.$$

Thus, $|\mathcal{H}| < s \left\lfloor \frac{(n-s)^2}{4} \right\rfloor$.

Subcase 3.3. $|T| > \frac{s}{2}$ and $L(x, \bar{S}) \cap L(y, \bar{S}) \neq \emptyset$ for any $x, y \in T$.

Fix distinct $x, y \in T$, and choose

$$uv \in L(x, \bar{S}) \cap L(y, \bar{S}).$$

If $xyz \in \mathcal{H}$ and $z \notin \{u, v\}$, then the three edges xuv, yuv, xyz form a copy of F_5 , a contradiction. Therefore every $z \in N_{\mathcal{H}}(x, y)$ belongs to $\{u, v\}$. It follows that $\deg(x, y) \leq 2$ for any $x, y \in T$. Thus,

$$\sum_{\{u,v\} \in \binom{A}{2}} \deg(u, v) \leq \binom{s-i}{2}n - \binom{t}{2}(n-2). \quad (15)$$

Claim 19. *We may assume $|A| = s$.*

Proof of the claim. Suppose for contradiction that $|A| \leq s-1$. Then by (11), (13) and (15),

$$|\mathcal{H}| \leq t \left\lfloor \frac{(n-s)^2}{4} \right\rfloor + (s-i-t)\frac{(n-s)^2}{5} + \binom{s-i}{2}n - \binom{t}{2}(n-2) + 6i^2n + isn =: f(t).$$

By $t \leq s-i$ and $n \geq 30(s+1)$,

$$f'(t) = \left\lfloor \frac{(n-s)^2}{4} \right\rfloor - \frac{(n-s)^2}{5} - \frac{2t-1}{2}(n-2) \geq \frac{(n-s)^2}{20} - 1 - \frac{(2s-1)(n-2)}{2} \geq 0.$$

Thus,

$$\begin{aligned} |\mathcal{H}| &\leq f(s-i) = (s-i) \left\lfloor \frac{(n-s)^2}{4} \right\rfloor + 2\binom{s-i}{2} + 6i^2n + isn \\ &\leq s \left\lfloor \frac{(n-s)^2}{4} \right\rfloor + \left(-i\frac{(n-s)^2}{4} + i + 2\binom{s-i}{2} + 6i^2n + isn \right). \end{aligned}$$

Since $|A| = s-i \leq s-1$, we have $1 \leq i$. Since the coefficient of i^2 is $6n+1 > 0$, the last expression is a convex quadratic in i , and hence it suffices to check $i = 1$ and $i = \frac{s}{2}$. For $n \geq 18s$, a direct computation shows that

$$-i\frac{(n-s)^2}{4} + i + 2\binom{s-i}{2} + 6i^2n + isn < 0.$$

Thus $|\mathcal{H}| < s \left\lfloor \frac{(n-s)^2}{4} \right\rfloor$. ◇

Now $S = A = \{1, 2, \dots, s\}$. Then $i = 0$ and by (11) we have

$$|\mathcal{H}| \leq \sum_{v \in [s]} \deg(v, [s+1, n]) + \sum_{\{u, v\} \in \binom{[s]}{2}} \deg(u, v). \quad (16)$$

Claim 20. *We may assume $T = S$.*

Proof of the claim. If $t \leq s-1$, then (16) and (15),

$$|\mathcal{H}| \leq t \left\lfloor \frac{(n-s)^2}{4} \right\rfloor + (s-t) \frac{(n-s)^2}{5} + \binom{s}{2} n - \binom{t}{2} (n-2) =: f(t).$$

By $t \leq s-1$ and $n \geq 30(s+1)$,

$$f'(t) = \left\lfloor \frac{(n-s)^2}{4} \right\rfloor - \frac{(n-s)^2}{5} - \frac{2t-1}{2} (n-2) \geq 0.$$

Thus,

$$\begin{aligned} |\mathcal{H}| &\leq f(s-1) \leq (s-1) \left\lfloor \frac{(n-s)^2}{4} \right\rfloor + \frac{(n-s)^2}{5} + 2 \binom{s-1}{2} + sn \\ &\leq s \left\lfloor \frac{(n-s)^2}{4} \right\rfloor - \frac{(n-s)^2}{20} + 1 + 2 \binom{s-1}{2} + sn \\ &< s \left\lfloor \frac{(n-s)^2}{4} \right\rfloor \end{aligned}$$

and we are done. $\diamond$

Recall that $T = \{v \in A : \deg(v, \bar{S}) > \frac{(n-s)^2}{5}\} = A = S = \{1, 2, \dots, s\}$. Recall also that the square graph of G , denoted by G^2 , has the same vertex set as G and $xy \in E(G^2)$ if and only if there exists z such that $xz, yz \in E(G)$.

Claim 21. *We may assume that every edge of $\mathcal{H}$ meets T in at most one vertex.*

Proof of the claim. Suppose that $\deg_{\mathcal{H}}(x, y) > 0$ for some distinct $x, y \in T$. By relabeling, let $x = 1, y = 2$, and choose $v \in N_{\mathcal{H}}(1, 2)$. Since $T = S$ and $s \geq 3$, choose $3 \in T \setminus \{1, 2\}$. Then by the F_5 -free property, for any $x_1 x_2 \in L(1, \bar{S}) \cap L(2, \bar{S})$ we have $\{v\} \cap \{x_1, x_2\} \neq \emptyset$. It follows that $|L(1, \bar{S}) \cap L(2, \bar{S})| \leq n-s$. Define G_3 as the square graph of $L(3, \bar{S})$ on the vertex set $\bar{S}$. By the F_5 -free property, we infer that $E(G_3) \cap (L(1, \bar{S}) \cup L(2, \bar{S})) = \emptyset$. By (2) we obtain that

$$\begin{aligned} |E(G_3)| &> \frac{1}{n-s} \cdot \sum_{v \in \bar{S}} \binom{\deg_{L(3, \bar{S})}(v)}{2} = \frac{1}{2(n-s)} \sum_{v \in \bar{S}} (\deg_{L(3, \bar{S})}(v)^2 - \deg_{L(3, \bar{S})}(v)) \\ &\geq \frac{1}{2(n-s)} \left(\frac{4|L(3, \bar{S})|^2}{n-s} - 2|L(3, \bar{S})| \right) \\ &= \frac{2|L(3, \bar{S})|^2}{(n-s)^2} - \frac{|L(3, \bar{S})|}{n-s}. \end{aligned}$$

Consequently,

$$\begin{aligned}
|L(1, \bar{S})| + |L(2, \bar{S})| &\leq |L(1, \bar{S}) \cup L(2, \bar{S})| + |L(1, \bar{S}) \cap L(2, \bar{S})| \\
&\leq \binom{n-s}{2} - |E(G_3)| + (n-s) \\
&< \binom{n-s}{2} - \frac{2|L(3, \bar{S})|^2}{(n-s)^2} + \frac{|L(3, \bar{S})|}{n-s} + (n-s).
\end{aligned}$$

We first deal separately with the case $s = 3$. By (16),

$$\begin{aligned}
|\mathcal{H}| &\leq \sum_{j=1}^3 |L(j, \bar{S})| + 6 \\
&< \binom{n-3}{2} - \frac{2|L(3, \bar{S})|^2}{(n-3)^2} + \frac{|L(3, \bar{S})|}{n-3} + (n-3) + |L(3, \bar{S})| + 6.
\end{aligned}$$

Since

$$|L(3, \bar{S})| - \frac{2|L(3, \bar{S})|^2}{(n-3)^2} = \frac{(n-3)^2}{8} - \frac{2}{(n-3)^2} \left(|L(3, \bar{S})| - \frac{(n-3)^2}{4} \right)^2 \leq \frac{(n-3)^2}{8},$$

and

$$\frac{|L(3, \bar{S})|}{n-3} \leq \frac{1}{n-3} \binom{n-3}{2} < \frac{n-3}{2}.$$

Consequently,

$$|\mathcal{H}| < \frac{5(n-3)^2}{8} + (n-3) + 6.$$

Since $n \geq 30(3+1)$, we have

$$|\mathcal{H}| < \frac{5(n-3)^2}{8} + (n-3) + 6 < 3 \left\lfloor \frac{(n-3)^2}{4} \right\rfloor.$$

Assume now that $s \geq 4$. By Theorem 2,

$$\sum_{j=3}^s |L(j, \bar{S})| \leq (s-2) \left\lfloor \frac{(n-s)^2}{4} \right\rfloor.$$

Together with $|L(3, \bar{S})| > \frac{(n-s)^2}{5}$, we obtain that

$$\begin{aligned}
|L(1, \bar{S})| + |L(2, \bar{S})| &\leq \binom{n-s}{2} - \frac{2}{25}(n-s)^2 + \frac{n-s}{5} + n-s \\
&\leq \frac{(n-s)^2}{2} - \frac{2}{25}(n-s)^2 + n-s.
\end{aligned}$$

By $n \geq 7s$, it follows that

$$\begin{aligned}
|\mathcal{H}| &< (s-2) \left\lfloor \frac{(n-s)^2}{4} \right\rfloor + 2 \binom{s}{2} + \frac{(n-s)^2}{2} - \frac{2}{25}(n-s)^2 + n - s \\
&< s \left\lfloor \frac{(n-s)^2}{4} \right\rfloor + 1 + n + s^2 - 2s - \frac{2}{25}(n-s)^2 \\
&< s \left\lfloor \frac{(n-s)^2}{4} \right\rfloor.
\end{aligned}$$

$\diamond$

Finally, by (16) and Theorem 2 we conclude that

$$|\mathcal{H}| = \sum_{v \in [s]} \deg(v, [s+1, n]) \leq s \left\lfloor \frac{(n-s)^2}{4} \right\rfloor$$

with equality holding if and only if $\mathcal{H} = \mathcal{H}_3(n, s)$ up to isomorphism. $\square$

5 Concluding remarks

In this paper, we have determined the Turán number of $\{F_5, M_{s+1}^3\}$ for $n \geq 30(s+1)$. It is natural to ask whether the same result remains valid throughout the full range in which an $(s+1)$ -matching can occur.

Conjecture 22. For $n \geq 3(s+1)$ and $s \geq 3$,

$$\text{ex}_3(n, \{F_5, M_{s+1}^3\}) = s \left\lfloor \frac{(n-s)^2}{4} \right\rfloor.$$

We conclude with a hypergraph analogue of Theorem 2. Let $\mathcal{H}_1, \dots, \mathcal{H}_p \subset \binom{[n]}{3}$. Define $\mathcal{H}_1 + \dots + \mathcal{H}_p$ to be the *edge-colored multi-3-graph* obtained by giving each copy of an edge from $\mathcal{H}_i$ color i . An F_5 in $\mathcal{H}_1 + \dots + \mathcal{H}_p$ is called *2-colored* if its three selected edge copies use exactly two colors. We say that $\mathcal{H}_1 + \dots + \mathcal{H}_p$ is *2-colored F_5 -free* if it contains no such copy.

Conjecture 23. Let $p \geq 2$, and let $\mathcal{H}_1, \dots, \mathcal{H}_p \subset \binom{[n]}{3}$. If $\mathcal{H}_1 + \dots + \mathcal{H}_p$ is 2-colored F_5 -free, then

$$\sum_{i=1}^p |\mathcal{H}_i| \leq \max \left\{ \binom{n}{3}, p \left\lfloor \frac{n}{3} \right\rfloor \left\lfloor \frac{n+1}{3} \right\rfloor \left\lfloor \frac{n+2}{3} \right\rfloor \right\}.$$

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