

Generalized Ramsey numbers via conflict-free hypergraph matchings

Andrew Lane^a Natasha Morrison^b

Submitted: Aug 12, 2025; Accepted: May 3, 2026; Published: Oct 9, 2026

© The authors. Released under the CC BY-ND license (International 4.0).

Abstract

Given graphs G, H and an integer $q \geq 2$, the *generalized Ramsey number*, denoted $r(G, H, q)$, is the minimum number of colours needed to edge-colour G such that every copy of H receives at least q colours. In this paper, we prove that for a fixed integer $k \geq 3$, we have $r(K_n, C_k, 3) = n/(k-2) + o(n)$. This generalizes the work of Joos and Mubayi, who proved $r(K_n, C_4, 3) = n/2 + o(n)$. We also provide an upper bound on $r(K_{n,n}, C_k, 3)$, which generalizes a result of Joos and Mubayi that $r(K_{n,n}, C_4, 3) = 2n/3 + o(n)$. Both of our results are in fact specific cases of more general theorems concerning families of cycles.

Mathematics Subject Classifications: 05C55, 05C65

1 Introduction

The classical Ramsey problem asks the question: given a graph H and $k \geq 2$, what is the minimum number $n = R_k(H)$ such that any edge-colouring of K_n using k colours contains a monochromatic copy of H ? This fundamental question has been widely studied, along with many variants; see the excellent survey of Conlon, Fox and Sudakov [9]. A very natural generalization of the classical Ramsey problem is to vary the number of colours k instead of the number of vertices n . For graphs G, H and $q \geq 2$, let $r(G, H, q)$ be the minimum number of colours needed to colour G such that every copy of H receives at least q colours. Determining $r(K_n, H, 2)$ for all n is equivalent to determining the classical Ramsey number $R_k(H)$ for all k . A colouring of G in which every copy of H receives at least q colours is called a (H, q) -colouring of G . The function $r(G, H, q)$ is called the *generalized Ramsey number* or the *Erdős-Gyárfás function*.

The study of the function $r(G, H, q)$ was initiated by Erdős and Shelah [15] in 1975. In 1997, Erdős and Gyárfás [16] considered the case where G and H are cliques and

^aFakultät für Informatik und Mathematik, Universität Passau, Passau, Germany
(andrew.lane@uni-passau.de).

^bDepartment of Mathematics and Statistics, University of Victoria, Canada (nmorrison@uvic.ca).

proved $r(K_n, K_p, q) = O\left(n^{(p-2)/((\binom{p}{2}-q+1)}\right)$. This was generalized by Axenovich, Füredi, and Mubayi [1] to give a general upper bound of $r(G, H, q) = O\left(n^{\frac{|V(H)|-2}{|E(H)|-q+1}}\right)$. Bennett, Delcourt, Li, and Postle [6] improved and generalized this bound.

Theorem 1 (Bennett, Delcourt, Li, Postle [6]). *Let H be a fixed graph with $|V(H)| \geq 3$ and let q be a positive integer with $q \leq |E(H)|$ and $|V(H)|-2$ not divisible by $|E(H)|-q+1$. If G is a graph on n vertices, then*

$$r(G, H, q) = O\left(\left(\frac{n^{|V(H)|-2}}{\log n}\right)^{\frac{1}{|E(H)|-q+1}}\right).$$

In fact, Theorem 1 is a consequence of a much stronger result in [6] concerning a list-colouring analogue. Precise results for $r(G, H, q)$ for specific H have also been obtained. In particular, Bennett, Cushman, Dudek, and Prałat [5] proved that $r(K_n, K_4, 5) = 5n/6 + o(n)$, and Joos and Mubayi [25] proved that $r(K_{n,n}, C_4, 3) = 2n/3 + o(n)$ and $r(K_n, C_4, 3) = n/2 + o(n)$.

Our main result provides a generalization to all cycles of the work of Joos and Mubayi [25]. This stronger result implies both theorems and gives a full characterization of the asymptotic behaviour of the number $r(K_n, C_k, 3)$ for fixed k . To state this result in full generality, we define the following: for a graph G , a family of graphs $\mathcal{F}$, and $q \geq 2$, an $(\mathcal{F}, q)$ -colouring of G is a colouring of G that is an (F, q) -colouring of G for all $F \in \mathcal{F}$. Let $r(G, \mathcal{F}, q)$ be the minimum number of colours in an $(\mathcal{F}, q)$ -colouring of G . Let $C_{[k, \ell]} := \{C_k, C_{k+1}, \dots, C_\ell\}$.

Theorem 2. *For fixed integers $3 \leq k \leq \ell$ and n sufficiently large,*

$$r(K_n, C_{[k, \ell]}, 3) = n/(k-2) + o(n).$$

Note that if $\mathcal{F}' \subseteq \mathcal{F}$, then $r(G, \mathcal{F}', q) \leq r(G, \mathcal{F}, q)$. Using this, we immediately obtain the following result.

Corollary 3. *For a fixed integer $k \geq 3$ and n sufficiently large,*

$$r(K_n, C_k, 3) = n/(k-2) + o(n).$$

We also prove a bipartite analogue of Theorem 2 that generalizes the bound of Joos and Mubayi [25] on $r(K_{n,n}, C_4, 3)$.

Theorem 4. *For any fixed integer $k \geq 2$, even number $\ell \geq k^2 - k + 2$, and n sufficiently large, we have*

$$r(K_{n,n}, C_{[k^2-k+2, \ell]}, 3) \leq \frac{2n}{k^2-1} + o(n).$$

Corollary 5. *For any fixed even number $k \geq 4$ and n sufficiently large,*

$$\frac{n}{k-2} + o(n) \leq r(K_{n,n}, C_k, 3) \leq \frac{2n}{\left\lfloor \frac{1+\sqrt{4k-7}}{2} \right\rfloor^2 - 1} + o(n).$$

Note that the lower bound in Corollary 5 is almost certainly not tight: it only gives $r(K_{n,n}, C_4, 3) \geq n/2 + o(n)$, while [25] gives $r(K_{n,n}, C_4, 3) = 2n/3 + o(n)$. There is no reason to expect that this situation improves as k grows larger. However, it is possible that the upper bound is tight, since the proof is analogous to that of Theorem 2, which gives a tight upper bound on $r(K_n, C_k, 3)$, and we can recover the tight upper bound $r(K_{n,n}, C_4, 3) \leq 2n/3 + o(n)$ of Joos and Mubayi [25].

Theorems 2 and 4 are both applications of the so-called “conflict-free hypergraph matching” method. This method was introduced by Glock, Joos, Kim, Kühn, and Lichev in [20] and independently in [12] by Delcourt and Postle as the “forbidden submatching method.” Roughly speaking, the method finds matchings in hypergraphs in which certain sets of edges are forbidden. The main idea is that for a hypergraph $\mathcal{H}$ and a hypergraph $\mathcal{C}$ with $V(\mathcal{C}) = E(\mathcal{H})$ whose edges are forbidden sets of edges in $\mathcal{H}$, if $\mathcal{C}$ has various small degree conditions, then it is possible to find a (nearly) perfect matching in $\mathcal{H}$.

Both foundational papers [12, 20] developed the method in order to prove asymptotic results about ‘high-girth’ Steiner-systems, thus substantially generalizing work from [8, 22] and settling approximate versions of conjectures from [19, 22, 27]. Since then, this method has already had a plethora of other exciting applications; see [10, 11, 13, 14, 18, 21, 28]. In particular, the results stated above of Bennett, Delcourt, Li, and Postle [6] and those of Joos and Mubayi [25] were also obtained via the conflict-free hypergraph matching method, which was also used for several other related results related to generalized Ramsey numbers; see [4, 7, 23].

Although our results are obtained via the same broad strategy as those of Joos and Mubayi [25], new ideas and arguments are required in order to deal with the additional structures at play when considering longer cycles. We give more detail regarding this in Section 3.

The paper is structured as follows. In Section 2 we introduce the conflict-free hypergraph matching method and state the main result (Theorem 8) which will be applied in our proofs. Theorem 2 is proved in Section 3. In Section 4 we present the proofs of Theorem 4 and Corollary 5. As the proof of Theorem 4 follows very similar lines to that of Theorem 2, we give only the key points here and present the more tedious details in the appendix included in the arXiv version of this paper. We provide some closing remarks and open problems in Section 5.

2 Conflict-Free Hypergraph Matchings

In this section, we briefly recall the simplified version of the conflict-free hypergraph matching theorem from [20] as presented in [25]. This will be applied in the proofs of Theorems 2 and 4.

Roughly speaking, these theorems show that if $\mathcal{H}$ is an approximately regular hypergraph, $\mathcal{C}$ is a hypergraph whose edges are forbidden configurations of edges of $\mathcal{H}$, and $\mathcal{C}$ satisfies various small degree conditions, then there exists an almost perfect matching in $\mathcal{H}$ that does not contain any edge of $\mathcal{C}$. In particular, for graphs G, H and $q \geq 2$, if $\mathcal{H}$ corresponds to possible edge-colourings of subgraphs of a graph G , and $\mathcal{C}$ corresponds to

arrangements of these subgraphs in which H receives fewer than q colours, then such a matching yields an (H, q) -colouring of almost all edges of G .

We require some definitions for a hypergraph $\mathcal{H}$. For a set S , let $S^{(k)} := \{T \subseteq S : |T| = k\}$.

- For $k \in \mathbb{N}$, say $\mathcal{H}$ is k -uniform if $|e| = k$ for all $e \in E(\mathcal{H})$.
- For $v \in V(\mathcal{H})$, let $d_{\mathcal{H}}(v) := |\{e \in E(\mathcal{H}) : v \in e\}|$ denote the *degree of v in $\mathcal{H}$* .
- Let $\delta(\mathcal{H}) := \min_{v \in V(\mathcal{H})} d_{\mathcal{H}}(v)$ and $\Delta(\mathcal{H}) := \max_{v \in V(\mathcal{H})} d_{\mathcal{H}}(v)$.
- For $i \in \mathbb{N}$, let $\mathcal{H}^{(i)}$ be the spanning sub-hypergraph of $\mathcal{H}$ comprising all edges of size exactly i .
- For $i \geq 2$, let $\Delta_i(\mathcal{H}) := \max_{S \in V(\mathcal{H})^{(i)}} |\{e \in E(\mathcal{H}) : S \subseteq e\}|$.

For hypergraphs $\mathcal{H}$ and $\mathcal{C}$, we refer to $\mathcal{C}$ as a *conflict system* if $E(\mathcal{H}) = V(\mathcal{C})$. We say a set $S \subseteq E(\mathcal{H})$ is $\mathcal{C}$ -free if S does not contain any element of $E(\mathcal{C})$ as a subset.

Definition 6 (Glock, Joos, Kim, Kühn, Lichev [20]). For integers $d \geq 1$, $\ell \geq 3$ and $\varepsilon \in (0, 1)$, say that a hypergraph $\mathcal{C}$ is (d, ℓ, ε) -bounded if the following holds.

- (C1) $3 \leq |S| \leq \ell$ for all $S \in E(\mathcal{C})$;
- (C2) $\Delta(\mathcal{C}^{(i)}) \leq \ell d^{i-1}$ for all $2 \leq i \leq \ell$;
- (C3) $\Delta_j(\mathcal{C}^{(i)}) \leq d^{i-j-\varepsilon}$ for all $3 \leq i \leq \ell$ and $2 \leq j \leq i-1$.

This means that all uniform sub-hypergraphs of $\mathcal{C}$ have small degrees and codegrees, which we use as a necessary condition for the conflict-free hypergraph matching theorem.

Glock, Joos, Kim, Kühn, and Lichev [20] also introduced a class of weight functions for which the matching admits quasirandom properties. We now introduce some notation and terminology required to describe these. A *test function* for $\mathcal{H}$ is a function $w : E(\mathcal{H})^{(j)} \rightarrow [0, \ell]$ where $j \in \mathbb{N}$ such that $w(S) = 0$ whenever S is not a matching. We say that w is j -uniform and refer to j as the uniformity of w . For $X \subseteq E(\mathcal{H})^{(j)}$, we define $w(X) := \sum_{x \in X} w(x)$. We also extend w to arbitrary subsets $S \subseteq E(\mathcal{H})$ by defining $w(S) := w(S^{(j)})$. We define $w(\mathcal{H}) := w(E(\mathcal{H}))$. For a hypergraph $\mathcal{C}$ and $e \in V(\mathcal{C})$, define $\mathcal{C}_e := \{S \setminus \{e\} : e \in S \in E(\mathcal{C})\}$.

Definition 7 (Glock, Joos, Kim, Kühn, Lichev [20]). For $j, d \in \mathbb{N}$, $\varepsilon > 0$ and hypergraphs $\mathcal{H}$ and $\mathcal{C}$ with $V(\mathcal{C}) = E(\mathcal{H})$ and $\ell := \max_{S \in E(\mathcal{C})} |S|$, we say that a j -uniform test function w for $\mathcal{H}$ is $(d, \varepsilon, \mathcal{C})$ -trackable if the following holds.

- (W1) $w(\mathcal{H}) \geq d^{j+\varepsilon}$;
- (W2) $w(\{S \in E(\mathcal{H})^{(j)} : S \supseteq S'\}) \leq w(\mathcal{H})/d^{j'+\varepsilon}$ for all $j' \in [j-1]$ and $S' \in E(\mathcal{H})^{(j')}$;
- (W3) $|\{\mathcal{C}_e\}^{(j')} \cap \{\mathcal{C}_f\}^{(j')}\}| \leq d^{j'-\varepsilon}$ for all distinct $e, f \in E(\mathcal{H})$ with $w(\{S \in E(\mathcal{H})^{(j)} : e, f \in S\}) > 0$ and all $j' \in [\ell-1]$;

(W4) $w(S) = 0$ for all $S \in E(\mathcal{H})^{(j)}$ that are not $\mathcal{C}$ -free.

Theorem 8 (Glock, Joos, Kim, Kühn, Lichev [20]). *For all $k, \ell \geq 2$, there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, there exists d_0 such that the following holds for all $d \geq d_0$. Suppose $\mathcal{H}$ is a k -uniform hypergraph on $n \leq \exp(d^{\varepsilon^3})$ vertices with $(1 - d^{-\varepsilon})d \leq \delta(\mathcal{H}) \leq \Delta(\mathcal{H}) \leq d$ and $\Delta_2(\mathcal{H}) \leq d^{1-\varepsilon}$, and suppose $\mathcal{C}$ is a (d, ℓ, ε) -bounded conflict system for $\mathcal{H}$. Suppose $\mathcal{W}$ is a set of $(d, \varepsilon, \mathcal{C})$ -trackable test functions for $\mathcal{H}$ of uniformity at most ℓ with $|\mathcal{W}| \leq \exp(d^{\varepsilon^3})$. Then there exists a $\mathcal{C}$ -free matching $\mathcal{M} \subseteq E(\mathcal{H})$ of size at least $(1 - d^{-\varepsilon^3})n/k$ with $w(\mathcal{M}) = (1 \pm d^{-\varepsilon^3})d^{-j}w(\mathcal{H})$ for all j -uniform $w \in \mathcal{W}$.*

3 Proof of Theorem 2

In this section, we prove Theorem 2, restated here for convenience.

Theorem 2. *For fixed integers $3 \leq k \leq \ell$ and n sufficiently large,*

$$r(K_n, C_{[k, \ell]}, 3) = n/(k-2) + o(n).$$

The lower bound in Theorem 2 will follow directly from the following results of Axenovich, Füredi, and Mubayi [1] and Faudree and Schelp [17]. For a graph G and $n \geq |V(G)|$, let $\text{ex}(n, G)$ be the maximum number of edges in an n -vertex graph that does not contain a copy of G .

Theorem 9 (Axenovich, Füredi, Mubayi [1]). *Let H be a connected graph. Then for any spanning tree T of H ,*

$$r(K_n, H, |E(H)| - |V(H)| + 3) \geq \frac{\binom{n}{2}}{\text{ex}(n, T)} = \Omega(n).$$

Theorem 10 (Faudree, Schelp [17]). *For all $n, k \in \mathbb{N}$,*

$$\text{ex}(n, P_{k+1}) \leq \frac{(k-1)n}{2}.$$

The upper bound in Theorem 2 is more involved, and this is where we apply the conflict-free hypergraph matching method. Before getting into the technical details of the proof, let us first give a brief overview.

Our goal is to colour K_n with at most $n/(k-2) + o(n)$ colours in such a way that no cycle with length in $[k, \ell]$ has fewer than 3 colours. One might hope to do this randomly via the local lemma, but there are simply too many dependencies to do this outright. Hence our strategy is to first colour most of the graph using at most $n/(k-2)$ colours with a very structured colouring in such a way that we can then colour the rest randomly with a set of $o(n)$ new colours using the local lemma.

One obstacle to overcome in generalizing the C_4 result to longer cycles is the wider variety of forbidden structures that appear. This means that at essentially every stage of the proof, we require more complex and careful arguments. For example, in the proof of

Lemma 11 below, we use a richer family of test functions and show that large structures, like the arrangements that produce the cycles mentioned in (IV), can be “tracked” using these test functions. In addition, we require the asymmetric Lovász local lemma [29] (see Lemma 12, below) to control the larger range of bad events that arise in the more general setting.

We now give some definitions and explain the properties of this structured colouring (given in Lemma 11 below).

For a hypergraph $\mathcal{H}$, a *cycle of length ℓ* is a sequence $e_1, e_2, \dots, e_\ell$ of distinct edges in $E(\mathcal{H})$ such that there exist distinct vertices $v_1, v_2, \dots, v_\ell \in V(\mathcal{H})$ where, taking indices modulo ℓ , we have $e_i \cap e_{i+1} = \{v_i\}$ for all $i \in [\ell]$. The *girth* of a hypergraph $\mathcal{H}$ is the minimum ℓ such that $\mathcal{H}$ has a cycle of length ℓ (or ∞ if it has no such cycle). In what follows, to simplify the calculations, we will treat cycles $a_1, \dots, a_k$ and $b_1, \dots, b_k$ in a (hyper)graph as distinct if $a_i \neq b_i$ for some $i \in [k]$, even if they represent the same circular arrangement. Note that this does not quite match the standard notation $v_1, v_2, \dots, v_k, v_1$ we use for graphs. For $x \in \mathbb{R}^+$, denote $\lfloor x \rfloor := \{1, 2, \dots, \lfloor x \rfloor\}$.

We are now ready to state our technical lemma, which is of a similar spirit to the technical lemma of Joos and Mubayi in [25].

Lemma 11. *For each $\ell \geq k \geq 3$, there exists $\delta > 0$ depending only on ℓ and k such that for all sufficiently large n , there exists an edge-colouring of a spanning subgraph G of K_n with at most $n/(k-2)$ colours and the following properties:*

- (I) *Every colour class consists of vertex-disjoint copies of K_{k-1} .*
- (II) *Given any two colours i, j , the $(k-1)$ -uniform hypergraph with vertex set $V(G)$ where each edge is formed by the vertex set of a copy of K_{k-1} in colour i or j has girth at least $2\lfloor \frac{\ell}{2} \rfloor + 1$.*
- (III) *The graph $L := K_n - E(G)$ has maximum degree at most $n^{1-\delta}$.*
- (IV) *For each $2 \leq m \leq \lfloor \frac{\ell}{2} \rfloor$ and $u_1 v_m \in E(K_n)$, the number of cycles $u_1, v_1, u_2, v_2, \dots, u_m, v_m, u_1$ such that $v_t u_{t+1} \in E(L)$ for all $t \in [m-1]$ with disjoint monochromatic copies $F_1, \dots, F_m$ of K_{k-1} in G , all of the same colour, such that $v_t u_t \in E(F_t)$ for all $t \in [m]$, is at most $n^{(m-1)(1-\delta)}$.*

Lemma 11 is proved in Subsection 3.1 below using the conflict-free hypergraph matching method. The conditions (I)-(IV) will ensure that the subgraph G has no cycle with length in $[k, \ell]$ that is coloured with fewer than 3 colours. Moreover, the colouring of G is structured such that there are only particular ways that colouring $L = K_n - E(G)$ could cause problems and that there are few enough dependencies to be able to use the local lemma to colour L so that no such cycle is created.

We will use the following generalization of the Lovász Local Lemma. For a set of events $\mathcal{E}$, say that a graph G is a *dependency graph* for $\mathcal{E}$ if $V(G) = \mathcal{E}$ and for all $E \in \mathcal{E}$, E is mutually independent of all events not in its G -neighbourhood.

Lemma 12 (Asymmetric Lovász Local Lemma [29]). *Let $E_1, \dots, E_n$ be events with dependency graph G . If there exist $x_1, \dots, x_n \in (0, 1)$ such that for all $i \in [n]$,*

$$\mathbb{P}(E_i) \leq x_i \prod_{E_j \in N_G(E_i)} (1 - x_j),$$

then $\mathbb{P}(\bigcap_{i=1}^n \overline{E_i}) > 0$.

We now formalise the steps in this overview to prove Theorem 2.

Proof of Theorem 2. We begin with the lower bound. By definition, a $(C_{[k,\ell]}, 3)$ -colouring of K_n is a $(C_k, 3)$ -colouring of K_n , so $r(K_n, C_{[k,\ell]}, 3) \geq r(K_n, C_k, 3)$. By combining Theorems 9 and 10, we obtain the following lower bound:

$$r(K_n, C_{[k,\ell]}, 3) \geq r(K_n, C_k, 3) \geq \frac{\binom{n}{2}}{\text{ex}(n, P_k)} \geq \frac{\frac{n(n-1)}{2}}{\frac{k-2}{2}n} = \frac{n}{k-2} + o(n).$$

We now turn our attention to the upper bound. We will show, for $3 \leq k \leq \ell$,

$$r(K_n, C_{[k,\ell]}, 3) \leq \frac{n}{k-2} + o(n).$$

First, apply Lemma 11 to obtain a coloured graph G and a number δ as stated in Lemma 11. We will first show that G is $(C_{[k,\ell]}, 3)$ -coloured. Then we will use the local lemma to colour $L = K_n - E(G)$.

Claim 13. *G is $(C_{[k,\ell]}, 3)$ -coloured.*

Proof of Claim. Suppose that G is not $(C_h, 3)$ -coloured for some $k \leq h \leq \ell$. Let $\mathcal{H}$ be the hypergraph described in Lemma 11(II), and let $C = v_1, v_2, \dots, v_h, v_1$ be a 2-coloured h -cycle in G . Partition C into maximal-length monochromatic paths $P_1, \dots, P_t$ such that (setting $P_{t+1} = P_1$) $V(P_i) \cap V(P_{i+1}) \neq \emptyset$ for all $i \in [t]$, and for each $i \in [t]$, let $u_i \in V(P_i) \cap V(P_{i+1})$. By the definition of $\mathcal{H}$, each P_i is a subgraph of a monochromatic copy of K_{h-1} , and since the P_i have maximal length, the colours of the P_i alternate. Thus if h is odd, then at least one of the P_i has length greater than 1, else there exist P_i, P_{i+1} that receive the same colour, contradicting the maximality of their lengths. This implies $t \leq 2\lfloor \frac{h}{2} \rfloor \leq 2\lfloor \frac{\ell}{2} \rfloor$.

For each $i \in [t]$, let F_i be the monochromatic copy of K_{h-1} that contains P_i as a subgraph. Then since each F_i, F_j are edge-disjoint for $i \neq j$ and vertex-disjoint for $i \neq j$, $i - j \equiv 0 \pmod{2}$, the u_i are distinct and $V(F_i) \cap V(F_{i+1}) = \{u_i\}$ for all i , so $V(F_1), V(F_2), \dots, V(F_t)$ is a 2-coloured cycle of length $t \leq 2\lfloor \frac{\ell}{2} \rfloor$ in $\mathcal{H}$, contradicting Lemma 11(II). Therefore, G is $(C_h, 3)$ -coloured for all $k \leq h \leq \ell$, i.e., G is $(C_{[k,\ell]}, 3)$ -coloured. $\square$

To complete the proof, we need to colour $L = K_n - E(G)$ in such a way as to not create a cycle with length in $[k, \ell]$ with fewer than 3 colours. Let $0 < \alpha < \min\{\frac{\delta}{4}, \frac{1}{2}\}$.

We will colour edges in L randomly such that there is a non-zero probability that the $(C_{[k,\ell]}, 3)$ -colouring is preserved. Let P be a set of $r := n^{1-\alpha}$ new colours that do

not colour edges in G . Colour each edge of L independently with a colour from P with uniform probability $1/r$.

For any pair e, f of adjacent edges in L and $i \in P$, let $A_{e,f,i}$ be the event that both e and f receive colour i . Then $\mathbb{P}[A_{e,f,i}] = r^{-2} = n^{-(2(1-\alpha))}$. For $\lceil \frac{k}{2} \rceil \leq m \leq \lfloor \frac{\ell}{2} \rfloor$ and a $2m$ -cycle D in L , define B_D to be the event that D is properly 2-coloured. Then once we have fixed the colours of two adjacent edges, there is only one choice for the colour of each remaining edge, so $\mathbb{P}[B_D] \leq r^{-(2m-2)} = n^{-(2m-2)(1-\alpha)}$. For $2 \leq m \leq \lfloor \frac{\ell}{2} \rfloor$ and a $2m$ -cycle $D = u_1, v_1, \dots, u_m, v_m, u_1$ in K_n where there are disjoint monochromatic copies of $F_1, \dots, F_m$ of K_{k-1} in G , all of the same colour, such that $u_i v_i \in E(F_i)$ for all $i \in [m]$ and $v_i u_{i+1} \in E(L)$ (taken mod m) for all $i \in [m]$, and for a colour $j \in P$, let $C_{D,j}$ be the event that every $v_i u_{i+1}$ (mod m) receives the colour j . Then $\mathbb{P}[C_{D,j}] = r^{-m} = n^{-m(1-\alpha)}$. Let $\mathcal{A}$ be the set of all defined $A_{e,f,i}$, and for all m , let $\mathcal{B}_m$, and $\mathcal{C}_m$ be the sets of all defined B_D and $C_{D,j}$, respectively, where D has length $2m$. Let

$$\mathcal{E} := \mathcal{A} \cup \bigcup_{m=\lceil k/2 \rceil}^{\lfloor \ell/2 \rfloor} \mathcal{B}_m \cup \bigcup_{m=2}^{\lfloor \ell/2 \rfloor} \mathcal{C}_m.$$

We now use the local lemma to show we can colour L avoiding the events in $\mathcal{E}$; we will then show that such a colouring will not create a cycle in K_n with length in $[k, \ell]$ coloured in fewer than 3 colours.

Claim 14. *There exists a colouring of L such that none of the events in $\mathcal{E}$ occurs.*

Proof of Claim. Say two events in $\mathcal{E}$ are *edge-disjoint* if the edges in L associated with each of them are distinct. For an event E and a set of events $\mathcal{X}$, let $d_{\mathcal{X}}(E)$ be the number of events in $\mathcal{X}$ that are not edge-disjoint from E . Then by Lemma 12, it suffices to prove that there exist $x, y_{\lceil k/2 \rceil}, \dots, y_{\lfloor \ell/2 \rfloor}, z_2, \dots, z_{\lfloor \ell/2 \rfloor} \in (0, 1)$ such that for all $A \in \mathcal{A}$, $B_m \in \mathcal{B}_m$ ($\lceil \frac{k}{2} \rceil \leq m \leq \lfloor \frac{\ell}{2} \rfloor$), and $C_m \in \mathcal{C}_m$ ($2 \leq m \leq \lfloor \frac{\ell}{2} \rfloor$), we have

$$\mathbb{P}(A) \leq x f(A), \quad \mathbb{P}(B_m) \leq y_m f(B_m), \quad \mathbb{P}(C_m) \leq z_m f(C_m), \quad (1)$$

where for all $E \in \mathcal{E}$,

$$f(E) := (1-x)^{d_{\mathcal{A}}(E)} \prod_{m=\lceil k/2 \rceil}^{\lfloor \ell/2 \rfloor} (1-y_m)^{d_{\mathcal{B}_m}(E)} \prod_{m=2}^{\lfloor \ell/2 \rfloor} (1-z_m)^{d_{\mathcal{C}_m}(E)}.$$

Fix $E \in \mathcal{E}$, and let $S = \{e_1, \dots, e_t\}$ be the edges in L associated with E .

For each $e_i \in S$ and end-vertex v of e_i , there are at most $\Delta(L)$ edges in L that share the vertex v with e_i , and there are r colours in P . Thus by Lemma 11(III),

$$d_{\mathcal{A}}(E) \leq 2t\Delta(L)r \leq 2tn^{1-\delta}r = 2tn^{2-\delta-\alpha}.$$

For each $e_i = uv \in S$ and $\lceil \frac{k}{2} \rceil \leq m \leq \lfloor \frac{\ell}{2} \rfloor$, to choose a cycle $D = u, v, u_2, v_2, \dots, u_m, v_m, u$ containing e_i , we choose $u_i \in N_L(v_{i-1})$ and $v_i \in N_L(u_i)$ for all $2 \leq i \leq m$, so there are at most $\Delta(L)^{2m-2}$ choices for D . Thus by Lemma 11(III),

$$d_{\mathcal{B}_m}(E) \leq t\Delta(L)^{2m-2} \leq tn^{(2m-2)(1-\delta)} \leq tn^{(2m-2)-(2m-2)\delta}.$$

By Lemma 11(IV), for $2 \leq m \leq \lfloor \frac{\ell}{2} \rfloor$,

$$d_{\mathcal{C}_m}(E) \leq trn^{(m-1)(1-\delta)} \leq tn^{1-\alpha}n^{(m-1)(1-\delta)} = tn^{m-(m-1)\delta-\alpha}.$$

Set

$$x = n^{-2+3\alpha}, \quad y_m = n^{-(2m-2)+(2m-1)\alpha}, \quad z_m = n^{-m+(m+1)\alpha}.$$

Thus, since $(1 - a^{-1})^a \geq \frac{1}{4}$ for all $a \geq 2$, n is arbitrarily large, and $\alpha < \frac{2}{3}$, we have

$$(1 - x)^{d_{\mathcal{A}}(E)} \geq (1 - n^{-2+3\alpha})^{2tn^{2-\delta-\alpha}} = (1 - n^{-(2-3\alpha)})^{(n^{2-3\alpha})^{2t(n^{2\alpha-\delta})}} \geq \left(\frac{1}{4}\right)^{2tn^{2\alpha-\delta}}.$$

Similarly, for all m , we have

$$(1 - y_m)^{d_{\mathcal{B}_m}(E)} \geq \left(\frac{1}{4}\right)^{tn^{(2m-1)\alpha-(2m-2)\delta}} \quad \text{and} \quad (1 - z_m)^{d_{\mathcal{C}_m}(E)} \geq \left(\frac{1}{4}\right)^{t(n^{(m+2)\alpha-(m-1)\delta})}.$$

In the right-hand side of each of these inequalities, the power of n is negative, since $\alpha < \delta/4$ implies $2\alpha < \delta$, $(2m-1)\alpha < (2m-2)\delta$ for all $\lceil \frac{k}{2} \rceil \leq m \leq \lfloor \frac{\ell}{2} \rfloor$, and $(m+2)\alpha < (m-1)\delta$ for all $2 \leq m \leq \lfloor \frac{\ell}{2} \rfloor$. Thus, each term of $f(E)$ is greater than or equal to $(\frac{1}{4})^{cn^b}$ for some constants $b < 0 < c$. So

$$\lim_{n \rightarrow \infty} f(E) = 1.$$

Thus for $A \in \mathcal{A}$,

$$xf(A) = n^{-2+3\alpha}f(A) > n^{-2(1-\alpha)} = \mathbb{P}(A),$$

for $B_m \in \mathcal{B}_m$ ($\lceil \frac{k}{2} \rceil \leq m \leq \lfloor \frac{\ell}{2} \rfloor$),

$$y_m f(B_m) = n^{-(2m-2)+(2m-1)\alpha} f(B_m) > n^{-(2m-2)(1-\alpha)} = \mathbb{P}(B_m),$$

and for $C_m \in \mathcal{C}_m$ ($2 \leq m \leq \lfloor \frac{\ell}{2} \rfloor$),

$$z_m f(C_m) = n^{-m+(m+1)\alpha} f(C_m) > n^{-m(1-\alpha)} = \mathbb{P}(C_m),$$

proving (1). This completes the proof of Claim 14. $\square$

We claim that a colouring of L from Claim 14, combined with the colouring of G , is a $(C_{[k,\ell]}, 3)$ -colouring of K_n . Let $k \leq h \leq \ell$, and let C be an h -cycle in K_n . We will show that C receives at least 3 colours.

If $E(C) \subseteq E(G)$, then we are done by Claim 13. If $E(C) \subseteq E(L)$, then C receives at least 3 colours because events in $\mathcal{A}$ and $\bigcup_{m=\lceil k/2 \rceil}^{\lfloor \ell/2 \rfloor} \mathcal{B}_m$ do not occur (C is properly coloured but not properly 2-coloured). If $E(C)$ contains two adjacent edges in L and an edge in G , then since events in $\mathcal{A}$ do not occur, C receives at least 2 colours in P and at least 1 colour not in P , so it receives at least 3 colours.

We have reduced to the case that $E(C) \cap E(L)$ is a matching and C receives at most 2 colours, say $c_1 \notin P$ and $c_2 \in P$, so it suffices to prove that this is impossible. Suppose for contradiction that it is possible. Let $m := |E(C) \cap E(L)|$. In this case, C can be decomposed into paths and single edges $P_1, e_1, P_2, e_2, \dots, P_m, e_m$, where e_i is incident with an endvertex v_i of P_i and an endvertex u_{i+1} of P_{i+1} for all i (taken mod m), each P_i is a monochromatic path in colour c_1 , and each e_i receives the colour c_2 . So $u_1, v_1, u_2, v_2, \dots, u_m, v_m, u_1$ is a cycle with alternating colours c_1, c_2 . Let $D = x_1, y_1, x_2, y_2, \dots, x_{m'}, y_{m'}, x_1$ be a shortest cycle with alternating colours c_1, c_2 in K_n , where $x_1 y_1$ receives colour c_1 .

For each $i \in [m']$, let F_i be the monochromatic copy of K_{k-1} that contains the edge $x_i y_i$. We claim that the F_i are all distinct (and thus disjoint by Lemma 11(I)). Suppose not, and let $1 \leq i < j \leq m'$ such that $F_i = F_j$. Then since $y_i, x_j \in V(F_i)$, $y_i x_j$ receives colour c_1 , so $y_i, x_{i+1}, y_{i+1}, \dots, x_{j-1}, y_{j-1}, x_j, y_i$ is a cycle with alternating colours c_1, c_2 in K_n of length less than $2m'$, contradicting minimality of m' .

This confirms that the event C_{D, c_2} occurs. But this contradicts Claim 14. Therefore, K_n is $(C_{[k, \ell]}, 3)$ -coloured with $n/(k-2) + n^{1-\alpha} = n/(k-2) + o(n)$ colours, so $r(K_n, C_{[k, \ell]}, 3) \leq n/(k-2) + o(n)$, as desired. $\square$

3.1 Proof of Lemma 11

Proof of Lemma 11. Let $\varepsilon = \varepsilon(k) \in (0, \frac{1}{k-2})$. We will choose n to be sufficiently large and δ to be sufficiently small compared to ε .

Let $E = E(K_n)$ and $V = V(K_n)$. We refer to $C := [\frac{n}{k-2}]$ as the set of colours. We will define the $\binom{k}{2}$ -uniform hypergraph $\mathcal{H}$ as follows. Let $V(\mathcal{H}) := E \cup (V \times C)$. Let

$$E(\mathcal{H}) := \{A^{(2)} \cup (A \times \{i\}) : A \in V^{(k-1)}, i \in C\}.$$

There is a natural one-to-one correspondence between $E(\mathcal{H})$ and $V^{(k-1)} \times C$, so we will denote an edge $A^{(2)} \cup (A \times \{i\}) \in E(\mathcal{H})$ simply by (A, i) .

For each matching $\mathcal{M}$ of $\mathcal{H}$, let $G(\mathcal{M})$ be the edge-coloured graph with $V(G) = V(K_n)$, $E(G) = \{e \in E : e \in A^{(2)} \text{ for some } (A, i) \in \mathcal{M}\}$, and edge-colouring ϕ with $e \mapsto i$ for all $(A, i) \in \mathcal{M}$ and $e \in A^{(2)}$. Note that by the definition of $\mathcal{H}$, any $G(\mathcal{M})$ satisfies condition (I).

Let

$$d := \frac{n^{k-2}}{(k-2)!}.$$

Our goal is to apply Theorem 8 on the hypergraph $\mathcal{H}$ to obtain a matching $\mathcal{M} \subseteq E(\mathcal{H})$ such that $G(\mathcal{M})$ satisfies conditions (I)–(IV). To this end, we will define a $(d, O(1), \varepsilon)$ -conflict system $\mathcal{C}$ such that whenever $\mathcal{M}$ is $\mathcal{C}$ -free, $G(\mathcal{M})$ satisfies (II). We will also define a set $\mathcal{W}$ of $(d, \varepsilon, \mathcal{C})$ -trackable test functions such that Theorem 8 yields a matching $\mathcal{M}$ for which $G(\mathcal{M})$ additionally satisfies (III) and (IV).

To apply Theorem 8, we must prove that $\mathcal{H}$ is approximately d -regular and bound $\Delta_2(\mathcal{H})$.

Claim 15. $d(1 - d^{-\varepsilon}) \leq \delta(\mathcal{H}) \leq \Delta(\mathcal{H}) \leq d$ and $\Delta_2(\mathcal{H}) \leq d^{1-\varepsilon}$.

Proof of Claim. There are $\binom{n-2}{k-3} \lfloor \frac{n}{k-2} \rfloor$ edges in $\mathcal{H}$ containing a fixed $e \in E$. There are $\binom{n-1}{k-2}$ edges in $\mathcal{H}$ containing a fixed $(v, i) \in V \times C$. So for any $u \in V(\mathcal{H})$, as $\varepsilon \in (0, \frac{1}{k-2})$, we have $d(1 - d^{-\varepsilon}) \leq d(u) \leq d$, as required.

Now let $u, v \in V(\mathcal{H})$ be distinct vertices contained in an edge. There are at most n^{k-3} ways to extend such a pair to an edge in $\mathcal{H}$. Indeed, to pick an edge we must choose $k-1$ vertices and a colour, and the pair u, v fixes at least 3 of these choices. As $\varepsilon \in (0, \frac{1}{k-2})$, we have $n^{k-3} \leq d^{1-\varepsilon}$, as required. $\square$

We next define a conflict system $\mathcal{C}$ for $\mathcal{H}$. We set $V(\mathcal{C}) := E(\mathcal{H})$. Let $E(\mathcal{C})$ be the set of edges that form a $2m$ -cycle in $\mathcal{H}$ in alternating colours for some $2 \leq m \leq \lfloor \frac{\ell}{2} \rfloor$. More precisely, given $2 \leq m \leq \lfloor \frac{\ell}{2} \rfloor$, distinct vertices $v_1, \dots, v_m, u_1, \dots, u_m$, two distinct colours $i, j \in C$, and $A_1, \dots, A_m \in V^{(k-1)}$ and $B_1, \dots, B_m \in V^{(k-1)}$ with $v_i \in A_i \cap B_i$ and $u_i \in B_i \cap A_{i+1}$ for all $i \in [m]$ (setting $A_{m+1} := A_1$), say

$$S := \{(A_1, i), (B_1, j), (A_2, i), (B_2, j), \dots, (A_m, i), (B_m, j)\}$$

is in $E(\mathcal{C})$ whenever S is a matching in $\mathcal{H}$.

Note that for any $\mathcal{C}$ -free matching $\mathcal{M}$ in $\mathcal{H}$, condition (II) holds for $G(\mathcal{M})$. Indeed, any cycle in the hypergraph from (II) must have alternating colours because of condition (I), and any cycle with alternating colours of length at most $2 \lfloor \frac{\ell}{2} \rfloor$ forms an edge in $\mathcal{C}$.

Claim 16. $\mathcal{C}$ is a $(d, O(1), \varepsilon)$ -bounded conflict system for $\mathcal{H}$.

Proof of Claim. (C1) holds trivially since $|S| \leq \ell = O(1)$ for all $S \in E(\mathcal{C})$.

We will first prove (C2). If we fix $2 \leq m \leq \lfloor \frac{\ell}{2} \rfloor$ and $(A, i) \in E(\mathcal{H})$, all edges in $\mathcal{C}^{(2m)}$ containing (A, i) correspond to cycles of the form $(A, i), (B_1, j), (A_1, i), (B_2, j), \dots, (A_{m-1}, i), (B_m, j)$, and there are $O(n)$ choices for j , $O(n^{k-2})$ choices for each of $B_t \setminus A_{t-1}$ and $A_t \setminus B_t$ for each $1 \leq t \leq m-1$ (where $A_0 = A$), and $O(n^{k-3})$ choices for $B_m \setminus (A_{m-1} \cup A)$, for a total of at most

$$O(n \cdot n^{2(m-1)(k-2)} n^{k-3}) = O(n^{(k-2)(2m-1)}) = O(d^{2m-1})$$

choices; this proves (C2).

We will now prove (C3). Let $2 \leq m \leq \lfloor \frac{\ell}{2} \rfloor$. To show $\Delta_t(\mathcal{C}^{(2m)}) \leq d^{2m-t-\varepsilon}$ for all $2 \leq t < 2m$, let $T = \{(A_1, i), \dots, (A_a, i), (B_1, j), \dots, (B_b, j)\} \subseteq E(\mathcal{H})$ with $a + b = t$ and $a \geq b$; we will count the ways to extend T to a cycle S in $E(\mathcal{C}^{(2m)})$. We will consider the vertices sequentially around the cycle starting at $(A_1, i) \in T$, making choices for the vertices that are not in T . First choose the positions in S of $T \setminus \{(A_1, i)\}$. There are $O(1)$ ways to do this. Let $(S_1, c_1), (S_2, c_2), (S_3, c_1)$ be a subpath of this sequential ordering. As $|S_1 \cap S_2| = 1$, given S_1 there are $O(n^{k-2})$ choices for S_2 . In fact, if $(S_3, c_1) \in T$, then there are $O(n^{k-3})$ choices for S_2 . In this second case, call S_2 restricted. Letting r be the number of restricted set choices, there are $O(n^{(k-2)(2m-t)-r})$ ways to choose the vertices in K_n induced by S , given T . Note that if $b = 0$, then $r \geq 2$ as $|T| \geq 2$. In this case we

must also choose the second colour for the cycle; there are $O(n)$ ways to do this. If $b > 0$, then both colours are determined by T and $r \geq 1$. In either case, as $\varepsilon \in (0, \frac{1}{k-2})$, we have at most

$$O(n^{(k-2)(2m-t)-1}) < d^{2m-t-\varepsilon}$$

edges in $\mathcal{C}^{(2m)}$ containing T .

Therefore, $\mathcal{C}$ is a $(d, O(1), \varepsilon)$ -bounded conflict system for $\mathcal{H}$, proving Claim 16. $\square$

We will next define a set of test functions. For each $v \in V$, let S_v be the set of edges in E incident to v . Let $w_v : E(\mathcal{H}) \rightarrow [0, k-2]$ be the 1-uniform test function defined by $S \mapsto |S \cap S_v|$.

Claim 17. w_v is $(d, \varepsilon, \mathcal{C})$ -trackable for all $v \in V$.

Proof of Claim. Note that

$$w_v(\mathcal{H}) = \sum_{S \in E(\mathcal{H})} |S \cap S_v| = \sum_{e \in S_v} d_{\mathcal{H}}(e) = nd \pm O(n^{k-2}) = d^{1+\frac{1}{k-2}} \pm O(n^{k-2}) > d^{1+\varepsilon},$$

which proves (W1). Observe that (W2), (W3), and (W4) hold vacuously because w_v is 1-uniform. This proves the claim. $\square$

For each $2 \leq m \leq \lfloor \frac{\ell}{2} \rfloor$ and $uv \in E(K_n)$, define

$$\mathcal{P}_{u,v,m} := \{ \{(A_1, i), \dots, (A_m, i)\} \in E(\mathcal{H})^{(m)} : i \in C, A_t \cap A_s = \emptyset \ \forall t \neq s, u \in A_1, v \in A_m \}.$$

Then by choosing $A_1 \setminus \{u\}$ and $A_m \setminus \{v\}$, then $\{A_2, \dots, A_{m-1}\}$, then i , we have

$$\begin{aligned} |\mathcal{P}_{u,v,m}| &= \binom{n}{k-2}^2 \frac{1}{(m-2)!} \binom{n}{k-1}^{m-2} \frac{n}{k-2} \pm O(n^{mk-m-2}) \\ &= \frac{d^m n^{m-1}}{(m-2)!(k-2)(k-1)^{m-2}} \pm O(n^{mk-m-2}). \end{aligned} \quad (2)$$

Let $w_{u,v,m}$ be the indicator function defined by

$$w_{u,v,m}(S) = \begin{cases} 1, & S \in \mathcal{P}_{u,v,m}, \\ 0, & S \notin \mathcal{P}_{u,v,m} \end{cases}$$

for all matchings $S \in E(\mathcal{H})^{(m)}$, extended to all subsets of $E(\mathcal{H})$ as described in Section 2. Note that $w_{u,v,m}$ is an m -uniform test function.

Claim 18. For all $uv \in E(K_n)$ and $2 \leq m \leq \lfloor \frac{\ell}{2} \rfloor$, $w_{u,v,m}$ is $(d, \varepsilon, \mathcal{C})$ -trackable.

Proof of Claim. Take any $uv \in E$ and $2 \leq m \leq \lfloor \frac{\ell}{2} \rfloor$. We will show that $w_{u,v,m}$ is $(d, \varepsilon, \mathcal{C})$ -trackable. By (2),

$$w_{u,v,m}(\mathcal{H}) = |\mathcal{P}_{u,v,m}| = \Theta\left(d^{m+\frac{m-1}{k-2}}\right) > d^{m+\varepsilon},$$

proving (W1).

To see (W2), take any $j' \in [m-1]$ and $S' \in E(\mathcal{H})^{(j')}$, and note that when we count the number of $S \in \mathcal{P}_{u,v,m}$ that contain S' , we choose at least $j'(k-2)$ fewer vertices and do not choose a colour, so

$$w_{u,v,m}(\{S \in E(\mathcal{H})^{(m)} : S \supseteq S'\}) = O\left(\frac{|\mathcal{P}_{u,v,m}|}{n^{j'(k-2)+1}}\right) = O\left(\frac{w_{u,v,m}(\mathcal{H})}{d^{j'+\frac{1}{k-2}}}\right) < \frac{w_{u,v,m}(\mathcal{H})}{d^{j'+\varepsilon}}.$$

To see (W3), first recall that for $e \in E(\mathcal{H})$, $\mathcal{C}_e = \{S \setminus \{e\} : e \in S \in E(\mathcal{C})\}$. Take any distinct $e, f \in E(\mathcal{H})$ with $w_{u,v,m}(\{S \in E(\mathcal{H})^{(m)} : e, f \in S\}) > 0$. Then $e = (A, i)$ and $f = (A', i)$ for some disjoint $A, A' \in V^{(k-1)}$ and $i \in C$. Since sizes of edges in $\mathcal{C}$ are even numbers ranging from 4 to ℓ , sizes of elements of $\mathcal{C}_e$ and $\mathcal{C}_f$ are odd numbers ranging from 3 to $\ell-1$, so take $2 \leq m' \leq \lfloor \frac{\ell}{2} \rfloor$ and let $j' = 2m' - 1$.

So any $S \in \mathcal{C}_e \cap \mathcal{C}_f$ is a “path” $\{(B_1, j), (A_2, i), (B_2, j), \dots, (A_{m'}, i), (B_{m'}, j)\}$, where $(A, i), (B_1, j), (A_2, i), \dots, (B_{m'}, j)$ and $(A', i), (B_1, j), (A_2, i), \dots, (B_{m'}, j)$ are both cycles in $\mathcal{H}$. In this case, two vertices in each of $B_1, B_{m'}$ are fixed by intersection with A and A' , and i is fixed. So choosing j , the A_t ($2 \leq t \leq m'$), the B_t ($2 \leq t \leq m' - 1$), and then $B_1, B_{m'}$, since $\varepsilon < \frac{2}{k-2}$, we have

$$|\mathcal{C}_e^{(j')} \cap \mathcal{C}_f^{(j')}| = O\left(n \cdot n^{(m'-1)(k-1)} n^{(m'-2)(k-3)} n^{2(k-4)}\right) = O\left(n^{(k-2)(2m'-1-\frac{2}{k-2})}\right) \leq d^{j'-\varepsilon},$$

proving (W3).

(W4) holds vacuously because $m < 2m$. Therefore, $w_{u,v,m}$ is $(d, \varepsilon, \mathcal{C})$ -trackable. $\square$

For each $2 \leq m \leq \lfloor \frac{\ell}{2} \rfloor$ and $uv \in E(K_n)$, define

$$\mathcal{T}_{u,v,m} := \{ \{(A_1, i), \dots, (A_m, i), (B, j)\} \in E(\mathcal{H})^{(m+1)} : \{(A_1, i), \dots, (A_m, i)\} \in \mathcal{P}_{u,v,m}, \\ j \neq i, u \in A_1 \setminus B, v \in A_m \setminus B, |B \cap A_t| \leq 1 \forall t \in [m], |B \cap A_1| = |B \cap A_2| = 1 \}.$$

Let

$$a := \begin{cases} (m-2)(k-1)(k-2), & m \geq 3, \\ (k-2)^2, & m = 2. \end{cases}$$

Note that every set in $\mathcal{T}_{u,v,m}$ contains exactly one set in $\mathcal{P}_{u,v,m}$. When counting sets $S \cup \{(B, j)\} \in \mathcal{T}_{u,v,m}$ for fixed $S \in \mathcal{P}_{u,v,m}$, we choose $(A_2, i) \in S \setminus \{(A_1, i), (A_m, i)\}$ if $m \geq 3$, and we choose $B \cap A_1, B \cap A_2, B \setminus (A_1 \cup A_2)$, and $j \in C \setminus \{i\}$.

So we have

$$\begin{aligned} |\mathcal{T}_{u,v,m}| &= \left(a \binom{n}{k-3} \frac{n}{k-2} \pm O(n^{k-3}) \right) |\mathcal{P}_{u,v,m}| \\ &= \frac{ad^{m+1}n^{m-1}}{(m-2)!(k-2)(k-1)^{m-2}} \pm O(n^{mk-m+k-4}). \end{aligned} \quad (3)$$

Let $w'_{u,v,m}$ be the indicator function defined by

$$w'_{u,v,m}(S) = \begin{cases} 1, & S \in \mathcal{T}_{u,v,m}, \\ 0, & S \notin \mathcal{T}_{u,v,m}, \end{cases}$$

for all matchings $S \in E(\mathcal{H})^{(m+1)}$, extended to all subsets of $E(\mathcal{H})$. Note that $w'_{u,v,m}$ is an $(m+1)$ -uniform test function.

Claim 19. For all $uv \in E(K_n)$ and $2 \leq m \leq \lfloor \frac{\ell}{2} \rfloor$, $w'_{u,v,m}$ is $(d, \varepsilon, \mathcal{C})$ -trackable.

Proof of Claim. Take any $uv \in E$ and $2 \leq m \leq \lfloor \frac{\ell}{2} \rfloor$. We will show that $w'_{u,v,m}$ is $(d, \varepsilon, \mathcal{C})$ -trackable. By (3),

$$w'_{u,v,m}(\mathcal{H}) = |\mathcal{T}_{u,v,m}| = \Omega\left(d^{m+1+\frac{m-1}{k-2}}\right) > d^{m+1+\varepsilon},$$

proving (W1).

To see (W2), take any $j' \in [m]$ and $S' \in E(\mathcal{H})^{(j')}$. If every edge in S' receives the same colour, then when we count the number of $S \in \mathcal{T}_{u,v,m}$ that contain S' , we choose at least $j'(k-2)$ fewer vertices and one fewer colour. If an edge in S' receives a different colour from the rest, then we choose at least $(j'-1)(k-2) + k-3$ fewer vertices and two fewer colours. In either case,

$$w'_{u,v,m}(\{S \in E(\mathcal{H})^{(m+1)} : S \supseteq S'\}) = O\left(\frac{w'_{u,v,m}(\mathcal{H})}{d^{j'+\frac{1}{k-2}}}\right) < \frac{w'_{u,v,m}(\mathcal{H})}{d^{j'+\varepsilon}},$$

so (W2) holds.

To see (W3), take any distinct $e, f \in E(\mathcal{H})$ with $w'_{u,v,m}(\{S \in E(\mathcal{H})^{(m)} : e, f \in S\}) > 0$, take $2 \leq m' \leq \lfloor \frac{\ell}{2} \rfloor$, and let $j' = 2m'$. The case in which $e = (A, i)$ and $f = (A', i)$ for some disjoint $A, A' \in V^{(k-1)}$ and $i \in C$ follows exactly the same as in Claim 18.

We claim that the other case, where $e = (A, i)$ and $f = (A', j)$ for $A, A' \in V^{(k-1)}$ and $i \neq j$, cannot occur. Suppose otherwise, and take any $S \in \mathcal{C}_e \cap \mathcal{C}_f$ with $|S| = 2m' - 1$. Then since colours alternate on edges in $\mathcal{C}$, the edge $S \cup \{e\} \in E(\mathcal{C})$ has exactly m' edges of colour i , so S has exactly $m' - 1$ edges of colour i . But the edge $S \cup \{f\} \in E(\mathcal{C})$ has exactly m' edges of colour i , so S has exactly m' edges of colour i , a contradiction. This proves (W3).

(W4) holds vacuously because $m+1 < 2m$. Therefore, $w'_{u,v,m}$ is $(d, \varepsilon, \mathcal{C})$ -trackable. $\square$

Let $\mathcal{W}$ be the collection of test functions consisting of all defined w_v , $w_{u,v,m}$, and $w'_{u,v,m}$. By Claims 17, 18 and 19, every function in $\mathcal{W}$ is $(d, \varepsilon, \mathcal{C})$ -trackable. Using Claims 15 and 16, apply Theorem 8 to obtain a matching $\mathcal{M}$ satisfying (I) and (II).

In addition, because δ can be chosen to be arbitrarily small compared to ε , for each $v \in V$,

$$d_{G(\mathcal{M})}(v) = w_v(\mathcal{M}) \geq (1 - d^{-\varepsilon^3})d^{-1}w_v(\mathcal{H}) > (1 - n^{-\delta})n = n - n^{1-\delta},$$

where the first equality follows from the fact that $\mathcal{M}$ is a matching. Therefore, we have $\Delta(K_n - E(G(\mathcal{M}))) \leq n^{1-\delta}$, which gives (III).

Using (2) and (3), we also obtain

$$|\mathcal{M}^{(m)} \cap \mathcal{P}_{u,v,m}| = w_{u,v,m}(\mathcal{M}) \leq (1 + d^{-\varepsilon^3})d^{-m}|\mathcal{P}_{u,v,m}| \leq \frac{(1 + n^{-m\delta})n^{m-1}}{(m-2)!(k-2)(k-1)^{m-2}}$$

and

$$|\mathcal{M}^{(m)} \cap \mathcal{T}_{u,v,m}| = w'_{u,v,m}(\mathcal{M}) \geq (1 - d^{-\varepsilon^3})d^{-(m+1)}|\mathcal{T}_{u,v,m}| \geq \frac{(1 - n^{-m\delta})an^{m-1}}{(m-2)!(k-2)(k-1)^{m-2}}.$$

Let $S_{u,v,m}$ be the set of cycles $u, v_1, u_2, v_2, \dots, u_m, v, u$ such that, setting $u_1 = u$ and $v_m = v$, there exist distinct (and thus disjoint) $(A_1, i), \dots, (A_m, i) \in \mathcal{M}$ such that $v_t u_t \in A_t$ for all $t \in [m]$. Let $S'_{u,v,m}$ be the subset of $S_{u,v,m}$ that consists of such cycles with $v_1 u_2 \in E(G(\mathcal{M}))$. Then (IV) holds for $G(\mathcal{M})$ if $|S_{u,v,m} \setminus S'_{u,v,m}| \leq n^{(m-1)(1-\delta)}$.

To obtain a cycle in $S_{u,v,m}$ from a set in $\mathcal{P}_{u,v,m}$, choose an ordering for $\{(A_2, i), \dots, (A_{m-1}, i)\}$, choose $u_t \in A_t$ for all $2 \leq t \leq m-1$, choose $v_t \in A_t \setminus \{u_t\}$ for all $1 \leq t \leq m-1$ ($u_1 = u$), and choose $u_m \in A_t \setminus \{v\}$. So

$$|S_{u,v,m}| \leq |\mathcal{M}^{(m)} \cap \mathcal{P}_{u,v,m}|(m-2)!(k-1)^{m-2}(k-2)^m \leq (1 + n^{-m\delta})(k-2)^{m-1}n^{m-1}.$$

To obtain a cycle in $S'_{u,v,m}$ from a set in $\mathcal{T}_{u,v,m}$, choose an ordering for $\{(A_3, i), \dots, (A_{m-1}, i)\}$, set $v_1 \in A_1 \cap B$ and $u_2 \in A_2 \cap B$, choose $u_t \in A_t$ for all $3 \leq t \leq m-1$, choose $v_t \in A_t \setminus \{u_t\}$ for all $2 \leq t \leq m-1$, and choose $u_m \in A_m \setminus \{v\}$. So for $m \geq 3$,

$$|S'_{u,v,m}| \geq |\mathcal{M}^{(m+1)} \cap \mathcal{T}_{u,v,m}|(m-3)!(k-1)^{m-3}(k-2)^{m-1} = (1 - n^{-m\delta})(k-2)^{m-1}n^{m-1},$$

and for $m = 2$,

$$|S'_{u,v,2}| \geq |\mathcal{M}^{(3)} \cap \mathcal{T}_{u,v,2}| = \frac{(1 - n^{-m\delta})(k-2)^2 n^{m-1}}{k-2} = (1 - n^{-m\delta})(k-2)^{m-1}n^{m-1}.$$

Therefore, for all m ,

$$|S_{u,v,m} \setminus S'_{u,v,m}| \leq 2n^{-m\delta}(k-2)^{m-1}n^{m-1} = O(n^{m-1-m\delta}) < n^{m-1-m\delta+\delta} = n^{(m-1)(1-\delta)}.$$

This proves (IV) for $G(\mathcal{M})$.

Therefore, $G := G(\mathcal{M})$ is a spanning subgraph of K_n with at most $n/(k-2)$ colours that satisfies (I)–(IV). This completes the proof of Lemma 11. $\square$

4 Proofs of Theorem 4 and Corollary 5

In this section, we prove Theorem 4 and Corollary 5, restated here for convenience.

Theorem 4. *For any fixed integer $k \geq 2$, even number $\ell \geq k^2 - k + 2$, and n sufficiently large, we have*

$$r(K_{n,n}, C_{[k^2-k+2, \ell]}, 3) \leq \frac{2n}{k^2-1} + o(n).$$

Corollary 20. *For any fixed even number $k \geq 4$ and n sufficiently large,*

$$\frac{n}{k-2} + o(n) \leq r(K_{n,n}, C_k, 3) \leq \frac{2n}{\left\lceil \frac{1+\sqrt{4k-7}}{2} \right\rceil^2 - 1} + o(n).$$

For disjoint sets X and Y , define $B(X, Y)$ to be the complete bipartite graph with partite sets X and Y .

We can obtain the lower bound in Corollary 5 easily using the bipartite extremal number $\text{ex}(m, n; H)$, as Axenovich, Füredi, and Mubayi [1] noted in the C_4 case.

Definition 21. For a bipartite graph H , let $\text{ex}(m, n; H)$ be the maximum number of edges in a subgraph of $K_{m,n}$ that contains no copy of H .

Theorem 22 (Gyárfás, Rousseau, Schelp [24]). *For all $m, n, k \in \mathbb{N}$ with $m \leq n$, we have*

$$\text{ex}(m, n; P_{2(k+1)}) = \begin{cases} mn, & m \leq k \\ nk, & k < m < 2k \\ (m + n - 2k)k, & m \geq 2k. \end{cases}$$

We can now prove Corollary 5 assuming Theorem 4 holds.

Proof of Corollary 5 (Assuming Theorem 4). For the lower bound, note that no $(C_k, 3)$ -colouring of $K_{n,n}$ can contain a monochromatic copy of P_k , since any such path in $K_{n,n}$ can be completed to a cycle of length k with at most 2 colours. Thus by Theorem 22, we have

$$r(K_{n,n}, C_k, 3) \geq \frac{|E(K_{n,n})|}{\text{ex}(n, n; P_k)} = \frac{n^2}{(k-2)n \pm O(1)} = \frac{n}{k-2} + o(n),$$

proving the lower bound.

We now turn our attention to the upper bound. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined by $f(x) := x^2 - x + 2$ and let $g : \mathbb{R} \rightarrow \mathbb{R}$ be defined by $g(x) := f(x) - k$. Note that g has a positive root at $x_0 := \frac{1+\sqrt{4k-7}}{2}$, so $0 = f(x_0) - k$. So as f is increasing on $[1, \infty)$, we have $k = f(x_0) \geq f(\lfloor x_0 \rfloor)$.

Thus, by Theorem 4,

$$r(K_{n,n}, C_k, 3) \leq r(K_{n,n} C_{[f(\lfloor x_0 \rfloor), k]}, 3) \leq \frac{2n}{\lfloor x_0 \rfloor^2 - 1} + o(n),$$

proving the upper bound. □

The proof of Theorem 4 follows very closely to the proof of Theorem 2. That is, we apply Theorem 8 to prove a technical lemma (see Lemma 23 below) that gives a very structured $(C_{[k, \ell]}, 3)$ -colouring of a large subgraph $G \subseteq K_{n,n}$. We are then able to show that adding a suitable random colouring of $K_{n,n} - G$ will not create a cycle with length in $[k, \ell]$ that has fewer than 3 colours.

The following is the technical lemma we will use to prove Theorem 4.

Lemma 23. *For each $k \geq 2$ and even $\ell \geq k^2 - k + 2$, there exists $\delta > 0$ depending only on k and ℓ such that for all sufficiently large n , there exists an edge-colouring of a spanning subgraph G of $B(X, Y) \cong K_{n,n}$ with at most $\frac{2n}{k^2-1}$ colours and the following properties:*

(I) *Every colour class consists of vertex-disjoint copies of $K_{\binom{k}{2}, \binom{k+1}{2}}$.*

- (II) Given any two colours i, j , the k^2 -uniform hypergraph with vertex set $V(G)$ where each edge is formed by the vertex set of a copy of $K_{\binom{k}{2}, \binom{k+1}{2}}$ in colour i or j has girth at least $\ell + 1$.
- (III) The graph $L := B(X, Y) - E(G)$ has maximum degree at most $n^{1-\delta}$.
- (IV) For each $2 \leq m \leq \ell/2$ and $(x_1, y_m) \in X \times Y$, the number of cycles $x_1, y_1, x_2, y_2, \dots, x_m, y_m, x_1$ such that $x_i y_{i+1} \in E(L)$ for all $i \in [m-1]$ with disjoint monochromatic copies $F_1, \dots, F_m$ of $K_{\binom{k}{2}, \binom{k+1}{2}}$, all of the same colour, such that $x_i y_i \in E(F_i)$ for all $i \in [m]$, is at most $n^{(m-1)(1-\delta)}$.

Before discussing the proof of Lemma 23 (in Subsection 4.1 below), we first apply it to prove Theorem 4. As the proof is incredibly similar to the proof of Theorem 2 above, we omit the proofs of the claims and include them for completeness in the appendix of the arXiv version of our paper.

Proof of Theorem 4. Apply Lemma 23 to obtain a coloured graph G and a number δ as stated in Lemma 23.

Claim 24. G is $(C_{[k^2-k+2, \ell]}, 3)$ -coloured.

Let $0 < \alpha < \min\{\frac{\delta}{4}, \frac{1}{2}\}$. We will colour edges in L randomly such that there is a non-zero probability that the $(C_{[k^2-k+2, \ell]}, 3)$ -colouring is preserved. Let P be a set of $r := n^{1-\alpha}$ new colours that do not colour edges in G . Colour each edge of L independently with a colour from P with uniform probability $1/r$.

For any pair e, f of adjacent edges in L and $i \in P$, let $A_{e,f,i}$ be the event that both e and f receive colour i . Then $\mathbb{P}[A_{e,f,i}] = r^{-2} = n^{-2(1-\alpha)}$. For $\binom{k}{2} + 1 = \frac{k^2-k+2}{2} \leq m \leq \frac{\ell}{2}$ and a $2m$ -cycle D in L , define B_D to be the event that D is properly 2-coloured.

Then once we have fixed the colours of two adjacent edges, there is only one choice for the colour of each remaining edge, so $\mathbb{P}[B_D] \leq r^{-(2m-2)} = n^{-(2m-2)(1-\alpha)}$. For $2 \leq m \leq \frac{\ell}{2}$ and a $2m$ -cycle $D = u_1, v_1, \dots, u_m, v_m, u_1$ in K_n where there are disjoint monochromatic copies of $F_1, \dots, F_m$ of K_{k-1} in G , all of the same colour, such that $u_i v_i \in E(F_i)$ for all $i \in [m]$ and $v_i u_{i+1} \in E(L)$ (taken mod m) for all $i \in [m]$, and for a colour $j \in P$, let $C_{D,j}$ be the event that every $v_i u_{i+1} \pmod{m}$ receives the same colour. Then $\mathbb{P}[C_{D,j}] = r^{-m} = n^{-m(1-\alpha)}$. Let $\mathcal{A}$ be the set of all defined $A_{e,f,i}$, and for all m , let $\mathcal{B}_m$, and $\mathcal{C}_m$ be the sets of all defined B_D and $C_{D,j}$, respectively, where D has length $2m$. Let

$$\mathcal{E} := \mathcal{A} \cup \bigcup_{m=(k^2-k+2)/2}^{\ell/2} \mathcal{B}_m \cup \bigcup_{m=2}^{\ell/2} \mathcal{C}_m.$$

Claim 25. There exists a colouring of L such that none of the events in $\mathcal{E}$ occurs.

We claim that a colouring of L from Claim 25, combined with the colouring of G , is a $(C_{[k^2-k+2, \ell]}, 3)$ -colouring of $K_{n,n}$. Let $k^2 - k + 2 \leq h \leq \ell$, and let C be an h -cycle in $K_{n,n}$. We will show that C receives at least 3 colours.

If $E(C) \subseteq E(G)$, then we are done by Claim 24. If $E(C) \subseteq E(L)$, then C receives at least 3 colours because events in $\mathcal{A}$ and $\bigcup_{m=(k^2-k+2)/2}^{\ell/2} \mathcal{B}_m$ do not occur (C is properly coloured but not properly 2-coloured). If $E(C)$ contains two adjacent edges in L and an edge in G , then since events in $\mathcal{A}$ do not occur, C receives at least 2 colours in P and at least 1 colour not in P , so it receives at least 3 colours.

We have reduced to the case that $E(C) \cap E(L)$ is a matching and C receives at most 2 colours, say $c_1 \notin P$ and $c_2 \in P$, so it suffices to prove that this is impossible. Suppose for contradiction that it is possible. Let $m := |E(C) \cap E(L)|$. In this case, C can be decomposed into paths and single edges $P_1, e_1, P_2, e_2, \dots, P_m, e_m$, where e_i is incident with an endvertex v_i of P_i and an endvertex u_{i+1} of P_{i+1} for all i (taken mod m), each P_i is a monochromatic path in colour c_1 , and each e_i receives the colour c_2 . So $u_1, v_1, u_2, v_2, \dots, u_m, v_m, u_1$ is a cycle with alternating colours c_1, c_2 . Let $D = x_1, y_1, x_2, y_2, \dots, x_{m'}, y_{m'}, x_1$ be a shortest cycle with alternating colours c_1, c_2 in $K_{n,n}$, where $x_1 y_1$ receives colour c_1 .

For each $i \in [m']$, let F_i be the monochromatic copy of $K_{\binom{k}{2}, \binom{k+1}{2}}$ that contains the edge $x_i y_i$. We claim that the F_i are all distinct (and thus disjoint by Lemma 23(I)). Suppose not, and let $1 \leq i < j \leq m'$ such that $F_i = F_j$. Then since $y_i, x_j \in V(F_i)$, $y_i x_j$ receives colour c_1 , so $y_i, x_{i+1}, y_{i+1}, \dots, x_{j-1}, y_{j-1}, x_j, y_i$ is a cycle with alternating colours c_1, c_2 in $K_{n,n}$ of length less than $2m'$, contradicting minimality of m' .

This confirms that the event C_{D, c_2} occurs. But this contradicts Claim 25. Therefore, $K_{n,n}$ is $(C_{[k^2-k+2, \ell]}, 3)$ -coloured with $\frac{2n}{k^2-1} + n^{1-\alpha} = \frac{2n}{k^2-1} + o(n)$ colours, so

$$r(K_{n,n}, C_{[k^2-k+2, \ell]}, 3) \leq \frac{2n}{k^2-1} + o(n),$$

as desired. □

4.1 Proof of Lemma 23

Here we give the proof of Lemma 23. It follows analogously to the proof of Lemma 11. As the proofs of the claims are essentially some technical counting that follows very similarly to arguments presented in the proof of Lemma 11 above, we omit these here. For completeness, they can be found in the appendix of the arXiv version of our paper.

Proof of Lemma 23. Let $\varepsilon = \varepsilon(k) \in (0, \frac{1}{k^2-1})$. We will choose n to be sufficiently large and δ to be sufficiently small compared to ε .

We define the $k(k+1)$ -uniform hypergraph $\mathcal{H}$ as follows. Let X and Y be two disjoint vertex sets of size n . Let $C := [\frac{2n}{k^2-1}]$; we will refer to C as the set of colours. Let

$$V(\mathcal{H}) := (X \cup Y)^{(2)} \cup ((X \cup Y) \times C),$$

let

$$\mathcal{T} := \left\{ A \in (X \cup Y)^{(k^2)} : |A \cap X| \in \left\{ \binom{k}{2}, \binom{k+1}{2} \right\} \right\},$$

and let

$$E(\mathcal{H}) := \{A^{(2)} \cup (A \times \{i\}) : A \in \mathcal{T}, i \in C\}.$$

There is a natural bijection between $E(\mathcal{H})$ and $\mathcal{T} \times C$, so denote an edge $A^{(2)} \cup (A \times \{i\}) \in E(\mathcal{H})$ simply by (A, i) .

For each matching $\mathcal{M}$ of $\mathcal{H}$, let $G = G(\mathcal{M})$ be the edge-coloured graph with $V(G) = X \cup Y$, $E(G) = \{e \in E(B(X, Y)) : e \in A^{(2)} \text{ for some } (A, i) \in \mathcal{M}\}$, and edge-colouring ϕ with $e \mapsto i$ for all $(A, i) \in \mathcal{M}$ and $e \in A^{(2)}$. Note that by the definition of $\mathcal{H}$, any $G(\mathcal{M})$ satisfies condition (I).

Let

$$d := \frac{k^2 n^{k^2-1}}{\binom{k}{2}! \binom{k+1}{2}!}.$$

Claim 26. $(1 - d^{-\varepsilon})d \leq \delta(\mathcal{H}) \leq \Delta(H) \leq d$.

Claim 27. $\Delta_2(\mathcal{H}) \leq d^{1-\varepsilon}$.

We next define a conflict system $\mathcal{C}$ for $\mathcal{H}$. We set $V(\mathcal{C}) := E(\mathcal{H})$. Let $E(\mathcal{C})$ be the set of edges that form a $2m$ -cycle in $\mathcal{H}$ in alternating colours for some $2 \leq m \leq \ell/2$. More precisely, given $2 \leq m \leq \ell/2$, distinct vertices $x_1, \dots, x_m \in X$ and $y_1, \dots, y_m \in Y$, two distinct colours $i, j \in C$, and $A_1, \dots, A_m, B_1, \dots, B_m \in \mathcal{T}$ with $x_i \in A_i \cap B_i$ and $y_i \in B_i \cap A_{i+1}$ for all $i \in [m]$ (setting $A_{m+1} := A_1$), say

$$S := \{(A_1, i), (B_1, j), (A_2, i), (B_2, j), \dots, (A_m, i), (B_m, j)\}$$

is in $E(\mathcal{C})$ whenever S is a matching in $\mathcal{H}$.

Claim 28. $\mathcal{C}$ is a $(d, O(1), \varepsilon)$ -bounded conflict system for $\mathcal{H}$.

We next define a set of test functions. For each $v \in X \cup Y$, let S_v be the set of edges in $B(X, Y)$ incident to v . Let $w_v : E(\mathcal{H}) \rightarrow [0, \binom{k+1}{2}]$ be the 1-uniform test function defined by $S \mapsto |S \cap S_v|$.

Claim 29. w_v is $(d, \varepsilon, \mathcal{C})$ -trackable for all $v \in V$.

For each $2 \leq m \leq \ell/2$, $(x, y) \in X \times Y$, and $a, b \in \{0, 1\}$, let

$$\mathcal{P}_{x,y,a,b,m} := \left\{ \{(A_1, i), \dots, (A_m, i)\} \in E(\mathcal{H})^{(m)} : i \in C, A_s \cap A_t = \emptyset \ \forall s \neq t, \right. \\ \left. x \in A_1, |A_1 \cap X| = \binom{k+a}{2}, y \in A_m, |A_m \cap Y| = \binom{k+b}{2} \right\}.$$

Then by choosing $A_1 \setminus \{x\}, A_m \setminus \{y\}$, then $\{A_2, \dots, A_{m-1}\}$, then i , we have

$$\begin{aligned} |\mathcal{P}_{x,y,a,b,m}| &= \binom{n}{\binom{k+a}{2}} - 1 \binom{n}{\binom{k+1-a}{2}} \binom{n}{\binom{k+b}{2}} - 1 \binom{n}{\binom{k+1-b}{2}} \frac{1}{(m-2)!} \\ &\quad \cdot \left(2 \binom{n}{\binom{k}{2}} \binom{n}{\binom{k+1}{2}} \right)^{m-2} \frac{2n}{k^2-1} \pm O(n^{mk^2-2}) \\ &= \frac{2^{m-1} \binom{k+a}{2} \binom{k+b}{2} d^m n^{m-1}}{k^{2m} (k^2-1)(m-2)!} \pm O(n^{mk^2-2}). \end{aligned} \quad (4)$$

Let $w_{x,y,a,b,m}$ be the indicator function defined by

$$w_{x,y,a,b,m}(S) = \begin{cases} 1, & S \in \mathcal{P}_{x,y,a,b,m}, \\ 0, & S \notin \mathcal{P}_{x,y,a,b,m} \end{cases}$$

for all matchings $S \in E(\mathcal{H})^{(m)}$, extended to all subsets of $E(\mathcal{H})$ as described in Section 2. Note that $w_{x,y,a,b,m}$ is an m -uniform test function.

Claim 30. For all $(x, y) \in X \times Y$, $2 \leq m \leq \ell/2$, and $a, b \in \{0, 1\}$, $w_{x,y,a,b,m}$ is (d, ε, C) -trackable.

For each $2 \leq m \leq \ell$, $(x, y) \in X \times Y$, and $a, b, c \in \{0, 1\}$, let $\mathcal{T}_{x,y,a,b,c,m}$ be the set of sets $S \cup \{(B, j)\} \in E(\mathcal{H})^{(m+1)}$ where $S = \{(A_1, i), \dots, (A_m, i)\} \in \mathcal{P}_{x,y,a,b,m}$ such that $j \neq i$, $x \in A_1 \setminus B$, $y \in A_m \setminus B$, $|B \cap A_k| \leq 1$ for all $k \in [m]$, $|A_2 \cap X| = \binom{k+c}{2}$, and $|B \cap A_1 \cap Y| = |B \cap A_2 \cap X| = 1$.

Let

$$z := \begin{cases} \frac{m-2}{2}, & m \geq 3, \\ 1, & m = 2. \end{cases}$$

Then when counting sets $\{(A_1, i), (A_2, i), \dots, (A_m, i)(B, j)\} \in \mathcal{T}_{x,y,a,b,c,m}$, we choose $A_1 \setminus \{x\}$, $A_m \setminus \{y\}$, A_2 , $\{A_3, \dots, A_{m-1}\}$, $B \cap A_1 \cap Y$, $B \cap A_2 \cap X$, $B \setminus (A_1 \cup A_2)$, $i \in C$, and $j \in C \setminus \{i\}$. The factor z accounts for the fact that when $m \geq 3$, the set A_2 is distinguished from $A_3, \dots, A_{m-1}$ by intersection with B , and $|A_2 \cap X|$ is fixed by c .

So we have

$$\begin{aligned} |\mathcal{T}_{x,y,a,b,c,m}| &= \binom{n}{\binom{k+a}{2}} - 1 \binom{n}{\binom{k+1-a}{2}} \binom{n}{\binom{k+b}{2}} - 1 \binom{n}{\binom{k+1-b}{2}} \frac{z}{(m-2)!} \left(2 \binom{n}{\binom{k}{2}} \binom{n}{\binom{k+1}{2}} \right)^{m-2} \\ &\quad \cdot \binom{k+1-a}{2} \binom{k+c}{2} 2 \binom{n}{\binom{k+1}{2}} - 1 \binom{n}{\binom{k}{2}} - 1 \left(\frac{2n}{k^2-1} \right)^2 \pm O(n^{(m+1)k^2-3}) \\ &= \frac{2^{m-3} \binom{k+c}{2} \binom{k+b}{2} z d^{m+1} n^{m-1}}{k^{2m-2} (m-2)!} \pm O(n^{(m+1)k^2-3}). \end{aligned} \quad (5)$$

Let $w'_{x,y,a,b,c,m}$ be the indicator function defined by

$$w'_{x,y,a,b,c,m}(S) = \begin{cases} 1, & S \in \mathcal{T}_{x,y,a,b,c,m}, \\ 0, & S \notin \mathcal{T}_{x,y,a,b,c,m} \end{cases}$$

for all matchings $S \in E(\mathcal{H})^{(m+1)}$, extended to all subsets of $E(\mathcal{H})$. Note that $w'_{x,y,a,b,c,m}$ is an $(m+1)$ -uniform test function.

Claim 31. *For all $(x, y) \in X \times Y$, $2 \leq m \leq \ell$, and $a, b, c \in \{0, 1\}$, $w'_{x,y,a,b,c,m}$ is $(d, \varepsilon, \mathcal{C})$ -trackable.*

Let $\mathcal{W}$ be the collection of test functions consisting of all defined w_v , $w_{x,y,a,b,m}$, and $w'_{x,y,a,b,c,m}$. By Claims 29, 30 and 31, every function in $\mathcal{W}$ is $(d, \varepsilon, \mathcal{C})$ -trackable. Using Claims 26 and 27, apply Theorem 8 to obtain a matching $\mathcal{M}$ satisfying (I) and (II).

In addition, because δ can be chosen to be arbitrarily small compared to ε , for each $v \in X \cup Y$,

$$d_{G(\mathcal{M})}(v) = w_v(\mathcal{M}) \geq (1 - d^{-\varepsilon^3})d^{-1}w_v(\mathcal{H}) > (1 - n^{-\delta})n = n - n^{1-\delta},$$

where the first equality follows from the fact that $\mathcal{M}$ is a matching. Therefore, we have $\Delta(B(X, Y) - E(G(\mathcal{M}))) \leq n^{1-\delta}$, which gives (III).

Since δ is arbitrarily small compared to ε , using (4) we obtain

$$\begin{aligned} |\mathcal{M}^{(m)} \cap \mathcal{P}_{x,y,a,b,m}| &= w_{x,y,a,b,m}(\mathcal{M}) \leq (1 + d^{-\varepsilon^3})d^{-m}|\mathcal{P}_{x,y,a,b,m}| \\ &\leq (1 + n^{-m\delta}) \frac{2^{m-1} \binom{k+a}{2} \binom{k+b}{2} n^{m-1}}{k^{2m}(k^2 - 1)(m-2)!}, \end{aligned} \quad (6)$$

and using (5), we obtain

$$\begin{aligned} |\mathcal{M} \cap \mathcal{T}_{x,y,a,b,c,m}| &= w'_{x,y,a,b,c,m}(\mathcal{M}) \geq (1 - d^{-\varepsilon^3})d^{-(m+1)}|\mathcal{T}_{x,y,a,b,c,m}| \\ &\geq (1 - n^{-m\delta}) \frac{2^{m-3} \binom{k+c}{2} \binom{k+b}{2} zn^{m-1}}{k^{2(m-1)}(m-2)!}. \end{aligned} \quad (7)$$

For $(x, y) \in X \times Y$, let $S_{x,y,m}$ be the set of cycles $x, y_1, x_2, y_2, \dots, x_m, y, x$ such that, setting $x_1 = x$ and $y_m = y$, there exist distinct (and thus disjoint) $(A_i, i), \dots, (A_m, i) \in \mathcal{M}$ such that $x_i y_i \in A_i$ for all i . Let $S'_{x,y,m}$ be the subset of $S_{u,v,m}$ that consists of such cycles with $x_1 y_2 \in E(G(\mathcal{M}))$. Then (IV) holds for $G(\mathcal{M})$ if $|S_{x,y,m} \setminus S'_{x,y,m}| \leq n^{(m-1)(1-\delta)}$ for all $(x, y) \in X \times Y$.

To obtain such a cycle in $S_{x,y,m}$ from a set in $\mathcal{M}^{(m)} \cap \bigcup_{(a,b) \in \{0,1\}^2} \mathcal{P}_{x,y,a,b,m}$, choose $a, b \in \{0, 1\}$, choose an ordering for $\{(A_2, i), \dots, (A_{m-1}, i)\}$, choose $y_1 \in A_1 \cap Y$ and $x_m \in A_m \cap X$, and choose $(x_k, y_k) \in (A_k \cap X) \times (A_k \cap Y)$ for each $2 \leq k \leq m-1$. So using (6), we obtain

$$\begin{aligned} |S_{u,v,m}| &\leq \sum_{(a,b) \in \{0,1\}^2} |\mathcal{M}^{(m)} \cap \mathcal{P}_{x,y,a,b,m}| (m-2)! \binom{k+1-a}{2} \binom{k+1-b}{2} \left(\binom{k}{2} \binom{k+1}{2} \right)^{m-2} \\ &\leq \sum_{a,b \in \{0,1\}} \frac{(1 + n^{-m\delta}) 2^{m-1} \binom{k+a}{2} \binom{k+b}{2} n^{m-1}}{k^{2m}(k^2 - 1)(m-2)!} \frac{(m-2)! \binom{k+1-a}{2} \binom{k+1-b}{2} (k^2(k^2 - 1))^{m-2}}{2^{2(m-2)}} \\ &= \frac{(1 + n^{-m\delta})(k^2 - 1)^{m-1} n^{m-1}}{2^{m-1}}. \end{aligned}$$

To obtain a cycle in $S'_{x,y,m}$ from a set in $\mathcal{M}^{(m+1)} \cap \bigcup_{(a,b,c) \in \{0,1\}^3} \mathcal{T}_{x,y,a,b,c,m}$, choose an ordering for $\{(A_3, i), \dots, (A_{m-1}, i)\}$, set $y_1 \in A_1 \cap B$ and $x_2 \in A_2 \cap B$, choose $y_2 \in A_2 \cap Y$, choose $x_m \in A_m \cap X$, and choose $(x_k, y_k) \in (A_k \cap X) \times (A_k \cap Y)$ for all $3 \leq k \leq m-1$. So for $m \geq 3$, using (7), we obtain

$$\begin{aligned} |S'_{x,y,m}| &\geq \sum_{(a,b,c) \in \{0,1\}^3} |\mathcal{M}^{(m+1)} \cap \mathcal{T}_{x,y,a,b,c,m}| (m-3)! \binom{k+1-c}{2} \binom{k+1-b}{2} \left(\binom{k}{2} \binom{k+1}{2} \right)^{m-3} \\ &\geq \sum_{(a,b,c) \in \{0,1\}^3} \frac{(1-n^{-m\delta}) 2^{m-3} \binom{k+c}{2} \binom{k+b}{2} (m-2) n^{m-1}}{2k^{2(m-2)} (m-2)!} \frac{(m-3)! \binom{k+1-c}{2} \binom{k+1-b}{2} (k^2(k^2-1))^{m-3}}{2^{2(m-3)}} \\ &= \frac{(1-n^{-m\delta})(k^2-1)^{m-1} n^{m-1}}{2^{m-1}}, \end{aligned}$$

and for $m=2$, we have $A_m = A_2$, so $c = 1-b$, and so

$$\begin{aligned} |S'_{x,y,2}| &\geq \sum_{(a,b) \in \{0,1\}^2} |\mathcal{M}^{(3)} \cap \mathcal{T}_{x,y,a,b,(1-b),2}| \\ &= \sum_{(a,b) \in \{0,1\}^2} \frac{(1-n^{-m\delta}) 2^{m-3} \binom{k+1-b}{2} \binom{k+b}{2} n^{m-1}}{k^{2(m-1)} (m-2)!} \\ &= \frac{(1-n^{-m\delta})(k^2-1)^{m-1} n^{m-1}}{2^{m-1}}. \end{aligned}$$

Therefore, for all m ,

$$|S_{x,y,m} \setminus S'_{x,y,m}| \leq \frac{2n^{-m\delta}(k^2-1)^{m-1} n^{m-1}}{2^{m-1}} = O(n^{m-1-m\delta}) < n^{m-1-m\delta+\delta} = n^{(m-1)(1-\delta)}.$$

This proves (IV) for $G(\mathcal{M})$. This completes the proof of Lemma 11. $\square$

5 Concluding Remarks

The problem of characterizing the asymptotics of $r(K_{n,n}, C_k, 3)$ for $k \geq 6$ remains open.

We wonder whether the upper bound in Theorem 4 is tight.

Question 32. For any fixed integer $k \geq 2$ and even number $\ell \geq k^2 - k + 2$, is it the case that

$$r(K_{n,n}, C_{[k^2-k+2, \ell]}, 3) = \frac{2n}{k^2-1} + o(n)?$$

This prompts the question of whether the upper bound in Theorem 4 can be interpolated to a bound on $r(K_{n,n}, C_k, 3)$ that improves Corollary 5 when k is not of the form $\ell^2 - \ell + 2$ for some $\ell \in \mathbb{N}$.

Question 33. Is it the case that for any fixed even $k \geq 4$,

$$r(K_{n,n}, C_k, 3) \leq \frac{2n}{\left(\frac{1+\sqrt{4k-7}}{2}\right)^2 - 1} + o(n)?$$

It could also be interesting to improve the error term in Theorem 2.

Question 34. Is $r(K_n, C_k, 3) = n/(k-2) + O(1)$? Is $r(K_n, C_k, 3) = (n-1)/(k-2)$ for infinitely many n ?

Notes: During the final stages of preparing this manuscript for submission we became aware that Theorem 2 and Theorem 4 had simultaneously and independently been proved by Bal, Bennett, Heath, and Zerbib [2]. Like us, they apply the conflict-free hypergraph matching method, but our applications are significantly different. They prove a slightly stronger version of Theorem 2, where ℓ is allowed to grow slowly with n .

Since submission of our original manuscript, further progress has been made. In particular, Bal and Bennett [3] recently determined $r(K_{n,n}, C_6, 3)$ asymptotically, and answered Question 5.2. In particular, it is now known that neither bound in Corollary 5 is tight.

It has also been brought to our attention that the proof method used here (conflict-free hypergraph matchings followed by the local lemma to color the leftovers) has more recently been turned into a black box result by Joos, Mubayi and Smith [26].

Acknowledgements

Natasha Morrison was supported by NSERC Discovery Grant RGPIN-2021-02511 and NSERC Early Career Supplement DGECR-2021-00047.

References

- [1] Maria Axenovich, Zoltán Füredi, and Dhruv Mubayi, *On generalized Ramsey theory: the bipartite case*, J. Combin. Theory Ser. B **79** (2000), no. 1, 66–86.
- [2] Deepak Bal, Patrick Bennett, Emily Heath, and Shira Zerbib, *Generalized Ramsey numbers of cycles, paths, and hypergraphs*, (2024), [arXiv:2405.15904](#).
- [3] Deepak Bal and Patrick Bennett, *Edge-coloring $K_{n,n}$ with no 2-colored C_{2k}* , (2025), [arXiv:2507.13329](#).
- [4] Patrick Bennett, Ryan Cushman, and Andrzej Dudek, *Generalized Ramsey numbers at the linear and quadratic thresholds*, Electron. J. Combin. **32** (2025), no. 1, Paper No. 1.16, 18.
- [5] Patrick Bennett, Ryan Cushman, Andrzej Dudek, and Paweł Prałat, *The Erdős-Gyárfás function $f(n, 4, 5) = \frac{5}{6}n + o(n)$ —so Gyárfás was right*, J. Combin. Theory Ser. B **169** (2024), 253–297.
- [6] Patrick Bennett, Michelle Delcourt, Lina Li, and Luke Postle, *On generalized Ramsey numbers in the non-integral regime*, (2022), [arXiv:2212.10542](#).
- [7] Patrick Bennett, Emily Heath, and Shira Zerbib, *Edge-coloring a graph G so that every copy of a graph H has an odd color class*, (2023), [arXiv:2307.01314](#).
- [8] Tom Bohman and Lutz Warnke, *Large girth approximate Steiner triple systems*, J. Lond. Math. Soc. (2) **100** (2019), no. 3, 895–913.

- [9] David Conlon, Jacob Fox, and Benny Sudakov, *Recent developments in graph Ramsey theory*, Surveys in combinatorics 2015, London Math. Soc. Lecture Note Ser., vol. 424, Cambridge Univ. Press, Cambridge, 2015, pp. 49–118.
- [10] Michelle Delcourt, Tom Kelly, and Luke Postle, *Clique decompositions in random graphs via refined absorption*, (2024), [arXiv:2204.08981](#).
- [11] Michelle Delcourt, Tom Kelly, and Luke Postle, *Thresholds for $(n, q, 2)$ -Steiner systems via refined absorption*, (2024), [arXiv:2402.17858](#).
- [12] Michelle Delcourt and Luke Postle, *Finding an almost perfect matching in a hypergraph avoiding forbidden submatchings*, (2022), [arXiv:2204.08981](#).
- [13] Michelle Delcourt and Luke Postle, *Proof of the high girth existence conjecture via refined absorption*, (2024), [arXiv:2402.17856](#).
- [14] Michelle Delcourt and Luke Postle, *Refined absorption: A new proof of the existence conjecture*, (2024), [arXiv:2402.17855](#).
- [15] Paul Erdős, *Problems and results on finite and infinite graphs*, Recent advances in graph theory (Proc. Second Czechoslovak Sympos., Prague, 1974), Academia, Prague, 1975, pp. 183–192.
- [16] Paul Erdős and András Gyárfás, *A variant of the classical Ramsey problem*, Combinatorica **17** (1997), no. 4, 459–467.
- [17] R. J. Faudree and R. H. Schelp, *Path Ramsey numbers in multicolorings*, J. Combinatorial Theory Ser. B **19** (1975), no. 2, 150–160.
- [18] Cristina G. Fernandes, Guilherme Oliveira Mota, and Nicolás Sanhueza-Matamala, *Separating Path Systems in Complete Graphs*, Random Structures Algorithms **66** (2025), no. 3, Paper No. e70006.
- [19] Zoltán Füredi and Miklós Ruszinkó, *Uniform hypergraphs containing no grids*, Adv. Math. **240** (2013), 302–324.
- [20] Stefan Glock, Felix Joos, Jaehoon Kim, Marcus Kühn, and Lyuben Lichev, *Conflict-free hypergraph matchings*, J. Lond. Math. Soc. (2) **109** (2024), no. 5, Paper No. e12899, 78.
- [21] Stefan Glock, Felix Joos, Jaehoon Kim, Marcus Kühn, Lyuben Lichev, and Oleg Pikhurko, *On the $(6, 4)$ -problem of Brown, Erdős, and Sós*, Proc. Amer. Math. Soc. Ser. B **11** (2024), 173–186.
- [22] Stefan Glock, Daniela Kühn, Allan Lo, and Deryk Osthus, *On a conjecture of Erdős on locally sparse Steiner triple systems*, Combinatorica **40** (2020), no. 3, 363–403.
- [23] Enrique Gomez-Leos, Emily Heath, Alex Parker, Coy Schwieder, and Shira Zerbib, *New bounds on the generalized Ramsey number $f(n, 5, 8)$* , Discrete Math. **347** (2024), no. 7, Paper No. 114012, 12.
- [24] A. Gyárfás, C. C. Rousseau, and R. H. Schelp, *An extremal problem for paths in bipartite graphs*, J. Graph Theory **8** (1984), no. 1, 83–95.
- [25] Felix Joos and Dhruv Mubayi, *Ramsey theory constructions from hypergraph matchings*, Proc. Amer. Math. Soc. **152** (2024), no. 11, 4537–4550.

- [26] Felix Joos, Dhruv Mubayi, and Zak Smith, *Conflict-free hypergraph matchings and coverings*, Combinatorics, Probability and Computing **35** (2025), no. 2, 230–254.
- [27] Peter Keevash and Jason Long, *The Brown-Erdős-Sós conjecture for hypergraphs of large uniformity*, (2020), [arXiv:2007.14824](https://arxiv.org/abs/2007.14824).
- [28] Miao Liu and Chong Shangguan, *Approximate generalized Steiner systems and near-optimal constant weight codes*, J. Combin. Theory Ser. A **209** (2025), Paper No. 105955, 19.
- [29] Joel Spencer, *Asymptotic lower bounds for Ramsey functions*, Discrete Math. **20** (1977/78), no. 1, 69–76.