

Det-extremal bipartite graphs

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Abstract

Let G be a balanced bipartite graph with a bipartite adjacency matrix A . We call G det-extremal if $|\det(A)| = \text{per}(A)$. If every edge of G belongs to a perfect matching, then G is matching covered. Det-extremal cubic bipartite graphs of connectivity 3 and 2 have been characterized by McCuaig and Funk et al., respectively. In this paper, we extend the above results by investigating the structure of det-extremal bipartite graphs. Due to the fact that only the edges belonging to perfect matchings contribute to the det-extremal property, we focus on det-extremal bipartite graphs that are matching covered. Firstly, we derive that a matching covered plane bipartite graph is det-extremal if and only if it has a plane bipartite ear decomposition such that each interior face is of size two modulo four. Then we turn to general matching covered bipartite graph, and we associate its bipartite ear decomposition with a standard perfect matching M^* . By establishing the properties of its M^* -alternating cycles, we provide the characterization of a general matching covered det-extremal bipartite graph in terms of standard bipartite ear decomposition.

Mathematics Subject Classifications: 05C70, 05C75

1 Introduction

The graphs considered here are finite and simple. A *matching* in a graph is a set of edges without common vertices. A *perfect matching* of a graph G is a matching covering all the vertices of G . An edge of a graph is *admissible* if it lies in a perfect matching of this graph and *forbidden* otherwise. A nontrivial graph is *matching covered* (or *1-extendable*) if it is connected and each of its edges is admissible. It is easy to see that a matching covered graph is 2-connected. A graph G is said to be *elementary* if its admissible edges form a connected subgraph of G . For a bipartite graph G , G is elementary if and only if G is matching covered [6]. For terminology and notation not defined here, we refer the reader to [1].

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Let G be a bipartite graph with the bipartition (X, Y) . A bipartite graph G is said to be *balanced* if $|X| = |Y| = n$. Denote the vertex set of a balanced bipartite graph G by $V(G) = \{x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n\}$. The *bipartite adjacency matrix* of G is $A = (a_{ij})_{n \times n}$, where $a_{ij} = 1$ if there is an edge between x_i and y_j and $a_{ij} = 0$ otherwise. A balanced bipartite graph G is said to be *det-extremal* if $|\det(A)| = \text{per}(A)$. A subgraph H of G is *central* if $G - V(H)$ has a perfect matching. A path of even (resp. odd) length is called an *even path* (resp. *odd path*). Similarly for even cycles and odd cycles. One of the possible ways to say it, it is that a graph is *Pfaffian* if it admits an orientation under which every even central cycle has an odd number of edges directed in either direction of the cycle. In particular, a balanced bipartite graph G is Pfaffian if and only if there exists a matrix B obtained by changing some 1's in A to -1 's such that $|\det(B)| = \text{per}(A)$ [11]. It is easy to see that a det-extremal bipartite graph is a special Pfaffian bipartite graph.

The recognition of Pfaffian bipartite graphs has been done in polynomial time by McCuaig [7] and independently by Robertson, Seymour and Thomas [9]. Prior to these, in 1986 Thomassen [10] showed that every Pfaffian bipartite graph has a vertex of degree at most three, and so k -regular det-extremal bipartite graphs can exist only for $k = 2$ or 3 . In [8], McCuaig gave a structural characterization for 3-connected det-extremal cubic bipartite graphs. Following this direction, Funk et. al [3] determined the structure of det-extremal cubic bipartite graphs of connectivity 2. Cavenagh and Wanless [2] proved the det-extremal identity of a planar matching matrix, and used it to show that every spherical latin trade embeds in a finite abelian group. The purpose of this paper is to extend the research on det-extremal cubic bipartite graphs and derive the structure of det-extremal bipartite graphs.

If a bipartite graph contains no perfect matching, it is obvious that $|\det(A)| = \text{per}(A) = 0$. This is because that edges lying in no perfect matching of a bipartite graph do not contribute to the det-extremal property. After removing all the forbidden edges of a det-extremal bipartite graph, the resulting graph is matching covered. Thus, to investigate the structure of a det-extremal bipartite graph, we may suppose that it is matching covered.

Ear decompositions of matching covered graphs are important for understanding their structure. Here we introduce some definitions and notations. An *ear* of a graph G is a path that is maximal with respect to internal vertices of degree 2 in G and it is contained in a cycle of G . An *ear decomposition* of G is a decomposition $(e, P_1, \dots, P_r)$ such that e is an edge and P_i for $i \geq 1$ is an ear of $e \cup P_1 \cup \dots \cup P_i$, denoted as $G = e + P_1 + P_2 + \dots + P_r$. In an ear decomposition, we denote $e + P_1 + \dots + P_i$ by G_i . For a bipartite graph, an ear decomposition is said to be a *bipartite ear decomposition*, if each ear is of odd length. For a plane graph G , an ear decomposition $e + P_1 + P_2 + \dots + P_r$ is called a *plane ear decomposition*, if $G_0 = e + P_1$ is the boundary of an interior face of G and the i th ear P_i ($i \geq 2$) lies in the exterior of G_{i-1} such that P_i and a part of the periphery of G_{i-1} surround an interior face of G for $1 \leq i \leq r$ [5]. If a bipartite ear decomposition is a plane ear decomposition, it is called a *plane bipartite ear decomposition*.

The following result for det-extremal bipartite graph is well known [3].

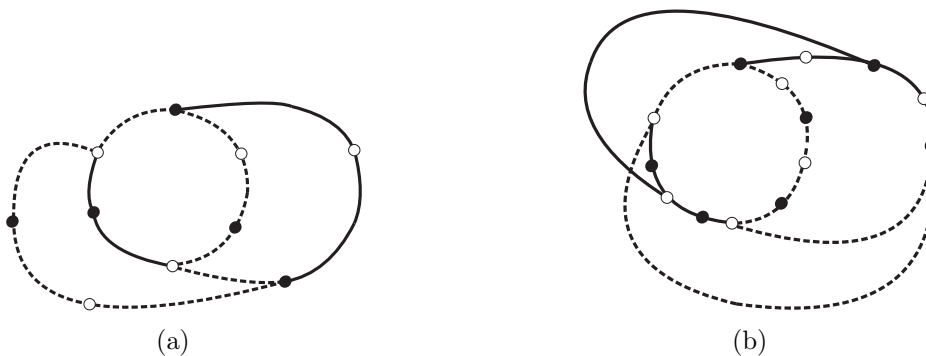


Figure 1: (a) A planar det-extremal graph; (b) a non-planar det-extremal graph.

Lemma 1. [3] *Let G be a bipartite graph. Then G is det-extremal if and only if every central cycle of G has length congruent to two modulo four.*

Lemma 1 provides an equivalent condition for det-extremal bipartite graphs. Following this, to describe the det-extremal graphs, we only need to characterize the balanced bipartite graph G whose central cycles have length congruent to two modulo four. If G contains no cycle of length divisible by four, then it must be det-extremal. However, as illustrated in Fig. 1(a) and Fig. 1(b), a det-extremal graph may contain cycles of length divisible by four. The bipartite graphs without cycles of length divisible by four have been successfully established by the plane ear decompositions [5]. Motivated by this idea, we try to describe the det-extremal graphs in terms of ear decompositions.

In Section 2, we show that a matching covered plane bipartite graph is det-extremal if and only if it has a plane bipartite ear decomposition such that each interior face is of size two modulo four. In Section 3, we turn to general matching covered det-extremal bipartite graphs. We prove that a matching covered det-extremal bipartite graph is det-extremal if and only if it admits a standard bipartite ear decomposition without singular ears, which is if and only if it admits a standard bipartite ear decomposition and contains no singular graph as a central subgraph.

2 The structure of plane det-extremal graphs

Every plane graph is Pfaffian [4], but not every plane bipartite graph is det-extremal. In this section, we focus on the matching covered plane bipartite graphs which are det-extremal, and characterize such graphs in terms of plane bipartite ear decompositions.

The following is a classical result of matching covered bipartite graphs [6].

Theorem 2. [6] *A bipartite graph is matching covered if and only if it has a bipartite ear decomposition starting with an edge.*

Given a central subgraph H of a matching covered bipartite graph, we can find an ear decomposition $G = H + P_1 + \cdots + P_i$ such that each ear P_i is of odd length [6]. Therefore,

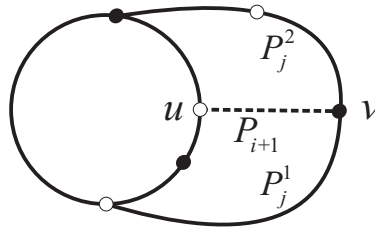


Figure 2: The vertex u lies on C_{j-1} and v lies on P_j .

the following result holds.

Corollary 3. *Let G be a matching covered bipartite graph and H a central subgraph of G . Then G has a bipartite ear decomposition starting with H .*

Inspired by the idea of constructing a reducible face decomposition given in [12], the author proposed the plane ear decomposition to investigate the bipartite graphs without cycles of length divisible by four [5]. Following this direction, here we will apply the plane bipartite ear decomposition to describe the structure of a plane matching covered bipartite graph.

Let $G = x + P_1 + P_2 + \cdots + P_r$ be an ear decomposition of G starting with an edge x and $G_i = x + P_1 + \cdots + P_i$. Set $G_{i+1} = G_i + P_{i+1}$. For a plane ear decomposition $x + P_1 + P_2 + \cdots + P_r$, we use ∂f_i to denote the boundary of the interior face f_i rounded by G_{i-1} and P_i . We call ∂f_i a *face cycle*. The length of ∂f_i is termed as *the size of the interior face f_i* , and we use $|\partial f_i|$ to denote the length of ∂f_i .

Theorem 4. *Let G be a matching covered plane bipartite graph. Then it admits a plane bipartite ear decomposition starting with an edge of G .*

Proof. By Theorem 2, G admits a bipartite ear decomposition $x + P_1 + P_2 + \cdots + P_r$ starting with an edge x of G . We use induction on the number of ears to show that G has a plane bipartite ear decomposition starting with x . If $r = 1$, there is exactly one finite face in G . It is obvious that it is a plane bipartite ear decomposition. Suppose that the assertion holds for $r \leq i$. Now, we consider the case that $r = i + 1$. Let $G_i = x + P_1 + P_2 + \cdots + P_i$ be a plane bipartite ear decomposition. We add the ear P_{i+1} to G_i . If P_{i+1} lies on the exterior of G_i , then the boundary of G_i and P_{i+1} surround a face of G_{i+1} . It follows that $x + P_1 + P_2 + \cdots + P_{i+1}$ is a plane ear decomposition of G_{i+1} .

Let R_j be the region rounded by a finite face cycle ∂f_j in G_i for $j \leq i$. The graph G is a plane graph. If the ear P_{i+1} does not lie on the exterior of G_i , then P_{i+1} lies in some region R_j . Now, we suppose that the ear P_{i+1} belongs to some region R_j . By induction, $x + P_1 + \cdots + P_j$ is a plane ear decomposition. Let C_{j-1} be the outer boundary of G_{j-1} and ∂f_j the face cycle rounded by C_{j-1} and P_j . Denote by u and v the two end-vertices of P_{i+1} . If u and v are both on C_{j-1} , then we add the ear P_{i+1} to G_{j-1} before P_j and attach the remaining ears one by one. The new ear decomposition is a plane bipartite ear decomposition.

It remains to consider the case in which at least one end-vertex of P_{i+1} does not lie on C_{j-1} . Subcase (a): u lies on C_{j-1} and v lies on P_j (as shown in Fig. 2). The vertex v divides P_j into two paths P_j^1 and P_j^2 . Since P_j is of odd length, then exactly one of P_j^1 and P_j^2 is of odd length. Without loss of generality, suppose that P_j^1 is of odd length. Let $P'_j = P_j^2 \cup P_{i+1}$ and $P'_{j+1} = P_j^1$. Subcase (b): the vertices u and v are both on P_j . Then the path L on P_j joining u and v is of odd length. Let $P'_j = (P_j - L) \cup P_{i+1}$, where $P_j - L$ is the graph obtained from P_j by deleting all the internal vertices and edges of L . Set $P'_{j+1} = L$. Under these two subcases (a) and (b), let $P'_k = P_k$ for $1 \leq k < j$ and let $P'_k = P_{k-1}$ for $j+2 \leq k \leq i+1$. The ears P'_j and P'_{j+1} are defined as above. Then the ear decomposition $x + P'_1 + P'_2 + \cdots + P'_{i+1}$ is a plane bipartite ear decomposition. $\square$

Our approach is to build an ear decomposition of a matching covered det-extremal plane bipartite graph. Before we show the main consequence, we need to prove the fundamental result below. The *symmetric difference* of two cycles C_1 and C_2 , denoted by $C_1 \triangle C_2$, is the cycle with edges in either C_1 or C_2 , but not in both of them.

Theorem 5. *Let G be a graph admitting a plane bipartite ear decomposition. Then each face cycle in this ear decomposition is a central cycle of G .*

Proof. Let $G = x + P_1 + P_2 + \cdots + P_r$ be a plane bipartite ear decomposition of G , where x is an edge of G . We use induction on the number of ears. If $r = 1$, $x + P_1$ is a face cycle of G , denoted by ∂f_1 . It is obvious that ∂f_1 is a central cycle. If $r = 2$, then there are two interior face cycles ∂f_1 and ∂f_2 , and the intersection of them is a path of odd length. Thus, ∂f_1 , ∂f_2 and the exterior face cycle $\partial f_1 \triangle \partial f_2$ are all central cycles of G .

Suppose that each face cycle of a plane bipartite ear decomposition is a central cycle of G for $r \leq i$. Set $r = i + 1$. Each face cycle ∂f_k ($1 \leq k \leq i$) is a central cycle of G_i by induction, and so it is a central cycle of G_{i+1} , since P_{i+1} is of odd length. We shall show first that ∂f_{i+1} , the face cycle formed by attaching P_{i+1} to G_i , is a central cycle of G_{i+1} . The induction implies that the exterior face cycle C_i of G_i is a central cycle of G_i . Thus, $G_i - C_i$ has a perfect matching, denoted by M_1 . Set $L_1 = C_i \cap \partial f_{i+1}$ and let L_2 be the path obtained from C_i by deleting all the vertices of L_1 . Since P_{i+1} is of odd length, L_1 is a path of odd length. Hence, the path L_2 is of odd length. Let M_2 be the perfect matching of L_2 . We can see that $M_1 \cup M_2$ is a perfect matching of $G_{i+1} - \partial f_{i+1}$. Therefore, ∂f_{i+1} is a central cycle of G_{i+1} . Denote by u and v the end-vertices of L_1 . Let M_3 be the perfect matching of $L_1 - u - v$. Then $M_1 \cup M_3$ is a perfect matching of $G_{i+1} - C_{i+1}$, where C_{i+1} is the face cycle of the exterior face of G_{i+1} . Thus, each face cycle of G_{i+1} is a central cycle of G_{i+1} . $\square$

Let G be a graph with a plane bipartite ear decomposition $x + P_1 + P_2 + \cdots + P_r$. Let C be a cycle in G , which is the symmetric differences of a series of faces cycles, denoted as $C = \partial f_{k_1} \triangle \partial f_{k_2} \triangle \cdots \triangle \partial f_{k_p}$ with $1 \leq k_1 < k_2 < \cdots < k_p \leq r$. Then the intersection $\partial f_{k_j} \cap (\partial f_{k_1} \triangle \cdots \triangle \partial f_{k_{j-1}})$ is a path, we call such an intersection *the iterative-intersection path* relative to C for $2 \leq j \leq p$.

Our main result is as below.

Theorem 6. *A matching covered plane bipartite graph G is det-extremal if and only if G has a plane bipartite ear decomposition such that each interior face is of size two modulo four.*

Proof. Suppose that G is a matching covered plane bipartite graph and it is det-extremal. by Theorem 4, G admits a plane bipartite ear decomposition $x + P_1 + P_2 + \cdots + P_r$ starting with an edge x of G . By the result of Theorem 5, each face cycle of this ear decomposition is a central cycle of G . Thus, each interior face must be of size two modulo four by Lemma 1.

For the converse, we show that if G admits a plane bipartite ear decomposition $x + P_1 + P_2 + \cdots + P_r$ such that each interior face is of size 2 (mod 4), then each central cycle of G is of length 2 (mod 4). Let C be a central cycle of G . If C is a face cycle, it is obvious that C is of length two modulo four. Thus, we suppose that C is not a face cycle. Then C is the symmetric differences of a series of faces cycles of G . Denote it as $C = \partial f_{k_1} \triangle \partial f_{k_2} \triangle \cdots \triangle \partial f_{k_p}$, where $1 \leq k_1 < k_2 < \cdots < k_p \leq r$. As C is a central cycle of the plane graph G , the cycle C must enclose an even number of vertices of G . If an odd number of iterative-intersection paths relative to C is of even length, then the total number of vertices enclosed by C is odd. It is a contradiction. Thus, an even number of iterative-intersection paths relative to C are of even length. Since each face cycle is of length two modulo four, after a series of symmetric difference operations, the cycle C is of length two modulo four. Therefore, G is det-extremal by Lemma 1. $\square$

3 The structure of general det-extremal graphs

A det-extremal graph may be planar or not (cf. Fig. 1). In this section, we pay attention to general matching covered det-extremal bipartite graphs and characterize the structure of such graphs concerning ear decomposition.

Let G be a matching covered bipartite graph and let $x + P_1 + P_2 + \cdots + P_r$ be a bipartite ear decomposition of G . Let M_1 be a perfect matching of $x + P_1$. Denote by u_i and v_i the end-vertices of the ear P_i and let M_i be the perfect matching of $P_i - u_i - v_i$ for $2 \leq i \leq r$. Then $M^* = \cup_{i=1}^r M_i$ is a perfect matching of G . We refer to M^* as the *standard perfect matching* of G relative to the bipartite ear decomposition.

Let M be a matching of a graph G . A *matched vertex* in G is a vertex incident with an edge in M . A *free vertex* is a vertex that is not matched. An *alternating path* (resp. *alternating cycle*) with respect to M is a path (resp. cycle) in which edges alternate between those in M and those not in M . An *M -augmenting path* (resp. *M -saturating path*) is an alternating path that starts and ends with a free vertex (resp. matched vertex).

In trying to establish the structure of matching covered det-extremal bipartite graphs, we prove the following result.

Theorem 7. *Let G be a matching covered bipartite graph and M^* the standard perfect matching of G relative to a bipartite ear decomposition. Then each central cycle of G is of length two modulo four if and only if each M^* -alternating cycle of G is of length two modulo four.*

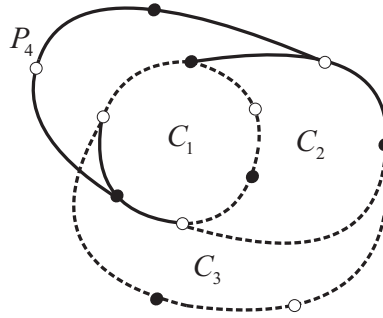


Figure 3: A singular graph.

Proof. An M^* -alternating cycle of G is a central cycle. The necessity holds obviously. We only need to prove the sufficiency. Let C be a central cycle of G . If C is an M^* -alternating cycle, then the assertion holds. Thus, we may suppose that C is not an M^* -alternating cycle. There will be a set S of vertices on C such that the two edges on C incident with any vertex in S do not belong to M^* . Let k_C be the number of vertices on C which are not incident with edges in $M^* \cap C$. It is obvious that k_C is even. Let M' be the perfect matching of $G - V(C)$. Then there will be an (M^*, M') -alternating path F in G such that $F \cap C = \{u, v\}$, where $u, v \in S$. Let L and Q be the two paths on C with end-vertices u and v . Let $C_1 = L \cup F$ and $C_2 = Q \cup F$. As F is an M' -augmenting path, C_1 and C_2 are both central cycles of G .

In the following, we will use induction on k_C to show that the central cycle C is of length two modulo four. If $k_C = 2$, then C_1 and C_2 are both M^* -alternating cycles, and so they are of length two modulo four. As F is a path of odd length and $C = C_1 \Delta C_2$, C is of length two modulo four. Suppose that the assertion holds for the case in which $k_C \leq t$. Now we suppose that $k_C = t + 2$. By construction, $k_{C_1} \leq t$ and $k_{C_2} \leq t$. As C_1 and C_2 are both central cycles of G , the induction implies that C_1 and C_2 are of length two modulo four. Analogously, C is of length two modulo four. $\square$

For describing the structure of matching covered det-extremal bipartite graph, we require the following definitions and notions.

Let G be a matching covered bipartite graph and M^* the standard perfect matching of G with respect to the bipartite ear decomposition $x + P_1 + P_2 + \cdots + P_r$. Every time attaching an ear P_i to G_{i-1} , there is an M^* -alternating cycle of G_i containing P_i . If there exists an M^* -alternating cycle of length two modulo four containing P_i in G_i , then P_i is said to be a *standard ear*. We say that a bipartite ear decomposition is *standard* if every ear in this decomposition is a standard ear. Let C_1 , C_2 and C_3 be three cycles of length two modulo four in G such that the intersection condition holds for C_1 , C_2 and C_3 , i.e. $C_1 \cap C_2$ is an odd path, $C_1 \cap C_3$ is an even path and $C_2 \cap C_3$ is an odd path. Moreover, if M^* restricted to $C_1 \cup C_2 \cup C_3$ is a perfect matching of $C_1 \cup C_2 \cup C_3$, we call $C_1 \cup C_2 \cup C_3$ the *intersecting graph* with respect to the ear decomposition, and call the path $C_1 \cap C_3$ the *critical path*.

Let P be a path on s internal vertices. Label all the internal vertices of P from 1 to s

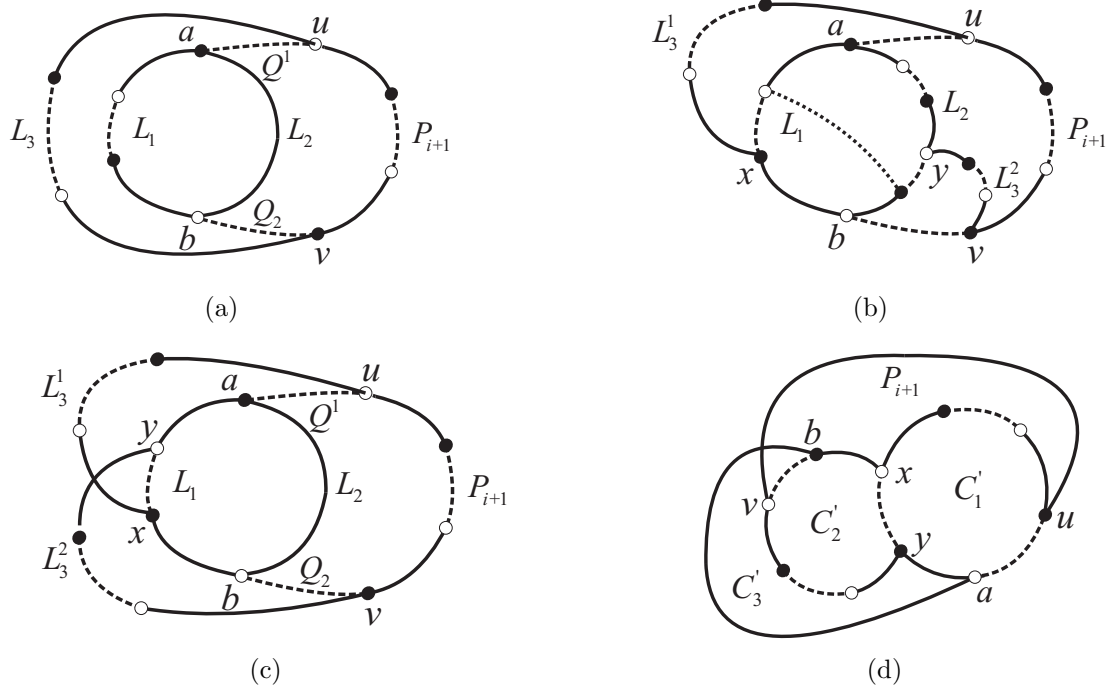


Figure 4: The subgraphs used in the proof of Lemma 8.

sequentially. Then an internal vertex of P is called an *odd vertex* if its labeling is an odd number. Let G be a matching covered bipartite graph with a bipartite ear decomposition $x + P_1 + P_2 + \cdots + P_r$. Let H be an intersecting subgraph of G with respect to this ear decomposition. If a bipartite ear P_i joins an odd vertex of the critical path $C_1 \cap C_3$ on H to a vertex of H not on $C_1 \cup C_3$, and P_i has no common edges with H , then P_i is said to be a *singular ear*. Otherwise, P_i is called a *non-singular ear*. As shown in Fig. 3, the ear P_4 is a singular ear. If P_i is a singular ear attached to H , then the graph isomorphic to $H \cup P_i$ is called a *singular graph*.

The following lemma is important for deriving the structure of det-extremal bipartite graphs.

Lemma 8. *Let G be a matching covered bipartite graph and M^* the standard perfect matching with respect to a bipartite ear decomposition of G . If this ear decomposition is a standard bipartite ear decomposition without singular ears, then any M^* -alternating cycle of G is of length two modulo four.*

Proof. Let $G = x + P_1 + \cdots + P_r$ be a standard bipartite ear decomposition of G . We prove by induction on the number of ears. It is trivial for the case $r = 1$. Suppose that the result holds for $r \leq i$. Now, we consider the case $r = i + 1$. Any M^* -alternating cycle of G_{i+1} that does not contain the ear P_{i+1} is an M^* -alternating cycle of G_i , and it is of length two modulo four by induction. It remains to consider the case in which G_{i+1} contains an M^* -alternating cycle C such that $P_{i+1} \subseteq C$. Since P_{i+1} is a standard ear,

there is an M^* -alternating cycle C^* containing P_{i+1} in G_{i+1} and it is of length two modulo four. If C^* is the unique M^* -alternating cycle in G_{i+1} containing P_{i+1} , then $C = C^*$ and we are done. Now, we suppose that there is another M^* -alternating cycle C' containing P_{i+1} in G_{i+1} . Assume to the contrary that C' is of length divisible by four. As both C and C' are M^* -alternating cycles, each component of $C \cap C'$ is an M^* -saturating path.

Firstly, we may assume that $C \cap C'$ has exactly one component. That is $C \cap C'$ is exactly an M^* -saturating path. Let $Q = C \cap C'$. Denote by a and b the end-vertices of Q . Let L_1 and L_2 be the paths, other than Q , on C and C' joining a and b , respectively. Then L_1 and L_2 are both M^* -augmenting paths. Let u and v be the end-vertices of the ear P_{i+1} . As $P_{i+1} \subseteq C \cap C'$, there is an M^* -saturating path Q_1 (resp. Q_2) on Q from a to u (resp. b to v). See Fig. 4(a). Set L_3 to be an M^* -augmenting path in G_i joining u and v . Let $C_1 = L_1 \cup L_2$. Since $C_1 = C \Delta C'$, C_1 is of length divisible by four. If L_3 is disjoint from L_1 and L_2 , then we will find that C_1 is a central cycle of the subgraph $C_1 \cup Q \cup L_3$, and so it is a central cycle of G_i . By Theorem 7, C_1 is of length two modulo four. A contradiction occurs.

Now, we suppose that $L_3 \cap C_1 \neq \emptyset$. Traveling along L_3 starting from u (resp. v), let x (resp. y) be first vertex of L_3 , which meets C_1 . Let L_3^1 (resp. L_3^2) be the path on L_3 from u to x (resp. from v to y). We assume first that $x \in L_1$ and $y \in L_2$ (cf. Fig. 4(b)). Let C_2 (resp. C_3) be the cycle enclosed by $Q_1 \cup L_3^1$ (resp. $Q_2 \cup L_3^2$) and a part of L_1 (resp. L_2). Then $C_1 \cup C_2 \cup C_3$ is a central subgraph of G_i and $C_1 \Delta C_2 \Delta C_3$ is a central cycle of G_i . Thus, $C_1 \Delta C_2 \Delta C_3$ is of length two modulo four by induction and Theorem 7. As C_2 and C_3 are M^* -alternating cycles, they are of length two modulo four. Moreover, $C_2 \cap C_1$ and $C_3 \cap C_1$ are all paths of even length. Hence, $C_1 \Delta C_2 \Delta C_3$ is of length divisible by four. This is a contradiction. If $x \in L_2$ and $y \in L_1$, then by a similar argument we can get a contradiction. Thus, it remains to consider the case in which $x \in L_1$ and $y \in L_1$. Let C'_1 be the cycle rounded by a part of L_1 and $Q_1 \cup L_3^1$. Let C'_2 be the cycle rounded by a part of L_1 and $Q_2 \cup L_3^2$. Denote by C'_3 be the cycle rounded by a part of L_1 and $L_2 \cup Q_2 \cup L_3^2$. See Fig. 4(c). As C'_1, C'_2 are all M^* -alternating cycles in G_i , the induction implies that they are all of length two modulo four. The cycle C_1 is of length divisible by four by assumption. Since $C'_3 = C'_2 \Delta C_1$ and $C'_2 \cap C_1$ is a path of even length, C'_3 is of length two modulo four. The construction implies that M^* restricted to $C'_1 \cup C'_2 \cup C'_3$ is a perfect matching of $C'_1 \cup C'_2 \cup C'_3$. Thus, $C'_1 \cup C'_2 \cup C'_3$ is an intersecting graph with the critical path $Q_2 \cup L_3^2$ (i.e. $C'_2 \cap C'_3$), and P_{i+1} is a singular ear as shown in Fig. 4(d). This contradicts that the ear decomposition of G_{i+1} contains no singular ear. Thus, the M^* -alternating cycle C' of G_{i+1} is of length two modulo four.

If $C \cap C'$ has more than one component, then the graph $C \cup C'$ will admit two M^* -alternating cycles C_s and C_t both containing P_{i+1} such that C_s is of length divisible by four and C_t is of length two modulo four, and $C_s \cap C_t$ is exactly an M^* -saturating path. Then we can turn to the cycles C_s and C_t , and use the analysis above to derive a contradiction. $\square$

Now we are ready to establish the characterization of matching covered det-extremal bipartite graphs in terms of ear decompositions.

Theorem 9. *Let G be a matching covered bipartite graph. Then G is det-extremal if and only if G admits a standard bipartite ear decomposition without singular ears.*

Proof. Let us first prove the necessity. We know that G admits a bipartite ear decomposition $x + P_1 + P_2 + \cdots + P_r$ starting with an edge x of G . Let M^* be the standard perfect matching with respect to this ear decomposition. Since G is det-extremal, it follows from Lemma 1 that each central cycle of G is of length two modulo four. Therefore, each ear in this ear decomposition is standard.

Suppose to the contrary that this ear decomposition admits a singular ear, denoted by P_{i+1} . Then there will be an intersecting graph H in G such that $H \cap P_{i+1}$ contains exactly the end-vertices of P_{i+1} and M^* restricted to H is a perfect matching of H . Thus, $H \cup P_{i+1}$ is a central subgraph of G . Suppose that $H = C_1 \cup C_2 \cup C_3$ and $C_1 \cap C_3$ is the critical path of H . We can see that $C_1 \triangle C_3$ is a central cycle of $H \cup P_{i+1}$, and so $C_1 \triangle C_3$ is a central subgraph of G . As $C_1 \triangle C_3$ is of length divisible by four. It contradicts that G is det-extremal.

Now, we prove the sufficiency. Suppose that the matching covered bipartite graph G admits a standard bipartite ear decomposition without singular ears. It follows from Lemma 8 that each M^* -alternating cycle of G is of length two modulo four. By the result of Theorem 7, each central cycle of G is of length two modulo four. Thus, G is det-extremal by Lemma 1. $\square$

Based on Theorem 9, we have the following result.

Theorem 10. *Let G be a matching covered bipartite graph. Then G is det-extremal if and only if G admits a standard bipartite ear decomposition and contains no singular graph as a central subgraph.*

Proof. Sufficiency. Let M^* be the standard perfect matching with respect to a standard bipartite ear decomposition of G . Firstly, we will show that the standard bipartite ear decomposition of G contains no singular ears. Suppose to the contrary that the standard bipartite ear decomposition of G admits a singular ear P_i . Let H be the intersecting graph that P_i is attached to. Then $H \cup P_i$ is a singular subgraph. As the restriction of M^* to $H \cup P_i$ is a perfect matching of $H \cup P_i$, $H \cup P_i$ is a central subgraph of G . This contradicts that G contains no singular graph as a central subgraph. Therefore, G admits a standard bipartite ear decomposition without singular ears. Theorem 9 implies that G is det-extremal.

Necessity. Theorem 9 shows that G has a standard bipartite ear decomposition. Suppose to the contrary that G contains a singular graph W as a central subgraph. Set $W = C_1 \cup C_2 \cup C_3 \cup P$, where C_1 , C_2 and C_3 are all cycles of length two modulo four and they satisfy the intersection condition, and $|C_1 \cap C_3|$ is even among $|C_i \cap C_j|$ for $1 \leq i, j \leq 3$ and $i \neq j$. Moreover, P is an odd path joining an odd vertex of $C_1 \cap C_3$ to a vertex of W not on $C_1 \cap C_3$. Moreover, we can verify that $C_1 \triangle C_3$ is a central cycle of W . Therefore, $C_1 \triangle C_3$ is a central subgraph of G . As $C_1 \triangle C_3$ is of length divisible by four. It contradicts that G is det-extremal. $\square$

A singular graph is non-planar. Thus, a matching covered plane bipartite graph is det-extremal if and only if it has a standard bipartite ear decomposition. This gives another characterization of a matching covered plane bipartite graph.

Corollary 11. *Let G be a matching covered plane bipartite graph. Then G is det-extremal if and only if G admits a standard bipartite ear decomposition.*

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