

On the minimum number of Hamilton cycles in Hamiltonian threshold graphs: a simple proof

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Abstract

A graph is called a threshold graph if it is induced- $\{P_4, C_4, 2K_2\}$ free. In 2020, Qiao and Zhan [J. Graph Theory 93 (2020) 222–229] determined the minimum number of Hamilton cycles in a Hamiltonian threshold graph and characterized the extremal graphs attaining this minimum. We show that every Hamiltonian threshold graph contains a special spanning subgraph, from which their two results follow directly.

Mathematics Subject Classifications: 05C38, 05C45

1 Introduction

We consider finite simple graphs, and use standard terminology and notations from [6, 21] throughout this article. The order of a graph is its number of vertices, and the size its number of edges. The neighborhood of a vertex x in G is denoted by $N_G(x)$, and its closed neighborhood is $N_G[x] = N_G(x) \cup \{x\}$. The degree of x is denoted by $\deg_G(x)$. The notation $|S|$ denotes the cardinality of a set S . A Hamilton cycle in G is a cycle containing every vertex of G . A graph G is Hamiltonian if it contains a Hamilton cycle. It is a natural problem to consider the number of cycles in a graph, which also has applications in coding theory according to [1, 2, 3]. Although the exact minimum number of Hamilton cycles has been determined for several specific graph classes, many cases remain open. For instance, it is a well-known fact that every Hamiltonian cubic graph contains at least three Hamilton cycles [6, p. 493]. In contrast, Sheehan's celebrated conjecture [20], stating that every Hamiltonian 4-regular graph contains at least two Hamilton cycles, remains open. The planar case has received considerable attention; for some results concerning the exact minimum number of Hamilton cycles in planar graphs, we refer the reader to [4, 13, 14, 15].

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For a graph H , we say that G is induced- H free if G has no induced subgraph isomorphic to H . Let $\mathcal{F}$ be a family of graphs. We say that G is induced- $\mathcal{F}$ free if G is induced- F free for every member F of $\mathcal{F}$.

Threshold graphs were introduced by Chvátal and Hammer in connection with the aggregation of set-packing inequalities in integer programming [7]. They form a basic hereditary graph class and are precisely the induced- $\{P_4, C_4, 2K_2\}$ -free graphs, or equivalently, the graphs that can be constructed from a single vertex by repeatedly adding either an isolated vertex or a dominating vertex [17]. Their nested-neighborhood structure makes exact structural and algorithmic analysis possible. Threshold graphs are also extremal in degree-sequence theory: their degree sequences are precisely the extreme points of the polytope of graphical degree sequences [12, 18], and a threshold graph is uniquely determined up to isomorphism by its degree sequence [17]. They also arise in synchronization, resource allocation, and scheduling [17]. More recently, Kahl [10] used graph compression to show that, for a broad family of graph parameters derived from the Tutte polynomial, extremal simple connected graphs of prescribed order and size may be chosen to be threshold graphs.

Hamiltonicity is especially natural in this class. Although the longest-cycle problem is NP-hard and deciding Hamiltonicity is NP-complete for general graphs, the nested-neighborhood structure of a threshold graph yields exact characterizations of Hamiltonicity and a linear-time algorithm for constructing a longest cycle [9, 16, 17]. Moreover, every Hamiltonian threshold graph is pancyclic [16]. Hamiltonicity also has a concrete scheduling interpretation: in Koop's cyclic off-weekend scheduling problem with ratio constraint $A = 2$, feasibility is equivalent to Hamiltonicity of an associated threshold graph [11, 17]. Recent work gives optimal certifying fully dynamic algorithms for Hamilton paths and cycles in threshold and chain graphs [5]. These structural, algorithmic, and applied features make counting Hamilton cycles in Hamiltonian threshold graphs a natural refinement of the existence problem.

The problem of determining the number of Hamilton cycles in a threshold graph was initiated by Qiao and Zhan [19], who determined the minimum number of Hamilton cycles and characterized the extremal graphs attaining this minimum. To state their result, we need to introduce a special class of threshold graphs defined as follows.

Definition 1 (Qiao-Zhan [19]). For each integer $n \geq 3$, let G_n denote the graph on n vertices whose degree sequence is

$$n-1, n-1, n-2, \dots, \left\lceil \frac{n}{2} \right\rceil, \left\lfloor \frac{n}{2} \right\rfloor, \dots, 3, 2,$$

which consists of $n-2$ distinct degrees. In particular, when $n = 3$, the displayed degree sequence is $2, 2, 2$, and hence $G_3 = K_3$.

The graph G_n was introduced by Qiao and Zhan [19]. It is well defined, uniquely determined by its degree sequence, and is a Hamiltonian threshold graph. It is readily observed that G_n is no longer a Hamiltonian threshold graph upon the deletion of any single edge. An illustration of G_8 , an example of this class of graphs, is given in Figure 1. Let us now state the result of Qiao and Zhan.

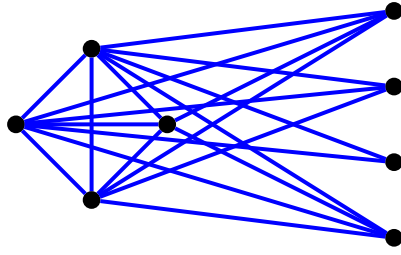


Figure 1: The threshold graph G_8 .

Theorem 2 (Qiao-Zhan [19]). *The minimum number of Hamilton cycles in a Hamiltonian threshold graph of order n is $2^{\lfloor \frac{n-3}{2} \rfloor}$, and this minimum is attained uniquely by the graph G_n .*

In addition to minimizing the number of Hamilton cycles, Qiao and Zhan also determined the minimum size among Hamiltonian threshold graphs, as follows.

Theorem 3 (Qiao-Zhan [19]). *The minimum size of a Hamiltonian threshold graph of order n is*

$$\begin{cases} \frac{n^2+2n-3}{4}, & \text{if } n \text{ is odd,} \\ \frac{n^2+2n-4}{4}, & \text{if } n \text{ is even,} \end{cases}$$

and this minimum is attained uniquely by the graph G_n .

In this paper, we give a short proof of Theorems 2 and 3. Our approach relies on the concept of graph toughness. For a graph G with $S \subseteq V(G)$, denote by $G[S]$ the subgraph of G induced by S . Let $G - S = G[V(G) - S]$. The number of components of G is denoted by $c(G)$. The toughness of a graph G , denoted by $\tau(G)$, is defined as

$$\tau(G) = \min \left\{ \frac{|S|}{c(G - S)} : S \subseteq V(G), c(G - S) \geq 2 \right\}$$

if G is not a complete graph and $\tau(K_n) = \infty$. For a positive real number t , a graph G is called t -tough if $\tau(G) \geq t$, that is, $|S| \geq t \cdot c(G - S)$ for each $S \subseteq V(G)$ with $c(G - S) \geq 2$. Clearly, every Hamiltonian graph is 1-tough. A set $S \subseteq V(G)$ is called a duplication class if $N_G(x) = N_G(y)$ for all $x, y \in S$, and a coduplication class if $N_G[x] = N_G[y]$ for all $x, y \in S$. The following lemma plays a crucial role in our proof.

Lemma 4 (Harary [8]). *The vertices of a threshold graph G can be partitioned into t coduplication classes $V_1, \dots, V_t$ and $t + 1$ duplication classes $U_0, U_1, \dots, U_t$ such that the vertices in $V_1 \cup V_2 \cup \dots \cup V_t$ form a clique, the vertices in $U_0 \cup U_1 \cup U_2 \cup \dots \cup U_t$ form an independent set and for $1 \leq i \leq t$,*

$$N(u) = V_1 \cup V_2 \cup \dots \cup V_i \text{ for any } u \in U_i,$$

$$N(v) = V_1 \cup \dots \cup (V_i \setminus \{v\}) \cup \dots \cup V_t \cup U_i \cup \dots \cup U_t \text{ for any } v \in V_i.$$

Moreover, U_i and V_i are non-empty for each i with $1 \leq i \leq t$ and G is connected if and only if U_0 is empty.

We now state the main result.

Theorem 5. *Every Hamiltonian threshold graph of order n contains G_n as a spanning subgraph.*

Proof. Let G be a Hamiltonian threshold graph. If G is complete, then the assertion is immediate. Hence, assume that G is not complete. Since G is Hamiltonian, it is 2-connected and 1-tough. By Lemma 4, $V(G)$ can be partitioned into t non-empty coduplication classes $V_1, \dots, V_t$ and t non-empty duplication classes $U_1, \dots, U_t$ which satisfy the condition in Lemma 4. Since G is 1-tough, we have $|V_1| \geq 2$ and for any i with $1 \leq i \leq t$,

$$\sum_{1 \leq j \leq i} |V_j| \geq \sum_{1 \leq j \leq i} |U_j|.$$

Let $U = U_1 \cup \dots \cup U_t$ and $V = V_1 \cup \dots \cup V_t$. By Lemma 4, every vertex of U has degree at most $|V|$, whereas every vertex of V has degree at least $|V|$. Hence, we may label $V(G) = \{v_1, v_2, \dots, v_n\}$ so that $d_i = \deg_G(v_i) \leq \deg_G(v_{i+1}) = d_{i+1}$ for all i and $U = \{v_1, \dots, v_{|U|}\}$. In particular, we have $|V| \geq |U|$.

Suppose first that $|U| < n/2$. In this case, in order to prove that G contains a spanning subgraph G_n , it suffices to show that $d_j \geq j + 1$ for all $j \leq |U|$. To the contrary, assume that $d_k \leq k \leq |U|$, with k as small as possible. As G is 2-connected, we have $d_1 \geq 2$. Thus $k \geq 2$. By the minimality of k , we have $k \geq d_k \geq d_i \geq i + 1$ for $1 \leq i \leq k - 1$. In particular, $d_k \geq d_{k-1} \geq k$, and so $d_k = k$. By Lemma 4, we have $N_G(v_i) \subseteq N_G(v_{i+1})$ for all i with $1 \leq i \leq k - 1$. Combining this with the fact that $k \leq |U| < n/2$, we have $V(G) \setminus (N_G(v_k) \cup \bigcup_{1 \leq i \leq k} \{v_i\}) \neq \emptyset$ and so

$$c(G - N_G(v_k)) \geq k + 1 > k = |N_G(v_k)|,$$

which contradicts the fact that G is 1-tough.

Suppose next that $|U| = n/2$. This implies that $|V| = |U|$ and n is even. Similar to the above argument, we also have $d_j \geq j + 1$ for all $j \leq n/2 - 1$, and so $d_{n/2} \geq d_{n/2-1} \geq n/2$. Thus G contains a spanning subgraph G_n . This completes the proof of Theorem 5. $\square$

Remark 6. Theorem 3 follows directly from Theorem 5. To derive Theorem 2, let C be a Hamilton cycle of G_n . Let $u, v \in V(G_n)$ with $uv \notin E(G_n)$, orient C , and denote the successors of u and v on C by u^+ and v^+ , respectively. Since G_n is induced- $\{P_4, C_4, 2K_2\}$ free, it follows that $u^+v^+ \in E(G_n)$. Replacing uu^+ and vv^+ in C by uv and u^+v^+ produces a Hamilton cycle of $G_n + uv$ containing uv . Together with Theorem 5, this proves Theorem 2.

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