A Robust Bollobás–Nikiforov Bound for Graphs Close to Triangle-Free Graphs

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Abstract

The Bollobás–Nikiforov (BN) conjecture, in its clique-number form, asserts that every non-complete graph G of order n with m edges and clique number $\omega(G)$ satisfies

$$\lambda_1(G)^2 + \lambda_2(G)^2 \leqslant 2\left(1 - \frac{1}{\omega(G)}\right) m,$$

where $\lambda_1(G) \geqslant \lambda_2(G) \geqslant \cdots \geqslant \lambda_n(G)$ are the adjacency eigenvalues of G. We prove an unconditional perturbative form for graphs that can be made triangle-free by deleting few edges. Let $\eta_\triangle(G)$ be the minimum number of edges whose deletion destroys all triangles of G. Every non-complete graph G on at least two vertices satisfies

$$\lambda_1(G)^2 + \lambda_2(G)^2 \leqslant 2\left(1 - \frac{1}{\omega}\right) m + C_\omega \eta_\triangle(G),$$

where $\omega = \omega(G)$ and

$$C_\omega = 2 + \frac{2}{\omega} + \frac{4\omega}{\omega - 2} \qquad (\omega \geqslant 3).$$

For $\omega = 1, 2$, no additive error is needed. Since $C_\omega \leqslant 44/3$, a uniform error term $(44/3)\eta_\triangle(G)$ is valid. The emphasis of the result is the explicit finite error that is linear in the edge-deletion distance from the triangle-free class. We also show that the vertex-disjoint triangle packing number cannot replace $\eta_\triangle(G)$ in this perturbative framework, and we clarify the relation with bounds expressed in terms of the total number of triangles.

Mathematics Subject Classifications: 05C50, 05C35

1 Introduction

Extremal graph theory asks how dense a graph can be when a prescribed subgraph is forbidden. Turán's theorem is the basic result of this type: an n-vertex graph with no K_{r+1}

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has at most $e(T_r(n))$ edges, where $T_r(n)$ is the balanced complete r-partite graph [14]. A major theme in spectral extremal graph theory is to replace edge density by eigenvalue information. This viewpoint goes back to classical work such as the Motzkin–Straus theorem, Wilf's spectral clique bound, and Nosal's triangle theorem [11, 15, 13].

Let $G = (V, E)$ be a finite simple graph of order n, let $A(G)$ be its adjacency matrix, and write

$$\lambda_1(G) \geqslant \lambda_2(G) \geqslant \cdots \geqslant \lambda_n(G)$$

for its adjacency eigenvalues. Put $m = e(G)$ and let $\omega(G)$ be the clique number. Nikiforov proved the spectral Turán inequality

$$\lambda_1(G)^2 \leqslant 2\left(1 - \frac{1}{\omega(G)}\right) m, \tag{1}$$

which is a sharp spectral analogue of Turán's theorem [12].

Bollobás and Nikiforov proposed the following two-eigenvalue strengthening [1]: if G is non-complete, then

$$\lambda_1(G)^2 + \lambda_2(G)^2 \leqslant 2\left(1 - \frac{1}{\omega(G)}\right) m. \tag{2}$$

We call (2) the Bollobás–Nikiforov inequality. It is substantially stronger than (1), since it retains the same right-hand side while adding the contribution of $\lambda_2(G)^2$. In particular, the problem is sensitive to disconnected graphs and to graphs with more than one large spectral component.

A multi-eigenvalue extension was proposed by Elphick, Linz and Wocjan [4]. Let $n^+(G)$ denote the number of positive eigenvalues of $A(G)$, counted with multiplicity, and set

$$\ell(G) := \min\{n^+(G), \omega(G)\}.$$

Their conjecture asserts that

$$\sum_{i=1}^{\ell(G)} \lambda_i(G)^2 \leqslant 2\left(1 - \frac{1}{\omega(G)}\right) m.$$

It is known for several structured graph families [4].

The ordinary BN conjecture is known in a number of important cases. Lin, Ning and Wu proved the triangle-free case and characterized equality [8]. Zhang proved the conjecture for regular graphs [17]. Giacomelli treated complete multipartite graphs and dense K_4-free graphs [5]. Liu and Bu proved that the conjecture holds asymptotically almost surely for $G(n, p)$ with fixed edge probability [9]. Zeng and Zhang established it for line graphs and for graphs satisfying a suitable total-triangle condition [16]. Coutinho, Spier and Zhang developed a conic-programming approach and obtained universal weak forms [2]. For a broader collection of related problems, see the survey of Liu and Ning [10].

The strongest base case for our purpose is the triangle-free theorem of Lin, Ning and Wu:

$$\lambda_1(H)^2 + \lambda_2(H)^2 \leqslant e(H) \tag{3}$$

for every triangle-free graph H with at least three vertices [8]. We ask for an explicit perturbative form of (3) when a graph is close to being triangle-free.

We use the following three triangle parameters:

$$\nu_{\triangle}(G) \leqslant \eta_{\triangle}(G) \leqslant t_{\triangle}(G),$$

where $\nu_{\triangle}(G)$ is the maximum number of pairwise vertex-disjoint triangles, $\eta_{\triangle}(G)$ is the minimum size of an edge set meeting every triangle, and $t_{\triangle}(G)$ is the total number of triangles. The inequalities are immediate: an edge can meet at most one triangle in a vertex-disjoint packing, and choosing one edge from each triangle gives a triangle edge-transversal.

For perturbation from the triangle-free class, $\eta_{\triangle}(G)$ is the natural parameter. If $\eta_{\triangle}(G) = q$, then G is obtained from a triangle-free graph by adding back exactly q edges. Weyl's inequality and the Frobenius norm then transfer (3) to G with an explicit linear error. By contrast, $\nu_{\triangle}(G)$ does not control this edge-deletion distance: a graph can have only one vertex-disjoint triangle and nevertheless require arbitrarily many edge deletions to become triangle-free.

1.1 Main result and scope

Definition 1 (Triangle edge-transversal and its number)**.** An edge subset $F \subseteq E(G)$ is called a triangle edge-transversal of G if every triangle of G contains at least one edge from F. For a graph G, the triangle edge-transversal number is defined as

$$\eta_{\triangle}(G) := \min\{|F| : F \subseteq E(G) \text{ is a triangle edge-transversal of } G\}.$$

Equivalently, $\eta_{\triangle}(G)$ is the minimum size of an edge set meeting every triangle of G.

Our main theorem, proved as Theorem 8, states that every non-complete graph G on at least two vertices, with $m = e(G)$ and $\omega = \omega(G)$, satisfies

$$\lambda_1(G)^2 + \lambda_2(G)^2 \leqslant 2\left(1 - \frac{1}{\omega}\right) m + C_\omega \eta_{\triangle}(G),$$

where

$$C_\omega := 2 + \frac{2}{\omega} + \frac{4\omega}{\omega - 2} \qquad (\omega \geqslant 3),$$

and $C_1 = C_2 = 0$. Since C_ω is decreasing for $\omega > 2$, one has $C_\omega \leqslant C_3 = 44/3$.

The main point is the explicit finite dependence $O(\eta_{\triangle}(G))$, with a constant independent of n and m. Thus bounded edge perturbations of triangle-free graphs incur only a bounded additive loss, and logarithmic edge perturbations incur only a logarithmic loss.

Kumar and Pragada proved that every graph with $m \geqslant 2$ edges satisfies

$$\lambda_1(G)^2 + \lambda_2(G)^2 \leqslant m + \left(3t_{\triangle}(G)\right)^{2/3}, \tag{4}$$

with strict inequality when $t_{\triangle}(G) > 0$ [7]. Their theorem and ours use different input parameters and have different functional forms. A small value of $\eta_{\triangle}(G)$ can coexist with

many highly overlapping triangles, but this parameter comparison alone does not imply that the resulting numerical upper bound is smaller than (4). In Section 3.4, we prove a quantitative relation showing that $\eta_{\triangle}(G) = o(m)$ already implies $t_{\triangle}(G) = o(m^{3/2})$. Consequently, the asymptotic consequence under $\eta_{\triangle}(G) = o(m)$ is also covered, and in fact sharpened, by (4). The distinct contribution here is therefore the explicit linear error in the edge-deletion distance.

The rest of the paper is organized as follows. Section 2 records the spectral perturbation tools. Section 3 proves the transfer lemma and the main theorem. Sections 3.1–3.3 discuss the role of $\eta_{\triangle}$, the failure of packing-based deletion arguments, and representative examples. Section 3.4 gives a precise comparison with the total-triangle parameter. Section 4 states two open problems.

2 Preliminaries

Set

$$s_2(G) := \lambda_1(G)^2 + \lambda_2(G)^2.$$

Whenever this notation is used, the graph has at least two vertices.

Theorem 2 (Nikiforov [12])**.** *Let G be a graph with m edges and clique number ω. Then*

$$\lambda_1(G)^2 \leqslant 2\left(1 - \frac{1}{\omega}\right) m.$$

Theorem 3 (Lin–Ning–Wu [8])**.** *If H is triangle-free and has at least three vertices, then*

$$\lambda_1(H)^2 + \lambda_2(H)^2 \leqslant e(H).$$

The lower-order exception K_2 is the reason for the vertex assumption: its eigenvalues are $1, -1$, so $s_2(K_2) = 2 > 1 = e(K_2)$. This causes no difficulty below. If $\omega(G) \geqslant 3$, the triangle-free auxiliary graph has the same vertex set as G and therefore at least three vertices. If $\omega(G) = 2$ and G is non-complete, then G also has at least three vertices.

Lemma 4 (Frobenius bound for edge perturbations [6])**.** *Let G and H be graphs on the same vertex set whose edge sets differ in exactly r edges. Then*

$$\|A(G) - A(H)\|_2 \leqslant \|A(G) - A(H)\|_F = \sqrt{2r}.$$

Lemma 5 (Weyl's inequality [6])**.** *Let M, N be real symmetric $n \times n$ matrices, with eigenvalues arranged in nonincreasing order. Then*

$$|\lambda_i(M) - \lambda_i(N)| \leqslant \|M - N\|_2$$

for every i.

Lemma 6 (Two-eigenvalue perturbation estimate)**.** *Suppose that G is obtained from H by adding r edges. Then*

$$s_2(G) \leqslant s_2(H) + 2\sqrt{2r}\Big(|\lambda_1(H)| + |\lambda_2(H)|\Big) + 4r. \tag{5}$$

In particular, if $s_2(H) \leqslant \beta e(H)$ for some $\beta > 0$, then

$$s_2(G) \leqslant \beta e(H) + 4\sqrt{\beta r e(H)} + 4r. \tag{6}$$

Proof. Set $E = A(G) - A(H)$. By Lemma 4, $\|E\|_2 \leqslant \sqrt{2r}$. Lemma 5 gives

$$|\lambda_i(G) - \lambda_i(H)| \leqslant \sqrt{2r}, \qquad i = 1, 2.$$

Hence

$$|\lambda_i(G)| \leqslant |\lambda_i(H)| + \sqrt{2r}, \qquad i = 1, 2.$$

Squaring and summing yields (5). If $s_2(H) \leqslant \beta e(H)$, then Cauchy–Schwarz gives

$$|\lambda_1(H)| + |\lambda_2(H)| \leqslant \sqrt{2s_2(H)} \leqslant \sqrt{2\beta e(H)}.$$

Substitution into (5) gives (6). □

3 A transfer lemma and the main theorem

The following lemma is useful beyond the triangle-free setting. It retains the exact identity $e(H) = e(G) - r$ rather than replacing $e(H)$ by $e(G)$ prematurely.

Lemma 7 (Robust transfer lemma)**.** *Let G be obtained from a graph H by adding r edges. Assume that*

$$s_2(H) \leqslant \beta e(H)$$

for some $\beta > 0$. If $\alpha > \beta$, then

$$s_2(G) \leqslant \alpha e(G) + \left(4 + \frac{4\beta}{\alpha - \beta} - \alpha\right) r. \tag{7}$$

Proof. By Lemma 6,

$$s_2(G) \leqslant \beta e(H) + 4\sqrt{\beta r e(H)} + 4r.$$

Put $c = \alpha - \beta > 0$. For every $x \geqslant 0$,

$$4\sqrt{\beta r x} \leqslant cx + \frac{4\beta}{c} r, \tag{8}$$

indeed,

$$cx + \frac{4\beta}{c} r - 4\sqrt{\beta r x} = \left(\sqrt{cx} - 2\sqrt{\frac{\beta r}{c}}\right)^2.$$

Applying (8) with $x = e(H) = e(G) - r$ gives

$$\begin{aligned} s_2(G) &\leqslant \beta(e(G) - r) + c(e(G) - r) + \frac{4\beta}{c} r + 4r \\ &= \alpha e(G) + \left(4 + \frac{4\beta}{\alpha - \beta} - \alpha\right) r. \end{aligned}$$

□

Theorem 8 (Explicit robust BN theorem)**.** *Let G be a non-complete graph on at least two vertices, with $m = e(G)$ and clique number $\omega = \omega(G)$. Then*

$$s_2(G) \leqslant 2\left(1 - \frac{1}{\omega}\right) m + C_\omega \eta_\triangle(G), \tag{9}$$

where

$$C_\omega := 2 + \frac{2}{\omega} + \frac{4\omega}{\omega - 2} \qquad (\omega \geqslant 3),$$

and $C_1 = C_2 = 0$. Consequently,

$$s_2(G) \leqslant 2\left(1 - \frac{1}{\omega}\right) m + \frac{44}{3} \eta_\triangle(G). \tag{10}$$

Proof. Let $r = \eta_\triangle(G)$, and choose $F \subseteq E(G)$ with $|F| = r$ such that $H = G - F$ is triangle-free. Then G is obtained from H by adding r edges.

Suppose first that $\omega \geqslant 3$. The graph H has the same vertex set as G, so it has at least three vertices. By Theorem 3,

$$s_2(H) \leqslant e(H).$$

Apply Lemma 7 with

$$\beta = 1, \qquad \alpha = 2\left(1 - \frac{1}{\omega}\right).$$

Since

$$\alpha - \beta = 1 - \frac{2}{\omega} > 0,$$

we obtain

$$\begin{aligned} s_2(G) &\leqslant 2\left(1 - \frac{1}{\omega}\right) m + \left(4 + \frac{4}{1 - 2/\omega} - 2\left(1 - \frac{1}{\omega}\right)\right) r \\ &= 2\left(1 - \frac{1}{\omega}\right) m + \left(2 + \frac{2}{\omega} + \frac{4\omega}{\omega - 2}\right) r. \end{aligned}$$

This proves (9) for $\omega \geqslant 3$.

If $\omega = 2$, then G is triangle-free and Theorem 3 gives (9) with $C_2 = 0$. If $\omega = 1$, then G has no edges and $s_2(G) = 0$.

Finally,

$$\frac{d}{dx}\left(2 + \frac{2}{x} + \frac{4x}{x - 2}\right) = -\frac{2}{x^2} - \frac{8}{(x - 2)^2} < 0 \qquad (x > 2).$$

Thus C_ω is decreasing for $\omega > 2$, and

$$C_\omega \leqslant C_3 = 2 + \frac{2}{3} + 12 = \frac{44}{3}.$$

This proves (10). □

Remark 9 (On the constant). The coefficient 44/3 arises from the scalar absorption step (8). That inequality is sharp as an inequality in r and x, but the graph-theoretic constraints may prevent equality throughout the full argument. Hence the coefficient is explicit, not claimed to be optimal. For example,

$$C_4 = 2 + \frac{1}{2} + 8 = \frac{21}{2}, \qquad \lim_{\omega\to\infty} C_\omega = 6.$$

Corollary 10 (Fixed edge-transversal number)**.** *For every fixed integer $q \geqslant 0$, every non-complete graph G on at least two vertices with $\eta_\triangle(G) \leqslant q$ satisfies*

$$s_2(G) \leqslant 2\left(1 - \frac{1}{\omega(G)}\right) e(G) + \frac{44}{3}q.$$

Remark 11 (A fixed-k extension). For a graph H and an integer $2 \leqslant k \leqslant |V(H)|$, define

$$s_k(H) := \sum_{i=1}^{k} \lambda_i(H)^2.$$

When $k = 2$, this is exactly $s_2(H)$. Suppose a graph class satisfies $s_k(H) \leqslant \beta e(H)$ for a fixed $\beta > 0$. If G is obtained from such an H by adding r edges, then for every $\alpha > \beta$,

$$s_k(G) \leqslant \alpha e(G) + \left(2k + \frac{2k\beta}{\alpha - \beta} - \alpha\right) r. \tag{11}$$

Indeed, Weyl's inequality gives

$$s_k(G) \leqslant s_k(H) + 2\sqrt{2r} \sum_{i=1}^{k} |\lambda_i(H)| + 2kr.$$

Cauchy–Schwarz yields

$$\sum_{i=1}^{k} |\lambda_i(H)| \leqslant \sqrt{k s_k(H)} \leqslant \sqrt{k\beta e(H)},$$

so

$$s_k(G) \leqslant \beta e(H) + 2\sqrt{2k\beta r e(H)} + 2kr.$$

With $c = \alpha - \beta$,

$$2\sqrt{2k\beta r x} \leqslant cx + \frac{2k\beta}{c} r,$$

and (11) follows after setting $x = e(H) = e(G) - r$.

Remark 12 (Limitation for the generalized BN conjecture). Remark 11 is a fixed-k statement. It does not prove a robust version of the generalized BN conjecture, because

$$\ell(G) = \min\{n^+(G), \omega(G)\}$$

may increase when edges are added. A fixed-k estimate for the base graph controls only the first k eigenvalues of the perturbed graph, not necessarily all eigenvalues appearing in the generalized conjecture for G.

3.1 Why vertex-disjoint triangles are not enough

Let

$$\nu_\triangle(G) := \max\{\text{number of pairwise vertex-disjoint triangles in } G\}.$$

Proposition 13 (A basic false step)**.** *For every integer $t \geqslant 1$, there exists a graph G with $\nu_\triangle(G) \leqslant t$ such that more than t edge deletions are needed to destroy all its triangles.*

Proof. Fix $t \geqslant 1$ and let G be the disjoint union of t copies of K_4. Each copy has triangle packing number 1. Deleting any one edge of K_4 leaves triangles, whereas deleting two nonadjacent edges destroys all triangles. Thus $\nu_\triangle(G) = t$ and $\eta_\triangle(G) = 2t > t$. □

Proposition 14 (Unbounded gap between $\boldsymbol{\nu_\triangle}$ and $\boldsymbol{\eta_\triangle}$)**.** *For every integer $k \geqslant 2$, let W_k be obtained from $K_{k,k}$, with bipartition $X \cup Y$, by adding a vertex z adjacent to every vertex of $X \cup Y$. Then*

$$\nu_\triangle(W_k) = 1, \qquad \eta_\triangle(W_k) = k.$$

Proof. Every triangle has the form zxy with $x \in X$ and $y \in Y$. Thus all triangles contain z, and hence $\nu_\triangle(W_k) = 1$.

Deleting all k edges from z to X destroys every triangle, so $\eta_\triangle(W_k) \leqslant k$. Conversely, let F be any edge set meeting every triangle. Put

$$A := \{x \in X : zx \in F\}, \qquad B := \{y \in Y : zy \in F\},$$

and write $a = |A|$, $b = |B|$. If $x \in X \setminus A$ and $y \in Y \setminus B$, then both zx and zy remain, so the edge xy must lie in F. Therefore

$$|F| \geqslant a + b + (k - a)(k - b).$$

Set $\alpha = k - a$ and $\beta = k - b$. If $\alpha = 0$ or $\beta = 0$, the right-hand side is at least k. If $\alpha, \beta \geqslant 1$, then

$$2k - \alpha - \beta + \alpha\beta = k + \Big((\alpha - 1)(\beta - 1) + k - 1\Big) \geqslant k.$$

Thus every triangle edge-transversal has size at least k. □

Remark 15. Proposition 14 does not rule out a BN-type inequality whose error depends only on $\nu_\triangle(G)$. It shows only that the present edge-deletion perturbation method cannot yield such an estimate with an error controlled solely by $\nu_\triangle(G)$.

3.2 Graph classes covered by the transfer theorem

Theorem 8 applies directly to any graph obtained from a triangle-free graph by adding a small number of edges. In particular:

- If G is obtained from a triangle-free graph by adding at most q edges, then $\eta_{\triangle}(G) \leqslant q$ and the additive error is at most $(44/3)q$.
- If G is obtained from $K_{a,b}$ by adding a set R of internal edges inside one or both parts, then $\eta_{\triangle}(G) \leqslant |R|$.
- If $\eta_{\triangle}(G) = O(1)$ and $e(G) = \Theta(n^2)$, then the theorem gives the BN right-hand side up to an additive constant.
- If $\eta_{\triangle}(G) = O(\log m)$, then the additive loss is $O(\log m)$.

The condition is genuinely different from a packing condition. For example, W_k has $\nu_{\triangle}(W_k) = 1$ but $\eta_{\triangle}(W_k) = k$. Similarly, bounded degeneracy or bounded tree-depth does not force $\eta_{\triangle}$ to be bounded independently of the number of vertices; disjoint unions of triangles already show this.

3.3 Examples and spectral checks

The cone over a complete bipartite graph. Consider $W_k = K_1 \vee K_{k,k}$ from Proposition 14. It has

$$m = k^2 + 2k, \qquad \omega(W_k) = 3, \qquad \eta_{\triangle}(W_k) = k.$$

By the standard theory of equitable partitions [3], the partition $\{z\}, X, Y$ is equitable, with quotient matrix

$$Q = \begin{pmatrix} 0 & k & k \\ 1 & 0 & k \\ 1 & k & 0 \end{pmatrix}.$$

The vector $(0, 1, -1)^T$ is a $(-k)$-eigenvector of Q, and the two remaining quotient eigenvalues are the roots of

$$\lambda^2 - k\lambda - 2k = 0.$$

To identify the full spectrum, observe that every vector supported on X whose coordinates sum to zero is a 0-eigenvector of $A(W_k)$, and the same is true for Y. These two spaces have total dimension $2k - 2$. Hence

$$\mathrm{Spec}(W_k) = \left\{ \frac{k + \sqrt{k^2 + 8k}}{2}, 0^{(2k-2)}, \frac{k - \sqrt{k^2 + 8k}}{2}, -k \right\}.$$

In particular,

$$\lambda_2(W_k) = 0, \qquad s_2(W_k) = \left(\frac{k + \sqrt{k^2 + 8k}}{2} \right)^2.$$

Theorem 8 gives

$$s_2(W_k) \leqslant \frac{4}{3}(k^2+2k)+\frac{44}{3}k.$$

For $k=2$, the exact value is approximately 10.472, while the displayed bound is 40; for $k=3$, the exact value is approximately 19.117, while the bound is 64. Thus this family is far from extremal for the constant.

Adding one internal edge to $K_{k,k}$. For $k \geqslant 3$, let B_k be obtained from $K_{k,k}$ by adding one edge inside one part. Then

$$\eta_{\triangle}(B_k)=1, \qquad \omega(B_k)=3,$$

and Theorem 8 gives

$$s_2(B_k) \leqslant \frac{4}{3}(k^2+1)+\frac{44}{3}.$$

Let A be the two endpoints of the added edge, let B be the remaining $k-2$ vertices in the same part, and let C be the opposite part. This is an equitable partition with quotient matrix

$$Q_k = \begin{pmatrix} 1 & 0 & k \\ 0 & 0 & k \\ 2 & k-2 & 0 \end{pmatrix},$$

whose characteristic polynomial is

$$p_k(\lambda) = \lambda^3 - \lambda^2 - k^2\lambda + k^2 - 2k.$$

The difference vector on the two vertices of A is a (-1)-eigenvector. Vectors supported on B with coordinate sum zero and vectors supported on C with coordinate sum zero are 0-eigenvectors, of total dimension

$$(k-3)+(k-1)=2k-4.$$

Thus the full spectrum consists of the three roots of p_k, together with -1 and $0^{(2k-4)}$.

Moreover,

$$p_k(k) = -2k < 0,$$

whereas

$$p_k\left(k+\frac{2}{k}\right) = 2k-4+\frac{12}{k}-\frac{4}{k^2}+\frac{8}{k^3} > 0 \qquad (k \geqslant 3).$$

Hence the largest root lies in $(k, k+2/k)$. Also,

$$p_k(1) = -2k < 0,$$

while

$$p_k\left(1-\frac{3}{k}\right) = k-\frac{3}{k}+\frac{18}{k^2}-\frac{27}{k^3} > 0 \qquad (k \geqslant 3).$$

Thus a second positive root lies in $(1-3/k, 1)$. Since the product of the three quotient roots is $-k(k-2) < 0$, the remaining quotient root is negative. Therefore

$$\lambda_1(B_k) = k + O(k^{-1}), \qquad \lambda_2(B_k) = 1 + O(k^{-1}),$$

and

$$s_2(B_k) = k^2 + O(1).$$

Disjoint triangle-free components. If G is the disjoint union of two copies of $K_{k,k}$, then G is triangle-free, $m = 2k^2$, and the two largest eigenvalues are both k. Hence

$$s_2(G) = 2k^2 = m.$$

This is an equality example for the triangle-free theorem and illustrates why the second eigenvalue is essential.

3.4 Comparison with the total number of triangles

The elementary relations

$$\nu_{\triangle}(G) \leqslant \eta_{\triangle}(G) \leqslant t_{\triangle}(G)$$

are supplemented by

$$t_{\triangle}(G) \leqslant \eta_{\triangle}(G)\,\Delta(G), \tag{12}$$

where $\Delta(G)$ is the maximum degree. Indeed, every triangle contains an edge of a minimum triangle edge-transversal, and each fixed edge lies in at most $\Delta(G)$ triangles. Let F be a minimum triangle edge-transversal of G. Summing the number of triangles incident to each edge of F, we have

$$t_{\triangle}(G) \leqslant \sum_{e\in F} \#\{\text{triangles containing } e\} \leqslant |F| \cdot \Delta(G) = \eta_{\triangle}(G)\Delta(G).$$

Consequently, when $\Delta(G) = O(1)$, the parameters $\eta_{\triangle}(G)$ and $t_{\triangle}(G)$ are comparable up to a constant factor. This statement concerns the parameters, not the numerical strength of the two spectral bounds, whose main terms and error exponents are different.

A stronger relation is useful for asymptotic comparison.

Proposition 16 (Bounding triangles by the edge-transversal number)**.** *Let G have m edges, put $q = \eta_{\triangle}(G)$, and set $h = m - q$. Then*

$$t_{\triangle}(G) \leqslant h\sqrt{2q} + q\sqrt{2h} + \frac{\sqrt{2}}{3}q^{3/2}. \tag{13}$$

In particular, if $q = o(m)$, then $t_{\triangle}(G) = o(m^{3/2})$.

Proof. Choose a minimum triangle edge-transversal $F \subseteq E(G)$ with $|F| = q$, and put $H = G - F$. Write $F^* = (V(G), F)$ for the spanning graph with edge set F. Let

$$B = A(H), \qquad C = A(F^*).$$

Since H is triangle-free,

$$\operatorname{tr}(B^3) = 0.$$

Using cyclicity of the trace,

$$6t_{\triangle}(G) = \operatorname{tr}\Big((B+C)^3\Big) = 3\operatorname{tr}(B^2C) + 3\operatorname{tr}(BC^2) + \operatorname{tr}(C^3).$$

By the Frobenius inner-product inequality,

$$\operatorname{tr}(B^2C) \leqslant \|B^2\|_F \|C\|_F.$$

Now

$$\|C\|_F = \sqrt{2q}, \qquad \|B^2\|_F^2 = \operatorname{tr}(B^4) \leqslant \Big(\operatorname{tr}(B^2)\Big)^2 = 4h^2,$$

so

$$\frac{1}{2}\operatorname{tr}(B^2C) \leqslant h\sqrt{2q}.$$

Similarly,

$$\|B\|_F = \sqrt{2h}, \qquad \|C^2\|_F^2 = \operatorname{tr}(C^4) \leqslant \Big(\operatorname{tr}(C^2)\Big)^2 = 4q^2,$$

which gives

$$\frac{1}{2}\operatorname{tr}(BC^2) \leqslant q\sqrt{2h}.$$

Finally,

$$6t_{\triangle}(F^*) = \operatorname{tr}(C^3) \leqslant \sum_i |\lambda_i(C)|^3 \leqslant \left(\sum_i \lambda_i(C)^2\right)^{3/2} = (2q)^{3/2},$$

where the second inequality follows from the monotonicity of ℓ_p norms. So

$$t_{\triangle}(F^*) \leqslant \frac{\sqrt{2}}{3} q^{3/2}.$$

Combining these estimates proves (13). If $q = o(m)$, then $h \leqslant m$ and each term on the right-hand side is $o(m^{3/2})$. □

Remark 17 (Asymptotic consequence). Proposition 16 and the Kumar–Pragada bound (4) imply

$$\eta_{\triangle}(G) = o(m) \quad \Longrightarrow \quad s_2(G) \leqslant m + o(m).$$

For $\omega(G) \geqslant 3$, this is asymptotically stronger than the direct consequence of Theorem 8. Accordingly, Theorem 8 should be viewed as a finite explicit perturbation theorem with an error linear in $\eta_{\triangle}(G)$, rather than as an asymptotic improvement over total-triangle estimates.

For the cone graph W_k, the corrected parameter values are

$$m = k^2 + 2k, \qquad \eta_{\triangle}(W_k) = k, \qquad t_{\triangle}(W_k) = k^2, \qquad \Delta(W_k) = 2k.$$

Here $\eta_{\triangle}(W_k) = o(m)$ while $t_{\triangle}(W_k) = \Theta(m)$. This shows that the two parameters can have very different magnitudes, although the total-triangle spectral bound is numerically stronger on this particular family. For a disjoint union of t copies of C_3, one has

$$m = 3t, \qquad \eta_{\triangle} = t, \qquad t_{\triangle} = t, \qquad \Delta = 2,$$

so the two triangle parameters have the same order.

4 Open problems

Problem 18 (Improve the constant)**.** Determine the least constant C^*_ω such that every non-complete graph G with clique number $\omega \geqslant 3$ satisfies

$$s_2(G) \leqslant 2\left(1 - \frac{1}{\omega}\right) e(G) + C^*_\omega \eta_\triangle(G).$$

The present argument gives

$$C^*_\omega \leqslant 2 + \frac{2}{\omega} + \frac{4\omega}{\omega - 2},$$

and in particular $C^*_3 \leqslant 44/3$.

Problem 19 (Packing version)**.** Does there exist a nondecreasing function $f : \mathbb{Z}_{\geqslant 0} \longrightarrow \mathbb{R}_{\geqslant 0}$ such that every non-complete graph G satisfies

$$s_2(G) \leqslant 2\left(1 - \frac{1}{\omega(G)}\right) e(G) + f(\nu_\triangle(G))?$$

Proposition 14 shows that such a statement, if true, cannot follow from a direct reduction to a triangle-free graph by deleting $O(\nu_\triangle(G))$ edges.

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