

Spectral generalized Turán problems for expansion hypergraphs

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Abstract

Let Q be an r -graph on q vertices and let $\alpha > 1$. The (α, Q) -spectral radius $\lambda_{\alpha, Q}(\mathcal{H})$ of an r -graph $\mathcal{H}$ is defined as the maximum value of the Q -Lagrangian polynomial over all vectors with unit α -norm. Given a family $\mathcal{F}$ of r -graphs, the (α, Q) -spectral Turán number $\text{spex}_{\alpha}(n, Q, \mathcal{F})$ of $\mathcal{F}$ is the maximum (α, Q) -spectral radius $\lambda_{\alpha, Q}(\mathcal{H})$ among all n -vertex $\mathcal{F}$ -free r -graphs $\mathcal{H}$. This spectral parameter provides a spectral counterpart of the generalized Turán problem and contains many classical spectral extremal problems as special cases.

In this paper, we first prove that for any $\alpha > 1$ and every 2-covered r -graph Q with $q \leq \ell$ vertices, the complete balanced ℓ -partite r -graph $T_{\ell, n}^r$ uniquely maximizes $\lambda_{\alpha, Q}(\mathcal{H})$ among all n -vertex ℓ -partite r -graphs. Based on this result and spectral stability theorems, we further determine the exact value of $\text{spex}_{\alpha}(n, Q, H_F^r)$ for all sufficiently large n , where H_F^r stands for the expansion of an edge-critical graph F with chromatic number $\ell + 1$. As corollaries, we obtain several Turán-type results for graphs and r -graphs, including spectral bounds for the clique spectral radius of F -free graphs and the α -spectral radius of H_F^r -free r -graphs, where F is an edge-critical graph.

Mathematics Subject Classifications: 05C50, 05C35

1 Introduction

For an integer $r \geq 2$, an r -uniform hypergraph (or simply r -graph) $\mathcal{H}$ is a pair (V, E) , where V denotes a finite set of vertices and E is a collection of r -subsets of V . We write $v(\mathcal{H})$ for the order of $\mathcal{H}$ and $|\mathcal{H}|$ for the number of edges in $\mathcal{H}$. A graph F is called *edge-critical* if there exists an edge e of F such that $\chi(F - e) < \chi(F)$. An r -graph Q is said to be *2-covered* if every pair of vertices of Q is contained in at least one edge of Q .

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For a graph F , the *expansion* H_F^r of F is the r -graph obtained by enlarging each edge of F with $(r - 2)$ new vertices.

Given a family $\mathcal{F}$ of r -graphs, an r -graph $\mathcal{H}$ is said to be $\mathcal{F}$ -free if it contains no member of $\mathcal{F}$ as a subgraph. A central problem in extremal combinatorics is the *Turán problem*, which asks for the maximum number of edges in an $\mathcal{F}$ -free r -graph on n vertices. The *Turán number* of $\mathcal{F}$ is defined as

$$\text{ex}(n, \mathcal{F}) := \max\{|\mathcal{H}| : v(\mathcal{H}) = n \text{ and } \mathcal{H} \text{ is } \mathcal{F}\text{-free}\}.$$

Turán first proved that $\text{ex}(n, K_{k+1}) = |T_{k,n}|$ in [12]. Here, the *Turán graph* $T_{k,n}$ is the complete k -partite graph on n vertices whose part sizes are as equal as possible. Following the seminal work of Turán, various results have been obtained. One of the most celebrated results is the Erdős-Stone-Simonovits theorem, which gives asymptotics of $\text{ex}(n, F)$ for all graphs F with chromatic number at least three.

Motivated by the classical Turán problem, Erdős [4] initiated the study of its generalized version, which was subsequently developed in a systematic manner by Alon and Shikhelman [1]. Let $\mathcal{N}(Q, \mathcal{H})$ denote the number of copies of Q in $\mathcal{H}$. The *generalized Turán problem* concerns the following function:

$$\text{ex}(n, Q, \mathcal{F}) := \max\{\mathcal{N}(Q, \mathcal{H}) : v(\mathcal{H}) = n \text{ and } \mathcal{H} \text{ is } \mathcal{F}\text{-free}\}.$$

When Q is the r -clique K_r , Alon and Shikhelman [1] proved the analogue of the Erdős-Stone-Simonovits theorem, by obtaining asymptotics on $\text{ex}(n, K_r, F)$ for all graphs F of chromatic number at least $r + 1$.

Analogous to the classical Turán problem, the spectral Turán-type problem has also emerged as a topic of considerable interest (see, e.g., [2, 9, 10]). This line of research, initiated by Nikiforov [10], aims to determine the maximum spectral radius of an n -vertex $\mathcal{F}$ -free graph.

For any real number $\alpha \geq 1$, the α -spectral radius of hypergraphs was introduced by Keevash, Lenz and Mubayi [6] and subsequently studied by Nikiforov [11]. Given an r -graph $\mathcal{H}$ of order n , and a vector $\mathbf{x} = (x_1, x_2, \dots, x_n)^T \in \mathbb{R}^n$, denote

$$P_{\mathcal{H}}(\mathbf{x}) = r! \sum_{e \in E(\mathcal{H})} \mathbf{x}^e,$$

where $\mathbf{x}^e = \prod_{v \in e} x_v$. The α -spectral radius of $\mathcal{H}$ is defined as

$$\lambda^{(\alpha)}(\mathcal{H}) := \max_{\|\mathbf{x}\|_{\alpha}=1} P_{\mathcal{H}}(\mathbf{x}) = \max_{\|\mathbf{x}\|_{\alpha}=1} r! \sum_{e \in E(\mathcal{H})} \mathbf{x}^e, \quad (1)$$

where $\|\mathbf{x}\|_{\alpha} := (|x_1|^{\alpha} + \dots + |x_n|^{\alpha})^{1/\alpha}$.

Recently, Liu [8] extended the results of Keevash, Lenz and Mubayi [6] to the spectral generalized Turán problem. To illustrate the problem, we introduce some necessary definitions and notations.

Let Q and $\mathcal{H}$ be two r -uniform hypergraphs. A mapping $\phi : V(Q) \rightarrow V(\mathcal{H})$ is said to be a *homomorphism* from Q to $\mathcal{H}$ if $\phi(e) \in E(\mathcal{H})$ for every edge $e \in E(Q)$. We denote by $\text{Inj}(Q, \mathcal{H})$ the set of all injective homomorphisms from Q to $\mathcal{H}$, and let $\text{inj}(Q, \mathcal{H}) = |\text{Inj}(Q, \mathcal{H})|$. The number of labeled copies of Q in $\mathcal{H}$ is then given by

$$\mathcal{N}(Q, \mathcal{H}) = \frac{\text{inj}(Q, \mathcal{H})}{|\text{Aut}(Q)|},$$

where $\text{Aut}(Q) = \text{Inj}(Q, Q)$ stands for the automorphism group of Q . For an n -vertex r -graph $\mathcal{H}$, we define its *Lagrangian Q -polynomial* as

$$P_{Q, \mathcal{H}}(x_1, \dots, x_n) = \sum_{\phi \in \text{Inj}(Q, \mathcal{H})} \prod_{i \in \phi(V(Q))} x_i.$$

The (α, Q) -spectral radius of $\mathcal{H}$ is given by

$$\lambda_{\alpha, Q}(\mathcal{H}) := \max_{\|\mathbf{x}\|_{\alpha}=1} P_{Q, \mathcal{H}}(x_1, \dots, x_n), \quad (2)$$

where $\|\mathbf{x}\|_{\alpha} := (|x_1|^{\alpha} + \dots + |x_n|^{\alpha})^{1/\alpha}$. If $\mathbf{x} \in \mathbb{R}^n$ is a vector with $\|\mathbf{x}\|_{\alpha} = 1$ such that $\lambda_{\alpha, Q}(\mathcal{H}) = P_{Q, \mathcal{H}}(\mathbf{x})$, then $\mathbf{x}$ is called an *eigenvector* corresponding to $\lambda_{\alpha, Q}(\mathcal{H})$. The (α, Q) -spectral Turán number of $\mathcal{F}$ is then defined as

$$\text{spex}_{\alpha}(n, Q, \mathcal{F}) := \max \{ \lambda_{\alpha, Q}(\mathcal{H}) : v(\mathcal{H}) = n \text{ and } \mathcal{H} \text{ is } \mathcal{F}\text{-free} \}.$$

Assume that $v(Q) = q$. Let

$$\text{inj}(n, Q, \mathcal{F}) := \max \{ \text{inj}(Q, \mathcal{H}) : v(\mathcal{H}) = n \text{ and } \mathcal{H} \text{ is } \mathcal{F}\text{-free} \}.$$

By an averaging argument, the ratio $\text{inj}(n, Q, \mathcal{F})/(n)_q$ is non-increasing in n , where $(n)_q = n(n-1)\dots(n-q+1)$. Together with $(n)_q/n^q \rightarrow 1$, the limit

$$\hat{\pi}(Q, \mathcal{F}) := \lim_{n \rightarrow \infty} \frac{\text{inj}(n, Q, \mathcal{F})}{n^q}$$

exists. The definitions below extend the relevant concepts introduced by Keevash, Lenz and Mubayi [6] to the framework of the generalized Turán problem. Fix an r -graph Q with $v(Q) = q$ and let $\mathcal{F}$ be a family of r -graphs. Let $\mathfrak{H}$ be a hereditary family of $\mathcal{F}$ -free r -graphs. Here, *hereditary* means that for every $\mathcal{H} \in \mathfrak{H}$, all subgraphs of $\mathcal{H}$ also belong to $\mathfrak{H}$. For every $n \in \mathbb{N}$, let

$$\lambda_{\alpha, Q}(n, \mathfrak{H}) := \max \{ \lambda_{\alpha, Q}(\mathcal{H}) : \mathcal{H} \in \mathfrak{H} \text{ and } v(\mathcal{H}) = n \}.$$

For an r -graph $\mathcal{H}$ and a vertex $v \in V(\mathcal{H})$, the Q -degree of v in $\mathcal{H}$ is defined as

$$d_{Q, \mathcal{H}}(v) := |\{ \phi \in \text{Inj}(Q, \mathcal{H}) : v \in \phi(V(Q)) \}|.$$

For convenience, we use $d_Q(\mathcal{H})$, $\delta_Q(\mathcal{H})$, $\Delta_Q(\mathcal{H})$ to denote the average, minimum, maximum Q -degree of $\mathcal{H}$, respectively.

Definition 1 ([8]). Let $\alpha \geq 1$ and $\delta \geq 0$ be real numbers. Let $N \geq 1$ be an integer.

(i) We say that $\mathcal{F}$ is (δ, N, Q) -smooth if for every $n \geq N$ the following inequality holds:

$$|\text{inj}(n, Q, \mathcal{F}) - \text{inj}(n-1, Q, \mathcal{F}) - q\hat{\pi}(Q, \mathcal{F}) \cdot n^{q-1}| \leq \delta n^{q-1}.$$

(ii) We say that $\mathcal{F}$ is (δ, N, Q) -degree stable with respect to $\mathfrak{H}$ if every $\mathcal{F}$ -free r -graph $\mathcal{H}$ on $n \geq N$ vertices with $\delta_Q(\mathcal{H}) \geq (1-\delta)q\hat{\pi}(Q, \mathcal{F})n^{q-1}$ is contained in $\mathfrak{H}$.

(iii) We say that $\mathfrak{H}$ is $(\alpha, \delta, N, Q, \mathcal{F})$ -balanced in spectral if for every $n \geq N$ the following inequality holds:

$$|\lambda_{\alpha, Q}(n, \mathfrak{H}) - \text{inj}(n, Q, \mathcal{F})/n^{q/\alpha}| \leq \delta n^{q-q/\alpha-1}.$$

Based on the definitions above, Liu [8] established the generalized α -spectral stability theorem, extending the stability theorem in [6].

Theorem 2 ([8]). Let $q \geq r \geq 2$ be integers and $\alpha > 1, \varepsilon > 0$ be real numbers. Let Q be an r -graph on q vertices, $\mathcal{F}$ be a family of r -graphs with $\hat{\pi}(Q, \mathcal{F}) > 0$, and $\mathfrak{H}$ be a hereditary family of $\mathcal{F}$ -free r -graphs. There exist constants $\delta > 0, M$, and N such that the following holds. Suppose that

- (i) $\mathcal{F}$ is (δ, M, Q) -smooth
- (ii) $\mathcal{F}$ is (ε, M, Q) -degree stable with respect to $\mathfrak{H}$, and
- (iii) $\mathfrak{H}$ is $(\alpha, \delta, M, Q, \mathcal{F})$ -balanced in spectral.

Then for every $\mathcal{F}$ -free r -graph $\mathcal{H}$ on $n \geq N$ vertices, we have

$$\lambda_{\alpha, Q}(\mathcal{H}) \leq \lambda_{\alpha, Q}(n, \mathfrak{H}),$$

with equality only if $\mathcal{H} \in \mathfrak{H}$. In particular, for every $n \geq N$,

$$\text{spex}_{\alpha}(n, Q, \mathcal{F}) = \lambda_{\alpha, Q}(n, \mathfrak{H}).$$

The application of the above theorem relies on degree stability, which is a fundamental and well-studied notion in extremal problems. The following degree stability result for expansion r -graphs will be needed later.

Theorem 3 ([3]). Let $\ell \geq k \geq r \geq 2$ be integers and Q be a blowup of some 2-covered k -vertex r -graph. Suppose that F is an edge-critical graph with $\chi(F) = \ell + 1$. Then there exist $\varepsilon > 0$ and $N_0 > 0$ such that the following statements hold for all $n \geq N_0$:

- (i) Every n -vertex H_F^r -free r -graph $\mathcal{H}$ with $\delta_Q(\mathcal{H}) \geq (1-\varepsilon)\frac{v(Q) \cdot \text{inj}(n, Q, H_F^r)}{n}$ is ℓ -partite.
- (ii) $\text{inj}(n, Q, H_F^r) = \text{inj}(Q, \mathcal{G})$ for some complete ℓ -partite r -graph $\mathcal{G}$ on n vertices.

Let $T_{\ell,n}^r$ be the complete ℓ -partite r -graph on n vertices where no two parts differ by more than one in size. The main results of this paper are as follows.

Theorem 4. *Let $n \geq \ell \geq q \geq r \geq 2$ be integers and let $\alpha > 1$. Let $\mathfrak{H}$ be the family of all ℓ -partite r -graphs and Q be a 2-covered q -vertex r -graph. For every n -vertex r -graph $\mathcal{H} \in \mathfrak{H}$, we have*

$$\lambda_{\alpha,Q}(\mathcal{H}) \leq \lambda_{\alpha,Q}(T_{\ell,n}^r),$$

equality holds if and only if $\mathcal{H} = T_{\ell,n}^r$.

Theorem 5. *Let $\ell \geq q \geq r \geq 2$ be integers and let $\alpha > 1$. Let Q be a 2-covered q -vertex r -graph. Suppose that F is an edge-critical graph with $\chi(F) = \ell + 1$. Then, for all sufficiently large n , every H_F^r -free r -graph $\mathcal{H}$ on n vertices satisfies*

$$\lambda_{\alpha,Q}(\mathcal{H}) \leq \lambda_{\alpha,Q}(T_{\ell,n}^r),$$

equality holds if and only if $\mathcal{H} = T_{\ell,n}^r$.

When $r = 2$, $\alpha = s$ and $Q = K_s$, $\lambda_{\alpha,Q}(G)$ becomes the s -clique spectral radius $\rho_s(G)$, which was introduced by Liu and Bu [7]. Theorem 5 immediately implies the following conclusion.

Corollary 6. *Suppose that F is an edge-critical graph with $\chi(F) = \ell + 1$. Let G be an n -vertex F -free graph. Then for any $2 \leq s \leq \ell$ and sufficiently large n ,*

$$\rho_s(G) \leq \rho_s(T_{\ell,n}),$$

equality holds if and only if $G = T_{\ell,n}$.

Note that $K_{\ell+1}$ is an edge-critical graph with chromatic number $\ell + 1$.

Corollary 7 ([13]). *Let G be an n -vertex $K_{\ell+1}$ -free graph. Then for any $2 \leq s \leq \ell$ and sufficiently large n ,*

$$\rho_s(G) \leq \rho_s(T_{\ell,n}),$$

equality holds if and only if $G = T_{\ell,n}$.

Remark: In fact, the above result in [13] holds for all integers $n \geq s$ (not only for sufficiently large n).

Note that if Q consists of a single hyperedge, then $\lambda_{\alpha,Q}(\mathcal{H})$ coincides with the α -spectral radius of $\mathcal{H}$. The following corollaries follow directly from Theorem 5.

Corollary 8. *Suppose that F is an edge-critical graph with $\chi(F) = \ell + 1$. For any $\ell \geq r \geq 3$ and $\alpha > 1$, there exists n_0 , such that for any H_F^r -free r -graph $\mathcal{H}$ on $n > n_0$ vertices, $\lambda_{\alpha}(\mathcal{H}) \leq \lambda_{\alpha}(T_{\ell,n}^r)$, with equality if and only if $\mathcal{H} = T_{\ell,n}^r$.*

Corollary 9 ([14]). *For any $\ell \geq r \geq 3$ and $\alpha > 1$, there exists n_0 , such that for any $H_{K_{\ell+1}}^r$ -free r -graph $\mathcal{H}$ on $n > n_0$ vertices, $\lambda_{\alpha}(\mathcal{H}) \leq \lambda_{\alpha}(T_{\ell,n}^r)$, with equality if and only if $\mathcal{H} = T_{\ell,n}^r$.*

The rest of the paper is organized as follows. In Section 2 we introduce some preliminary lemmas for the proofs of our main results. The proofs of Theorem 4 and Theorem 5 are presented in Sections 3 and 4, respectively.

2 Preliminaries

Throughout the proofs, we shall use several classical inequalities, including the Power Mean inequality, the Arithmetic Mean–Geometric Mean inequality, Bernoulli’s inequality, Hölder’s inequality, and Maclaurin’s inequality. A standard reference for these inequalities is [5].

Fix integers $\ell \geq q \geq r \geq 2$, and let Q be a 2-covered r -graph on q vertices. Let $\alpha > 0$ and $\mathbf{x} = (x_1, \dots, x_n)$ be a vector with $\|\mathbf{x}\|_\alpha = 1$. We present several results that will be used in the subsequent proofs.

Fact 10. ([8]) *Let $\mathcal{H}$ be an n -vertex r -graph with $E(\mathcal{H}) \neq \emptyset$. Then there exists a non-negative eigenvector corresponding to $\lambda_{\alpha,Q}(\mathcal{H})$.*

Fix an ordering $(u_1, \dots, u_q)$ on the vertices of Q . For a vertex $v \in V(\mathcal{H})$, we define the *ordered Q -link* $L_{Q,\mathcal{H}}^o(v)$ to be the ordered multiset consisting of all tuples $\phi - v$, where $\phi \in \text{Inj}(Q, \mathcal{H})$ and $v \in \phi(V(Q))$. Here $\phi - v$ stands for the ordered tuple obtained by removing v from $(\phi(u_1), \dots, \phi(u_q))$. For $w \in V(\mathcal{H})$, let $L_{Q,\mathcal{H}}(w)$ denote the multiset of sets $\phi(V(Q)) \setminus \{w\}$, indexed by $\phi \in \text{Inj}(Q, \mathcal{H})$ with $w \in \phi(V(Q))$.

Lemma 11. *Let $\mathcal{H}$ be an n -vertex r -graph containing a copy of Q . Suppose that $u, v \in V(\mathcal{H})$ are distinct vertices satisfying $L_{Q,\mathcal{H}}(u) = L_{Q,\mathcal{H}}(v)$. If $\alpha > 1$ and $\mathbf{x}$ is an eigenvector corresponding to $\lambda_{\alpha,Q}(\mathcal{H})$, then $x_u = x_v$.*

Proof. Since $L_{Q,\mathcal{H}}(u) = L_{Q,\mathcal{H}}(v)$, no term in $P_{Q,\mathcal{H}}(\mathbf{x})$ contains both x_u and x_v . We may therefore write

$$P_{Q,\mathcal{H}}(\mathbf{x}) = \sum_{\phi \in \text{Inj}(Q, \mathcal{H})} \prod_{i \in \phi(V(Q))} x_i = x_u A + x_v A + B,$$

where A and B are independent of x_u and x_v .

Suppose $x_u \neq x_v$. Define a vector $\mathbf{x}' = (x'_1, \dots, x'_n)$ by

$$x'_u = x'_v = \frac{x_u + x_v}{2}, \quad \text{and} \quad x'_i = x_i \quad \text{for all } i \in [n] \setminus \{u, v\}.$$

It is clear that $P_{Q,\mathcal{H}}(\mathbf{x}') = P_{Q,\mathcal{H}}(\mathbf{x})$. Since $\mathcal{H}$ contains a copy of Q , this implies that $\lambda_{\alpha,Q}(\mathcal{H}) = P_{Q,\mathcal{H}}(\mathbf{x}) = P_{Q,\mathcal{H}}(\mathbf{x}') > 0$. Hence $\mathbf{x}' \neq \mathbf{0}$. Since $\alpha > 1$, the power mean (PM) inequality implies

$$|x'_1|^\alpha + \dots + |x'_n|^\alpha < |x_1|^\alpha + \dots + |x_n|^\alpha = 1.$$

Set $\mathbf{y} = \frac{\mathbf{x}'}{\|\mathbf{x}'\|_\alpha}$. Then $\|\mathbf{y}\|_\alpha = 1$. Moreover, since $0 < \|\mathbf{x}'\|_\alpha < 1$, we have $\|\mathbf{x}'\|_\alpha^{-q} > 1$. Consequently,

$$\lambda_{\alpha,Q}(\mathcal{H}) \geq P_{Q,\mathcal{H}}(\mathbf{y}) = \frac{P_{Q,\mathcal{H}}(\mathbf{x}')}{(\|\mathbf{x}'\|_\alpha)^q} > P_{Q,\mathcal{H}}(\mathbf{x}') = P_{Q,\mathcal{H}}(\mathbf{x}) = \lambda_{\alpha,Q}(\mathcal{H}).$$

This leads to a contradiction. The result follows. □

Lemma 12. Let $\ell \geq q \geq r \geq 2$ be integers and let $\alpha > 1$. Suppose $\mathcal{H}$ is a complete ℓ -partite r -graph. Then every nonnegative eigenvector corresponding to $\lambda_{\alpha,Q}(\mathcal{H})$ is strictly positive.

Proof. Let $\mathcal{H}$ be a complete ℓ -partite r -graph. By Fact 10, there exists a nonnegative eigenvector $\mathbf{x}$ corresponding to $\lambda_{\alpha,Q}(\mathcal{H})$. Suppose $\mathbf{x}$ contains at least one zero entry. By Lemma 11, the entries of $\mathbf{x}$ are equal within each partition class. Hence, if $x_v = 0$ for some vertex v , then $x_w = 0$ for every vertex w in the same partition class. Let $\mathcal{H}'$ denote the induced subgraph of $\mathcal{H}$ on the vertices corresponding to positive entries of $\mathbf{x}$. Then $\mathcal{H}'$ is a complete k -partite r -graph with $r \leq k < \ell$, and

$$\lambda_{\alpha,Q}(\mathcal{H}) = \lambda_{\alpha,Q}(\mathcal{H}').$$

If every partition class of $\mathcal{H}'$ has size one, then $\mathcal{H}' = K_k^r$. Combining $\ell \geq q$ and $\ell > k$, we obtain

$$\lambda_{\alpha,Q}(\mathcal{H}) \geq \lambda_{\alpha,Q}(K_\ell^r) > \lambda_{\alpha,Q}(K_k^r) = \lambda_{\alpha,Q}(\mathcal{H}'),$$

which is a contradiction. Consequently, $\mathcal{H}'$ contains a partition class of size at least two.

Take a vertex i from a partition class of $\mathcal{H}'$ with size at least 2, and let j be a vertex satisfying $x_j = 0$. Clearly $j \notin V(\mathcal{H}')$. We define a new vector $\mathbf{x}'$ by setting $x'_j = x_i$, $x'_i = 0$, and $x'_k = x_k$ for all $k \notin \{i, j\}$. It is easy to see that $\|\mathbf{x}'\|_\alpha = 1$. We now prove that

$$P_{Q,\mathcal{H}}(\mathbf{x}') > P_{Q,\mathcal{H}}(\mathbf{x}),$$

which yields a contradiction.

Let i^* be another vertex in the same partition class as i . Since Q is 2-covered, no term in $P_{Q,\mathcal{H}}(\mathbf{x})$ contains both x_i and x_{i^*} . In contrast, $P_{Q,\mathcal{H}}(\mathbf{x}')$ includes extra terms that do not appear in $P_{Q,\mathcal{H}}(\mathbf{x})$, and all such additional terms are strictly positive. Hence $P_{Q,\mathcal{H}}(\mathbf{x}') > P_{Q,\mathcal{H}}(\mathbf{x})$, as desired. This completes the proof. $\square$

Lemma 13. Let $\ell \geq q \geq r \geq 2$ be integers and $\alpha > 1$. Let $\mathcal{H}$ be a complete ℓ -partite r -graph. If $\mathcal{H}'$ is a subgraph of $\mathcal{H}$, then $\lambda_{\alpha,Q}(\mathcal{H}') < \lambda_{\alpha,Q}(\mathcal{H})$, unless $\mathcal{H}' = \mathcal{H}$.

Proof. Suppose that $\lambda_{\alpha,Q}(\mathcal{H}') = \lambda_{\alpha,Q}(\mathcal{H})$. We will show $\mathcal{H}' = \mathcal{H}$. By Fact 10, we can choose a nonnegative vector $\mathbf{x}$ attaining $\lambda_{\alpha,Q}(\mathcal{H}')$ and satisfying $\|\mathbf{x}\|_\alpha = 1$.

We first claim that $V(\mathcal{H}') = V(\mathcal{H})$. Suppose for contradiction that $V(\mathcal{H}') \subsetneq V(\mathcal{H})$. We extend $\mathbf{x}$ to a vector $\tilde{\mathbf{x}}$ defined on $V(\mathcal{H})$ by setting $\tilde{x}_v = 0$ for each vertex $v \in V(\mathcal{H}) \setminus V(\mathcal{H}')$. It then follows that

$$P_{Q,\mathcal{H}}(\tilde{\mathbf{x}}) = P_{Q,\mathcal{H}'}(\mathbf{x}) = \lambda_{\alpha,Q}(\mathcal{H}'),$$

which implies that $\tilde{\mathbf{x}}$ achieves the spectral radius $\lambda_{\alpha,Q}(\mathcal{H})$. This yields a contradiction to Lemma 12, as every nonnegative eigenvector associated with $\lambda_{\alpha,Q}(\mathcal{H})$ is strictly positive. We therefore conclude that $V(\mathcal{H}') = V(\mathcal{H})$. By an identical argument, any eigenvector corresponding to $\lambda_{\alpha,Q}(\mathcal{H}')$ must also be strictly positive.

We now claim that $E(\mathcal{H}') = E(\mathcal{H})$. Suppose for contradiction that $\mathcal{H}'$ is a proper subgraph of $\mathcal{H}$. Since $\mathcal{H}$ is a complete ℓ -partite r -graph with $\ell \geq q$, every edge of $\mathcal{H}$

is contained in a copy of Q . It follows that removing any edge will strictly reduce the number of copies of Q . Since $\mathbf{x}$ is strictly positive, we have

$$P_{Q,\mathcal{H}'}(\mathbf{x}) < P_{Q,\mathcal{H}}(\mathbf{x}),$$

which is a contradiction. Therefore $E(\mathcal{H}') = E(\mathcal{H})$. This completes the proof. $\square$

Lemma 14. *For sufficiently large n , we have*

$$\text{inj}(n, Q, H_F^r) = \text{inj}(Q, T_{\ell,n}^r).$$

Proof. Let $\mathcal{G}$ be an H_F^r -free r -graph on n vertices satisfying $\text{inj}(n, Q, H_F^r) = \text{inj}(Q, \mathcal{G})$. We will prove that $\mathcal{G} = T_{\ell,n}^r$. By Theorem 3 (ii), $\mathcal{G}$ is a complete ℓ -partite r -graph with partition classes $V_1, \dots, V_\ell$ of respective sizes $n_1, \dots, n_\ell$. Since Q is 2-covered, we have

$$\text{inj}(Q, \mathcal{G}) = q! \sum_{1 \leq i_1 < \dots < i_q \leq \ell} n_{i_1} \cdots n_{i_q}.$$

We next prove that this quantity is maximized only if $n_1, \dots, n_\ell$ are as equal as possible, which implies that $\mathcal{G} = T_{\ell,n}^r$. Indeed, suppose there exist i, j with $n_i - n_j \geq 2$. Write

$$\text{inj}(Q, \mathcal{G}) = q! \cdot (An_i n_j + B(n_i + n_j) + C),$$

where nonnegative integers A, B and C are independent of n_i and n_j . Let $\mathcal{G}'$ be the complete ℓ -partite r -graph obtained from $\mathcal{G}$ by moving one vertex from V_i to V_j . Then we have

$$\text{inj}(Q, \mathcal{G}') - \text{inj}(Q, \mathcal{G}) = q! \cdot A((n_i - 1)(n_j + 1) - n_i n_j),$$

which is strictly positive since $n_i - n_j \geq 2$ and $A > 0$. This leads to a contradiction. Consequently, $\mathcal{G} = T_{\ell,n}^r$. $\square$

The following fact follows from the definition of $\hat{\pi}(Q, H_F^r)$.

Fact 15. *For any $\delta > 0$, there exists an integer $N_1(\delta)$ such that for all $n > N_1$,*

$$|d_Q(T_{\ell,n}^r) - q\hat{\pi}(Q, H_F^r)n^{q-1}| \leq \frac{\delta}{2}n^{q-1}.$$

Lemma 16. *For any $\delta > 0$, there exists an integer $N_2(\delta)$ such that for all $n > N_2$,*

$$\Delta_Q(T_{\ell,n}^r) - \delta_Q(T_{\ell,n}^r) \leq \frac{\delta}{2}n^{q-1}.$$

Proof. Let $V_1, \dots, V_\ell$ denote the partition classes of $T_{\ell,n}^r$, with $|V_i| = n_i$ and $n_1 \geq n_2 \geq \dots \geq n_\ell$. Take $u \in V_1$ and $v \in V_\ell$. For convenience, we set

$$A = \mathcal{N}(Q, T_{\ell,n}^r), \text{ and } B_i = \mathcal{N}(Q, T_{\ell-1, n-n_i}^r) \quad \text{for } i \in \{1, \ell\}.$$

We then obtain

$$\begin{aligned}
\Delta_Q(T_{\ell,n}^r) - \delta_Q(T_{\ell,n}^r) &= |d_Q(u) - d_Q(v)| \\
&= |\text{Aut}(Q)| \left| \frac{A - B_1}{n_1} - \frac{A - B_\ell}{n_\ell} \right| \\
&= |\text{Aut}(Q)| \left| \left(\frac{1}{n_1} - \frac{1}{n_\ell} \right) A + \left(\frac{1}{n_\ell} - \frac{1}{n_1} \right) B_\ell + \frac{B_\ell - B_1}{n_1} \right| \\
&\leq |\text{Aut}(Q)| \left(\left| \frac{1}{n_1} - \frac{1}{n_\ell} \right| A + \left| \frac{1}{n_\ell} - \frac{1}{n_1} \right| B_\ell + \frac{|B_\ell - B_1|}{n_1} \right) \\
&\leq |\text{Aut}(Q)| \left(\frac{A}{n_\ell^2} + \frac{B_\ell}{n_\ell^2} + \frac{|B_\ell - B_1|}{n_1} \right) \\
&\leq \frac{\delta}{2} n^{q-1},
\end{aligned}$$

where the last inequality holds since A and B_i are $O(n^q)$, and $|B_\ell - B_1|$ is $O(n^{q-1})$. $\square$

Lemma 17. *Let $\mathcal{H}$ be an n -vertex ℓ -partite r -graph. If $\alpha = 1$, then*

$$\lambda_{1,Q}(\mathcal{H}) \leq q! \binom{\ell}{q} \ell^{-q}.$$

Proof. Let $\mathbf{x}$ be a nonnegative n -dimensional vector satisfying $\|\mathbf{x}\|_1 = 1$ and $\lambda_{1,Q}(\mathcal{H}) = P_{Q,\mathcal{H}}(\mathbf{x})$. We further choose such an $\mathbf{x}$ to have the minimal number of positive entries, and let m denote this number. We claim $m \leq \ell$.

Suppose for contradiction that $m > \ell$. Then $\mathbf{x}$ has two positive entries x_i and x_j corresponding to vertices in the same partition class. Since Q is 2-covered, we have

$$P_{Q,\mathcal{H}}(\mathbf{x}) = x_i \frac{\partial P_{Q,\mathcal{H}}(\mathbf{x})}{\partial x_i} + x_j \frac{\partial P_{Q,\mathcal{H}}(\mathbf{x})}{\partial x_j} + S,$$

where S is independent of x_i and x_j .

Assume $\frac{\partial P_{Q,\mathcal{H}}(\mathbf{x})}{\partial x_i} \geq \frac{\partial P_{Q,\mathcal{H}}(\mathbf{x})}{\partial x_j}$, and define a new vector $\mathbf{x}'$ by

$$x'_i = x_i + x_j, \quad x'_j = 0, \quad \text{and} \quad x'_s = x_s \quad \text{for all } s \in [n] \setminus \{i, j\}.$$

Clearly $\|\mathbf{x}'\|_1 = 1$, and

$$P_{Q,\mathcal{H}}(\mathbf{x}') - P_{Q,\mathcal{H}}(\mathbf{x}) = x_j \left(\frac{\partial P_{Q,\mathcal{H}}(\mathbf{x})}{\partial x_i} - \frac{\partial P_{Q,\mathcal{H}}(\mathbf{x})}{\partial x_j} \right) \geq 0.$$

Hence $P_{Q,\mathcal{H}}(\mathbf{x}') = P_{Q,\mathcal{H}}(\mathbf{x})$, whereas $\mathbf{x}'$ has exactly $m - 1$ positive entries. This contradicts the minimality of m . We therefore conclude that $m \leq \ell$.

Without loss of generality, let $x_1, \dots, x_m$ denote the positive entries of $\mathbf{x}$. By Maclaurin's inequality,

$$\begin{aligned} P_{Q, \mathcal{H}}(\mathbf{x}) &\leq q! \sum_{1 \leq i_1 < \dots < i_q \leq m} x_{i_1} \cdots x_{i_q} \\ &\leq q! \binom{m}{q} \left(\frac{1}{m} \sum_{i=1}^m x_i \right)^q \\ &= q! \binom{m}{q} m^{-q} \leq q! \binom{\ell}{q} \ell^{-q}. \end{aligned}$$

This completes the proof. $\square$

Lemma 18. *Let $\mathcal{H}$ be an n -vertex ℓ -partite r -graph. If $\alpha > 1$, then*

$$\lambda_{\alpha, Q}(\mathcal{H}) \leq (q!)^{1/\alpha} \binom{\ell}{q}^{1/\alpha} \ell^{-q/\alpha} \text{inj}(Q, \mathcal{H})^{1-1/\alpha}. \quad (3)$$

Moreover, we have

$$\lambda_{\alpha, Q}(\mathcal{H}) \leq q! \binom{\ell}{q} \ell^{-q} n^{q-q/\alpha}, \quad (4)$$

equality holds if and only if $\ell \mid n$ and $\mathcal{H} = T_{\ell, n}^r$.

Proof. First, we prove inequality (3). Let $\mathbf{x}$ be a nonnegative eigenvector corresponding to $\lambda_{\alpha, Q}(\mathcal{H})$. By the power-mean (PM) inequality, we have

$$\lambda_{\alpha, Q}(\mathcal{H}) = \sum_{\phi \in \text{Inj}(Q, \mathcal{H})} \prod_{i \in \phi(V(Q))} x_i \leq \text{inj}(Q, \mathcal{H})^{1-1/\alpha} \left(\sum_{\phi \in \text{Inj}(Q, \mathcal{H})} \prod_{i \in \phi(V(Q))} x_i^\alpha \right)^{1/\alpha}. \quad (5)$$

Let $\mathbf{y} = (y_1, \dots, y_n) = (x_1^\alpha, \dots, x_n^\alpha)$. By Lemma 17,

$$\sum_{\phi \in \text{Inj}(Q, \mathcal{H})} \prod_{i \in \phi(V(Q))} x_i^\alpha = \sum_{\phi \in \text{Inj}(Q, \mathcal{H})} \prod_{i \in \phi(V(Q))} y_i \leq \lambda_{1, Q}(\mathcal{H}) \leq q! \binom{\ell}{q} \ell^{-q}. \quad (6)$$

Hence inequality (3) follows immediately.

We now prove inequality (4). Let $\mathcal{H}_2$ be the graph with vertex set $V(\mathcal{H}_2) = V(\mathcal{H})$, and let its edge set be defined by

$$E(\mathcal{H}_2) = \{uv \mid \exists \phi \in \text{Inj}(Q, \mathcal{H}) \text{ such that } u, v \in \phi(V(Q))\}.$$

Then each copy of Q in $\mathcal{H}$ corresponds to a unique q -clique in $\mathcal{H}_2$. Consequently,

$$\text{inj}(Q, \mathcal{H}) \leq q! \cdot k_q(\mathcal{H}_2).$$

By Zykov's theorem [15], we have

$$k_q(\mathcal{H}_2) \leq k_q(T_{\ell,n}) \leq \binom{\ell}{q} \left(\frac{n}{\ell}\right)^q,$$

equality holds if and only if $\ell \mid n$ and $\mathcal{H}_2 = T_{\ell,n}$. Substituting this bound gives

$$\text{inj}(Q, \mathcal{H}) \leq q! \binom{\ell}{q} \left(\frac{n}{\ell}\right)^q, \quad (7)$$

with equality if and only if $\ell \mid n$ and $\mathcal{H} = T_{\ell,n}^r$.

Combining (5), (6) and (7), we obtain

$$\lambda_{\alpha,Q}(\mathcal{H}) \leq (q!)^{1/\alpha} \binom{\ell}{q}^{1/\alpha} \ell^{-q/\alpha} \left(q! \binom{\ell}{q} \left(\frac{n}{\ell}\right)^q \right)^{1-1/\alpha} = q! \binom{\ell}{q} \ell^{-q} n^{q-q/\alpha}.$$

This establishes inequality (4). If equality holds in (4), then equality must hold in (7). Consequently, $\ell \mid n$ and $\mathcal{H} = T_{\ell,n}^r$. $\square$

Lemma 19. Let $(\ell)_q = \ell(\ell-1)\cdots(\ell-q+1)$. For sufficiently large n , we have

$$\text{inj}(Q, T_{\ell,n}^r) = \frac{(\ell)_q}{\ell^q} n^q + O(n^{q-2}). \quad (8)$$

Proof. Let $n = \ell t + s$ with integers $\ell > s \geq 0$. Since Q is 2-covered,

$$\begin{aligned} \text{inj}(Q, T_{\ell,n}^r) &= q! \sum_{i=0}^q \binom{s}{i} \binom{\ell-s}{q-i} (t+1)^i t^{q-i} \\ &= q! \sum_{i=0}^q \binom{s}{i} \binom{\ell-s}{q-i} (t^q + it^{q-1} + O(t^{q-2})) \\ &= q! \left(\sum_{i=0}^q \binom{s}{i} \binom{\ell-s}{q-i} \left(\frac{n-s}{\ell}\right)^q \right. \\ &\quad \left. + \sum_{i=0}^q i \binom{s}{i} \binom{\ell-s}{q-i} \left(\frac{n-s}{\ell}\right)^{q-1} \right) + O(n^{q-2}). \end{aligned}$$

By Vandermonde's identity and the combinatorial identity $i \binom{s}{i} = s \binom{s-1}{i-1}$, we have

$$\begin{aligned} \frac{\text{inj}(Q, T_{\ell,n}^r)}{q!} &= \left(\binom{\ell}{q} \left(\frac{n-s}{\ell}\right)^q + \sum_{i=1}^q s \binom{s-1}{i-1} \binom{\ell-s}{q-i} \left(\frac{n-s}{\ell}\right)^{q-1} \right) + O(n^{q-2}) \\ &= \left(\frac{(\ell)_q}{q! \ell^q} n^q - \frac{sq}{\ell^q} \binom{\ell}{q} n^{q-1} + s \sum_{i=0}^{q-1} \binom{s-1}{i} \binom{\ell-s}{q-1-i} \left(\frac{n}{\ell}\right)^{q-1} \right) + O(n^{q-2}) \\ &= \left(\frac{(\ell)_q}{q! \ell^q} n^q - \frac{sq}{\ell^q} \binom{\ell}{q} n^{q-1} + \frac{s}{\ell^{q-1}} \binom{\ell-1}{q-1} n^{q-1} \right) + O(n^{q-2}) \\ &= \left(\frac{(\ell)_q}{q! \ell^q} n^q + O(n^{q-2}) \right). \end{aligned}$$

This completes the proof. $\square$

3 Proof of Theorem 4

Let $\mathcal{H} \in \mathfrak{H}$ be an r -graph achieving $\lambda_{\alpha, Q}(n, \mathfrak{H})$. By Lemma 13, $\mathcal{H}$ is a complete ℓ -partite r -graph with partition classes $V_1, \dots, V_\ell$. Let $n_i = |V_i|$ for $i \in [\ell]$, with $n_1 \leq n_2 \leq \dots \leq n_\ell$ and $\sum_{i=1}^\ell n_i = n$. We proceed by contradiction and assume that $n_\ell - n_1 \geq 2$. By Lemma 12, $\lambda_{\alpha, Q}(\mathcal{H})$ admits a positive eigenvector $\mathbf{x}$. Furthermore, Lemma 11 implies that all entries of $\mathbf{x}$ corresponding to vertices in the same partition class are equal. For each $i \in [\ell]$, let a_i be the value of the entries of $\mathbf{x}$ on V_i . For convenience, set $c = n_1 a_1^\alpha + n_\ell a_\ell^\alpha$.

Define

$$\begin{aligned} S_0 &= \sum_{1 < i_1 < \dots < i_q < \ell} n_{i_1} a_{i_1} \cdots n_{i_q} a_{i_q}, \\ S_1 &= \sum_{1 < i_1 < \dots < i_{q-1} < \ell} n_{i_1} a_{i_1} \cdots n_{i_{q-1}} a_{i_{q-1}}, \\ S_2 &= \sum_{1 < i_1 < \dots < i_{q-2} < \ell} n_{i_1} a_{i_1} \cdots n_{i_{q-2}} a_{i_{q-2}}. \end{aligned}$$

Since Q is 2-covered, $P_{Q, \mathcal{H}}(\mathbf{x})$ can be written as follows.

$$\frac{P_{Q, \mathcal{H}}(\mathbf{x})}{q!} = (n_1 a_1 + n_\ell a_\ell) S_1 + (n_1 a_1) (n_\ell a_\ell) S_2 + S_0.$$

Obviously, S_0, S_1 and S_2 are independent of a_1, a_ℓ, n_1, n_ℓ .

We split the argument into two cases. In each case we construct a hypergraph $\mathcal{H}'$ satisfying

$$\lambda_{\alpha, Q}(\mathcal{H}') > \lambda_{\alpha, Q}(\mathcal{H}).$$

Case 1. $n_\ell + n_1 = 2m$ for an integer m .

Let $\mathcal{H}'$ denote the complete ℓ -partite r -graph with vertex partition

$$V(\mathcal{H}') = V'_1 \cup \dots \cup V'_\ell,$$

satisfying $|V'_1| = |V'_\ell| = m$, $V'_1 \cup V'_\ell = V_1 \cup V_\ell$, and $V'_i = V_i$ for all $1 < i < \ell$. Define a vector $\mathbf{y} = (y_1, \dots, y_n)$ by

$$y_i = \left(\frac{c}{2m}\right)^{1/\alpha} \triangleq b \quad \text{for } i \in V'_1 \cup V'_\ell,$$

and $y_i = x_i$ for all $i \in \bigcup_{k=2}^{\ell-1} V'_k$. Observe that $y_1^\alpha + \dots + y_n^\alpha = 1$. Moreover,

$$\frac{P_{Q, \mathcal{H}'}(\mathbf{y}) - P_{Q, \mathcal{H}}(\mathbf{x})}{q!} = (m^2 b^2 - n_1 n_\ell a_1 a_\ell) S_2 + (2mb - (n_1 a_1 + n_\ell a_\ell)) S_1.$$

We claim that

$$m^2b^2 - n_1n_\ell a_1a_\ell > 0, \text{ and } 2mb - (n_1a_1 + n_\ell a_\ell) \geq 0,$$

which implies that $P_{Q,\mathcal{H}'}(\mathbf{y}) > P_{Q,\mathcal{H}}(\mathbf{x})$.

Since $m^2 > n_1n_\ell$ and $\alpha > 1$, applying the AM-GM inequality yields

$$\begin{aligned} m^2b^2 &= m^2 \left(\frac{n_1a_1^\alpha + n_\ell a_\ell^\alpha}{2m} \right)^{2/\alpha} \geq m^{2-\frac{2}{\alpha}} \left(\sqrt{n_1a_1^\alpha n_\ell a_\ell^\alpha} \right)^{2/\alpha} \\ &= \left(\frac{m^2}{n_1n_\ell} \right)^{1-1/\alpha} n_1n_\ell a_1a_\ell \\ &> n_1n_\ell a_1a_\ell. \end{aligned}$$

For the second estimate, the power-mean (PM) inequality implies

$$2mb = 2m \left(\frac{n_1a_1^\alpha + n_\ell a_\ell^\alpha}{2m} \right)^{1/\alpha} \geq 2m \left(\frac{n_1a_1 + n_\ell a_\ell}{2m} \right) = n_1a_1 + n_\ell a_\ell.$$

Therefore,

$$\lambda_{\alpha,Q}(\mathcal{H}') \geq P_{Q,\mathcal{H}'}(\mathbf{y}) > P_{Q,\mathcal{H}}(\mathbf{x}) = \lambda_{\alpha,Q}(\mathcal{H}),$$

which is a contradiction. This completes the proof for the case that $n_1 + n_\ell$ is even.

Case 2. $n_1 + n_\ell = 2m + 1$ for an integer m .

Let $\mathcal{H}'$ be the complete ℓ -partite r -graph with vertex partition

$$V(\mathcal{H}') = V'_1 \cup \cdots \cup V'_\ell,$$

satisfying $|V'_1| = m$, $|V'_\ell| = m + 1$, $V'_1 \cup V'_\ell = V_1 \cup V_\ell$ and $V'_i = V_i$ for all $1 < i < \ell$. Define a vector $\mathbf{y} = (y_1, \dots, y_n)$ by

$$y_i = \left(\frac{c}{2m+1} \right)^{1/\alpha} \triangleq b \quad \text{for } i \in V'_1 \cup V'_\ell,$$

and $y_i = x_i$ for all $i \in \bigcup_{k=2}^{\ell-1} V'_k$. Observe that $y_1^\alpha + \cdots + y_n^\alpha = 1$. Further, we obtain

$$\frac{P_{Q,\mathcal{H}'}(\mathbf{y}) - P_{Q,\mathcal{H}}(\mathbf{x})}{q!} = (m(m+1)b^2 - n_1n_\ell a_1a_\ell) S_2 + ((2m+1)b - (n_1a_1 + n_\ell a_\ell)) S_1.$$

First, we claim the lower-bound estimate:

$$(2m+1)b - (n_1a_1 + n_\ell a_\ell) \geq 0.$$

This inequality follows directly from the power mean (PM) inequality. Indeed,

$$\begin{aligned}(2m+1)b &= (2m+1) \left(\frac{n_1 a_1^\alpha + n_\ell a_\ell^\alpha}{2m+1} \right)^{1/\alpha} \\ &\geq (2m+1) \cdot \frac{n_1 a_1 + n_\ell a_\ell}{2m+1} \\ &= n_1 a_1 + n_\ell a_\ell.\end{aligned}$$

We then proceed to verify

$$m(m+1)b^2 - n_1 n_\ell a_1 a_\ell > 0,$$

which contradicts the extremal optimality of $\mathcal{H}$.

We split the proof into two subcases by considering $\alpha > 9/8$ and $1 < \alpha \leq 9/8$. Here, the artifactual threshold $9/8$ arises from the method used in our proof.

Subcase 2.1. $\alpha > 9/8$.

Since $\alpha > 9/8$, applying Bernoulli's inequality gives

$$\begin{aligned}\left(\frac{(2m+1)^2}{4(m-1)(m+2)} \right)^{1/\alpha} &= \left(1 + \frac{9}{4(m-1)(m+2)} \right)^{1/\alpha} \\ &\leq 1 + \frac{9}{4\alpha(m-1)(m+2)} \\ &< \frac{m(m+1)}{(m-1)(m+2)}.\end{aligned}\tag{9}$$

Recall that $n_1 n_\ell \leq (m-1)(m+2)$. By the AM-GM inequality, we have

$$\begin{aligned}m(m+1)b^2 &= m(m+1) \left(\frac{n_1 a_1^\alpha + n_\ell a_\ell^\alpha}{2m+1} \right)^{2/\alpha} \\ &\geq m(m+1) \left(\frac{1}{2m+1} \right)^{2/\alpha} (2\sqrt{n_1 a_1^\alpha n_\ell a_\ell^\alpha})^{2/\alpha} \\ &= m(m+1) \left(\frac{4}{(2m+1)^2} \right)^{1/\alpha} \left(\frac{1}{n_1 n_\ell} \right)^{1-1/\alpha} n_1 n_\ell a_1 a_\ell \\ &\geq m(m+1) \left(\frac{4}{(2m+1)^2} \right)^{1/\alpha} \left(\frac{1}{(m-1)(m+2)} \right)^{1-1/\alpha} n_1 n_\ell a_1 a_\ell \\ &= \frac{m(m+1)}{(m-1)(m+2)} \left(\frac{4(m-1)(m+2)}{(2m+1)^2} \right)^{1/\alpha} n_1 n_\ell a_1 a_\ell \\ &> n_1 n_\ell a_1 a_\ell,\end{aligned}$$

where the last inequality follows from (9).

Subcase 2.2. $1 < \alpha \leq 9/8$.

Assume for contradiction that $m(m+1)b^2 - n_1 n_\ell a_1 a_\ell \leq 0$. This immediately yields

$$a_1 a_\ell \geq \frac{m(m+1)}{n_1 n_\ell} \left(\frac{c}{2m+1} \right)^{2/\alpha}. \quad (10)$$

Define a vector $\mathbf{z} = (z_1, z_2, \dots, z_n)$ as follows:

$$\begin{cases} z_i = \frac{n_1 a_1}{m} \triangleq b_1, & \text{if } i \in V'_1, \\ z_j = \frac{n_\ell a_\ell}{m+1} \triangleq b_\ell, & \text{if } j \in V'_\ell, \\ z_k = x_k, & \text{for all } k \in V'_2 \cup V'_3 \cup \dots \cup V'_{\ell-1}. \end{cases}$$

One can readily verify that

$$\begin{aligned} \frac{P_{Q, \mathcal{H}'}(\mathbf{z}) - P_{Q, \mathcal{H}}(\mathbf{x})}{q!} &= (m(m+1)b_1 b_\ell - n_1 n_\ell a_1 a_\ell) S_2 \\ &\quad + (m b_1 + (m+1)b_\ell - n_1 a_1 - n_\ell a_\ell) S_1 \\ &= 0. \end{aligned}$$

Then

$$\lambda_{\alpha, Q}(\mathcal{H}) = P_{Q, \mathcal{H}}(\mathbf{x}) = P_{Q, \mathcal{H}'}(\mathbf{z}) \leq \lambda_{\alpha, Q}(\mathcal{H}') \|\mathbf{z}\|_\alpha^q. \quad (11)$$

Since $\lambda_{\alpha, Q}(\mathcal{H}') \leq \lambda_{\alpha, Q}(\mathcal{H})$, inequality (11) implies that

$$\|\mathbf{z}\|_\alpha \geq 1. \quad (12)$$

On the other hand, one has

$$z_1^\alpha + z_2^\alpha + \dots + z_n^\alpha = 1 - (n_1 a_1^\alpha + n_\ell a_\ell^\alpha) + m b_1^\alpha + (m+1) b_\ell^\alpha.$$

Combining this identity with (12), we arrive at

$$m b_1^\alpha + (m+1) b_\ell^\alpha \geq n_1 a_1^\alpha + n_\ell a_\ell^\alpha.$$

Substituting $b_1 = \frac{n_1 a_1}{m}$ and $b_\ell = \frac{n_\ell a_\ell}{m+1}$ into the above inequality gives

$$m \left(\frac{n_1 a_1}{m} \right)^\alpha + (m+1) \left(\frac{n_\ell a_\ell}{m+1} \right)^\alpha \geq n_1 a_1^\alpha + n_\ell a_\ell^\alpha.$$

Rearranging terms, we obtain

$$n_\ell a_\ell^\alpha \left(\left(\frac{n_\ell}{m+1} \right)^{\alpha-1} - 1 \right) \geq n_1 a_1^\alpha \left(1 - \left(\frac{n_1}{m} \right)^{\alpha-1} \right).$$

Given $\alpha > 1$, together with the conditions $(n_\ell/(m+1))^{\alpha-1} - 1 > 0$ and $1 - (n_1/m)^{\alpha-1} > 0$, we further deduce that

$$\frac{n_\ell a_\ell^\alpha}{n_1 a_1^\alpha} \geq \frac{1 - (n_1/m)^{\alpha-1}}{(n_\ell/(m+1))^{\alpha-1} - 1}. \quad (13)$$

For $1 < \alpha \leq \frac{9}{8}$, we apply Bernoulli's inequality to obtain

$$\left(\frac{n_1}{m}\right)^{\alpha-1} = \left(1 - \frac{m-n_1}{m}\right)^{\alpha-1} < 1 - \frac{(\alpha-1)(m-n_1)}{m}. \quad (14)$$

Similarly,

$$\left(\frac{n_\ell}{m+1}\right)^{\alpha-1} = \left(1 + \frac{n_\ell-m-1}{m+1}\right)^{\alpha-1} < 1 + \frac{(\alpha-1)(n_\ell-m-1)}{m+1}. \quad (15)$$

Note that $m - n_1 = n_\ell - m - 1$. Combining with (13), (14) and (15), we have

$$\frac{n_\ell a_\ell^\alpha}{n_1 a_1^\alpha} > \frac{m+1}{m}. \quad (16)$$

Using (16) and the identity $n_1 a_1^\alpha + n_\ell a_\ell^\alpha = c$, we derive

$$n_1 a_1^\alpha n_\ell a_\ell^\alpha < \frac{m(m+1)}{(2m+1)^2} c^2.$$

Rearranging this inequality yields

$$a_1 a_\ell < \left(\frac{m(m+1)}{n_1 n_\ell}\right)^{1/\alpha} \left(\frac{c}{2m+1}\right)^{2/\alpha}.$$

Combining the above estimate with (10), we conclude

$$\left(\frac{m(m+1)}{n_1 n_\ell}\right)^{1/\alpha} > \frac{m(m+1)}{n_1 n_\ell}.$$

However, this is a contradiction, since $m(m+1) > n_1 n_\ell$ and $1/\alpha < 1$ jointly imply that the reverse inequality must hold.

Therefore, whenever $n_\ell - n_1 \geq 2$, both Case 1 and Case 2 lead to $\lambda_{\alpha, Q}(\mathcal{H}') > \lambda_{\alpha, Q}(\mathcal{H})$, which contradicts the optimality of $\mathcal{H}$. Consequently, we must have $\mathcal{H} = T_{\ell, n}^r$. $\square$

4 Proof of Theorem 5

Fix integers $\ell \geq q \geq r \geq 2$ and a real number $\alpha > 1$. Let Q be a 2-covered r -graph on q vertices. Recall that H_F^r denotes the expansion of F , where F is an edge-critical graph

with chromatic number $\chi(F) = \ell + 1$. Let $\mathfrak{H}$ be a hereditary family of ℓ -partite r -graphs that are H_F^r -free.

We now apply Theorem 2 to establish Theorem 5. In view of Theorem 4, it is enough to verify conditions (i), (ii), and (iii) of Theorem 2.

We first verify condition (i). It suffices to prove that there exists an integer N such that for all $n \geq N$,

$$|\text{inj}(n, Q, H_F^r) - \text{inj}(n-1, Q, H_F^r) - q\hat{\pi}(Q, H_F^r) \cdot n^{q-1}| \leq \delta n^{q-1}.$$

By Lemmas 14 and 16, we obtain

$$\begin{aligned} & |\text{inj}(n, Q, H_F^r) - \text{inj}(n-1, Q, H_F^r) - d_Q(T_{\ell,n}^r)| \\ &= |\text{inj}(Q, T_{\ell,n}^r) - \text{inj}(Q, T_{\ell,n-1}^r) - d_Q(T_{\ell,n}^r)| \\ &\leq |\Delta_Q(T_{\ell,n}^r) - \delta_Q(T_{\ell,n}^r)| \\ &\leq \frac{\delta}{2} n^{q-1}. \end{aligned}$$

Combining this with Fact 15, we conclude that condition (i) is satisfied.

Condition (ii) follows immediately from Theorem 3.

For condition (iii), Theorem 4 yields

$$\lambda_{\alpha,Q}(n, \mathfrak{H}) = \lambda_{\alpha,Q}(T_{\ell,n}^r).$$

By Lemma 14, we have $\text{inj}(n, Q, H_F^r) = \text{inj}(Q, T_{\ell,n}^r)$. It therefore suffices to establish the following inequality:

$$\frac{\text{inj}(Q, T_{\ell,n}^r)}{n^{q/\alpha}} - \delta n^{q-q/\alpha-1} \leq \lambda_{\alpha,Q}(T_{\ell,n}^r) \leq \frac{\text{inj}(Q, T_{\ell,n}^r)}{n^{q/\alpha}} + \delta n^{q-q/\alpha-1}. \quad (17)$$

The lower bound is trivial. Indeed, let $\mathbf{x} = (n^{-1/\alpha}, \dots, n^{-1/\alpha})$ be the uniform vector. One readily verifies that

$$\lambda_{\alpha,Q}(T_{\ell,n}^r) \geq P_{Q, T_{\ell,n}^r}(\mathbf{x}) = \frac{\text{inj}(Q, T_{\ell,n}^r)}{n^{q/\alpha}} > \frac{\text{inj}(Q, T_{\ell,n}^r)}{n^{q/\alpha}} - \delta n^{q-q/\alpha-1}$$

for all $\delta > 0$. This proves the lower bound in (17).

We now consider the upper bound in (17). Combining (4) and (8), we obtain

$$\lambda_{\alpha,Q}(T_{\ell,n}^r) \leq \frac{\text{inj}(Q, T_{\ell,n}^r)}{n^{q/\alpha}} + O(n^{q-q/\alpha-2}).$$

For any fixed $\delta > 0$, there exists an integer N such that the inequality

$$\lambda_{\alpha,Q}(T_{\ell,n}^r) \leq \frac{\text{inj}(Q, T_{\ell,n}^r)}{n^{q/\alpha}} + \delta n^{q-q/\alpha-1}$$

holds for all $n > N$. This completes the proof of the upper bound in (17).

Since all three conditions of Theorem 2 are satisfied, combining Theorem 2 with Theorem 4 yields that for all $n \geq N$,

$$\text{spex}_\alpha(n, Q, H_F^r) = \lambda_{\alpha, Q}(n, \mathfrak{H}) = \lambda_{\alpha, Q}(T_{\ell, n}^r).$$

To establish uniqueness, suppose that an H_F^r -free r -graph $\mathcal{H}$ satisfies $\lambda_{\alpha, Q}(\mathcal{H}) = \lambda_{\alpha, Q}(T_{\ell, n}^r)$. Since $\lambda_{\alpha, Q}(\mathcal{H}) = \lambda_{\alpha, Q}(T_{\ell, n}^r) = \lambda_{\alpha, Q}(n, \mathfrak{H})$, equality in Theorem 2 yields $\mathcal{H} \in \mathfrak{H}$, so $\mathcal{H}$ is ℓ -partite. The equality case of Theorem 4 then forces $\mathcal{H} = T_{\ell, n}^r$. This completes the proof. $\square$

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References

- [1] N. Alon, C. Shikhelman. Many T copies in H -free graphs. *Journal of Combinatorial Theory, Series B*, 121: 146–172, 2016.
- [2] R. A. Brualdi, E. S. Solheid. On the spectral radius of complementary acyclic matrices of zeros and ones. *SIAM Journal on Algebraic Discrete Methods*, 7: 265–272, 1986.
- [3] W. Chen, X. Liu. Strong stability from vertex-extendability and applications in generalized Turán problems. [arXiv:2406.05748](https://arxiv.org/abs/2406.05748), 2024.
- [4] P. Erdős. Über ein Extremalproblem in der Graphentheorie. *Archiv der Mathematik (Basel)*, 13: 222–227, 1962.
- [5] G. H. Hardy, J. E. Littlewood, G. Pólya. Inequalities. 2nd edition, *Cambridge University Press*, vi+324, 1988.
- [6] P. Keevash, J. Lenz, D. Mubayi. Spectral extremal problems for hypergraphs. *SIAM Journal on Discrete Mathematics*, 28: 1838–1854, 2014.
- [7] C. Liu, C. Bu. A tensor’s spectral bound on the clique number. *Discrete Mathematics*, 349: 114694, 2026.
- [8] X. Liu. Spectral generalized Turán problems. [arXiv:2507.21689](https://arxiv.org/abs/2507.21689), 2025.
- [9] V. Nikiforov. Bounds on graph eigenvalues II. *Linear Algebra and its Applications*, 427: 183–189, 2007.
- [10] V. Nikiforov. The spectral radius of graphs without paths and cycles of specified length. *Linear Algebra and its Applications*, 432: 2243–2256, 2010.
- [11] V. Nikiforov. Analytic methods for uniform hypergraphs. *Linear Algebra and its Applications*, 457: 455–535, 2014.
- [12] P. Turán. On an extremal problem in graph theory. *Matematikai és Fizikai Lapok*, 48: 436–452, 1941.

- [13] L. Yu, Y. Peng. A Spectral Version of the Theorem of Zykov and Erdős. *The Electronic Journal of Combinatorics*, 32(4):#P4.15, 2025.
- [14] J. Zheng, H. Li, Y. Fan. Spectral Turán Problems for Nondegenerate Hypergraphs. *The Electronic Journal of Combinatorics*, 33(1):#P1.42, 2026.
- [15] A. A. Zykov. On some properties of linear complexes (in Russian). *Matematicheskii Sbornik*, 24: 163–188, 1949.