

Combinatorics of generalized orthogonal polynomials of type R_{II}

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Abstract

In 1995, Ismail and Masson introduced orthogonal polynomials of types R_I and R_{II} , which are defined by specific three-term recurrence relations with additional conditions. Recently, Kim and Stanton found a combinatorial interpretation for the moments of orthogonal polynomials of type R_I in the spirit of the combinatorial theory of orthogonal polynomials due to Flajolet and Viennot. In this paper, we push this combinatorial model further to orthogonal polynomials of type R_{II} . Moreover, we generalize orthogonal polynomials of type R_{II} by relaxing some of their conditions. Under a basis condition, we then prove a master theorem that generalizes combinatorial models for generalized moments of various types of orthogonal polynomials; ordinary orthogonal polynomials, Laurent biorthogonal polynomials, and orthogonal polynomials of types R_I and R_{II} . We also give a combinatorial interpretation of the generalized dual coefficients.

Mathematics Subject Classifications: 05A15, 33C45

1 Introduction

An *ordinary orthogonal polynomial sequence* $(P_n(x))_{n \geq 0}$ is a sequence of polynomials such that $\deg P_n(x) = n$ and $\mathcal{L}(P_n(x)P_m(x)) = \kappa_n \delta_{n,m}$ with $\kappa_n \neq 0$, for some linear functional $\mathcal{L}$ defined on the space of polynomials. It is well known that a monic polynomial sequence is orthogonal if and only if it satisfies a three-term recurrence relation

$$P_{n+1}(x) = (x - b_n)P_n(x) - \lambda_n P_{n-1}(x), \quad n \geq 0, \quad (1)$$

where $P_{-1}(x) = 0$, $P_0(x) = 1$ and $\lambda_n \neq 0$ for $n \geq 1$. Well-known examples of orthogonal polynomials include the Hermite, Laguerre, and Jacobi polynomials. These are usually called *classical orthogonal polynomials*. For more information, see [5].

In this paper, we focus on two quantities of orthogonal polynomials, namely moments and dual coefficients. The values $\mu_n = \mathcal{L}(x^n)$ are called the *moments* of the orthogonal

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polynomials. Here, we take $\mathcal{L}(1) = 1$. By the combinatorial theory of orthogonal polynomials due to Flajolet [4] and Viennot [14], the moment μ_n is a generating function for the Motzkin paths from $(0, 0)$ to $(n, 0)$.

For $m \geq 0$, let

$$d_m = \lambda_1 \cdots \lambda_m = \mathcal{L}(P_m(x)^2), \quad Q_m(x) = \frac{P_m(x)}{d_m}. \quad (2)$$

The *generalized moments* are defined by

$$\mu_{n,r,s} = \mathcal{L}(x^n P_r(x) Q_s(x)) = \frac{\mathcal{L}(x^n P_r(x) P_s(x))}{\mathcal{L}(P_s(x)^2)}. \quad (3)$$

Viennot [14] showed that $\mu_{n,r,s}$ is a generating function for the Motzkin paths from $(0, r)$ to (n, s) .

For any polynomial sequence $(P_n(x))_{n \geq 0}$ with $\deg P_n(x) = n$, its coefficients $\rho_{n,k}$ and *dual coefficients* $\tau_{n,k}$ are defined by

$$P_n(x) = \sum_{k \geq 0} \rho_{n,k} x^k, \quad x^n = \sum_{k \geq 0} \tau_{n,k} P_k(x).$$

The two lower triangular matrices $(\rho_{n,k})$ and $(\tau_{n,k})$ are mutual inverses, which explains the term dual coefficients. The *generalized dual coefficients* $\tau_{n,r,s}$ are defined by

$$x^n P_r(x) = \sum_{s \geq 0} \tau_{n,r,s} P_s(x). \quad (4)$$

For an ordinary orthogonal polynomial sequence, orthogonality gives $\mathcal{L}(P_k Q_s) = \delta_{k,s}$, and hence $\mu_{n,r,s} = \tau_{n,r,s}$. However, generalized moments and generalized dual coefficients need not coincide for orthogonal polynomials of types R_I and R_{II} . In this paper, we give combinatorial interpretations of generalized moments and generalized dual coefficients for orthogonal polynomials of type R_{II} and their generalizations. We first recall the definitions and known results for types R_I and R_{II} .

In 1995, Ismail and Masson [6] introduced orthogonal polynomials of types R_I and R_{II} . *Orthogonal polynomials* $P_n^I(x)$ of type R_I are defined by the three-term recurrence

$$P_{n+1}^I(x) = (x - b_n)P_n^I(x) - (a_n x + \lambda_n)P_{n-1}^I(x), \quad n \geq 0, \quad (5)$$

where $P_{-1}^I(x) = 0$ and $P_0^I(x) = 1$, and $a_n \neq 0$ and $P_n^I(-\lambda_n/a_n) \neq 0$ for $n \geq 1$. Let

$$d_m^I(x) = \prod_{i=1}^m (a_i x + \lambda_i), \quad Q_m^I(x) = \frac{P_m^I(x)}{d_m^I(x)}.$$

Note that $d_m^I(x)$ is the product of the recurrence coefficients $(a_i x + \lambda_i)$ in (5), so the rational function $Q_m^I(x)$ is a natural generalization of the polynomial $Q_m(x)$ in (2).

Ismail and Masson [6, Theorem 2.1] (see also [11, Theorem 2.1]) showed that there is a unique linear functional $\mathcal{L}^I$ on the vector space of rational functions $W^I = \text{span}\{x^n Q_m^I(x) : n, m \geq 0\}$ such that $\mathcal{L}^I(1) = 1$ and $\mathcal{L}^I(x^n Q_m^I(x)) = 0$ for $0 \leq n < m$.

Kim and Stanton [11] presented a combinatorial theory of orthogonal polynomials of type R_I . In particular, they found combinatorial interpretations of the moments $\mu_n^I = \mathcal{L}^I(x^n)$ and the generalized moments $\mu_{n,r,s}^I = \mathcal{L}^I(x^n P_r^I(x) Q_s^I(x))$ using Motzkin–Schröder paths. This generalizes a result of Kamioka [9] that the moments of Laurent biorthogonal polynomials are generating functions for Schröder paths.

Orthogonal polynomials $P_n^{II}(x)$ of type R_{II} are defined by $P_{-1}^{II}(x) = 0$, $P_0^{II}(x) = 1$, and

$$P_{n+1}^{II}(x) = (x - b_n)P_n^{II}(x) - (c_n x^2 + a_n x + \lambda_n)P_{n-1}^{II}(x), \quad n \geq 0, \quad (6)$$

with the assumptions that $c_n \neq 0$, $P_n^{II}(\alpha_n) \neq 0$, and $P_n^{II}(\beta_n) \neq 0$ for $n \geq 1$, where α_n and β_n are the zeros of $c_n x^2 + a_n x + \lambda_n$. Let

$$d_m^{II}(x) = \prod_{i=1}^m (c_i x^2 + a_i x + \lambda_i), \quad Q_m^{II}(x) = \frac{P_m^{II}(x)}{d_m^{II}(x)}.$$

Ismail and Masson [6, Theorem 3.5] showed that, for fixed N_0 and N_1 , there is a unique linear functional $\mathcal{L}^{II}$ on the vector space of rational functions

$$V^{II} = \text{span}\{x^n Q_m^{II}(x) : 0 \leq n \leq m\} \quad (7)$$

such that $\mathcal{L}^{II}(1) = N_0$, $\mathcal{L}^{II}(x Q_1^{II}(x)) = N_1$, and $\mathcal{L}^{II}(x^n Q_m^{II}(x)) = 0$ for $0 \leq n < m$. See [7, 12] for recent work on specific orthogonal polynomials of type R_{II} .

In contrast to the type R_I case, no analogous combinatorial theory has been developed for orthogonal polynomials of type R_{II} . To give a unified theory, we consider the larger class of *generalized orthogonal polynomials of type R_{II}* . These are polynomials $P_n(x)$ satisfying the same recurrence and initial conditions as in (6), with only the condition that $c_n x^2 + a_n x + \lambda_n$ is not the zero polynomial for $n \geq 1$. This class contains ordinary orthogonal polynomials, Laurent biorthogonal polynomials, and orthogonal polynomials of types R_I and R_{II} .

There are two differences from the ordinary and type R_I cases. First, $P_n(x)$ need not be monic; for example, the coefficient of x^2 in $P_2(x)$ is $1 - c_1$. Second, the orthogonality conditions alone need not determine the generalized moments. For example, in the original type R_{II} case, the orthogonality conditions are stated with respect to a linear functional $\mathcal{L}$ on $V = \text{span}\{x^n Q_m(x) : 0 \leq n \leq m\}$, which does not contain x^n for $n \geq 1$, so $\mu_n = \mathcal{L}(x^n)$ is not even defined. To define the generalized moments $\mu_{n,r,s} = \mathcal{L}(x^n P_r(x) Q_s(x))$, we consider a linear functional on the larger space

$$W = \text{span}\{x^n Q_m(x) : n, m \geq 0\}. \quad (8)$$

Our master theorem, Theorem 21, expresses the generalized moments $\mu_{n,r,s}$ for the four classes above as sums of the weights of R_{II} paths from $(0, r)$ to (n, s) , with the

corresponding specializations of the weights. For orthogonal polynomials of type R_{II} , the restriction of $\mathcal{L}$ to V satisfies the orthogonality conditions above, with $\mathcal{L}(1) = N_0$ and $\mathcal{L}(xQ_1(x)) = N_1$ given by the sums of the weights of R_{II} paths from $(0, 0)$ to $(0, 0)$ and $(1, 1)$, respectively; see Section 3.

Since the generalized dual coefficients coincide with the generalized moments for ordinary orthogonal polynomials, the Motzkin path model shows that the generalized dual coefficients are polynomials with nonnegative integer coefficients in $(b_0, b_1, \dots)$ and $(\lambda_1, \lambda_2, \dots)$. See [3, 2] for recent work on generalized dual coefficients.

For generalized orthogonal polynomials $P_n(x)$ of type R_{II} , however, the generalized dual coefficients $\tau_{n,r,s}$ need not be polynomials. We regard a_n, b_n, c_n, λ_n as indeterminates and expand $\tau_{n,r,s}$ as a formal power series. For example, if $b_n = 1$, $a_n = \lambda_n = 0$, and $c_n = c$, where c is an indeterminate, then

$$\tau_{4,0,3} = \frac{4 - 6c}{(1 - 2c)(1 - 3c + c^2)} = 4 + 14c + 42c^2 + 120c^3 + \dots.$$

The nonnegativity of the coefficients is not clear from this expression. Combinatorially, we show that the generalized dual coefficient $\tau_{n,r,s}$ is the sum of the weights of restricted R_{II} paths from $(0, r)$ to (n, s) . It follows that $\tau_{n,r,s}$ is a formal power series in a_i, b_i, c_i, λ_i with nonnegative integer coefficients. The paths and their weights are defined in Section 2.

The structure of the paper is as follows. In Section 2, we provide basic definitions and review some known results. In Section 3, we construct a linear functional on W for orthogonal polynomials of type R_{II} . We then give a combinatorial interpretation for the generalized moments $\mu_{n,r,s}$ in terms of R_{II} paths. In Section 4, we generalize this construction and prove a master theorem that includes the four classes of orthogonal polynomials discussed above. In Section 5, we give a combinatorial interpretation for the generalized dual coefficients of generalized orthogonal polynomials of type R_{II} in terms of restricted R_{II} paths. In Section 6, we give sufficient conditions for the absolute convergence of the sums of weights of R_{II} paths and restricted R_{II} paths. Finally, in Section 7, we show that, when the recurrence coefficients are constant, the type R_{II} moments coincide, up to a common factor, with moments of certain ordinary orthogonal polynomials and of certain orthogonal polynomials of type R_I .

2 Preliminaries

In this section, we introduce basic definitions and known results. We start with the definition of orthogonal polynomials of type R_{II} , introduced by Ismail and Masson [6].

Definition 1. We say that $(P_n(x))_{n \geq 0}$ is a sequence of *orthogonal polynomials of type R_{II}* if $P_{-1}(x) = 0$, $P_0(x) = 1$, and

$$P_{n+1}(x) = (x - b_n)P_n(x) - (c_n x^2 + a_n x + \lambda_n)P_{n-1}(x), \quad n \geq 0, \quad (9)$$

for some sequences $\mathbf{b} = (b_0, b_1, \dots)$, $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \dots)$, $\mathbf{a} = (a_1, a_2, \dots)$, and $\mathbf{c} = (c_1, c_2, \dots)$ such that $c_n \neq 0$, $P_n(\alpha_n) \neq 0$, and $P_n(\beta_n) \neq 0$ for $n \geq 1$, where α_n and β_n are the roots of $c_n x^2 + a_n x + \lambda_n$.

Given the polynomials $P_n(x)$, we will use the following notation:

$$\begin{aligned} d_n(x) &= \prod_{i=1}^n (c_i x^2 + a_i x + \lambda_i), & Q_n(x) &= \frac{P_n(x)}{d_n(x)}, \\ e_n(x) &= \prod_{i=1}^n (x - \alpha_i), & f_n(x) &= \prod_{i=1}^n (x - \beta_i). \end{aligned}$$

Here, we use the standard convention that the empty product is defined to be 1; for example, $d_0(x) = 1$. We also define the vector space

$$V = \text{span}\{x^n Q_m(x) : 0 \leq n \leq m\},$$

and its extension

$$W = \text{span}\{x^n Q_m(x) : n, m \geq 0\}. \quad (10)$$

Ismail and Masson [6] showed that orthogonal polynomials of type R_{II} have partial orthogonality as follows.

Theorem 2 ([6, Theorem 3.5]). *For the polynomials $P_n(x)$ defined in Definition 1 and for any numbers N_0 and N_1 , there is a unique linear functional $\mathcal{L}$ on V such that $\mathcal{L}(1) = N_0$, $\mathcal{L}(xQ_1(x)) = N_1$, and $\mathcal{L}(x^n Q_m(x)) = 0$ for $0 \leq n < m$.*

Now we review known combinatorial properties of these orthogonal polynomials, which will be generalized to orthogonal polynomials of type R_{II} in this paper. To put them all in one framework, we introduce R_{II} paths.

A *lattice path* is a sequence $p = (p_0, p_1, \dots, p_n)$ of points $p_i \in \mathbb{Z} \times \mathbb{Z}_{\geq 0}$. For $i \geq 1$, the i th *step* s_i of p is defined by $s_i = p_i - p_{i-1}$. We will also consider the path p as the sequence $s_1 \cdots s_n$ of its steps (together with its starting point p_0).

Definition 3. An R_{II} *path* is a lattice path consisting of *up steps* $U = (1, 1)$, *horizontal steps* $H = (1, 0)$, *down steps* $D = (1, -1)$, *vertical-down steps* $V = (0, -1)$, and *backward-down steps* $B = (-1, -1)$. Denote by $R_{n,r,s}$ the set of R_{II} paths from $(0, r)$ to (n, s) . We also write $R_{k,m} = R_{k,0,m}$ and $R_k = R_{k,0,0}$.

For later use, we also define restricted R_{II} paths.

Definition 4. A *restricted R_{II} path* is an R_{II} -path in $R_{n,r,s}$ that immediately ends when it touches the line $x = n$. We denote by $\tilde{R}_{n,r,s}$ the set of restricted R_{II} paths in $R_{n,r,s}$. We call the condition that the path ends when it first touches the line $x = n$ the *restricted condition*.

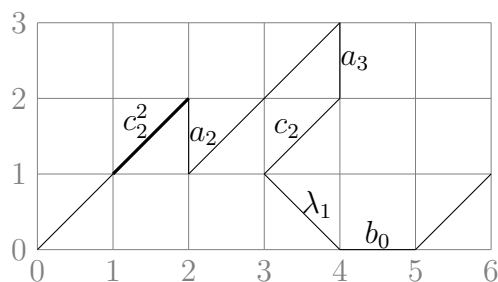


Figure 1: An R_{II} path $P = UUBUBUVUUVBDHU \in \mathbf{R}_{6,1}$ with $\text{wt}(P) = a_2 a_3 b_0 c_2^3 \lambda_1$. The thick segment from $(1, 1)$ to $(2, 2)$ represents $UBUBU$.

For an R_{II} path $p = s_1 \cdots s_k$, we define the *weight* $\text{wt}(p)$ of p by $\text{wt}(p) = \prod_{t=1}^k \text{wt}(s_t)$, where

$$\text{wt}(s_t) = \begin{cases} 1 & \text{if } s_t = U, \\ b_i & \text{if } s_t = H \text{ starting at height } i, \\ \lambda_i & \text{if } s_t = D \text{ starting at height } i, \\ a_i & \text{if } s_t = V \text{ starting at height } i, \\ c_i & \text{if } s_t = B \text{ starting at height } i. \end{cases} \quad (11)$$

See Figure 1. Note that steps U and B can be overlapped in an R_{II} path as shown in Figure 1. Thus, there can be infinitely many elements in $\mathbf{R}_{n,r,s}$ and $\sum_{p \in \mathbf{R}_{n,r,s}} \text{wt}(p)$ is a formal power series in the variables in $\mathbf{b}, \mathbf{\lambda}, \mathbf{a}$, and $\mathbf{c}$.

A *Motzkin path* is an R_{II} path without vertical-down steps and backward-down steps. A *Schröder path* is an R_{II} path without down steps and backward-down steps. A *Motzkin–Schröder path* is an R_{II} path without backward-down steps. Let $\mathbf{Mot}_{n,r,s}$ (resp. $\mathbf{Sch}_{n,r,s}$ and $\mathbf{MS}_{n,r,s}$) denote the set of Motzkin paths (resp. Schröder paths and Motzkin–Schröder paths) from $(0, r)$ to (n, s) .

We are now ready to state known results on various orthogonal polynomials. Note that for the case of ordinary orthogonal polynomials, by the well-known fact $\mathcal{L}(P_n(x)^2) = \lambda_1 \cdots \lambda_n$, we have

$$Q_n(x) = \frac{P_n(x)}{d_n(x)} = \frac{P_n(x)}{\lambda_1 \cdots \lambda_n} = \frac{P_n(x)}{\mathcal{L}(P_n(x)^2)}.$$

Theorem 5 ([14, Proposition 17, pp. I-15]). *Let $P_n(x)$ be the ordinary orthogonal polynomials given by (9) with $\mathbf{c} = \mathbf{a} = \mathbf{0}$. Then there is a linear functional $\mathcal{L}$ on W such that*

$$\mathcal{L}(x^n P_r(x) Q_s(x)) = \sum_{p \in \mathbf{Mot}_{n,r,s}} \text{wt}(p).$$

Theorem 6 ([9, Theorem 17]). *Let $P_n(x)$ be the Laurent biorthogonal polynomials given by (9) with $\mathbf{c} = \mathbf{\lambda} = \mathbf{0}$. Then there is a linear functional $\mathcal{L}$ on W such that*

$$\mathcal{L}(x^n P_r(x) Q_s(x)) = \sum_{p \in \mathbf{Sch}_{n,r,s}} \text{wt}(p).$$

We note that Theorem 6 is an equivalent statement of [9, Theorem 17]; see [11, Theorem 4.7].

Theorem 7 ([11, Theorem 3.12]). *Let $P_n(x)$ be the orthogonal polynomials of type R_I given by (9) with $\mathbf{c} = \mathbf{0}$. Then there is a linear functional $\mathcal{L}$ on W such that*

$$\mathcal{L}(x^n P_r(x) Q_s(x)) = \sum_{p \in \text{MS}_{n,r,s}} \text{wt}(p).$$

Finally, we introduce dual coefficients of polynomials.

Definition 8. Let $(P_n(x))_{n \geq 0}$ be a sequence of polynomials such that $\deg P_n(x) = n$ for all $n \geq 0$. The *coefficients* $\rho_{n,k}$ and *dual coefficients* $\tau_{n,k}$ of the polynomials $P_n(x)$ are defined by

$$P_n(x) = \sum_{k \geq 0} \rho_{n,k} x^k, \quad x^n = \sum_{k \geq 0} \tau_{n,k} P_k(x). \quad (12)$$

We also define the (*generalized*) *dual coefficients* $\tau_{n,r,s}$ by

$$x^n P_r(x) = \sum_{s \geq 0} \tau_{n,r,s} P_s(x). \quad (13)$$

Let $P_n(x)$ be the ordinary orthogonal polynomials given by (9) with $\mathbf{c} = \mathbf{a} = \mathbf{0}$, and let $\mathcal{L}$ be the corresponding linear functional. The orthogonality relation is

$$\mathcal{L}(P_r(x) Q_s(x)) = \delta_{r,s}.$$

Thus, multiplying both sides of (13) by $Q_s(x)$ and applying $\mathcal{L}$ yields

$$\mathcal{L}(x^n P_r(x) Q_s(x)) = \tau_{n,r,s}. \quad (14)$$

Thus, Theorem 5 gives the following corollary.

Corollary 9. *Let $P_n(x)$ be the ordinary orthogonal polynomials given by (9) with $\mathbf{c} = \mathbf{a} = \mathbf{0}$. Then*

$$\tau_{n,r,s} = \sum_{p \in \text{Mot}_{n,r,s}} \text{wt}(p).$$

We note that (14) no longer holds for Laurent biorthogonal polynomials or orthogonal polynomials of type R_I or R_{II} . In Section 5, we will find an analogous result of Corollary 9 for orthogonal polynomials of type R_{II} using restricted R_{II} paths.

3 A combinatorial interpretation for generalized moments

Throughout this section we follow the notation in Definition 1. Recall the vector space

$$W = \text{span}\{x^n Q_m(x) : n, m \geq 0\}.$$

3.1 A basis for W

In this subsection, we provide a basis for W as follows.

Theorem 10. *The vector space $W = \text{span}\{x^n Q_m(x) : n, m \geq 0\}$ has a basis*

$$\mathcal{B} = \{x^n : n \geq 0\} \cup \{Q_n(x) : n \geq 1\} \cup \{xQ_n(x) : n \geq 1\}.$$

To prove this theorem, we consider

$$W' = \text{span}\{x^n/d_m(x) : n, m \geq 0\}.$$

Note that since $Q_m = P_m/d_m$ and P_m is a polynomial, every $x^n Q_m$ belongs to W' (in particular, so do Q_m and xQ_m). Hence W is contained in W' .

The proof of Theorem 10 proceeds in three steps. First, we find a basis $\mathcal{B}'$ of W' . Second, we prove an equality of finite-dimensional spans that implies $W = W'$. Finally, we use these results to prove that $\mathcal{B}$ spans W and is linearly independent.

Lemma 11. *The vector space $W' = \text{span}\{x^n/d_m(x) : n, m \geq 0\}$ has a basis*

$$\mathcal{B}' = \{x^n : n \geq 0\} \cup \{1/d_m(x) : m \geq 1\} \cup \{x/d_m(x) : m \geq 1\}.$$

Proof. Since $1/d_m(x) \sim x^{-2m}$ and $x/d_m(x) \sim x^{-2m+1}$ asymptotically up to constants, $\mathcal{B}'$ is linearly independent. Since $\mathcal{B}' \subset W'$, it suffices to show that $\text{span}(\mathcal{B}') = W'$. To see this, we claim that $x^n/d_m(x) \in \text{span}(\mathcal{B}')$ for any $n, m \geq 0$. First, we write the fraction as

$$\frac{x^n}{d_m(x)} = q(x) + \frac{r(x)}{d_m(x)}, \quad (15)$$

where $q(x)$ and $r(x)$ are polynomials and $\deg(r(x)) < 2m$. Now it suffices to show that $r(x)/d_m(x) \in \text{span}(\mathcal{B}')$. If $\deg r(x) \leq 1$, then we have $r(x)/d_m(x) \in \text{span}(\mathcal{B}')$. Otherwise, we can write

$$\frac{r(x)}{d_m(x)} = \frac{(c_m x^2 + a_m x + \lambda_m)r_1(x) + s_1(x)}{d_m(x)} = \frac{r_1(x)}{d_{m-1}(x)} + \frac{s_1(x)}{d_m(x)}, \quad (16)$$

where $r_1(x)$ is a polynomial with $\deg r_1(x) < 2m - 2$ and $s_1(x)$ is a polynomial with $\deg s_1(x) \leq 1$. By iterating (16), we can express $r(x)/d_m(x)$ as a linear combination of the elements in $\mathcal{B}'$, which completes the proof. $\square$

Lemma 12. *For $M \geq 0$, let*

$$\begin{aligned} \mathcal{B}_M &= \{1\} \cup \{Q_m(x), xQ_m(x) : 1 \leq m \leq M\}, \\ \mathcal{B}'_M &= \{1\} \cup \{1/d_m(x), x/d_m(x) : 1 \leq m \leq M\}. \end{aligned}$$

Then

$$\text{span}(\mathcal{B}_M) = \text{span}(\mathcal{B}'_M).$$

In particular, $W = W'$.

Proof. For $m \geq 1$, we have $\deg P_m \leq m$ and $\deg d_m = 2m$. The polynomial division argument in the proof of Lemma 11 therefore expresses $Q_m(x)$ and $xQ_m(x)$ as linear combinations of 1 and $1/d_i(x), x/d_i(x)$ with $1 \leq i \leq m$. Thus $\text{span}(\mathcal{B}_M) \subseteq \text{span}(\mathcal{B}'_M)$. We prove the reverse inclusion by induction on M . The case $M = 0$ is clear since $\mathcal{B}_0 = \mathcal{B}'_0 = \{1\}$.

Now, let $M = m \geq 1$ and assume the equality holds for $M = m - 1$. It suffices to show that $1/d_m(x), x/d_m(x) \in \text{span}(\mathcal{B}_m)$. First, we claim that $1/d_m(x) \in \text{span}(\mathcal{B}_m)$. If $r_m(x)$ is the quotient of $P_m(x)$ when divided by $x - \beta_m$, then we have

$$\frac{P_m(x)}{d_{m-1}(x)(x - \beta_m)} = \frac{r_m(x)}{d_{m-1}(x)} + \frac{P_m(\beta_m)}{d_{m-1}(x)(x - \beta_m)}. \quad (17)$$

Since $\deg r_m \leq m - 1$, the same polynomial division argument gives $r_m(x)/d_{m-1}(x) \in \text{span}(\mathcal{B}'_{m-1}) = \text{span}(\mathcal{B}_{m-1})$ by the inductive hypothesis. Rewriting (17) for the last term on its right-hand side, substituting $c_m(x - \alpha_m)Q_m(x) = P_m(x)/(d_{m-1}(x)(x - \beta_m))$, and dividing by the nonzero number $P_m(\beta_m)$, we obtain

$$\frac{1}{d_{m-1}(x)(x - \beta_m)} = \frac{1}{P_m(\beta_m)} \left(c_m x Q_m(x) - c_m \alpha_m Q_m(x) - \frac{r_m(x)}{d_{m-1}(x)} \right). \quad (18)$$

Thus $1/(d_{m-1}(x)(x - \beta_m)) \in \text{span}(\mathcal{B}_m)$. We also have $1/(d_{m-1}(x)(x - \alpha_m)) \in \text{span}(\mathcal{B}_m)$ in the same way.

If $\alpha_m \neq \beta_m$, then

$$\frac{1}{d_m(x)} = \frac{1}{c_m(\beta_m - \alpha_m)} \left(\frac{1}{d_{m-1}(x)(x - \beta_m)} - \frac{1}{d_{m-1}(x)(x - \alpha_m)} \right).$$

Hence $1/d_m(x) \in \text{span}(\mathcal{B}_m)$. If $\alpha_m = \beta_m$, then we can write $Q_m(x)$ as

$$\begin{aligned} Q_m(x) &= \frac{P_m(x)}{d_m(x)} = \frac{c_m(x - \alpha_m)^2 R_{m-2}(x) + \gamma c_m(x - \alpha_m) + P_m(\alpha_m)}{d_m(x)} \\ &= \frac{R_{m-2}(x)}{d_{m-1}(x)} + \frac{\gamma}{d_{m-1}(x)(x - \alpha_m)} + \frac{P_m(\alpha_m)}{d_m(x)} \end{aligned}$$

for a polynomial $R_{m-2}(x)$ of degree at most $m - 2$ and a constant γ , with the convention that $R_{-1}(x) = 0$. As above, $R_{m-2}(x)/d_{m-1}(x) \in \text{span}(\mathcal{B}_{m-1})$, so

$$\frac{1}{d_m(x)} = \frac{1}{P_m(\alpha_m)} \left(Q_m(x) - \frac{R_{m-2}(x)}{d_{m-1}(x)} - \frac{\gamma}{d_{m-1}(x)(x - \alpha_m)} \right).$$

Thus $1/d_m(x) \in \text{span}(\mathcal{B}_m)$ in this case as well.

Finally, since $1/(d_{m-1}(x)(x - \alpha_m)) = c_m(x - \beta_m)/d_m(x)$, we obtain

$$\frac{x}{d_m(x)} = \frac{\beta_m}{d_m(x)} + \frac{1}{c_m d_{m-1}(x)(x - \alpha_m)} \in \text{span}(\mathcal{B}_m). \quad (19)$$

This completes the induction. Since every element of $\mathcal{B}'$ belongs to W , Lemma 11 gives $W' \subseteq W$. Together with $W \subseteq W'$, this proves $W = W'$. $\square$

Now we prove Theorem 10.

Proof of Theorem 10. By Lemma 12, $\mathcal{B}' \subseteq \text{span}(\mathcal{B})$. Since $\mathcal{B} \subseteq W = W'$, Lemma 11 shows that $\mathcal{B}$ spans W .

For linear independence, let $M, N \geq 0$. By Lemma 12, the sets

$$\mathcal{B}_M \cup \{x^n : 1 \leq n \leq N\} \quad \text{and} \quad \mathcal{B}'_M \cup \{x^n : 1 \leq n \leq N\}$$

have the same span. By Lemma 11, the second set is linearly independent and has $2M + N + 1$ elements. Since the first set has at most $2M + N + 1$ elements, it is also linearly independent. Every finite subset of $\mathcal{B}$ is contained in such a set, so $\mathcal{B}$ is linearly independent. $\square$

Remark 13. In [6, Theorem 3.5], Ismail and Masson implicitly use the fact (without providing a proof) that $\mathcal{B} \setminus \{x^n : n \geq 1\}$ is a basis for V . This follows from Lemmas 11 and 12 and Theorem 10. Consequently, $W = V \oplus \text{span}\{x^n : n \geq 1\}$.

3.2 Moments and generalized moments for orthogonal polynomials of type R_{II}

In this subsection, we construct a linear functional $\mathcal{L}$ on $W = \text{span}\{x^n Q_m(x) : n, m \geq 0\}$ that yields the partial orthogonality relations for orthogonal polynomials of type R_{II} . Moreover, we provide a combinatorial description of the generalized moments $\mathcal{L}(x^n P_r(x) Q_s(x))$. The linear functional on V considered by Ismail and Masson [6] does not have a unique extension to W , since its values on x^n , $n \geq 1$, must be specified separately. We choose these values combinatorially.

By Theorem 10, we can define a linear functional on W as follows.

Definition 14. The linear functional $\mathcal{L}$ on the vector space $\text{span}\{x^n Q_m(x) : n, m \geq 0\}$ is defined by

$$\begin{aligned} \mathcal{L}(x^n) &= \sum_{P \in R_n} \text{wt}(P), \quad n \geq 0, \\ \mathcal{L}(Q_n(x)) &= \mathcal{L}(x Q_{n+1}(x)) = 0, \quad n \geq 1, \\ \mathcal{L}(x Q_1(x)) &= \sum_{P \in R_{1,1}} \text{wt}(P). \end{aligned}$$

Definition 15. The *generalized moment* $\mu_{n,r,s}$ of orthogonal polynomials of type R_{II} is given by

$$\mu_{n,r,s} = \mathcal{L}(x^n P_r(x) Q_s(x)).$$

We also write $\mu_{n,m} = \mu_{n,0,m}$ and $\mu_n = \mu_{n,0,0}$.

By multiplying both sides of (9) by $1/d_n(x)$, we have

$$(c_{n+1}x^2 + a_{n+1}x + \lambda_{n+1})Q_{n+1}(x) = (x - b_n)Q_n(x) - Q_{n-1}(x). \quad (20)$$

The next proposition provides a combinatorial interpretation for the values of $\mathcal{L}$ on the set $\{x^n Q_m(x) : n, m \geq 0\}$. Recall that $c_i \neq 0$ for all $i \geq 1$ by Definition 1.

Proposition 16. For nonnegative integers n and m , we have

$$\mu_{n,m} = \sum_{p \in R_{n,m}} \text{wt}(p).$$

Proof. Let $\mu'_{n,m}$ be the right-hand side of the equation. We will prove $\mu_{n,m} = \mu'_{n,m}$ using induction on (n, m) . By Definition 14, the statement holds for $m = 0$, $n = 0$, or $n = 1$.

Now let $n \geq 2$ and $m \geq 1$, and suppose that the statement holds for all $(n', m') \neq (n, m)$ with $n' \leq n$ and $m' \leq m$. By (20),

$$(c_m x^2 + a_m x + \lambda_m) Q_m(x) = (x - b_{m-1}) Q_{m-1}(x) - Q_{m-2}(x).$$

Multiplying both sides by x^{n-2} and applying $\mathcal{L}$ yields

$$c_m \mu_{n,m} + a_m \mu_{n-1,m} + \lambda_m \mu_{n-2,m} = \mu_{n-1,m-1} - b_{m-1} \mu_{n-2,m-1} - \mu_{n-2,m-2}. \quad (21)$$

On the other hand, considering the last step of $p \in R_{n-1,m-1}$, we have

$$\mu'_{n-1,m-1} = c_m \mu'_{n,m} + a_m \mu'_{n-1,m} + \lambda_m \mu'_{n-2,m} + b_{m-1} \mu'_{n-2,m-1} + \mu'_{n-2,m-2},$$

which is the same recurrence relation as (21). Thus, applying the inductive hypothesis to every term in (21) except $c_m \mu_{n,m}$ gives $c_m \mu_{n,m} = c_m \mu'_{n,m}$. Since $c_m \neq 0$, the statement also holds for (n, m) , which completes the proof. $\square$

Remark 17. Using the recurrence (21) along with Definition 14, one can see that $\mu_{n,m} = 0$ for $0 \leq n < m$. This shows the orthogonality of orthogonal polynomials of type R_{II} .

We are now ready to obtain a combinatorial description of $\mu_{n,r,s}$.

Theorem 18. For nonnegative integers n, r , and s , we have

$$\mu_{n,r,s} = \sum_{P \in R_{n,r,s}} \text{wt}(P).$$

Proof. Let $\mu'_{n,r,s} = \sum_{P \in R_{n,r,s}} \text{wt}(P)$. We will prove $\mu_{n,r,s} = \mu'_{n,r,s}$ by induction on r . The base step $r = 0$ follows from Proposition 16. For the inductive step, let $k \geq 0$ and suppose that $\mu_{n,r,s} = \mu'_{n,r,s}$ holds for all $r \leq k$. Replacing n by k in (9), multiplying both sides of the resulting equation by $x^n Q_s(x)$, and taking $\mathcal{L}$, we obtain

$$\mu_{n,k+1,s} = \mu_{n+1,k,s} - b_k \mu_{n,k,s} - c_k \mu_{n+2,k-1,s} - a_k \mu_{n+1,k-1,s} - \lambda_k \mu_{n,k-1,s}. \quad (22)$$

On the other hand, considering the first step of $p \in R_{n+1,k,s}$, we have

$$\mu'_{n+1,k,s} = \mu'_{n,k+1,s} + b_k \mu'_{n,k,s} + c_k \mu'_{n+2,k-1,s} + a_k \mu'_{n+1,k-1,s} + \lambda_k \mu'_{n,k-1,s},$$

which is the same recurrence as (22). Thus, by the inductive hypothesis, $\mu_{n,r,s} = \mu'_{n,r,s}$ also holds for $r = k + 1$. By induction, we obtain the theorem. $\square$

4 Generalized orthogonal polynomials of type R_{II}

In this section, we extend the definition of orthogonal polynomials of type R_{II} by relaxing some of their conditions.

Definition 19. We say that $(P_n(x))_{n \geq 0}$ is a sequence of *generalized orthogonal polynomials of type R_{II}* , or *R_{II} polynomials* for short, if $P_{-1}(x) = 0$, $P_0(x) = 1$, and

$$P_{n+1}(x) = (x - b_n)P_n(x) - (c_n x^2 + a_n x + \lambda_n)P_{n-1}(x), \quad n \geq 0,$$

where $\mathbf{a} = (a_1, a_2, \dots)$, $\mathbf{b} = (b_0, b_1, \dots)$, $\mathbf{c} = (c_1, c_2, \dots)$, and $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \dots)$ are sequences such that, for all $n \geq 1$, at least one of λ_n , a_n , or c_n is nonzero. We will use the following notation:

$$d_n(x) = \prod_{i=1}^n (c_i x^2 + a_i x + \lambda_i), \quad Q_n(x) = \frac{P_n(x)}{d_n(x)}.$$

Note that R_{II} polynomials contain various types of orthogonal polynomials:

- Orthogonal polynomials of type R_{II} are R_{II} polynomials such that $c_n \neq 0$, $P_n(\alpha_n) \neq 0$, and $P_n(\beta_n) \neq 0$ for $n \geq 1$, where α_n and β_n are the roots of $c_n x^2 + a_n x + \lambda_n$.
- Orthogonal polynomials of type R_I are R_{II} polynomials such that $c_n = 0$, $a_n \neq 0$, and $P_n(-\lambda_n/a_n) \neq 0$ for $n \geq 1$.
- Laurent biorthogonal polynomials are R_{II} polynomials such that $c_n = \lambda_n = 0$, $a_n \neq 0$, and $P_n(0) \neq 0$ for $n \geq 1$.
- Ordinary orthogonal polynomials are R_{II} polynomials such that $c_n = a_n = 0$ and $\lambda_n \neq 0$ for $n \geq 1$.

Let

$$W = \text{span}\{x^n Q_m(x) : n, m \geq 0\}.$$

Recall from (20) that, for $k \geq 0$,

$$(c_{k+1}x^2 + a_{k+1}x + \lambda_{k+1})Q_{k+1}(x) = (x - b_k)Q_k(x) - Q_{k-1}(x), \quad (23)$$

where $Q_{-1}(x) = 0$. If we multiply both sides by x^ℓ , we obtain

$$\begin{aligned} c_{k+1}x^{\ell+2}Q_{k+1}(x) + a_{k+1}x^{\ell+1}Q_{k+1}(x) + \lambda_{k+1}x^\ell Q_{k+1}(x) \\ = x^{\ell+1}Q_k(x) - b_k x^\ell Q_k(x) - x^\ell Q_{k-1}(x). \end{aligned} \quad (24)$$

Let $S \subseteq \mathbb{Z}_{\geq 0}^2$ and $B = \{x^n Q_m(x) : (n, m) \in S\}$. We say that $x^n Q_m(x)$ is obtained from B by a single application of (24) if there exist $k, \ell \geq 0$ such that $x^n Q_m(x)$ appears in (24) with a nonzero coefficient and, for every term $x^{n'} Q_{m'}(x)$ in (24) with $(n', m') \neq (n, m)$, either its coefficient is zero or it is contained in B . We say that $x^n Q_m(x)$ is obtained from B by multiple applications of (24) if there is a sequence $(n_1, m_1), (n_2, m_2), \dots, (n_t, m_t)$ with $(n_t, m_t) = (n, m)$ such that $x^{n_i} Q_{m_i}(x)$ is obtained from B_{i-1} by a single application of (24) for all $i = 1, 2, \dots, t$, where $B_0 = B$ and $B_i = B_{i-1} \cup \{x^{n_i} Q_{m_i}(x)\}$.

Definition 20. A basis B of W is said to be *good* if $B = \{x^n Q_m(x) : (n, m) \in S\}$ for a subset $S \subseteq \mathbb{Z}_{\geq 0}^2$ and, for every $n, m \geq 0$, $x^n Q_m(x)$ is obtained from B by multiple applications of (24).

Theorem 21. Suppose that there is a good basis of W . Then there is a linear functional $\mathcal{L}$ on W such that, for all $n, r, s \geq 0$,

$$\mathcal{L}(x^n P_r(x) Q_s(x)) = \sum_{p \in R_{n,r,s}} \text{wt}(p).$$

Proof. Let $B = \{x^n Q_m(x) : (n, m) \in S\}$ be a good basis of W . Define $\mathcal{L}$ on B by

$$\mathcal{L}(x^n Q_m(x)) = R_{n,m}, \quad (25)$$

where $R_{n,m} = \sum_{p \in R_{n,m}} \text{wt}(p)$. Since B is a basis, we can extend the definition of $\mathcal{L}$ to W by linearity. We claim that (25) holds for all $n, m \geq 0$.

To prove the claim, first suppose that $x^n Q_m(x)$ is obtained from B by a single application of (24). Applying $\mathcal{L}$ to both sides of (24) gives

$$c_{k+1} R'_{\ell+2,k+1} + a_{k+1} R'_{\ell+1,k+1} + \lambda_{k+1} R'_{\ell,k+1} = R'_{\ell+1,k} - b_k R'_{\ell,k} - R'_{\ell,k-1}, \quad (26)$$

where $R'_{i,j} = \mathcal{L}(x^i Q_j(x))$. By the construction of $\mathcal{L}$, if $x^i Q_j(x) \in B$, then $R'_{i,j} = R_{i,j}$. Considering the last step of any path $p \in R_{\ell+1,k}$, we have

$$R_{\ell+1,k} = c_{k+1} R_{\ell+2,k+1} + a_{k+1} R_{\ell+1,k+1} + \lambda_{k+1} R_{\ell,k+1} + b_k R_{\ell,k} + R_{\ell,k-1}. \quad (27)$$

Since every term $R'_{i,j}$ with $(i, j) \neq (n, m)$ appearing in (26) satisfies $R'_{i,j} = R_{i,j}$, comparing (26) and (27) gives $R'_{n,m} = R_{n,m}$, hence $\mathcal{L}(x^n Q_m(x)) = R_{n,m}$.

Now consider $x^n Q_m(x)$ for arbitrary $n, m \geq 0$. Since B is a good basis, $x^n Q_m(x)$ is obtained from B by multiple applications of (24). Applying the previous argument iteratively, we obtain that $\mathcal{L}(x^n Q_m(x)) = R_{n,m}$. Therefore (25) holds for all $n, m \geq 0$, as claimed.

This shows the theorem for the case $r = 0$. The general case $r \geq 0$ can be proved by induction using the same arguments in the proof of Theorem 18. $\square$

Since $R_{n,0,m} = \emptyset$ for $n < m$, we obtain the following orthogonality.

Corollary 22. Suppose that there is a good basis of W . Then there is a linear functional $\mathcal{L}$ on W such that $\mathcal{L}(1)$ is a power series in $\mathbf{c} = (c_1, c_2, \dots)$ with constant term 1 and, for all $0 \leq n < m$,

$$\mathcal{L}(x^n Q_m(x)) = 0.$$

Since there are good bases as in Table 1, Theorem 21 implies the combinatorial models of generalized moments for ordinary orthogonal polynomials, Laurent biorthogonal polynomials, and orthogonal polynomials of types R_I and R_{II} in Theorems 5, 6, and 7. Theorem 21 also includes orthogonal polynomials that do not belong to any of these four classes of orthogonal polynomials; for example, one can consider the sequence $\mathbf{c}$ with $c_{2n+1} = 0$ and $c_{2n} \neq 0$.

Note that in Theorem 21 we have the assumption that W has a good basis. It would be interesting to characterize the existence of a good basis.

polynomials	good bases
orthogonal polynomials	$\{(n, m) : m = 0\}$
Laurent biorthogonal polynomials	$\{(n, m) : n = 0 \text{ or } m = 0\}$
orthogonal polynomials of type R_I	$\{(n, m) : n = 0 \text{ or } m = 0\}$
orthogonal polynomials of type R_{II}	$\{(n, m) : 0 \leq n \leq 1 \text{ or } m = 0\}$

Table 1: Orthogonal polynomials and their good bases.

Problem 23. Find a characterization of the sequences $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, and $\boldsymbol{\lambda}$ such that W has a good basis.

5 A combinatorial interpretation for dual coefficients

In this section, we provide a combinatorial interpretation of the (generalized) dual coefficients $\tau_{n,r,s}$ of R_{II} polynomials introduced in Section 4, which are orthogonal polynomials of type R_{II} with relaxed conditions.

We start with an observation on the combinatorial interpretation for generalized moments $\mu_{n,r,s}$ of R_{II} polynomials. Let $P_n(x)$ be R_{II} polynomials defined in the previous section. Recall that dual coefficients $\tau_{n,r,s}$ are defined by

$$x^n P_r(x) = \sum_{s \geq 0} \tau_{n,r,s} P_s(x). \quad (28)$$

Multiplying both sides of (28) by $Q_\ell(x)$ and applying $\mathcal{L}$ yields

$$\mu_{n,r,\ell} = \sum_{s \geq 0} \tau_{n,r,s} \mu_{0,s,\ell}. \quad (29)$$

By Theorem 21, $\mu_{n,r,\ell}$ is the generating function for all R_{II} paths from $(0, r)$ to (n, ℓ) . Hence, the identity (29) naturally suggests that $\tau_{n,r,s}$ is the generating function for the restricted R_{II} paths from $(0, r)$ to (n, s) defined in Definition 4. Indeed, such a generating function satisfies (29). For a fixed ℓ , however, this single identity does not determine all the coefficients $\tau_{n,r,s}$. For example, for $n = r = \ell = 0$, let F be any formal power series and set

$$\tau'_{0,0,0} = 1 + F\mu_{0,1,0}, \quad \tau'_{0,0,1} = -F\mu_{0,0,0}, \quad \tau'_{0,0,s} = 0 \quad (s \geq 2).$$

Then we also have

$$\mu_{0,0,0} = \sum_{s \geq 0} \tau'_{0,0,s} \mu_{0,s,0}.$$

The suggested generating function nevertheless turns out to be equal to $\tau_{n,r,s}$, which is the main result in this section.

Theorem 24. For nonnegative integers n , r , and s , we have

$$\tau_{n,r,s} = \sum_{p \in \tilde{R}_{n,r,s}} \text{wt}(p). \quad (30)$$

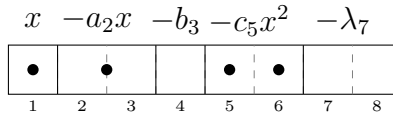


Figure 2: An example of $T \in \text{DT}_8$ with $\text{wt}(T) = a_2 b_3 c_5 \lambda_7 x^4$.

Note that in order to prove Theorem 24, it suffices to show that the right-hand side of (30), denoted $\tau'_{n,r,s}$, satisfies the same relation (28) as $\tau_{n,r,s}$, that is,

$$x^n P_r(x) = \sum_{s \geq 0} \tau'_{n,r,s} P_s(x). \quad (31)$$

We will prove (31) by giving combinatorial interpretations for both sides and finding a sign-reversing involution. To do this we introduce the following definition.

Definition 25. Consider a $1 \times n$ square board whose boxes are labeled 1 through n from left to right. Let M_i (resp. D_i) be a monomino (resp. domino) with i dots inside. A *dotted tiling of size n* is a tiling of the board using tiles in $\{M_0, M_1, D_0, D_1, D_2\}$. The set of dotted tilings of size n is denoted by DT_n .

For a dotted tiling T , we define the weight $\text{wt}(T)$ to be the product of the weights $\text{wt}(\tau)$ of the tiles τ in T , where

$$\text{wt}(\tau) = \begin{cases} x & \text{if } \tau = M_1, \\ -b_{i-1} & \text{if } \tau = M_0 \text{ with label } i, \\ -\lambda_{i-1} & \text{if } \tau = D_0 \text{ with label } (i-1, i), \\ -a_{i-1}x & \text{if } \tau = D_1 \text{ with label } (i-1, i), \\ -c_{i-1}x^2 & \text{if } \tau = D_2 \text{ with label } (i-1, i). \end{cases}$$

See Figure 2 for an example.

By (9), it is immediate that

$$P_n(x) = \sum_{T \in \text{DT}_n} \text{wt}(T). \quad (32)$$

We are now ready to prove Theorem 24.

Proof of Theorem 24. As discussed before, it suffices to show (31). By (32), we can rewrite (31) as

$$\sum_{k \geq 0} \tau'_{n,r,k} P_k(x) = \sum_{(p,T) \in X} \text{wt}(p) \text{wt}(T), \quad (33)$$

where X is the set of pairs (p, T) such that $p \in \tilde{\mathcal{R}}_{n,r,k}$ and $T \in \text{DT}_k$ for some $k \geq 0$. To prove (31), we establish a sign-reversing involution on X such that the sum over its fixed points is equal to $x^n P_r(x)$.

Consider $(p, T) \in X$ with $p \in \tilde{\mathbf{R}}_{n,r,k}$ and $T \in \text{DT}_k$. For $m \geq 1$, we denote by p^m the m th step from the end of p , and by T^m the m th tile from the end of T . Let i (resp. j) be the maximum number of consecutive U 's (resp. M_1 's) at the end of p (resp. T). Here, if $p^1 \neq U$, then $i = 0$, and if $T^1 \neq M_1$, then $j = 0$. We define $\phi(p, T) = (p', T')$ considering the two cases $i \leq j$ and $i > j$.

Case 1: $i \leq j$. Note that p^{i+1} does not exist if and only if p consists only of up steps, in which case $i = n = k - r$. We define p' and T' in the following subcases. In Cases 1-1 and 1-2, if T^{i+1} does not exist, we insert the new tile at the left end of T .

Case 1-1 $p^{i+1} = H$. Let p' be the path obtained from p by replacing p^{i+1} with U . Let T' be the dotted tiling obtained from T by inserting a new tile M_0 after T^{i+1} .

Case 1-2 $p^{i+1} = D$. Let p' be the path obtained from p by replacing p^{i+1} with U . Let T' be the dotted tiling obtained from T by inserting a new tile D_0 after T^{i+1} .

Case 1-3 $p^{i+1} = V$. In this case, we have $i \geq 1$. Let p' be the path obtained from p by removing p^{i+1} . Let T' be the dotted tiling obtained from T by replacing T^i with D_1 .

Case 1-4 $p^{i+1} = B$. In this case, we have $i \geq 2$. Let p' be the path obtained from p by removing the pair of steps (p^{i+1}, U) , where U is the step following p^{i+1} . Let T' be the dotted tiling obtained from T by replacing the pair of tiles (T^i, T^{i-1}) with a single tile D_2 .

Case 1-5 p^{i+1} does not exist. In this case, let $(p', T') = (p, T)$.

Case 2: $i > j$. Then p^{j+1} is an up step starting at height $k - j - 1$. If T^{j+1} is a domino, it occupies the boxes labeled $k - j - 1$ and $k - j$, so this height is at least 1. We define p' and T' in the following subcases.

Case 2-1 $T^{j+1} = M_0$. Let p' be the path obtained from p by replacing p^{j+1} with H . Let T' be the dotted tiling obtained from T by removing the tile T^{j+1} .

Case 2-2 $T^{j+1} = D_0$. Let p' be the path obtained from p by replacing p^{j+1} with D . Let T' be the dotted tiling obtained from T by removing the tile T^{j+1} .

Case 2-3 $T^{j+1} = D_1$. Let p' be the path obtained from p by inserting a step V before p^{j+1} . Let T' be the dotted tiling obtained from T by replacing T^{j+1} with M_1 .

Case 2-4 $T^{j+1} = D_2$. Let p' be the path obtained from p by inserting a pair of steps (B, U) before p^{j+1} . Let T' be the dotted tiling obtained from T by replacing T^{j+1} with a pair of tiles (M_1, M_1) .

For $h = 1, 2, 3, 4$, the operations in Cases 1- h and 2- h are inverse to each other.

See Figures 3 to 6. It is straightforward to check that ϕ is a sign-reversing involution on X whose fixed points are the pairs (p, T) , where p is the unique path in $\mathbf{R}_{n,r,n+r}$ consisting

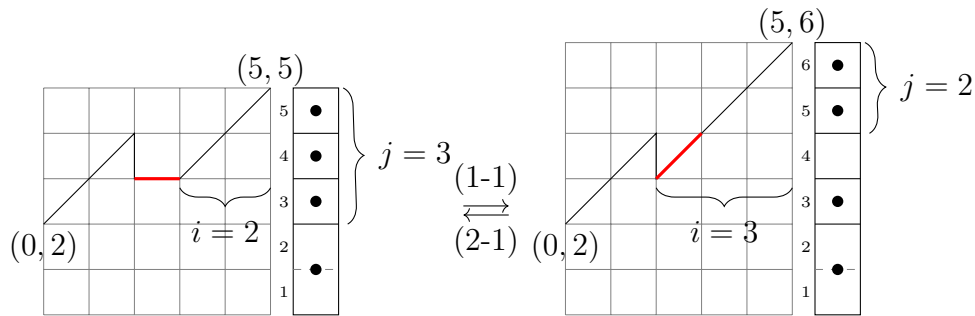


Figure 3: An example of $(P, T) \in \tilde{\mathbf{R}}_{5,2,5} \times \text{DT}_5$ for Case 1-1 and its image $(\phi(P), \phi(T)) \in \tilde{\mathbf{R}}_{5,2,6} \times \text{DT}_6$ corresponding to Case 2-1, described in the proof of Theorem 24.

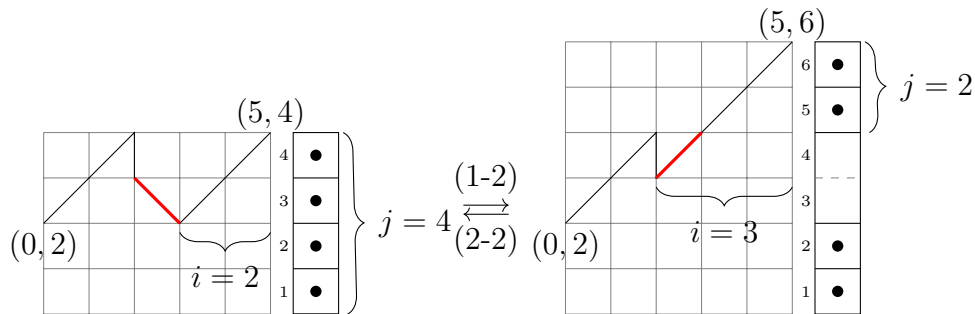


Figure 4: An example of $(P, T) \in \tilde{\mathbf{R}}_{5,2,4} \times \text{DT}_4$ for Case 1-2 and its image $(\phi(P), \phi(T)) \in \tilde{\mathbf{R}}_{5,2,6} \times \text{DT}_6$ corresponding to Case 2-2, described in the proof of Theorem 24.

only of up steps and T is a dotted tiling in DT_{n+r} such that the last n tiles are all equal to M_1 . Since $\text{wt}(p) \text{wt}(T) = x^n \text{wt}(T_0)$, where T_0 is the dotted tiling obtained from T by removing the last n tiles, we have

$$\sum_{(p,T) \in X} \text{wt}(p) \text{wt}(T) = x^n \sum_{T_0 \in \text{DT}_r} \text{wt}(T_0) = x^n P_r(x). \quad (34)$$

Then (31) follows from (33) and (34), which completes the proof. $\square$

If $c_n = 0$ in Theorem 24, we obtain the following corollary.

Corollary 26. *Let $(P_n^I(x))_{n \geq 0}$ be the sequence of orthogonal polynomials of type R_I defined in (5) and let $\tau_{n,r,s}^I$ be their dual coefficients. Then we have*

$$\tau_{n,r,s}^I = \sum_{p \in \tilde{\mathbf{R}}_{n,r,s}^I} \text{wt}(p),$$

where $\tilde{\mathbf{R}}_{n,r,s}^I$ is the set of restricted R_{II} paths from $(0, r)$ to (n, s) without backward-down steps.

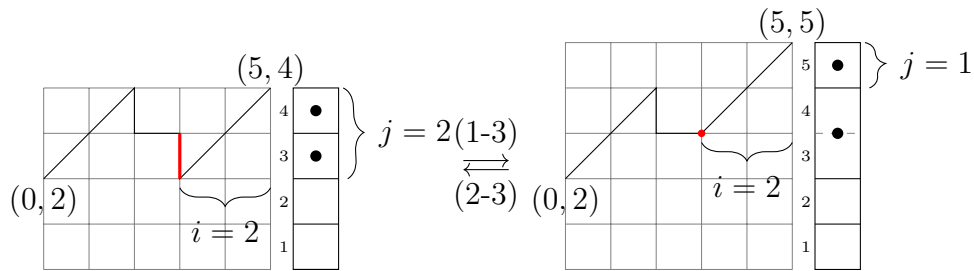


Figure 5: An example of $(P, T) \in \widetilde{\mathcal{R}}_{5,2,4} \times \text{DT}_4$ for Case 1-3 and its image $(\phi(P), \phi(T)) \in \widetilde{\mathcal{R}}_{5,2,5} \times \text{DT}_5$ corresponding to Case 2-3, described in the proof of Theorem 24.

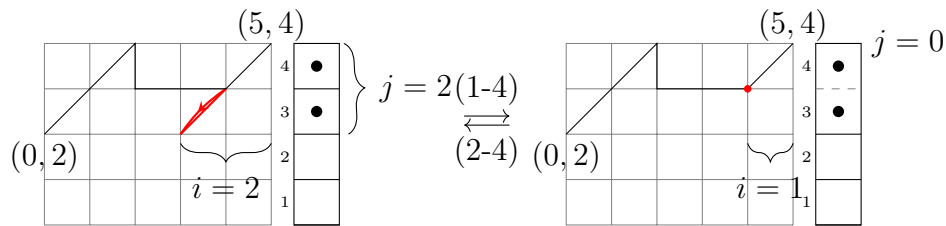


Figure 6: An example of $(P, T) \in \widetilde{\mathcal{R}}_{5,2,4} \times \text{DT}_4$ for Case 1-4 and its image $(\phi(P), \phi(T)) \in \widetilde{\mathcal{R}}_{5,2,4} \times \text{DT}_4$ corresponding to Case 2-4, described in the proof of Theorem 24.

Recall that a Motzkin path is an R_{II} path without vertical-down steps and backward-down steps. Thus, there is no difference between Motzkin paths and restricted Motzkin paths. Therefore, if $c_n = a_n = 0$ in Theorem 24, we obtain Corollary 9.

Remark 27. One may try to prove Theorem 24 by induction on n . Once the base steps $n = 0$ and $n = 1$ have been proved, we can easily apply the inductive argument as in the case of $\mu_{n,r,s}$. The case $n = 0$ is immediate from the definition. However, the other case $n = 1$ is not obvious. Note that, for the case of $\mu_{n,r,s}$, both cases $n = 0$ and $n = 1$ follow from the definition of $\mathcal{L}$.

6 Convergence of the moments and dual coefficients

In Section 3, we found a combinatorial description for generalized moments $\mu_{n,r,s}$ in terms of the weight sum of R_{II} paths. As there are infinitely many R_{II} paths from $(0, r)$ to (n, s) , we need to check whether the weight sum converges. In this section, we give sufficient conditions on the weights of steps for convergent weight sums of R_{II} paths and restricted R_{II} paths.

A *Dyck path of semilength n* is a lattice path from $(0, 0)$ to $(2n, 0)$ with steps $U = (1, 1)$ and $D = (1, -1)$ that never goes below the x -axis. Such paths are enumerated by the Catalan number $C_n = \frac{1}{n+1} \binom{2n}{n}$; see, for example, [13, Section 1.1].

Lemma 28. *Suppose that $b_m = b$ for $m \geq 0$, and that $a_m = a$, $c_m = c$, and $\lambda_m = \lambda$ for*

$m \geq 1$. Then

$$\mu_n = \sum_{\substack{i,j,k \geq 0 \\ 2i+j \leq n}} C_{i+j+k} \binom{n+j+2k}{n-2i-j} \binom{i+j+k}{j} \binom{i+k}{i} b^{n-2i-j} \lambda^i a^j c^k.$$

Proof. Suppose that $p \in R_n$ has u up steps, h horizontal steps, i down steps, j vertical-down steps, and k backward-down steps. Then

$$(n, 0) = u(1, 1) + h(1, 0) + i(1, -1) + j(0, -1) + k(-1, -1),$$

which implies $u = i + j + k$ and therefore $h = n - 2i - j$. Consequently

$$\mu_n = \sum_{p \in R_n} \text{wt}(p) = \sum_{\substack{i,j,k \geq 0 \\ 2i+j \leq n}} f_n(i, j, k) b^{n-2i-j} \lambda^i a^j c^k, \quad (35)$$

where $f_n(i, j, k)$ is the number of paths $p \in R_n$ with i down steps, j vertical-down steps, and k backward-down steps.

Given a path $p \in R_n$, define $\phi(p)$ to be the labeled Motzkin path obtained from p by replacing every vertical-down step with a down step labeled a , every backward-down step with a down step labeled c . If p has the given number of steps of each type as above, then $\phi(p)$ is a labeled Motzkin path from $(0, 0)$ to $(n + j + 2k, 0)$ with $n - 2i - j$ horizontal steps, i (unlabeled) down steps, j down steps labeled a , and k down steps labeled c . Such a labeled Motzkin path can be obtained from a Dyck path from $(0, 0)$ to $(2i + 2j + 2k, 0)$ by inserting $n - 2i - j$ horizontal steps in $\binom{n+j+2k}{n-2i-j}$ ways, labeling j down steps by a in $\binom{i+j+k}{j}$ ways, and labeling k down steps by c in $\binom{i+k}{k} = \binom{i+k}{i}$ ways. This implies that

$$f_n(i, j, k) = C_{i+j+k} \binom{n+j+2k}{n-2i-j} \binom{i+j+k}{j} \binom{i+k}{i}. \quad (36)$$

By (35) and (36), we obtain the desired formula. $\square$

Lemma 29. Suppose that $b_m = b$ for $m \geq 0$, and that $a_m = a$, $c_m = c$, and $\lambda_m = \lambda$ for $m \geq 1$, with $|c| < 1/4$. Then μ_n converges absolutely.

Proof. By Lemma 28, we have $\mu_n = \sum_{\substack{i,j,k \geq 0 \\ 2i+j \leq n}} f_n(i, j, k) b^{n-2i-j} \lambda^i a^j c^k$, where $f_n(i, j, k)$ is given in (36). There are finitely many pairs (i, j) in this sum. Thus it suffices to show that, for fixed i and j with $2i + j \leq n$, the sum

$$\sum_{k \geq 0} f_n(i, j, k) c^k \quad (37)$$

converges.

By Stirling's formula, for fixed m and large N , we have the asymptotic behaviors

$$C_N \sim \frac{4^N}{N^{3/2} \sqrt{\pi}}, \quad \binom{N}{m} \sim \frac{N^m}{m!}.$$

Thus, for fixed i, j, n and large k ,

$$\begin{aligned} f_n(i, j, k)c^k &\sim \frac{4^{i+j+k}}{(i+j+k)^{3/2}\sqrt{\pi}} \cdot \frac{(n+j+2k)^{n-2i-j}}{(n-2i-j)!} \cdot \frac{(i+j+k)^j}{j!} \cdot \frac{(i+k)^i}{i!} c^k \\ &\sim D \cdot (4c)^k k^{n-i-3/2}, \end{aligned}$$

where D is a constant with respect to k . Since $|4c| < 1$, by the ratio test, (37) converges. $\square$

Theorem 30. *Let $(a_m)_{m \geq 0}, (b_m)_{m \geq 0}, (c_m)_{m \geq 0}$ and $(\lambda_m)_{m \geq 0}$ be bounded sequences of complex numbers. Suppose $|c_m| < \frac{1}{4} - \epsilon$ for all m , where $\epsilon > 0$ is a fixed number. Then the generalized moments $\mu_{n,r,s}$ converge absolutely.*

Proof. Choose positive numbers a, b, λ such that $|a_m| \leq a$, $|b_m| \leq b$, and $|\lambda_m| \leq \lambda$ for all m , and set $R = \max\{\lambda, 1\}$ and $c = 1/4 - \epsilon$. Let $\widehat{\text{wt}}(p)$ be the weight of p after the substitution $(b_i, \lambda_i, a_i, c_i) \mapsto (b, R, a, c)$ for all i . Then $|\text{wt}(p)| \leq \widehat{\text{wt}}(p)$.

For $p \in R_{n,r,s}$, let p' be the R_{II} path obtained from p by adding r up steps at the beginning and s down steps at the end. Then $p' \in R_{n+r+s}$. Since the map $p \mapsto p'$ is injective and $\widehat{\text{wt}}(p) = R^{-s}\widehat{\text{wt}}(p') \leq \widehat{\text{wt}}(p')$, we have

$$\sum_{p \in R_{n,r,s}} |\text{wt}(p)| \leq \sum_{p' \in R_{n+r+s}} \widehat{\text{wt}}(p').$$

By Lemma 29, the right-hand side converges, which implies that $\mu_{n,r,s}$ converges absolutely. $\square$

For the convergence of $\tau_{n,r,s}$, we only need the condition on $\mathbf{c}$.

Corollary 31. *Suppose that $|c_m| < \frac{1}{4} - \epsilon$ for all m , where $\epsilon > 0$ is a fixed number. Then the dual coefficients $\tau_{n,r,s}$ converge absolutely.*

Proof. Recall that $\tau_{n,r,s} = \sum_{p \in \widetilde{R}_{n,r,s}} \text{wt}(p)$. Note that if (x, y) is a point in any R_{II} -path $p \in R_{n,r,s}$, then $y - x$ never increases along the path, which implies $y - x \leq r - 0 = r$. Moreover, if $p \in \widetilde{R}_{n,r,s}$, then we also have $x \leq n$. Therefore, each $p \in \widetilde{R}_{n,r,s}$ cannot go above the line $y = n + r$. Hence

$$\sum_{p \in \widetilde{R}_{n,r,s}} |\text{wt}(p)| \leq \sum_{p \in R_{n,r,s}} |\text{wt}'(p)|, \quad (38)$$

where $\text{wt}'(p)$ is the weight of p using the sequences $\mathbf{a}', \mathbf{b}', \mathbf{c}'$, and $\boldsymbol{\lambda}'$ obtained from $\mathbf{a}, \mathbf{b}, \mathbf{c}$, and $\boldsymbol{\lambda}$ by substituting $b_i = a_i = \lambda_i = c_i = 0$ for $i > n + r$. Then, by Theorem 30, we see that the right-hand side of (38) converges. $\square$

$$\mu(x) = 1 + \underbrace{bx\mu(x)}_{\text{arc}} + \underbrace{x\overbrace{\mu(x)}^c\mu(x)}_{\text{arc}} + \underbrace{x\overbrace{\mu(x)}^a\mu(x)}_{\text{arc}} + \underbrace{x\overbrace{\mu(x)}^{\lambda x}\mu(x)}_{\text{arc}}$$

Figure 7: A visualization of the functional equation for $\mu(x)$.

7 Moments with constant recurrence coefficients

In this section, we assume that $b_m = b$ for $m \geq 0$, and that $a_m = a$, $c_m = c$, and $\lambda_m = \lambda$ for $m \geq 1$. The goal is to find a formula for μ_n in this case by considering it as a formal power series in variables a, b, c, λ .

Let

$$\mu(x) = \sum_{n \geq 0} \mu_n x^n.$$

Considering the first time each R_{II} path returns to the x -axis as shown in Figure 7, we obtain the following functional equation:

$$\mu(x) = 1 + bx\mu(x) + (c + ax + \lambda x^2)\mu(x)^2.$$

Solving the equation yields

$$\begin{aligned} \mu(x) &= \frac{1 - bx - \sqrt{(1 - 4c) - (4a + 2b)x - (4\lambda - b^2)x^2}}{2(c + ax + \lambda x^2)} \\ &= \frac{2}{1 - bx + \sqrt{(1 - 4c) - (4a + 2b)x - (4\lambda - b^2)x^2}}. \end{aligned}$$

We aim to show that (up to a constant) μ_n can be expressed as the moment of certain ordinary or type R_I orthogonal polynomials. Recall that the moments of ordinary orthogonal polynomials (resp. orthogonal polynomials of type R_I) are equal to the weight sums of certain Motzkin paths (resp. Motzkin–Schröder paths). Thus, the goal is to reinterpret the values μ_n in the framework of Motzkin or Motzkin–Schröder path models. While there are infinitely many R_{II} paths from $(0, 0)$ to $(n, 0)$, the number of such Motzkin or Motzkin–Schröder paths is finite. From this point of view, our results give useful ways of calculating μ_n . Note that this section generalizes the result on moments of R_I polynomials in [11, Section 6.2].

Let $\mathcal{C}$ be the generating function for the Catalan numbers C_n :

$$\mathcal{C} = \sum_{n \geq 0} C_n c^n = \frac{1 - \sqrt{1 - 4c}}{2c} = \frac{2}{1 + \sqrt{1 - 4c}}.$$

Proposition 32. *Suppose that $b_m = b$ for $m \geq 0$, and that $a_m = a$, $c_m = c$, and $\lambda_m = \lambda$ for $m \geq 1$. We have*

$$\mu(x) = \mathcal{C} \sum_{n \geq 0} \mu_n(\mathbf{B}, \mathbf{\Lambda}) x^n.$$

Equivalently, for every $n \geq 0$,

$$\mu_n = \mathcal{C} \cdot \mu_n(\mathbf{B}, \Lambda).$$

Here, $\mu_n(\mathbf{B}, \Lambda)$ is the n th moment for ordinary orthogonal polynomials given by

$$p_{n+1}(x) = (x - B_n)p_n(x) - \Lambda_n p_{n-1}(x), n \geq 0,$$

where $p_{-1}(x) = 0, p_0(x) = 1$, and

$$\begin{aligned} B_0 &= \frac{\mathcal{C}a + b}{\sqrt{1 - 4c}}, & \Lambda_1 &= \frac{\mathcal{C}(b^2c + a^2 + ab - 4c\lambda + \lambda)}{\sqrt{1 - 4c}^3}, \\ B_n &= \frac{2a + b}{1 - 4c}, & \Lambda_{n+1} &= \frac{b^2c + a^2 + ab - 4c\lambda + \lambda}{(1 - 4c)^2}, \quad n \geq 1. \end{aligned}$$

Proof. By Flajolet's theory on continued fractions [4],

$$\sum_{n \geq 0} \mu_n(\mathbf{B}, \Lambda) x^n = \frac{1}{1 - B_0x - \Lambda_1 x^2 \cdot g(x)}, \quad (39)$$

where

$$g(x) = \frac{1}{1 - B_1x - \frac{\Lambda_2 x^2}{1 - B_2x - \dots}}.$$

Since $B_1 = B_2 = \dots$ and $\Lambda_2 = \Lambda_3 = \dots$, we see that

$$g(x) = \frac{1}{1 - B_1x - \Lambda_2 x^2 \cdot g(x)}.$$

Solving this functional equation gives

$$g(x) = \frac{2}{1 - B_1x + \sqrt{1 - 2B_1x + (B_1^2 - 4\Lambda_2)x^2}}. \quad (40)$$

By (39) and (40),

$$\begin{aligned} \mathcal{C} \sum_{n \geq 0} \mu_n(\mathbf{B}, \Lambda) x^n &= \frac{\mathcal{C}}{1 - B_0x - \frac{2\Lambda_1 x^2}{1 - B_1x + \sqrt{1 - 2B_1x + (B_1^2 - 4\Lambda_2)x^2}}} \\ &= \frac{2}{1 - bx + \sqrt{(1 - 4c) - (4a + 2b)x - (4\lambda - b^2)x^2}} \\ &= \mu(x), \end{aligned}$$

as desired. □

As a corollary, we obtain a formula for the Hankel determinant $\det(\mu_{i+j})_{i,j=0}^n$.

Corollary 33. Suppose that $b_m = b$ for $m \geq 0$, and that $a_m = a$, $c_m = c$, and $\lambda_m = \lambda$ for $m \geq 1$. Then, we have

$$\det(\mu_{i+j})_{i,j=0}^n = \frac{(b^2c + a^2 + ab - 4c\lambda + \lambda)^{\binom{n+1}{2}} \mathcal{C}^{2n+1}}{\sqrt{1-4c}^{\binom{2n+1}{2}}}.$$

Proof. By Proposition 32 and the fact that $\det(\mu_{i+j}(\mathbf{B}, \mathbf{\Lambda}))_{i,j=0}^n = \Lambda_1^n \Lambda_2^{n-1} \cdots \Lambda_n^1$, we have

$$\begin{aligned} \det(\mu_{i+j})_{i,j=0}^n &= \mathcal{C}^{n+1} \det(\mu_{i+j}(\mathbf{B}, \mathbf{\Lambda}))_{i,j=0}^n \\ &= \mathcal{C}^{n+1} \Lambda_1^n \Lambda_2^{n-1} \cdots \Lambda_n^1 \\ &= \frac{(b^2c + a^2 + ab - 4c\lambda + \lambda)^{\binom{n+1}{2}} \mathcal{C}^{2n+1}}{\sqrt{1-4c}^{\binom{2n+1}{2}}}. \quad \square \end{aligned}$$

Now we show that (up to a constant) μ_n can also be interpreted as the moment of orthogonal polynomials of type R_I .

Proposition 34. Suppose that $b_m = b$ for $m \geq 0$, and that $a_m = a$, $c_m = c$, and $\lambda_m = \lambda$ for $m \geq 1$. Then

$$\mu_n = \mathcal{C} \cdot \mu_n(\mathbf{B}, \mathbf{\Lambda}, \mathbf{A}).$$

Here, $\mu_n(\mathbf{B}, \mathbf{\Lambda}, \mathbf{A})$ is the n th moment for orthogonal polynomials of type R_I given by

$$p_{n+1}(x) = (x - B_n)p_n(x) - (A_n x + \Lambda_n)p_{n-1}(x), \quad n \geq 0,$$

where $p_{-1}(x) = 0, p_0(x) = 1$, and

$$\begin{aligned} B_0 &= \frac{b}{\sqrt{1-4c}}, & B_n &= \frac{b}{1-4c}, & n &\geq 1, \\ \Lambda_1 &= \frac{\mathcal{C}(b^2c - 4c\lambda + \lambda)}{\sqrt{1-4c}^3}, & \Lambda_{n+1} &= \frac{b^2c - 4c\lambda + \lambda}{(1-4c)^2}, & n &\geq 1, \\ A_1 &= \frac{\mathcal{C}a}{\sqrt{1-4c}}, & A_{n+1} &= \frac{a}{1-4c}, & n &\geq 1. \end{aligned}$$

Proof. The proof is similar to the proof of Proposition 32. By a direct computation, we obtain

$$\begin{aligned} \mathcal{C} \sum_{n \geq 0} \mu_n(\mathbf{B}, \mathbf{\Lambda}, \mathbf{A}) x^n &= \frac{\mathcal{C}}{1 - B_0 x - \frac{A_1 x + \Lambda_1 x^2}{1 - B_1 x - \frac{A_2 x + \Lambda_2 x^2}{1 - B_2 x - \cdots}}} \\ &= \frac{\mathcal{C}}{1 - B_0 x - \frac{2A_1 x + 2\Lambda_1 x^2}{1 - B_1 x + \sqrt{1 - (4A_2 + 2B_1)x - (4\Lambda_2 - B_1^2)x^2}}} \\ &= \mu(x). \quad \square \end{aligned}$$

The moment $\mu_n(\mathbf{B}, \mathbf{A})$ of ordinary orthogonal polynomials is the generating function for Motzkin paths from $(0, 0)$ to $(n, 0)$. Likewise, $\mu_n(\mathbf{B}, \mathbf{A}, \mathbf{A})$, the moment of orthogonal polynomials of type R_I , is the generating function for Motzkin–Schröder paths with the same endpoints. Both are finite sums. Hence, Propositions 32 and 34 provide practical methods for computing μ_n . At this point, we can pose natural questions:

1. Are the path models in Propositions 32 and 34 applicable for $\mu_{n,m}$ or $\mu_{n,r,s}$?
2. If not, are there Motzkin or Motzkin–Schröder paths with different weights that can be used for computing $\mu_{n,m}$ or $\mu_{n,r,s}$?

The answer to the first question is “no”. For instance, one can verify that the coefficient $[c]\mu_{2,1}$ of c in $\mu_{2,1}$ is equal to $10a + 8b$. In contrast, let $\hat{\mu}_{2,1}$ be the weight sum of Motzkin paths (resp. Motzkin–Schröder paths) from $(0, 0)$ to $(2, 1)$ with the weights given in Proposition 32 (resp. Proposition 34). Then $[c]\hat{\mu}_{2,1} = 11a + 6b$ (resp. $[c]\hat{\mu}_{2,1} = 7a + 6b$).

Regarding the second question, we observe that $\mu_{i,i} = \mu_{i,0,i} = \mathcal{C}^{i+1}$. This implies that it is impossible to find ordinary or type R_I orthogonal polynomials whose generalized moments $\tilde{\mu}_{n,r,s}$ are the same as $\mu_{n,r,s}$, since $\tilde{\mu}_{i,0,i} = 1$. One might consider introducing a scaling factor F such that $\mu_{n,r,s} = F \cdot \tilde{\mu}_{n,r,s}$ as we have $F = \mathcal{C}$ when $r = s = 0$. However, we note that it is not possible to interpret $\mu_{n,r,s}$ as a weight sum of Motzkin–Schröder paths. In fact, one can verify that

$$\mu_{0,1,0} = \mathcal{C}^2 \cdot \frac{a + 2bc}{\sqrt{1 - 4c}} \quad \text{and} \quad \mu_{0,2,1} = \mu_{0,1,0}(2\mathcal{C} - 1). \quad (41)$$

To interpret these using Motzkin–Schröder paths, $\mu_{0,i,i-1}$ must be the weight A_i of a vertical down step $(0, -1)$ starting at height i . This implies that $\mu_{0,2,0} = A_2 A_1 = \mu_{0,2,1} \mu_{0,1,0}$, which does not contain any λ by (41). This contradicts the fact that $\mu_{0,2,0}$ must contain a term λc , since DB is an R_{II} path from $(0, 2)$ to $(0, 0)$ whose weight is λc .

Therefore, we need to consider combinatorial models different from Motzkin–Schröder paths. In the literature, there are other lattice path models for orthogonal polynomials. For example, Jang and Song [8] provide lattice paths called gentle Motzkin paths for their study of orthogonal polynomials on the unit circle.

We end this section with the following open problem.

Problem 35. Find a lattice path model that can be used to calculate $\mu_{n,m}$ or $\mu_{n,r,s}$ in a finite process.

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