On Edge-Primitive 2-Arc-Transitive (Bi-)Cayley Graphs

Hao Nan Chen[a] Jing Jian Li[a] Ji Cheng Ma[b] Hao Yu[a]

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Abstract

A graph is called edge-primitive if its full automorphism group acts primitively on its edge set. In this paper, we give a complete classification of all types of finite edge-primitive 2-arc-transitive Cayley graphs and bi-Cayley graphs.

Mathematics Subject Classifications: 05C25, 05C45

1 Introduction

All graphs considered in this paper are assumed to be finite, simple, connected and undirected. An edge-transitive graph is called edge-primitive if some (and hence every) edge-stabilizer, the subgroup of the full automorphism group which fixes a given edge, is a maximal subgroup of the automorphism group.

The interest for edge-primitive graphs arises partially from the fact that many (almost) simple groups may be represented as the automorphism groups of edge-primitive graphs. For example, the sporadic Higman-Sims group HS is a group of automorphisms of a rank 3 graph (i.e., HS acts on the vertex set as a transitive permutation group of rank 3) with order 100 and valency 22, which is in fact an edge-primitive and 2-arc-transitive graph with full automorphism group HS.C_2. Many famous graphs are edge-primitive, for example, the Heawood graph, the Tutte-Coxeter graph and the Higman-Sims graph. It is well-known that edge-transitive graphs are either bipartite or vertex-transitive, and so are edge-primitive graphs. As a subclass of the edge-transitive graphs, edge-primitive graphs posses more restrictions on their symmetries and automorphism groups. By [8], a connected G-edge-primitive graph is either G-arc-transitive or a star graph, provided that it is not a cycle. Moreover, appealing to the O'Nan-Scott Theorem for (quasi)primitive groups, Giudici and Li investigated the structural properties of edge-primitive graphs, particularly, on their automorphism groups, in [8].

[a]School of Mathematics, Center for Applied Mathematics of Guangxi, Guangxi University, Nanning, Guangxi, 530004, P. R. China (hnchen2022@163.com, lijjhx@gxu.edu.cn, haoyu@gxu.edu.cn).
[b]Chongqing Key Lab. of Group & Graph Theories and Applications, Chongqing University of Arts and Sciences, Chongqing 402160, P. R. China (ma_jicheng@hotmail.com).

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An $(\mathrm{Aut}\Gamma, s)$-arc-transitive graph Γ (simply called an *s-arc-transitive* graph) means that the full automorphism graph $\mathrm{Aut}\Gamma$ acts transitively on all s-arcs. As for s-arc-transitive graphs, Li, etc. [15] studied s-arc-transitive graphs with $s \geqslant 2$, and gave a characterization of the actions of vertex-transitive normal subgroups. Finite 2-transitive permutation groups are classified, as a consequence of the classification of finite simple groups, and Li, etc. gave the structural information of their point stabilizers.

Lu [25] proved that a finite edge-primitive 2-arc-transitive graph has almost simple automorphism group if it is neither a cycle nor a complete bipartite graph. Furthermore, Lu also presented two examples of such graphs, which are 3-arc-transitive and have faithful vertex-stabilizers. With these results, we are curious to find a classification of edge-primitive and 2-arc-transitive Cayley graphs and bi-Cayley graphs, where the (bi-)Cayley graph means its full automorphism group contains a subgroup acting regularly (resp. semiregularly and having two same orbits) on its vertex set.

In this paper, we aim to determine all the Cayley graphs and bi-Cayley graphs under the assumption that they are edge-primitive and 2-arc-transitive. The main result of this paper is stated as follows.

Theorem 1. *Let Γ be an edge-primitive 2-arc-transitive Cayley graph or bi-Cayley graph with valency no less than three. Then Γ is isomorphic to either a complete graph K_n for $n \geqslant 5$, the Higman-Sims graph, a complete bipartite graph $\mathsf{K}_{m,m}$ for $m \geqslant 3$ integer, the Heawood graph, the point-line non-incidence graph, or a graph given in Examples 14–18.*

2 Preliminaries

For the group-theoretic terminology not defined here we refer the reader to [5, 32]. For two groups N and F, denoted by $N.F$ an arbitrary extension of N by F. For convenience, we denote by $[n]$ an (unspecified) group of order n.

Let Γ be a graph. The vertex set, edge set and full automorphism group of Γ will be denoted by $V\Gamma$, $E\Gamma$ and $\mathrm{Aut}\Gamma$, respectively. For $v \in V\Gamma$, the set $\Gamma(v) = \{u \in V | \{u, v\} \in E\Gamma\}$ is called the neighborhood of v in Γ, while $|\Gamma(v)|$ is called the valency of v. Note that the group $\mathrm{Aut}\Gamma$ has a natural action on the edge set $E\Gamma$. The graph Γ is called edge-transitive if $E\Gamma$ is not empty and for each pair of edges there exists some $g \in \mathrm{Aut}\Gamma$ mapping one of the two edges to another. Similarly, we may define vertex-transitive, arc-transitive and s-arc-transitive graphs. For a non-negative integer s, an $(s+1)$-tuple $(\alpha_0, \alpha_1, \cdots, \alpha_s)$ of vertices is called an *s-arc* if $\{\alpha_{i-1}, \alpha_i\} \in E\Gamma$ for $1 \leqslant i \leqslant s$ and $\alpha_{i-1} \neq \alpha_{i+1}$ for $1 \leqslant i \leqslant s-1$. The graph Γ is called *(G, s)-arc-transitive* if Γ has at least one s-arc, and $G \leqslant \mathrm{Aut}\Gamma$ is transitive on both of $V\Gamma$ and the set of s-arcs of Γ. Note that an $(G, 0)$-arc-transitive graph is just a *G-vertex-transitive* graph. In particular, an $(\mathrm{Aut}\Gamma, s)$-arc-transitive graph Γ will be simply called an *s-arc-transitive* graph.

A graph Γ is said to be a *cover* of a graph Σ, if there exists a surjection $\phi : V\Gamma \to V\Sigma$ which preserves adjacency and is a bijection from the neighbors of v to the neighbors of v^ϕ for each vertex v of Γ. Suppose that Γ is a (G, s)-arc-transitive graph with $s \geqslant 2$, and G has a normal subgroup N which has at least three orbits on the vertex set $V\Gamma$. Then

N induces a graph Γ_N, where $V\Gamma_N$ is the set of N-orbits on $V\Gamma$ and $E\Gamma_N$ is the set of N-orbits on $E\Gamma$, called the *normal quotient* of Γ induced by N. In this context, we call Γ a *normal cover* of Γ_N, and call Γ a *proper normal cover* of Γ_N if in addition $N \neq 1$.

Let Γ be a connected G-arc-transitive graph and Σ a G-invariant partition of $V\Gamma$. We define the *quotient graph* Γ_Σ to be the graph with vertex set Σ such that $U, W \in \Sigma$ are adjacent if and only if Γ has an edge $\{u, v\}$ with $u \in U$ and $v \in W$. It implies that Γ_Σ is arc-transitive. Further, if for an arc (U, W) of Γ_Σ, there is only one arc (u, v) of Γ with $u \in U$ and $v \in W$, then Γ is called a spread of Γ_Σ. Let $N \trianglelefteq G$ and Σ be an orbit of N. Then Γ_Σ is called the normal quotient of Γ denoted by Γ_N, and further, a spread of Γ_N is called a normal cover of Γ_N.

Let G be a finite group. For a subset S of G such that S does not contain the identity 1 of G and $S = S^{-1} := \{s^{-1} | s \in S\}$, the Cayley graph $\Gamma = \mathrm{Cay}(G, S)$ of G with respect to S is the graph with vertex set $V\Gamma = G$ and edge set $E\Gamma = \{\{g, sg\} | g \in G, s \in S\}$. Let $X = \mathrm{Aut}\Gamma$, and X_α be the point-stabilizer of $\alpha \in G$. It is familiar to us that $X = G_R X_\alpha$, where the group $G_R = \{\rho_g | g \in G\}$ of right translations $\rho_g : x \mapsto xg$ is a subgroup of the full automorphism group $\mathrm{Aut}\Gamma$ and acts regularly on $V\Gamma$. Without causing confusion, we also identify G with G_R. Given a group H and a subset S of H, the bi-Cayley graph over H with respect to S, denoted by $\mathrm{BiCay}(H, S)$, is defined as the bipartite graph with vertex set $H \times \{0, 1\}$ and edge set $\{\{(h, 0), (sh, 1)\} \mid h \in H, s \in S\}$.

Let Γ be a connected graph, $\alpha \in V\Gamma$, $G \leqslant \mathrm{Aut}\Gamma$ and $\{\alpha, \beta\} \in E\Gamma$. The point-stabilizer of α, the edge-stabilizer of $\{\alpha, \beta\}$ and the arc-stabilizer of (α, β) are denoted by G_α, $G_{\{\alpha,\beta\}}$ and $G_{\alpha\beta}$ respectively. Denote by $G_\alpha^{\Gamma(\alpha)}$ the permutation group induced by G_α on $\Gamma(\alpha)$ and $G_\alpha^{[1]}$ the kernel of G_α acting on $\Gamma(\alpha)$.

Assume that G is 2-arc-transitive on Γ. Then $G_\alpha^{\Gamma(\alpha)}$ is a 2-transitive group. In particular, the socle $\mathrm{soc}(G_\alpha^{\Gamma(\alpha)})$ is either a nonabelian simple group or an elementary abelian group of prime power order. Note that the arc-stabilizer $G_{\alpha\beta}$ acts transitively on $\Gamma(\beta)\backslash\{\alpha\}$ with kernel $G_\beta^{[1]}$. Since $G_\alpha^{[1]}$ is normal in $G_{\alpha\beta}$, we have

$$(G_\alpha^{[1]})^{\Gamma(\beta)} \trianglelefteq G_{\alpha\beta}^{\Gamma(\beta)} \cong G_{\alpha\beta}/G_\beta^{[1]}. \tag{1}$$

Let $G_{\alpha\beta}^{[1]} := G_\alpha^{[1]} \cap G_\beta^{[1]}$. Writing G_α in group extensions, we have

$$G_\alpha = G_{\alpha\beta}^{[1]}.(G_\alpha^{[1]})^{\Gamma(\beta)}.G_\alpha^{\Gamma(\alpha)}. \tag{2}$$

The next lemma is straightforward (see, for example, [21, Lemma 2.1]).

Lemma 2. *Let Γ be a k-valent $(X, 2)$-arc-transitive graph with $X \leqslant \mathrm{Aut}(\Gamma)$ and let $\alpha \in V\Gamma$. If X_α is almost simple, then $X_\alpha \cong X_\alpha^{\Gamma(\alpha)}$.*

The next result on 2-arc-transitive graphs is formulated from [28, 31, 30].

Lemma 3. *Let Γ be a connected graph of valency $k \geqslant 3$, and G be a 2-arc-transitive group of Γ. Assume that $G_{\alpha\beta}^{[1]} \neq 1$ for $\{\alpha, \beta\} \in E\Gamma$. Then $G_{\alpha\beta}^{[1]}$ is a p-group for some prime p, $G_{\alpha\beta}^{\Gamma(\alpha)}$ is a p-local, $\mathrm{PSL}_d(p^f) \trianglelefteq G_\alpha^{\Gamma(\alpha)}$, $k = \frac{p^{fd}-1}{p^f-1}$, and either $d \geqslant 3$ or one of the following holds:*

(1) $G_\alpha = [p^{2f}] : (\mathrm{C}_c.\mathrm{PGL}_2(p^f)).O$, *where* $c = \frac{p^f-1}{(3,p^f-1)}$ *and* O *has order a divisor of* $(3, p^f - 1)f$;

(2) $p = 2$ *and* $G_\alpha = [2^{3f}] : \mathrm{GL}_2(2^f).\mathrm{C}_e$, *where* e *is a divisor of* f;

(3) $p = 3$ *and* $G_\alpha = [3^{5f}] : \mathrm{GL}_2(3^f).\mathrm{C}_e$, *where* e *is a divisor of* f;

In particular, G_α *is solvable if and only if* $(k, d, p^f) = (3, 2, 2)$ *or* $(4, 2, 3)$.

Let H be a subgroup of a finite group G such that H is core-free, that is $\cap_{g\in G} H^g = 1$. For $x \in G\backslash H$, the *coset graph* $\mathrm{Cos}(G, H, H\{x, x^{-1}\}H)$ is defined on $[G : H]$, the set of right cosets of H in G, such that Hg_1 and Hg_2 are adjacent whenever $g_2g_1^{-1} \in H\{x, x^{-1}\}H$. In the case where $HxH = Hx^{-1}H$, we denote this coset graph by $\mathrm{Cos}(G, H, x)$. Note that G may be viewed as a subgroup of $\mathrm{AutCos}(G, H, H\{x, x^{-1}\}H)$, where G acts on $[G : H]$ by right multiplication. In particular, $\mathrm{Cos}(G, H, x)$ is a Cayley graph whenever $H = 1$.

Further, we have the following result (see [19, Lemma 3.2] for example).

Lemma 4. *Let* Γ *be a graph with* $E\Gamma \neq \varnothing$ *and* $G \leqslant \mathrm{Aut}\Gamma$. *If* Γ *is* G*-arc-transitive, then for* $\{u, v\} \in E\Gamma$ *and* $x \in G$ *with* $(u, v)^x = (v, u)$, *we have* $\Gamma \cong \mathrm{Cos}(G, G_u, x)$; *moreover, such an* x *may be chosen as a* 2*-element in* $\mathbf{N}_G(G_{uv}) \setminus \mathbf{N}_G(G_u)$ *and* $\langle x, G_u\rangle = G$.

For the exact factorization, we have the following result.

Lemma 5. *[2, Thoerem 3] Let* G *be an almost simple group with socle* G_0 *and* $G = HK$ *such that* H *is solvable,* K *is core-free and* $H \cap K = 1$. *Then one of the following holds:*

(i) $G_0 = \mathrm{A}_n$, H *is transitive on* $\{1, \cdots, n\}$ *and* $K = \mathrm{A}_{n-1}$ *or* S_{n-1}.

(ii) $G_0 = \mathrm{A}_n$, $n = p^a$ *is a prime power,* $H \leqslant \mathrm{A\Gamma L}_1(p^a)$ *is* 2*-homogeneous on* $\{1, \cdots, n\}$ *and* $\mathrm{A}_{n-2} \trianglelefteq K \leqslant \mathrm{S}_{n-2} \times \mathrm{S}_2$.

(iii) $G_0 = \mathrm{PSL}_n(q)$, $H \cap G_0 \leqslant \mathrm{C}_{\frac{q^n-1}{(q-1)d}} : \mathrm{C}_n$ *and* $\mathrm{C}_p^{(n-1)f} : \mathrm{SL}_{n-1}(q) \trianglelefteq K \cap G_0$ *with* $d = (n, q - 1)$.

(iv) $G_0 = \mathrm{PSp}_{2m}(q)$ *with* q *even,* $m \geqslant 3$, $H\cap G_0 \leqslant \mathrm{C}_p^{mf} : \mathrm{C}_{q^m-1}.\mathrm{C}_m$ *and* $K\cap G_0 = \Omega^-_{2m}(q)$.

(v) (G, H, K) *is one of the cases listed in [2, Table 4].*

For edge-primitive 2-arc-transitive graph, we have the following result which is critical for this paper.

Lemma 6. *[8, Lemma 3.5] Let* Γ *be an edge-primitive* 2*-arc-transitive graph. Then* Γ *is either vertex-quasiprimitive or vertex-biquasiprimitive.*

Remark 7. By examining the proof process of the above lemma, we get that Γ is vertex-quasiprimitive if it is not a bipartite graph.

Recall that a Cayley graph of finite group H is called a solvable graph if H is solvable. The following lemma is about solvable graphs.

Table 1: (G, S, Γ)

row	G	S	G_α	$G_{\alpha\beta}$	s	Γ	n	k
1	S_5	$5:4$	D_{12}	2^2	3	Petersen	10	3
2	S_6	$3^2:\mathrm{D}_8$	$\mathrm{AGL}_1(5)$	4	2		36	5
3	$\mathrm{PLL}_2(9)$	$3^2:\mathrm{D}_8$	$2\times\mathrm{AGL}_1(5)$	2×4	2	same as row 2	36	5
4	$\mathrm{PSL}_2(11)$	$11:5$	A_4	3	2		55	4
5	$\mathrm{PGL}_2(11)$	$11:5$	S_4	S_3	2	same as row 4	55	4
6	$\mathrm{PSL}_2(23)$	$23:11$	S_4	S_3	2		253	4
7	$\mathrm{PSL}_2(19)$	$19:9$	A_5	D_{10}	2		57	6
8	$\mathrm{PSL}_2(59)$	$59:29$	A_5	D_{10}	2		1711	6
9	$\mathrm{PSU}_3(5)$	$5_+^{1+2}:8$	A_7	A_6	3	Hoffman-Singleton	50	7
10	$\mathrm{P}_2(5)$	$5_+^{1+2}:8$	S_7	S_6	3	Hoffman-Singleton	50	7
11	HS	$5_+^{1+2}:8.2$	M_{22}	$\mathrm{PSL}_3(4)$	2	Higman-Sims	100	22
12	HS .2	$5^2:4$	$\mathrm{M}_{22}.2$	$\mathrm{P\Sigma L}_3(4)$	2	Higman-Sims	100	22

Lemma 8. *[17, Corollary 1.5] A connected non-bipartite 2-arc-transitive solvable Cayley graph of valency at least three is a normal cover of*

(1) *a complete graph, or*

(2) *a normal Cayley graph of* C_2^d*, or*

(3) *a* (G, s)*-arc-transitive graph* Γ *where* $G \leqslant \mathrm{Aut}\Gamma$ *and either*

  (3.1) $G = \mathrm{PSL}_2\left(p^f\right)\cdot\mathcal{O}$ *is almost simple with* $\mathcal{O} \leqslant \mathrm{C}_{(2,p-1)}\times\mathrm{C}_f$ *and* p *prime,* $G_\alpha = p^f : G_{\alpha\beta}, G_{\alpha\beta}$ *is a transitive subgroup of* $\Gamma\mathrm{L}_1\left(p^f\right), G$ *has a vertex-transitive subgroup* $S = \mathrm{D}_{(2,p)\left(p^f+1\right)}\cdot\mathcal{O}$*, and* $s = 2$*; or*

  (3.2) G *has a vertex-transitive solvable subgroup* S *such that* (G, S, Γ) *is described in Table 1.*

For edge-primitive 2-arc-transitive graphs, we have the following useful result.

Lemma 9. *[25, Theorem 1.1] Let* Γ *be an edge-primitive* d*-regular graph for some integer* $d \geqslant 3$*. If* Γ *is 2-arc-transitive, then either* Γ *is a complete bipartite graph, or* $\mathrm{Aut}\Gamma$ *is almost simple.*

Weiss [29] determined all edge-primitive cubic graphs. This result can also be found from [10, Page 492].

Lemma 10. *Let* Γ *be an edge-primitive cubic graph. Then* Γ *is isomorphic to the complete bi-partite graph* $\mathsf{K}_{3,3}$*, or the Heawood graph of order* 14*, or the Biggs-Smith cubic distance-transitive graph of order* 102 *or the Tutte-Coxeter graph of order* 30 *(also known as Tutte's* 8*-cage or the Levi graph).*

To complete the paper, we also need the following result.

Lemma 11. *Let $n \geqslant 2$ be a given positive integer, and k, m be positive integers such that $n = k + m$. Then the value of $k! \cdot m!$ is minimal if and only if $|k - m| \leqslant 1$.*

Proof. Without loss of generality, we assume that $k \geqslant m$.

Firstly, suppose that $n = 2t + 1$ is odd with $t \geqslant 1$ an integer. Let $k = t + 1 + a$ and $m = t - a$ for $a \geqslant 0$ integer. Then we have that

$$\begin{aligned}\frac{k! \cdot m!}{(t+1)! \cdot t!} &= \frac{(t+1+a)! \cdot (t-a)!}{(t+1)! \cdot t!} = \frac{(t+2)(t+3)\cdots(t+a)(t+1+a)}{t(t-1)\cdots(t-a+2)(t-a+1)} \\ &= \frac{t+2}{t} \cdot \frac{t+3}{t-1} \cdot \cdots \cdot \frac{t+a}{t-a+2} \cdot \frac{t+1+a}{t-a+1} \geqslant 1,\end{aligned}$$

and the equation holds if and only if $a = 0$ with $k - m = 1$, as desired.

Next, assume that $n = 2t$ is even. Let $k = t + a$ and $m = t - a$ for an integer $a \geqslant 0$. Then we have that

$$\begin{aligned}\frac{k! \cdot m!}{(t!)^2} &= \frac{(t+a)! \cdot (t-a)!}{(t!)^2} = \frac{(t+1)(t+2)\cdots(t-1+a)(t+a)}{t(t-1)\cdots(t-a+2)(t-a+1)} \\ &= \frac{t+1}{t} \cdot \frac{t+2}{t-1} \cdot \cdots \cdot \frac{t-1+a}{t-a+2} \cdot \frac{t+a}{t-a+1} \geqslant 1,\end{aligned}$$

and the equation holds if and only if $a = 0$ with $k - m = 0$, as desired. □

The next lemma is about the order of the maximal subgroups of S_n and A_n.

Lemma 12. *Let $X \cong \mathrm{S}_n$ or A_n and G a maximal subgroup of X for a positive integer $n \geqslant 5$. Then $|G| > n\frac{|X|}{|\mathrm{A}_n|}$.*

Proof. Note that the maximal subgroups of S_n or A_n are listed in [22, Theorem]. Assume that $n = 5$. Note that the maximal subgroup of $X \cong \mathrm{S}_5$ is isomorphic to S_4, $\mathrm{C}_5 : \mathrm{C}_4$ or $\mathrm{C}_2 \times \mathrm{S}_3$, and the maximal subgroup of $X \cong \mathrm{A}_5$ is isomorphic to A_4, D_{10} or S_3 (see [5, page 2] for example). Then $|G| > 2 \cdot 5$ whenever $X \cong \mathrm{S}_5$ and $|G| > 5$ whenever $X \cong \mathrm{A}_5$, and so Lemma 12 holds in this case.

Now we suppose that $n \geqslant 6$ in the following proof. Note that G is a maximal subgroup of X. Assume that G is the type (a) of [22, Theorem], i.e., $G \cong (\mathrm{S}_k \times \mathrm{S}_m) \cap X$ for $n = k + m$ and $k \neq m$. If $m = 1$ or $k = 1$, since $m \neq k$, then $(X, G) \cong (\mathrm{S}_n, \mathrm{S}_{n-1})$ or $(\mathrm{A}_n, \mathrm{A}_{n-1})$. Since $n \geqslant 6$, one yields that $n - 1 \geqslant 5$ and $(n-1)! > 3(n-1)$. On the other hand, since $3n - 3 - 2n = n - 3 \geqslant 3 > 0$, one yields that $(n-1)! > 2n$ and $|G| \geqslant (n-1)!\frac{|X|}{|\mathrm{S}_n|} > 2n\frac{|X|}{|\mathrm{S}_n|} = n\frac{|X|}{|\mathrm{A}_n|}$, and hence the Lemma 12 holds in this case. Now suppose that $k \geqslant 2$ and $m \geqslant 2$. Then $|G| = k! \cdot m!\frac{|X|}{|\mathrm{S}_n|}$ as $G \cong (\mathrm{S}_k \times \mathrm{S}_m) \cap X$. Suppose that n is odd. Then $n = 2t + 1$ for $t \geqslant 1$ an integer and $t \geqslant 3$ as $n \geqslant 6$. According to the Lemma 11, one yields that $k! \cdot m! \geqslant (t+1)! \cdot t! > 4t + 2 = 2n$, i.e., $|G| > n\frac{|X|}{|\mathrm{A}_n|}$. Assume that n is even. Then $n = 2t$ for $t \geqslant 1$ an integer and $t \geqslant 3$ as $n \geqslant 6$. According to the Lemma 11, one yields that $k! \cdot m! \geqslant (t!)^2 > 4t = 2n$, i.e., $|G| > n\frac{|X|}{|\mathrm{A}_n|}$. To sum up, we have $|G| > n\frac{|X|}{|\mathrm{A}_n|}$ and so the Lemma 12 holds.

Suppose that G is the type (b) of [22, Theorem], i.e., $G \cong (\mathrm{S}_k \wr \mathrm{S}_m) \cap X$ for $n = mk$ with $k \geqslant 2$ and $m \geqslant 2$. Then $|\mathrm{S}_k \wr \mathrm{S}_m| = |\mathrm{S}_k|^m \cdot |S_m| = (k!)^m \cdot m! = (m-1)!(k-1)!(k!)^{m-1} \cdot mk$.

Since $n \geqslant 6$, one yields that $|G| = |\mathrm{S}_k \wr \mathrm{S}_m|\frac{|X|}{|\mathrm{S}_n|} > 2mk\frac{|X|}{|\mathrm{S}_n|} = n\frac{|X|}{|\mathrm{A}_n|}$ and so Lemma 12 holds in this case. Assume that G is the type (c) of [22, Theorem], i.e., $G \cong \mathrm{AGL}_d(p) \cap X$, where $\mathrm{AGL}_d(p) \cong \mathrm{C}_p^d : \mathrm{GL}_d(p)$, $n = p^d$ for d some positive integer and p some prime. Since $n \geqslant 6$, $p^d \geqslant 6$ and $|\mathrm{GL}_d(p)| > 2$. Then $|\mathrm{AGL}_d(p)| = p^d \cdot |\mathrm{GL}_d(p)| > 2n$. It follows that $|G| = |\mathrm{AGL}_d(p)|\frac{|X|}{|\mathrm{S}_n|} > n\frac{|X|}{|\mathrm{A}_n|}$ and so the Lemma 12 holds in this case. Suppose that G is the type (d) of [22, Theorem], the so-called diagonal case, that is, $G \cong (T^m.(\mathrm{Out}(T)\times \mathrm{S}_m))\cap X$, where T is a nonabelian simple group, $m \geqslant 2$ is an integer and $n = |T|^{m-1}$. Then $|G| = |T^m.(\mathrm{Out}(T) \times \mathrm{S}_m)|\frac{|X|}{|\mathrm{S}_n|} > |T|^m\frac{|X|}{|\mathrm{S}_n|} = |T| \cdot |T|^{m-1}\frac{|X|}{|\mathrm{S}_n|} > 2n\frac{|X|}{|\mathrm{S}_n|} = n\frac{|X|}{|\mathrm{A}_n|}$ and so the Lemma 12 holds in this case. Assume that G is the type (e) of [22, Theorem], i.e., $G \cong (\mathrm{S}_k \wr \mathrm{S}_m) \cap X$, where $n = k^m$, $k \geqslant 2$ and $m \geqslant 5$ are integers. Then $|\mathrm{S}_k \wr \mathrm{S}_m| = |\mathrm{S}_k|^m|\mathrm{S}_m| = (k!)^m \cdot m! = m!((k-1)!)^m \cdot k^m$. Since $n \geqslant 6$ and $n = k^m$, it is easy to get that $|G| = |\mathrm{S}_k \wr \mathrm{S}_m|\frac{|X|}{|\mathrm{S}_n|} = m!((k-1)!)^m \cdot k^m\frac{|X|}{|\mathrm{S}_n|} > 2n\frac{|X|}{|\mathrm{S}_n|} = n\frac{|X|}{|\mathrm{A}_n|}$ and hence the Lemma 12 holds in this case.

Assume that G is the type (f) of [22, Theorem], the so-called almost simple case, i.e., $T \lhd G \leqslant \mathrm{Aut}(T)$, with T being a nonabelian simple group, $T \neq \mathrm{A}_n$ and G acts primitively on the set $\Omega = \{1, 2, \cdots, n\}$. Let $\gamma \in \Omega$. Deriving from the assumption that G is an almost simple primitive permutation group on Ω, we yield that G_γ is a maximal subgroup of G and $|G| = |G_\gamma|n$. If $|G_\gamma| = 2$, then $T \cap G_\gamma \leqslant \mathrm{C}_2$. For $T \cap G_\gamma = \mathrm{C}_2$, one yields that $G = T$ and $T_\gamma \cong \mathrm{C}_2$ is a maximal subgroup of T, a contradiction occurs as $4 \mid |T|$. Now that $T \cap G_\gamma = 1$ and T is regular on Ω contradicting the result of [23, Theorem]. Hence, $|G_\gamma| > 2$ and $|G| = |G_\gamma|n > 2n \geqslant n\frac{|X|}{|\mathrm{A}_n|}$. Therefore Lemma 12 holds. □

The next lemma is about the subgroup structure of $\mathrm{Out}(L)$ where L is a finite nonabelian simple group, which can be found from the [6, Lemma 5].

Lemma 13. *Let L be a finite non-abelian simple group with $\mathrm{Out}(L) \neq 1$. If H is a proper subgroup of $\mathrm{Out}(L)$, M is a maximal subgroup of H, then either $|H : M|$ is a prime, or $L = \mathrm{P}\Omega_8^+(q)$, where q is odd and $|H : M| = 4$.*

3 Examples

In this section, some examples of edge-primitive 2-arc-transitive Cayley graphs or bi-Cayley graphs are given.

Example 14. Let $a, b, c, d, e, f, g, h, i, t \in \mathrm{S}_{26}$ defined as follows:

$$
\begin{aligned}
a &= (1,2)(3,5)(4,6)(7,10)(8,9)(11,15)(12,16)(13,18)(14,20)(17,22)(19,24)(21,23)(25,26),\\
b &= (1,3)(2,4,7,11)(5,8,12,17)(6,9,13,19)(10,14)(15,21,25,22)(16,20)(18,23)(24,26),\\
c &= (1,11,20)(2,23,13)(3,14,8)(5,19,18)(7,24,15)(12,22,26),\\
d &= (3,25,15)(4,9,10)(5,11,26)(6,7,8)(12,20,19)(14,16,24),\\
e &= (1,22)(3,25)(4,10)(5,26)(6,14)(7,24)(8,16)(19,20),\\
f &= (1,10,18,9)(2,13)(3,15,24,14)(4,22)(5,20,11,19)(7,16,8,25)(12,26)(17,23),\\
g &= (1,22,18)(3,14,8)(5,11,26)(6,25,16)(7,15,24)(12,19,20),\\
h &= (1,20,22,12)(2,23)(3,8,15,24)(4,5)(6,14,16,7)(9,26,10,11)(13,17)(18,19),\\
i &= (1,5)(3,8)(7,24)(11,18)(13,23)(16,25)(19,20)(22,26), \text{ and}\\
t &= (1,3)(2,12)(4,17)(5,7)(6,21)(8,11)(9,25)(10,16)(13,22)(14,20)(15,19)(18,24)(23,26).
\end{aligned}
$$

Let $G = \langle a, b\rangle$, $H = \langle c, d, e, f, g, h, i\rangle$ and $\Gamma = \mathrm{Cos}(G, H, t)$. With the help of Magma, we have $G \cong \mathrm{PSL}_3(3) : \mathrm{C}_2$, $H \cong \mathrm{C}_3^2 : 2\mathrm{S}_4$, $\langle H, t\rangle = G$ and $|H : H^t \cap H| = 9$. Furthermore, we have that $A = \mathrm{Aut}(\Gamma) \cong G$, $A_\alpha \cong H \cong \mathrm{C}_3^2 : 2\mathrm{S}_4$, $A_{\alpha\beta} \cong 2\mathrm{S}_4$ and $\mathbf{N}_G(G_{\alpha\beta}) \cong 2\mathrm{S}_4 : \mathrm{C}_2$ for

$\beta \in \Gamma(\alpha)$. Further, we also can get $H \cong A_\alpha$ is a subgroup of $\mathrm{soc}(A) \cong \mathrm{PSL}_3(3)$, and so $|V\Gamma| = |A : H| = 2|\mathrm{soc}(A) : H|$ and $\mathrm{soc}(A)$ has exactly two orbits on $V\Gamma$. It follows that Γ is a bipartite graph with $A^+ = \mathrm{soc}(A) \cong \mathrm{PSL}_3(3)$ with $k = |H : H^t \cap H| = 9$. Noting that $A_{\{\alpha,\beta\}} \cong 2\mathrm{S}_4 : \mathrm{C}_2$ is a maximal subgroup of $A \cong \mathrm{PSL}_3(3) : \mathrm{C}_2$ (see [5, Page 13] for example), one yields that Γ is edge-primitive. On the other hand, due to $A_\alpha \cong \mathrm{C}_3^2 : 2\mathrm{S}_4$ and $A_{\alpha\beta} \cong 2\mathrm{S}_4$, by [3, Table 7.4], A_α is 2-transitive on $\Gamma(\alpha)$ ($|\Gamma(\alpha)| = 9$) and so Γ is 2-arc-transitive. Simple verifications show that A^+ has a regular subgroup $B \cong \mathrm{C}_{13}$ such that $A^+ = BH$ and $B \cap H = 1$ (confirmed by Magma), we observe that Γ is a bi-Cayley graph over $B \cong \mathrm{C}_{13}$.

Example 15. Let a, b, c, d, e, f, t_1 and t_2 be eight permutations in S_{22} defined as follows:

$$\begin{array}{l}
a = (1,2)(3,5)(4,6)(7,11)(8,12)(9,13)(10,14)(15,19)(16,20)(17,21)(18,22),\\
b = (2,3,4)(5,7,8)(6,9,10)(12,15,16)(14,17,18)(19,22,21),\\
c = (1,9)(2,11)(3,15)(4,18)(5,8)(7,19)(10,20)(16,17),\\
d = (1,20,6)(3,17,12)(4,14,16)(5,8,19)(7,22,21)(11,15,18),\\
e = (1,6)(3,11)(4,17)(7,9)(8,20)(10,22)(13,18)(14,15),\\
f = (2,4,16)(3,15,12)(5,8,21)(6,20,10)(7,22,19)(11,17,18),\\
t_1 = (1,17)(2,9)(3,20)(4,22)(5,18)(6,12)(7,14)(8,15)(10,13)(11,19)(16,21), \text{ and}\\
t_2 = (1,2)(3,8)(4,20)(5,15)(6,12)(7,17)(9,11)(10,18)(13,22)(14,21)(16,19).
\end{array}$$

Let $G = \langle a, b\rangle$, $H = \langle c, d\rangle$, $K = \langle e, f\rangle$ and $\Gamma_1 = \mathrm{Cos}(G, H, t_1)$, $\Gamma_2 = \mathrm{Cos}(G, H, t_2)$. With the help of Magma, we yield that $G \cong \mathrm{PSL}_2(11) : \mathrm{C}_2$, $H \cong \mathrm{A}_5$, $\mathrm{soc}(G) = K \cong \mathrm{PSL}_2(11)$, $\langle H, t_1\rangle = G$ and $|H : H^{t_1} \cap H| = 5$, and $\langle H, t_2\rangle = G$ and $|H : H^{t_2} \cap H| = 6$.

At first, we can also get that $A = \mathrm{Aut}(\Gamma_1) = G$, $A_\alpha \cong \mathrm{A}_5$, $A_{\alpha\beta} \cong \mathrm{A}_4$ and $A_{\{\alpha,\beta\}} = A_{\alpha\beta} : \mathrm{C}_2 \cong \mathrm{S}_4$, where $\alpha, \beta \in V\Gamma_1$ and $\beta \in \Gamma_1(\alpha)$. Noting that S_4 is a maximal subgroup of $\mathrm{PSL}_2(11) : \mathrm{C}_2$ (see [5, Page 7] for example), one concludes that Γ_1 is edge-primitive. Since $A_\alpha \cong \mathrm{A}_5$ is 2-transitive on $\Gamma_1(\alpha)$ ($|\Gamma_1(\alpha)| = 5$), one deduces that Γ_1 is 2-arc-transitive. On the other hand, computation by the Magma shows that A has a subgroup $B \cong \mathrm{D}_{22}$ which acts regular on $V\Gamma_1$ with $A = BH$ and $B \cap H = 1$, we can obtain that Γ_1 is a Cayley graph over B. Checking the list of the maximal subgroups of $K \cong \mathrm{PSL}_2(11)$ (see [5, Page 7] for example), we deduce that K has exactly two orbits on $V\Gamma_1 = [G : H]$ and so Γ_1 is a bipartite graph of valency 5.

In the meantime, we also have $A = \mathrm{Aut}(\Gamma_2) \cong G$, $A_\alpha \cong \mathrm{A}_5$, $A_{\alpha\beta} \cong \mathrm{D}_{10}$ and $A_{\{\alpha,\beta\}} = A_{\alpha\beta} : \mathrm{C}_2 \cong \mathrm{D}_{20}$, where $\alpha, \beta \in V\Gamma_2$ and $\beta \in \Gamma_2(\alpha)$. Noticing that D_{20} is a maximal subgroup of $\mathrm{PSL}_2(11) : \mathrm{C}_2$ (see [5, Page 7] for example), Γ_2 is edge-primitive. Since $A_\alpha \cong \mathrm{A}_5$ is 2-transitive on $\Gamma_2(\alpha)$ ($|\Gamma_2(\alpha)| = 6$), one yields that Γ_2 is 2-arc-transitive. On the other hand, A has a regular subgroup $B \cong \mathrm{D}_{22}$ such that $A = BH$ and $B \cap H = 1$, we obtain that Γ_2 is a Cayley graph over B. The subgroups structure of $\mathrm{PSL}_2(11)$ (see [5, Page 7] for example) shows that K is not transitive on $V\Gamma_2$ as $|V\Gamma_2| = 22$, one yields that Γ_2 is bipartite of valency 6. □

Example 16. Let a, b, c, d, t be five permutations in S_{30} defined as follows:

$$\begin{array}{l}
a = (3,17,23,5,21,13,6,14,19,27,20,12,28,10,16,30,15,26,22,29,9,8,25,24,4,11,7,18),\\
b = (1,8,2)(3,12,20)(4,23,24)(5,16,21)(6,7,26)(9,14,25)(10,18,27)(11,15,19)(13,30,17)(22,29,28),\\
c = (1,8,22)(2,21,17)(3,16,10)(4,9,30)(5,29,12)(6,11,15)(7,14,23)(13,28,24)(18,20,26)(19,25,27),\\
d = (1,26)(3,14)(4,18)(5,15)(6,20)(7,30)(8,25)(9,19)(10,21)(11,13)(12,27)(16,29)(17,24)(23,28), \text{ and}\\
t = (1,25)(2,7)(3,27)(4,20)(5,15)(6,28)(8,24)(9,12)(10,17)(11,18)(13,23)(14,29)(16,19)(21,26)(22,30).
\end{array}$$

Let $G = \langle a, b\rangle$, $H = \langle c, d\rangle$ and $\Gamma = \mathrm{Cos}(G, H, t)$. With the help of Magma, we have $G \cong \mathrm{PSL}_2(29).\mathrm{C}_2$, $H \cong \mathrm{A}_5$, $\langle H, t\rangle = G$ and $|H : H^t \cap H| = 5$, in particular, $H \leqslant$

$\mathrm{soc}(G) \cong \mathrm{PSL}_2(29)$. Furthermore, we have that $A = \mathrm{Aut}(\Gamma) \cong G$, $A_\alpha \cong \mathrm{A}_5$, $A_{\alpha\beta} \cong \mathrm{A}_4$ and $\mathbf{N}_G(G_{\alpha\beta}) \cong \mathrm{S}_4$ for $\alpha, \beta \in V\Gamma$ and $\beta \in \Gamma(\alpha)$. Since $H \leqslant \mathrm{soc}(A)$, we obtain that $|V\Gamma| = |G : H| = 2|\mathrm{soc}(A) : H|$ and $\mathrm{soc}(A)$ has exactly two orbits on $V\Gamma$. It follows that Γ is a bipartite graph with $A^+ = \mathrm{soc}(A) \cong \mathrm{PSL}_2(29)$. Noticing that $|H : H^t \cap H| = 5$, one yields that the valency of Γ is 5. However, $A_{\{\alpha,\beta\}} \cong \mathrm{S}_4$ is a maximal subgroup of $A \cong \mathrm{PSL}_2(29) : \mathrm{C}_2$ leading to that Γ is edge-primitive. Derived from $A_\alpha \cong \mathrm{A}_5$ is 2-transitive on $\Gamma(\alpha)$ (5 points), one yields that Γ is 2-arc-transitive. Again by Magma, one deduces that A has a regular subgroup $B \cong \mathrm{C}_{29} : \mathrm{C}_7$ such that $\mathrm{soc}(A) = BA_\alpha$ and $B \cap A_\alpha = 1$ and so Γ is a bi-Cayley graph over $B \cong \mathrm{C}_{29} : \mathrm{C}_7$. □

Example 17. Let a, b, c, d, t be five permutations in S_{60} defined as follows:

$a = (3, 32, 54, 24, 48, 41, 16, 57, 5, 43, 10, 55, 46, 29, 23, 11, 26, 18, 51, 14, 30, 44, 28, 7, 27, 13, 38, 42, 59, 4, 21, 25, 50, 36, 56, 35, 19, 33, 49, 12, 45, 37, 52, 40, 34, 17, 8, 53, 20, 58, 6, 47, 22, 15, 39, 9, 31, 60),$
$b = (1, 60, 2)(3, 31, 59)(4, 20, 57)(5, 42, 58)(6, 46, 28)(7, 26, 17)(8, 52, 39)(9, 30, 43)(10, 54, 23)(11, 25, 49)(12, 44, 27)(13, 37, 51)(14, 29, 22)(15, 38, 41)(16, 56, 34)(18, 50, 35)(19, 32, 53)(21, 24, 47)(33, 48, 40)(36, 55, 45),$
$c = (1, 18)(2, 36)(3, 7)(4, 24)(5, 17)(6, 42)(8, 12)(9, 33)(10, 58)(11, 51)(13, 39)(14, 57)(15, 29)(16, 19)(20, 27)(21, 37)(22, 31)(23, 34)(25, 40)(26, 28)(30, 43)(32, 45)(35, 50)(38, 54)(41, 52)(44, 53)(46, 60)(47, 49)(48, 55)(56, 59),$
$d = (1, 46, 12)(2, 58, 51)(3, 5, 40)(4, 19, 31)(6, 20, 36)(7, 33, 18)(8, 42, 26)(9, 45, 49)(10, 30, 34)(11, 53, 37)(13, 43, 59)(14, 16, 39)(15, 48, 60)(17, 28, 21)(22, 32, 50)(23, 24, 52)(25, 38, 35)(27, 55, 56)(29, 47, 57)(41, 54, 44),$
$t = (1, 38)(2, 35)(3, 49)(4, 52)(6, 30)(7, 24)(8, 11)(9, 12)(10, 40)(13, 56)(14, 50)(16, 28)(17, 31)(18, 45)(19, 42)(20, 27)(21, 22)(23, 48)(25, 43)(26, 41)(29, 33)(32, 57)(34, 44)(36, 46)(37, 55)(39, 54)(47, 51)(53, 60)(58, 59).$

Let $G = \langle a, b\rangle$, $H = \langle c, d\rangle$ and $\Gamma = \mathrm{Cos}(G, H, t)$. With the help of Magma, we have $G \cong \mathrm{PSL}_2(59).\mathrm{C}_2$, $H \cong \mathrm{A}_5$, $\langle H, t\rangle = G$ and $|H : H^t \cap H| = 5$. Furthermore, we have $A = \mathrm{Aut}(\Gamma) \cong G$, $A_\alpha \cong \mathrm{A}_5$, $A_{\alpha\beta} \cong \mathrm{A}_4$ and $A_{\{\alpha,\beta\}} = A_{\alpha\beta} : \mathrm{C}_2 \cong \mathrm{S}_4$. As S_4 is a maximal subgroup of $\mathrm{PSL}_2(59).\mathrm{C}_2$, one yields that Γ is edge-primitive. Since $A_\alpha \cong \mathrm{A}_5$ is 2-transitive on $\Gamma(\alpha)$ ($|\Gamma(\alpha)| = 5$), Γ is 2-arc-transitive. Note that A has a regular subgroup $B \cong \mathrm{C}_{59} : \mathrm{C}_{58}$ such that $A = BH$ and $B \cap H = 1$. Then Γ is a Cayley graph over B. Moreover, the fact that K is not transitive on $59 \times 58 = 3422$ points leads to that K has exact two orbits on $V\Gamma$ and hence Γ is a bipartite graph of valency 5. □

Example 18. Let a, b, c, d, t be five permutations in S_{62} defined as follows:

$a = (1, 4, 11, 20)(2, 3)(5, 6)(7, 8, 17, 30, 47, 49, 28, 14)(9, 10)(12, 13, 25, 42, 58, 53, 54, 29)(15, 16)(18, 19, 35, 36, 26, 27, 44, 60)(21, 22)(23, 24, 39, 57)(31, 32)(33, 34, 40, 41, 48, 55, 56, 59)(37, 38, 50, 43)(45, 46, 51, 52)(61, 62),$
$b = (1, 2)(3, 5)(4, 7)(6, 9)(8, 12)(10, 14)(11, 15)(13, 18)(16, 21)(17, 23)(19, 26)(20, 28)(22, 29)(24, 31)(25, 33)(27, 35)(30, 37)(32, 40)(34, 43)(36, 45)(38, 41)(39, 47)(42, 51)(44, 53)(46, 55)(48, 58)(49, 54)(50, 56)(52, 59)(57, 61)(60, 62),$
$c = (1, 28)(2, 49)(3, 54)(4, 19)(5, 29)(6, 45)(10, 11)(12, 17)(14, 21)(15, 43)(16, 59)(20, 61)(23, 56)(24, 26)(27, 30)(31, 53)(32, 58)(33, 42)(34, 62)(35, 41)(36, 46)(37, 47)(38, 48)(44, 51),$
$d = (1, 47, 62, 44)(2, 27)(3, 12, 51, 55)(4, 13, 60, 5)(6, 26, 50, 28)(7, 59)(8, 43)(9, 33, 48, 31)(10, 25, 57, 41)(11, 32)(14, 30, 52, 46)(15, 36, 53, 20)(17, 18)(21, 56)(22, 24)(29, 39)(34, 58)(35, 54)(38, 61)(40, 49),$
$t = (1, 4)(2, 7)(3, 14)(5, 6)(8, 11)(9, 10)(12, 31)(13, 50)(15, 41)(16, 40)(17, 49)(18, 61)(19, 45)(20, 28)(21, 22)(23, 34)(24, 39)(25, 53)(26, 36)(27, 35)(29, 32)(30, 47)(33, 55)(37, 38)(42, 58)(43, 54)(44, 46)(48, 57)(51, 52)(56, 59)(60, 62).$

Let $G = \langle a, b\rangle$, $H = \langle c, d\rangle$ and $\Gamma = \mathrm{Cos}(G, H, t)$. With the help of Magma, we have $G \cong \mathrm{PSL}_5(2) : \mathrm{C}_2$, $H \cong \mathrm{C}_2^6 : (\mathrm{S}_3 \times \mathrm{PSL}_3(2))$, $\langle H, t\rangle = G$ and $|H : H^t \cap H| = 64$, in particular, H is core-free in G and $H \leqslant \mathrm{soc}(G) \cong \mathrm{PSL}_5(2)$. And further, $G \leqslant \mathrm{Aut}(\Gamma)$, $G_\alpha \cong H$, $G_{\alpha\beta} \cong \mathrm{S}_3 \times \mathrm{PSL}_3(2)$ and $G_{\{\alpha,\beta\}} = G_{\alpha\beta} : \mathrm{C}_2 \cong (\mathrm{PSL}_3(2) \times \mathrm{S}_3) : \mathrm{C}_2$. Since $H \leqslant \mathrm{soc}(G)$, we conclude that $|V\Gamma| = |G : H| = 2|\mathrm{soc}(G) : H|$ and $\mathrm{soc}(G)$ has exactly two orbits on $V\Gamma$, and so Γ is a bipartite graph with $G^+ = \mathrm{soc}(G) \cong \mathrm{PSL}_5(2)$. Noticing that $G_{\{\alpha,\beta\}} \cong (\mathrm{PSL}_3(2) \times \mathrm{S}_3) : \mathrm{C}_2$ is a maximal subgroup of $G \cong \mathrm{PSL}_5(2) : \mathrm{C}_2$ (see [5, Page 70] for example), Γ is edge-primitive. Derived from the fact that $G_\alpha \cong \mathrm{C}_2^6 : (\mathrm{S}_3 \times \mathrm{PSL}_3(2))$ is 2-transitive on $\Gamma(\alpha)$ (64 points), one deduces that Γ is $(G, 2)$-arc-transitive. Again by Magma, G has a regular subgroup $B \cong \mathrm{C}_{31} : \mathrm{C}_5$ such that $G^+ = BH$ and $B \cap H = 1$. Then Γ is a bi-Cayley graph over B of valency 64. □

4 Cayley graphs

In this section, we will deal with the edge-primitive 2-arc-transitive Cayley graphs with valency no less than 3. In the rest of this paper, we will let $\mathcal{O}_1\mathcal{O}_2 = \mathcal{O} \lesssim \mathrm{C}_2$ and $q = p^f$ for some prime p and some positive integer f.

Let $\Gamma = \mathrm{Cay}(G, S)$ be an edge-primitive 2-arc-transitive Cayley graph of valency k for $k \geqslant 3$ an integer. Let $\alpha \in V\Gamma$ and $\beta \in \Gamma(\alpha)$. Suppose that Γ is not a complete bipartite graph. Then according to Lemma 9, A is almost simple. If Γ is a bipartite graph, then it is easy to deduce that Γ is a bi-Cayley graph of G^+, which will be dealt with in the next section. Now we assume that Γ satisfies the following condition in this section.

Condition 4.1. $\Gamma = \mathrm{Cay}(G, S)$ is a non-bipartite edge-primitive 2-arc-transitive Cayley graph of valency k for $k \geqslant 3$ an integer, its full automorphism group $A = \mathrm{Aut}\Gamma$ is almost simple and $T = \mathrm{soc}(A)$ is a nonabelian simple group.

Lemma 19. *Let Γ be a graph satisfied Condition 4.1. If G is solvable, then Γ is isomorphic to either a complete graph K_n or the Higman-Sims graph, where $n \geqslant 5$.*

Proof. Note that $\Gamma = \mathrm{Cay}(G, S)$ is a connected non-bipartite 2-arc-transitive solvable Cayley graph (G is solvable) of valency at least 3 and A acts primitively on $E\Gamma$ and quasiprimitively on $V\Gamma$. Noting that G acts regularly on $V\Gamma$, we deduce that G is core-free in A. Further, by Lemma 8, we can get all the candidates of Γ. So in what follows, we divide the proof into four cases.

Firstly, suppose that Lemma 8.(1) holds, that is Γ is a complete graph denoted by K_n. Then $A \cong \mathrm{S}_n$ for $n \geqslant 5$ as A is almost simple, and hence the first part of this lemma holds.

Secondly, suppose that Lemma 8.(2) holds, that is Γ is a normal Cayley graph of C_2^d. Then $A = \mathrm{C}_2^d : A_\alpha$, contradicts to the assumption that A is almost simple.

Thirdly, suppose that Lemma 8.(3.1) holds. Note if Γ is $(H, 2)$-arc-transitive, then it also is $(A, 2)$-arc-transitive, where $H \leqslant A$. Then $A \cong \mathrm{PSL}_2(p^f).O$ is almost simple with $O \leqslant \mathrm{C}_{(2,p-1)} \times \mathrm{C}_f$ and p prime, $A_\alpha \cong \mathrm{C}_p^f : A_{\alpha\beta}$, $A_{\alpha\beta}$ is a transitive subgroup of $\Gamma\mathrm{L}_1(p^f)$ and $G = \mathrm{D}_{(2,p)(p^f+1)}.O$. Then the valency $k = p^f$ and the exact factorization $H = GH_\alpha$ shows that $|A_\alpha| = |A|/|G|$ and $|A_{\alpha\beta}| = |A_\alpha|/k = \frac{p^f-1}{2}$ contradicting to that $A_{\alpha\beta}$ is a transitive subgroup of $\Gamma\mathrm{L}_1(p^f)$.

Finally, suppose that Lemma 8.(3.2) holds. Then $(A, G, A_\alpha, A_{\alpha\beta}, \Gamma)$ is described in Table 1. Note that the edge primitiveness leads to that $A_{\{\alpha,\beta\}} = A_{\alpha\beta} : \mathrm{C}_2$ is a maximal subgroup of A. Since $|A| = |G||A_\alpha|$, we can deduce that $A \cong \mathrm{HS}.\mathrm{C}_2$ and Γ is the Higman-Sims graph. Hence the Lemma 19 holds. □

Lemma 20. *If Γ satisfies the Condition 4.1 and G is insolvable, then A_α is not solvable.*

Proof. Assume on the contrary that A_α is solvable. Then $A_{\{\alpha,\beta\}} = A_{\alpha\beta} : \mathrm{C}_2$ is solvable, too. Noting that A acts primitively on the edges of Γ, one deduces that $A_{\{\alpha,\beta\}}$ is a solvable maximal subgroup of A. Followed from [18, Theorem 1.1 (ii)], one yields that there exists a normal subgroup of A denoted by L, which is minimal such that $L_{\{\alpha,\beta\}} = L \cap A_{\{\alpha,\beta\}}$ is

a maximal subgroup of L for $T = \mathrm{soc}(A) = \mathrm{soc}(L)$, and $(L, L_{\{\alpha,\beta\}})$ is given in [18, Tables 14-20]. Note that A_α has a 2-transitive permutation representation of degree k and we can get the structure of $(A_\alpha, A_{\alpha\beta})$ from [4, Tables 7.3 and 7.4]. In particular, both of L_α and T_α are transitive on $\Gamma(\alpha)$ as $T_\alpha \trianglelefteq L_\alpha \trianglelefteq A_\alpha$ and $T_\alpha \trianglelefteq A_\alpha$.

Followed from Lemma 6 and Remark 7, one yields that A is quasiprimitive on $V\Gamma$. Noting that G is regular on $V\Gamma$, we deduce that G is a core-free subgroup of A. On the other hand, since A is almost simple, $A = GA_\alpha$ is an exact factorization and A_α is solvable, one can get all the candidates of (A, G, A_α) in Lemma 5. So in what follows, we divide the proof into the following five cases and exclude them one by one.

(I). Assume that Lemma 5.(i) holds. Note that $T = \mathrm{soc}(A) \cong \mathrm{A}_n$, $G \cong \mathrm{A}_{n-1}$ or S_{n-1}, and A_α is a transitive subgroup of A on $\{1, 2, \cdots, n\}$ of order n. Since G is insolvable, we get $n \geqslant 6$. If $n = 6$, then $(A, G) \in \{(\mathrm{A}_6, \mathrm{A}_5)$, $(\mathrm{A}_6.\mathrm{C}_2, \mathrm{A}_5)$, $(\mathrm{A}_6.\mathrm{C}_2, \mathrm{S}_5)$, $(\mathrm{A}_6.\mathrm{C}_2^2, \mathrm{A}_5)$, $(\mathrm{A}_6.\mathrm{C}_2^2, \mathrm{S}_5)\}$. Derived from $|A_\alpha| = |A : G|$, one yields that $|\mathrm{A}_\alpha| = 6$, 12 or 24, which implies that $|A_{\{\alpha,\beta\}}| = 2|A_\alpha|/k \leqslant 16$ for $k \geqslant 3$. But derived from [18, Table 14], one concludes that this case does not occur, a contradiction. Now we assume that $n \geqslant 7$. Then $A \cong \mathrm{A}_n$ or S_n, and $(A, G) \in \{(\mathrm{A}_n, \mathrm{A}_{n-1})$, $(\mathrm{S}_n, \mathrm{A}_{n-1})$, $(\mathrm{S}_n, \mathrm{S}_{n-1})\}$. It leads to $|A_\alpha| = |A : G| = n$ or $2n$, and so $|A_{\{\alpha,\beta\}}| = 2|A_\alpha|/k \leqslant \frac{4n}{3}$ for $k \geqslant 3$. Also by [18, Table 14], one concludes that this case does not occur, a contradiction again.

(II). Assume that Lemma 5.(ii) holds. Note that $T = \mathrm{soc}(A) \cong \mathrm{A}_n$, $n = p^a$ is a prime power, $A_\alpha \leqslant \mathrm{A\Gamma L}_1(p^a) \cong \mathrm{C}_p^a : \mathrm{C}_{p^a-1}$ is 2-homogeneous on $\{1, 2, \cdots, n\}$ and $\mathrm{A}_{n-2} \trianglelefteq G \leqslant \mathrm{S}_{n-2} \times \mathrm{S}_2$. Then $\mathrm{C}_p^a = \mathrm{O}_p(A_\alpha)$ and $k = p^a$. Since A_α is a 2-transitive group of degree p^a, one deduces that $p^a \cdot (p^a - 1) \mid |A_\alpha|$ and $A_\alpha \cong \mathrm{C}_p^a : \mathrm{C}_{p^a-1}$, in particular, $G = \{\mathrm{A}_{n-2}, \mathrm{A}_{n-2} \times \mathrm{S}_2, \mathrm{S}_{n-2}, \mathrm{S}_{n-2} \times \mathrm{S}_2\}$. Since G is insolvable, one deduces $n \geqslant 7$. It leads to that $A \cong \mathrm{A}_n$ or S_n, and so $(A, G) \in \{(\mathrm{A}_n, \mathrm{A}_{n-2})$, $(\mathrm{S}_n, \mathrm{A}_{n-2} \times \mathrm{S}_2)$, $(\mathrm{S}_n, \mathrm{S}_{n-2})\}$ as $|A : G| = |A_\alpha| = n(n-1)$. It follows that $|A_{\{\alpha,\beta\}}| = 2|A_\alpha|/k \leqslant \frac{2n(n-1)}{3}$ for $k \geqslant 3$. Detail checking the [18, Table 14], one concludes that it is impossible and so this case does not occur.

(III). Assume that Lemma 5.(iii) holds. In this case, $T = \mathrm{soc}(A) \cong \mathrm{PSL}_n(q)$, $T_\alpha = A_\alpha \cap T \leqslant \mathrm{C}_{\frac{q^n-1}{(q-1)d}} : \mathrm{C}_n$ and $\mathrm{C}_p^{(n-1)f} : \mathrm{SL}_{n-1}(q) \trianglelefteq G \cap T$ with $d = (n, q-1)$ and $q = p^f$. Since $\frac{1}{2}|L_{\{\alpha,\beta\}}| = \frac{1}{k}|L_\alpha|$, i.e., $k = 2|L_\alpha|/|L_{\{\alpha,\beta\}}|$ for $k \geqslant 3$, we get that $k = |L_\alpha|/|L_{\alpha\beta}| = |T_\alpha|/|T_{\alpha\beta}|$ is a factor of the integer $|\mathrm{C}_{\frac{q^n-1}{(q-1)d}} : \mathrm{C}_n|/|T_{\alpha\beta}|$ as $T_{\alpha\beta} \leqslant T_\alpha \leqslant \mathrm{C}_{\frac{q^n-1}{(q-1)d}} : \mathrm{C}_n$. Let τ be the transpose-inverse automorphism, σ a field automorphism and δ the diagonal automorphism. In particular, $\langle\tau\rangle \cong \langle\delta\tau\rangle \cong \mathrm{C}_2$. Next, we deal with $(L, L_{\{\alpha,\beta\}})$ into the following four cases which has been given in [18, Table 16].

Firstly, suppose that $(L, L_{\{\alpha,\beta\}})$ is given in the line 1 of [18, Table 16]. In this case, $L = T \cong \mathrm{PSL}_2(q)$ and $L_{\{\alpha,\beta\}} = T_{\{\alpha,\beta\}} \cong A_4$, S_4, $\mathrm{C}_p^f : \mathrm{C}_{\frac{q-1}{d}}$ or $\mathrm{D}_{\frac{2(q\pm1)}{d}}$, where $d = (2, q-1)$. Since $T \cong \mathrm{PSL}_2(q)$ is simple, we get $q \geqslant 5$. Note that $T_\alpha \leqslant \mathrm{C}_{\frac{q^2-1}{(q-1)d}} : \mathrm{C}_2 \cong \mathrm{C}_{\frac{q+1}{d}} : \mathrm{C}_2$. At first, since A_4 has none subgroup with index 2, we can exclude the case where $T_{\{\alpha,\beta\}} \cong \mathrm{A}_4$. If $T_{\{\alpha,\beta\}} \cong \mathrm{S}_4$, then $\mathrm{A}_4 \cong T_{\alpha\beta} \leqslant T_\alpha \leqslant \mathrm{C}_{(q+1)/d} : \mathrm{C}_2$. But by the subgroup structure of A_4, one yields that it is impossible. If $T_{\{\alpha,\beta\}} \cong \mathrm{C}_p^f : \mathrm{C}_{\frac{q-1}{d}}$, then $k = |T_\alpha|/|T_{\alpha\beta}| \leqslant |\mathrm{C}_{\frac{q+1}{d}} : \mathrm{C}_2|/|T_{\alpha\beta}| = \frac{4(q+1)}{q(q-1)}$ is not an integer and this case is excluded. If $T_{\{\alpha,\beta\}} \cong \mathrm{D}_{\frac{2(q+1)}{d}}$, then

$|T_{\alpha\beta}| = \frac{(q+1)}{d}$ and $k = |T_\alpha|/|T_{\alpha\beta}| \leqslant 2$, contradicting to the assumption that $k \geqslant 3$. Finally, assume that $T_{\{\alpha,\beta\}} \cong \mathrm{D}_{\frac{2(q-1)}{d}}$. Then $|T_{\alpha\beta}| = \frac{(q-1)}{d}$ and $3 \leqslant k = |T_\alpha|/|T_{\alpha\beta}| \leqslant |\mathrm{C}_{\frac{q+1}{d}} : \mathrm{C}_2|/|T_{\alpha\beta}| = \frac{2(q+1)}{q-1}$. So $\frac{2(q+1)}{q-1} - 3 \geqslant 0$ and $\frac{5-q}{q-1} \geqslant 0$. Since $\frac{2(q+1)}{q-1}$ is an integer and $q \geqslant 5$, it follows that $q = 5$ and so $\frac{2(q+1)}{q-1} = 3$. Then $(q,k) = (5,3)$ and so $T_\alpha \cong \mathrm{S}_3$. In particular, $T \cong \mathrm{PSL}_2(5) \cong \mathrm{A}_5$ and $T_{\{\alpha,\beta\}} \cong \mathrm{D}_4 \cong \mathrm{C}_2^2$, a contradiction occurs as C_2^2 is not isomorphic to any maximal subgroup of A_5.

Secondly, suppose that $(L, L_{\{\alpha,\beta\}})$ is given in lines 2-6 of [18, Table 16]. Suppose further that the pair $(L, L_{\{\alpha,\beta\}})$ is given in the line 2 of [18, Table 16], that is $L \cong \mathrm{PGL}_2(p)$ and $L_{\{\alpha,\beta\}} \cong \mathrm{S}_4$ for $p \equiv \pm 11, \pm 19 \pmod{40}$, D_{12} or D_{16} with $p = 7$, or D_{20} for $p = 11$. Then p is odd and $L = T.\mathrm{C}_2 \cong \mathrm{PSL}_2(p).\mathrm{C}_2$ for $T \cong \mathrm{PSL}_2(p)$. If $L_{\{\alpha,\beta\}} \cong \mathrm{S}_4$, then $T_{\{\alpha,\beta\}} \cong \mathrm{A}_4$ has no subgroup of index 2, a contradiction. Assume that $L_{\{\alpha,\beta\}} \cong \mathrm{D}_{12}$ or D_{16} with $p = 7$. Then $A = L \cong \mathrm{PGL}_2(7)$ and $A_\alpha = L_\alpha = T_\alpha.\mathrm{C}_2 \leqslant (\mathrm{C}_4 : \mathrm{C}_2).\mathrm{C}_2$, which implies $|L_\alpha| \mid 16$. If $L_{\{\alpha,\beta\}} \cong \mathrm{D}_{12}$, then $|L_{\alpha\beta}| = 6$ can not divide $|L_\alpha|$, a contradiction. Further, if $L_{\{\alpha,\beta\}} \cong \mathrm{D}_{16}$, then $|L_{\alpha\beta}| = 8$ and $k = \frac{|L_\alpha|}{|L_{\alpha\beta}|} \leqslant \frac{16}{|L_{\alpha\beta}|} = 2$, contradicting with the assumption $k \geqslant 3$. Suppose that $L_{\alpha\beta} \cong \mathrm{D}_{20}$ with $p = 11$. Then $A = L \cong \mathrm{PGL}_2(11)$, $T \cong \mathrm{PSL}_2(11)$, $L_\alpha \leqslant (\mathrm{C}_6 : \mathrm{C}_2).\mathrm{C}_2$ and $|L_{\alpha\beta}| = 10$, which implies $|L_{\alpha\beta}| \nmid |L_\alpha|$, a contradiction. Hence this case does not occur. Next, by the complete similar methods with above cases, we can exclude the lines 3-6 of [18, Table 16].

Next, suppose that $(L, L_{\{\alpha,\beta\}})$ is given in the line 7 of [18, Table 16]. Then $L = T \cong \mathrm{PSL}_3(q)$ and $L_{\{\alpha,\beta\}} = T_{\{\alpha,\beta\}} \cong [\frac{(q-1)^2}{(3,q-1)}].\mathrm{S}_3$ with $q \neq 2, 4$, $\mathrm{C}_3^2 : \mathrm{Q}_8$ with $p = q \equiv 4, 7 \pmod 9$, $\mathrm{C}_3^2 : \mathrm{Q}_8 : \mathrm{C}_3$ with $p = q \equiv 1 \pmod 9$ or $\mathrm{C}_3^2 : \mathrm{Q}_8 : \mathrm{S}_3$ with $q = 3$. Remind $T_\alpha \leqslant \mathrm{C}_{\frac{q^3-1}{(q-1)(3,q-1)}} : \mathrm{C}_3$. Thus, $|T_\alpha| \mid \frac{q^3-1}{q-1} = q^2 + q + 1$ is odd, which implies that $|T_{\alpha\beta}|$ is odd. Since $|T_{\alpha\beta}| = |T_{\{\alpha,\beta\}}|/2$, we can exclude the cases where $L_{\{\alpha,\beta\}} \in \{\mathrm{C}_3^2 : \mathrm{Q}_8, \mathrm{C}_3^2 : \mathrm{Q}_8 : \mathrm{C}_3, \mathrm{C}_3^2 : \mathrm{Q}_8 : \mathrm{S}_3\}$. If $T_{\{\alpha,\beta\}} \cong [\frac{(q-1)^2}{(3,q-1)}].\mathrm{S}_3$ with $q \neq 2, 4$, then $|T_{\alpha\beta}| = \frac{3(q-1)^2}{(3,q-1)}$ and $k = |T_\alpha|/|T_{\alpha\beta}|$ is a factor of $\frac{q^3-1}{(q-1)^3}$. Consequently, k is not an integer, a contradiction.

Finally, suppose that $(L, L_{\{\alpha,\beta\}})$ is given in lines 8-29 of [18, Table 16]. Suppose further that $(L, L_{\{\alpha,\beta\}})$ is given in the lines 10 or 11 of [18, Table 16], that is either $L \cong \mathrm{PSL}_3(q).\langle\sigma\tau\rangle$ and $L_{\{\alpha,\beta\}} \cong [q^3] : [(q-1)^2/(3, q-1)].\mathrm{C}_2$ with $q = p^{2f}$ and $o(\sigma) = 2$, or $L \cong \mathrm{PGL}_3(q).\langle\tau\rangle$ and $L_{\{\alpha,\beta\}} \cong [q^3] : [(q-1)^2/(3, q-1)].\mathrm{S}_3$ with $3|(q-1)$. Then $T \cong \mathrm{PSL}_3(q)$ and $T_\alpha \leqslant \mathrm{C}_{\frac{q^3-1}{(q-1)(3,q-1)}} : \mathrm{C}_3$. Noting $o(\sigma) = o(\tau) = o(\sigma\tau) = 2$, one yields that $|L_\alpha| = 2|T_\alpha| \leqslant \frac{6(q^3-1)}{(q-1)(3,q-1)}$. However, $|L_{\alpha\beta}| = |L_{\{\alpha,\beta\}}|/2 \geqslant \frac{3q^3(q-1)^2}{(3,q-1)} > \frac{6(q^3-1)}{(q-1)(3,q-1)} \geqslant |L_\alpha|$, a contradiction. By the complete similar methods with above two cases, we can exclude the lines 8, 9, 12-29 of [18, Table 16].

(IV). Assume that Lemma 5.(iv) holds. In this case, $T \cong \mathrm{PSp}_{2m}(q)$, $T_\alpha \leqslant \mathrm{C}_p^{mf} : \mathrm{C}_{q^m-1}.\mathrm{C}_m$ and $G \cap T = \Omega_{2m}^-(q)$, where $q = 2^f$ for some positive integer f and $m \geqslant 3$. Then $(L, L_{\{\alpha,\beta\}})$ is given in [18, Table 17], which implies $L \cong \mathrm{PSp}_6(2)$ and $L_{\{\alpha,\beta\}} \cong \mathrm{C}_2.[2^6] : \mathrm{S}_3^2$. Since $\mathrm{Out}(\mathrm{PSp}_6(2)) = 1$, we have $A = L = T \cong \mathrm{PSp}_6(2)$. However, $L_\alpha = T_\alpha \leqslant \mathrm{C}_2^3 : \mathrm{C}_7.\mathrm{C}_3$ leads to that $|L_\alpha| \leqslant 168 < 2304 = |L_{\alpha\beta}|$, contradicting the fact that $L_{\alpha\beta} \leqslant L_\alpha$. Hence this case is excluded.

(V). Assume that Lemma 5.(v) holds. Note that (A, A_α, G) is one of the cases listed

in [2, Table 4]. On the other hand, the edge-primitive graphs with soluble edge-stabilizers are determined in [9, 10, 11, 18, 29]. By comparing the above results, we can conclude that all the candidates of (A, A_α) are isomorphic to $(\mathrm{PSL}_2(11).\mathcal{O}, \mathrm{C}_{11} : (\mathrm{C}_5 \times \mathcal{O}_1))$, $(\mathrm{PSL}_2(23).\mathcal{O}, \mathrm{C}_{23} : (\mathrm{C}_{11} \times \mathcal{O}))$ or $(\mathrm{PSL}_2(59).\mathcal{O}, \mathrm{C}_{59} : (\mathrm{C}_{29} \times \mathcal{O}))$ with graphs being isomorphic to K_{12}, K_{24} or K_{60} respectively. However, $\mathrm{Aut}(\mathsf{K}_{12}) \cong \mathrm{S}_{12}$, $\mathrm{Aut}(\mathsf{K}_{24}) \cong \mathrm{S}_{24}$ and $\mathrm{Aut}(\mathsf{K}_{60}) \cong \mathrm{S}_{60}$, a contradiction. □

Lemma 21. *Assume that Γ satisfies the Condition 4.1. If G is insolvable, then Γ is isomorphic to the complete graph K_n for $n \geqslant 60$.*

Proof. Since both of G and A_α are insolvable, the result of [16, Theorem 1.2] shows that the triples (A, G, A_α) are given in [16, Table 1]. Detail checking [16, Table 1.1], combining with [4, Tables 7.3 and 7.4], we can get all the candidates for the triples (A, G, A_α).

Suppose that the line 1 of [16, Table 1.1] holds, that is $A \cong \mathrm{A}_n.\mathcal{O}$ and $\{A_\alpha, G\} \cong \{[n].\mathcal{O}_1, \mathrm{A}_{n-1}.\mathcal{O}_2\}$. Since both of A_α and G are insolvable, one yields that $n \geqslant 60$. Assume that $A_\alpha \cong \mathrm{A}_{n-1}.\mathcal{O}_2$. Then $A_{\alpha\beta} \cong \mathrm{A}_{n-2}.\mathcal{O}_2$ and $A_{\{\alpha,\beta\}} \cong (\mathrm{A}_{n-2}.\mathcal{O}_2) : \mathrm{C}_2$. Since $A_{\{\alpha,\beta\}}$ is a maximal subgroup of A, by the maximal subgroups structure of A (see [22, Theorem] for example), $(A, A_\alpha, A_{\{\alpha,\beta\}}, A_{\alpha\beta}) \cong (\mathrm{A}_n, \mathrm{A}_{n-1}, \mathrm{S}_{n-2}, \mathrm{A}_{n-2})$ and $G \cong [n]$. It is clear that $\Gamma \cong \mathsf{K}_n$, the complete graph of order n and hence the first part of Lemma 21 holds. Now assume that $A_\alpha \cong [n].\mathcal{O}_1$. Then $G \cong \mathrm{A}_{n-1}.\mathcal{O}_2$. Since $|A_{\{\alpha,\beta\}}| = \frac{2|A_\alpha|}{k} < |A_\alpha| \leqslant 2n$ as $k \geqslant 3$, by Lemma 12, one yields that $A \ncong \mathrm{S}_n$ as $A_{\{\alpha,\beta\}}$ is a maximal subgroup of A. Therefore, $A \cong \mathrm{A}_n$, $A_\alpha \cong [n]$ and $G \cong \mathrm{A}_{n-1}$. It follows that $|A_{\{\alpha,\beta\}}| = \frac{2|A_\alpha|}{k} < |A_\alpha| \leqslant n$ and by Lemma 12, $A_{\{\alpha,\beta\}}$ is not a maximal subgroup of A, a contradiction. Hence the line 1 of [16, Table 1.1] does not occur.

Assume that $A \cong \mathrm{A}_{q+1}.\mathcal{O}$ and $A_\alpha \cong \mathrm{PSL}_2(q).(\mathrm{C}_{(2,q-1)} \times \mathcal{O}_1)$ or $\mathrm{A}_{q-2}.\mathcal{O}_2$, just as the line 2 of [16, Table 1.1]. For the former, by [4, Table 7.4], $A_{\alpha\beta} \cong (\mathrm{C}_p^f : \mathrm{C}_{\frac{q-1}{(2,q-1)}}).(\mathrm{C}_{(2,q-1)} \times \mathcal{O}_1)$ and $A_{\{\alpha,\beta\}} \cong ((\mathrm{C}_p^f : \mathrm{C}_{\frac{q-1}{(2,q-1)}}).(\mathrm{C}_{(2,q-1)} \times \mathcal{O}_1)) : \mathrm{C}_2$ is solvable. Combing with the [11, Theorem 1.1] and [7, Theorem 1], we yield that this case does not occur. Suppose that $A_\alpha \cong \mathrm{A}_{q-2}.\mathcal{O}_2$. Since A_α is insolvable, $q - 2 \geqslant 5$ and $q \geqslant 7$. If $q \leqslant 10$, then checking by Magma, one can get the following results.

q	k	$A_{\alpha\beta}$	$A_{\{\alpha\beta\}}$
7	5	$\mathrm{A}_4.\mathcal{O}_2$	$(\mathrm{A}_4.\mathcal{O}_2) : \mathrm{C}_2$
7	6	$\mathrm{D}_{10}.\mathcal{O}_2$	$(\mathrm{D}_{10}.\mathcal{O}_2) : \mathrm{C}_2$
8	6	$\mathrm{A}_5.\mathcal{O}_2$	$(\mathrm{A}_5.\mathcal{O}_2).\mathrm{C}_2$
8	10	$(\mathrm{C}_3^2 : \mathrm{C}_4).\mathcal{O}_2$	$((\mathrm{C}_3^2 : \mathrm{C}_4).\mathcal{O}_2).\mathrm{C}_2$
9	7	$\mathrm{A}_6.\mathcal{O}$	$(\mathrm{A}_6.\mathcal{O}).\mathrm{C}_2$
9	15	$\mathrm{PSL}_2(7).\mathcal{O}$	$(\mathrm{PSL}_2(7).\mathcal{O}).\mathrm{C}_2$
10	8	$\mathrm{A}_7.\mathcal{O}_2$	$(\mathrm{A}_7.\mathcal{O}_2).\mathrm{C}_2$
10	15	$(\mathrm{C}_2^3 : \mathrm{PSL}_3(2)).\mathcal{O}_2$	$((\mathrm{C}_2^3 : \mathrm{PSL}_3(2)).\mathcal{O}_2).\mathrm{C}_2$

Checking the list of the maximal subgroups of A_{q+1} or S_{q+1} (see [5] for the list), we yield that $A_{\{\alpha\beta\}} =$ is not a maximal subgroup of A in those cases, a contradiction. Now that $q - 2 \geqslant 9$. Then $k = q - 2$ and $A_{\alpha\beta} \cong \mathrm{A}_{q-3}.\mathcal{O}_2$ (see [4, Table 7.4] for example). It follows

that $A_{\{\alpha,\beta\}} \cong (\mathrm{A}_{q-3}.\mathcal{O}_2) : \mathrm{C}_2$ is insolvable. However, the maximal subgroups structure of A (see [22, Theorem] for example) shows that $A_{\{\alpha,\beta\}}$ is not a maximal subgroup of A, a contradiction. By a complete similarly method with above case, we can exclude the line 3 of [16, Table 1.1].

Suppose that $A \cong \mathrm{A}_{12}.\mathcal{O}$ and $A_\alpha \cong \mathrm{M}_{12}$ or $\mathrm{A}_7.\mathcal{O}$, just as the line 7 of [16, Table 1.1]. Then $A \cong \mathrm{A}_{12}$ or S_{12}. Followed from [4, Table 7.4], $(A_\alpha, A_{\alpha\beta})$ is isomorphic to $(\mathrm{M}_{12}, \mathrm{M}_{11})$ whenever $k = 12$, $(\mathrm{A}_7.\mathcal{O}, \mathrm{A}_6.\mathcal{O})$ whenever $k = 7$ or $(\mathrm{A}_7.\mathcal{O}, \mathrm{PSL}_2(7).\mathcal{O})$ whenever $k = 15$. However, the maximal subgroups structure of A (see [5, Page 91] for example) shows that $A_{\{\alpha,\beta\}} = A_{\alpha\beta} : \mathrm{C}_2$ is not a maximal subgroup of A, a contradiction. By a complete similar methods with above case, we can exclude the lines 4, 5, 6, 22 and 23 of [16, Table 1.1].

Assume that $A \cong \mathrm{A}_{3481}.\mathcal{O}$ and $A_\alpha \cong \mathrm{C}_{59}^2 : (\mathrm{SL}_2(5) \times \mathrm{C}_{29})$ or $\mathrm{A}_{3479}.\mathcal{O}$, just as the line 11 of [16, Table 1.1]. Followed from [4, Tables 7.3 and 7.4], $(A_\alpha, A_{\alpha\beta})$ is isomorphic to $(\mathrm{C}_{29}^2 : (\mathrm{SL}_2(5) \times \mathrm{C}_7), \mathrm{SL}_2(5) \times \mathrm{C}_7)$ whenever $k = 29^2$ or $(\mathrm{A}_{839}.\mathcal{O}, \mathrm{A}_{838}.\mathcal{O})$ whenever $k = 839$. However, by the maximal subgroups structure of A which is given in [22, Theorem], we deduce that $A_{\{\alpha,\beta\}}$ is not a maximal subgroup of A, a contradiction. Similarly, we can exclude the lines 8, 9 and 10 of [16, Table 1.1].

Assume that $A \cong \mathrm{SL}_4(4).(\mathrm{C}_2 \times \mathcal{O})$ and $A_\alpha \cong (\mathrm{SL}_2(16).\mathrm{C}_4) \times \mathcal{O}_1$ or $\mathrm{SL}_3(4).(\mathrm{C}_2 \times \mathcal{O}_2)$, just as the line 12 of the [16, Table 1.1]. If $A_\alpha \cong \mathrm{SL}_3(4).(\mathrm{C}_2 \times \mathcal{O}_2)$, then detail checking the [4, Tables 7.3 and 7.4], one yields that $A_\alpha^{\Gamma(\alpha)}$ is an almost simple group with $\mathrm{soc}(A_\alpha^{\Gamma(\alpha)}) \cong \mathrm{PSL}_3(4)$. Further, combining with the formulas (1) and (2) and the Lemma 3, one yields that A_α has no 2-transitive permutation representation, contradicting with the assumption that Γ is 2-arc-transitive. Now that $A_\alpha \cong (\mathrm{SL}_2(16).\mathrm{C}_4) \times \mathcal{O}_1$. By Lemma 2, $A_\alpha \cong A_\alpha^{\Gamma(\alpha)}$ is an almost simple group with $\mathrm{soc}(A_\alpha) \cong \mathrm{SL}_2(16)$. Followed from [4, Table 7.4], $k = 17$ and $A_{\alpha\beta} \cong ((\mathrm{C}_2^4 : \mathrm{C}_{15}).\mathrm{C}_4) \times \mathcal{O}_1$. However, by the maximal subgroups structure of A which is given in [1, Tables 8.8 and 8.9], we deduce that $A_{\{\alpha,\beta\}} = A_{\alpha\beta} : \mathrm{C}_2$ is not a maximal subgroup of A, a contradiction. By the complete similar methods with above case, we can exclude the lines 13, 14, 15, 16, 17, 18, 19, 20 and 21 of [16, Table 1.1].

Finally, we complete the proof of Lemma 21. □

5 Bi-Cayley graphs

In this section, we will deal with the edge-primitive 2-arc-transitive bi-Cayley graphs over group G with valency at least 3. Let Γ be a symmetric bi-Cayley graph over a finite group G with two parts Δ_1 and Δ_2, and $A = \mathrm{Aut}\Gamma$. Set A^+ is the stabilizer of A on Δ_1 and Δ_2, and $\{\alpha, \beta\} \in E\Gamma$. Then $A^+ = GA_\alpha^+$ is an exactly factorization. Now we always assume that Γ satisfies the following condition in this section:

Condition 5.1. Let $\Gamma = \mathrm{BiCay}(G, S)$ be an edge-primitive 2-arc-transitive bi-Cayley graph over group G of valency k for $k \geqslant 3$ integer and $A = \mathrm{Aut}\Gamma$. Assume that A is almost simple and $T = \mathrm{soc}(A)$ is a nonabelian simple group.

Under the above condition, the following properties are useful.

Lemma 22. *Let Γ satisfy Condition 5.1. Then A^+ is also almost simple with $\mathrm{soc}(A^+) = \mathrm{soc}(A)$, and in particular, G is core-free in A^+.*

Proof. Clearly, A^+ is almost simple. Let $T = \mathrm{soc}(A)$. Suppose that $T \nleqslant A^+$. Then the normality of T and A^+ leads to $T \cap A^+ \trianglelefteq T$ and so $T \cap A^+ = 1$. Thus, $A = T \times A^+$ shows that $T \cong \mathrm{C}_2$ as $|A : A^+| = 2$, a contradiction has arisen. Hence, $T \leqslant A^+$ and so the first part of Lemma 22 holds.

Note that Γ is an edge-primitive 2-arc-transitive bipartite graph. By Lemma 6, we can conclude that Γ is vertex-biquasiprimitive, that is A^+ acts quasiprimitively on Δ_1 and Δ_2. Since G is regular on Δ_1 and Δ_2, we deduce that either $G \trianglelefteq A^+$ or G is a core-free subgroup of A^+. Assume that $G \trianglelefteq A^+$. Since A^+ is almost simple with $T = \mathrm{soc}(A^+) = \mathrm{soc}(A)$, one can conclude that $T = G \trianglelefteq A$ and so the graph Γ is a normal bi-Cayley graph. Consequently, $A^+ = G : A_\alpha$ and $A_\alpha \lesssim A^+/G \leqslant \mathrm{Out}(T)$. Followed from the fact that A_α is 2-transitive on $\Gamma(\alpha)$ with $|\Gamma(\alpha)| = k \geqslant 3$, one yields that $A_{\alpha\beta}$ is maximal in A_α and T is a simple group of Lie type with $k(k-1) \mid |\mathrm{Out}(T)|$. Employing the result of Lemma 13, either $|A_\alpha : A_{\alpha\beta}|$ is prime, or $T \cong \mathrm{P}\Omega_8^+(q)$ for q odd and $|A_\alpha : A_{\alpha\beta}| = 4$. For the latter, one deduces that Γ is an edge-primitive tetravalent graph. By checking the list of edge-primitive tetravalent graphs (see [9, Theorem 1.1] for the list), we can rule out this case easily. Now we consider the case where $k = |A_\alpha : A_{\alpha\beta}|$ is prime.

Suppose that $k = 3$. Then Γ is a cubic edge-primitive graph, which is one of the graphs listed in Lemma 10. If Γ is isomorphic to the complete bipartite graph $\mathsf{K}_{3,3}$, then $A \cong \mathrm{S}_3 \wr \mathrm{C}_2$, contrary to that A is almost simple. If Γ is isomorphic to the Heawood graph of order 14, then $A \cong \mathrm{PGL}_2(7)$ (see [14, Proposition 1] for example), and so $A^+ = G = T \cong \mathrm{PSL}_2(7)$ and $A_\alpha = 1$ contradicting with the fact that Γ is 2-arc-transitive. If Γ is isomorphic to the Biggs-Smith cubic distance transitive graph of order 102, then $A \cong \mathrm{PSL}_2(17)$ contradicting to $A = A^+.\mathrm{C}_2$. If Γ is isomorphic to the Tutte-Coxeter graph of order 30, then $A \cong \mathrm{P\Gamma L}_2(9)$ and the assumption that $\mathrm{PSL}_2(9) \cong T = G \trianglelefteq A^+ \trianglelefteq A$ leads to that $A^+ \cong \mathrm{PSL}_2(9).\mathrm{C}_2$ and so $A_\alpha \leqslant \mathrm{C}_2$, a contradiction arisen as $|A_\alpha| = |A|/30 = 48$.

Now that $k \geqslant 5$. Note that $\mathrm{Out}(T)$ is solvable, so are A_α and $A_{\{\alpha,\beta\}}$. According to [20, Theorem 1.1], we can show that Γ is a connected s-transitive graph with $s = 2$ or 3. We want to show that Γ is not isomorphic to a complete graph K_{k+1} or a complete bipartite graph $\mathsf{K}_{k,k}$. Indeed, if not, then $A \cong \mathrm{S}_{k+1}$ or $\mathrm{S}_k \wr \mathrm{C}_2$ respectively. The latter case contradicts to the assumption that A is almost simple and so $A \cong \mathrm{S}_{k+1}$ with $T \cong \mathrm{A}_{k+1}$ and $A_\alpha \leqslant \mathrm{C}_2$, a contradiction arisen as Γ is 2-arc-transitive. Further, by checking the classification of edge-primitive graphs of prime valency with solvable edge stabilizers given in [26, Theorem 1.2], one yields that such Γ does not exist. To sum up, G is not normal in A^+ and so the Lemma 22 holds. □

Due to the above result, A^+ is an almost simple group with $\mathrm{soc}(A^+) = \mathrm{soc}(A) = T$ and $A^+ = GA_\alpha$ is an exact factorization with G being core-free in A^+. Then the candidates of (A^+, G, A_α) can be get in the [16, Theorem 1.2] (both of A_α and G are insolvable) and [2, Theorem 1.3] (one of A_α and G is solvable). On the other hand, we can easily deduce that $A_{\{\alpha,\beta\}} = A_{\alpha\beta} : \mathrm{C}_2$ is a maximal subgroup of $A = A^+ : \mathrm{C}_2$ with $A^+_{\alpha\beta} = A_{\alpha\beta}$ and $A^+_\alpha = A_\alpha$.

Lemma 23. *If Γ satisfies Condition 5.1, then one of G and A_α is solvable.*

Proof. Assume that both of G and A_α are insolvable. By Lemma 22, A^+ is also almost simple with $T = \mathrm{soc}(A^+) = \mathrm{soc}(A)$, in particular, $A^+ = GA_\alpha$ is an exact factorization

and G is core-free in A^+. Derived from [16, Theorem 1.2], one yields that the triples (A^+, G, A_α) are given in [16, Table 1.1]. With a detailed inspection in [16, Table 1.1], since A_α has a 2-transitive permutation representation of degree k and $A_{\{\alpha,\beta\}}$ is a maximal subgroup of A, combining with [4, Tables 7.3 and 7.4], we can get all the candidates for the triples (A^+, G, A_α). Since $|A : A^+| = 2$, by the consideration of the outer automorphism group of A^+, we can rule out immediately the lines 13-17, 18, 22 and 23 of [16, Table 1.1]. Further, by the proof of Lemma 21, we can deduce that the other lines of [16, Table 1.1] can be excluded, except for lines 1 and 2.

Suppose that $A^+ \cong \mathrm{A}_n.\mathcal{O}$ and $\{A_\alpha, G\} = \{[n].\mathcal{O}_1, \mathrm{A}_{n-1}.\mathcal{O}_2\}$, just as the line 1 of [16, Table 1.1]. Since both of A_α and G are insolvable, one yields that $n \geqslant 60$. Noticing that $A = A^+ : \mathrm{C}_2$, one yields that $A^+ \cong \mathrm{A}_n$, $A \cong \mathrm{S}_n$ and $A_\alpha \cong [n]$ or A_{n-1}. If $A_\alpha \cong \mathrm{A}_{n-1}$, then $A_{\alpha\beta} \cong \mathrm{A}_{n-2}$ and $A_{\{\alpha,\beta\}} \cong \mathrm{S}_{n-2}$. Note that $A \cong \mathrm{S}_n$ has no maximal subgroup isomorphic to S_{n-2}. Thus this case does not occur. Now assume that $A_\alpha \cong [n]$ and $G \cong \mathrm{A}_{n-1}$. Since $|A_{\{\alpha,\beta\}}| = 2|A_\alpha|/k < |A_\alpha| = n$ as $k \geqslant 3$ is an integer, and by referring to Lemma 12, one yields that $A_{\{\alpha,\beta\}}$ is not a maximal subgroup of A. Hence this case is excluded.

Assume that $A^+ \cong \mathrm{A}_{q+1}.\mathcal{O}$ and $A_\alpha \cong \mathrm{PSL}_2(q).(\mathrm{C}_{(2,q-1)} \times \mathcal{O}_1)$ or $\mathrm{A}_{q-2}.\mathcal{O}_2$, just as the line 2 of [16, Table 1.1]. Since both of G and A_α are insolvable, one yields that $q - 2 \geqslant 5$ and so $q \geqslant 7$ and $q + 1 \geqslant 8$. It follows that $(A^+, A) \cong (\mathrm{A}_{q+1}, \mathrm{S}_{q+1})$ as $A = A^+ : \mathrm{C}_2$, in particular, $\mathcal{O} = \mathcal{O}_1 = \mathcal{O}_2 = 1$. (I). At first, we suppose that $A_\alpha \cong \mathrm{PSL}_2(q).\mathrm{C}_{(2,q-1)}$. If $q = 8$, then $A \cong \mathrm{S}_9$ and $A_\alpha \cong \mathrm{PSL}_2(8)$. Derived from [4, Table 7.4] and [5, Page 6], one yields that $k = 9$ and $A_{\alpha\beta} \cong \mathrm{C}_2^3 : \mathrm{C}_7$ or $k = 28$ and $A_{\alpha\beta} \cong \mathrm{D}_{18}$. Then $A_{\{\alpha,\beta\}} \cong (\mathrm{C}_2^3 : \mathrm{C}_7) : \mathrm{C}_2$ or $\mathrm{D}_{18} : \mathrm{C}_2$, respectively. Checking the list of the maximal subgroups of S_9 (see [5, Page 37] for the list), we yield that neither $(\mathrm{C}_2^3 : \mathrm{C}_7) : \mathrm{C}_2$, nor $\mathrm{D}_{18} : \mathrm{C}_2$ is a maximal subgroup of A (up to isomorphic), a contradiction. Suppose that $q = 11$, then $A \cong \mathrm{S}_{12}$ and $A_\alpha \cong \mathrm{PSL}_2(11) : \mathrm{C}_2$. Derived from [4, Table 7.4] and [5, Page 7], one yields that $k = 12$ and $A_{\alpha\beta} \cong \mathrm{C}_{11} : \mathrm{C}_{10}$. Then $A_{\{\alpha,\beta\}} \cong (\mathrm{C}_{11} : \mathrm{C}_{10}) : \mathrm{C}_2$ and the list of the maximal subgroups of $A \cong \mathrm{S}_{12}$ (see [5, Page 91] for the list) shows that $A_{\{\alpha,\beta\}}$ is not a maximal subgroup of A, a contradiction. Now suppose that $q \geqslant 7$ and $q \neq 8, 11$. Note that $(A^+, A) \cong (\mathrm{A}_{q+1}, \mathrm{S}_{q+1})$ and $A_\alpha \cong \mathrm{PSL}_2(q).\mathrm{C}_{(2,q-1)}$. Followed from [4, Table 7.4], one yields that $A_{\alpha\beta} \cong (\mathrm{C}_p^f : \mathrm{C}_{\frac{q-1}{(2,q-1)}}).\mathrm{C}_{(2,q-1)}$ and $A_{\{\alpha,\beta\}} \cong ((\mathrm{C}_p^f : \mathrm{C}_{\frac{q-1}{(2,q-1)}}).\mathrm{C}_{(2,q-1)}) : \mathrm{C}_2$ are solvable. By [11, Theorem 1.1] and [7, Theorem 1], we yield that this case does not occur. (II). Assume that $A_\alpha \cong \mathrm{A}_{q-2}$. Note that $q - 2 \geqslant 5$. If $\mathrm{A}_{q-2} = \mathrm{A}_5 \cong \mathrm{PSL}_2(4) \cong \mathrm{PSL}_2(5)$, then $k = 5$ and $A_{\alpha\beta} \cong \mathrm{A}_4$, or $k = 6$ and $A_{\alpha\beta} \cong \mathrm{D}_{10}$, in particular, $q = 7$ and $A \cong \mathrm{S}_8$. It follows that $A_{\{\alpha,\beta\}} \cong \mathrm{S}_4$ or $\mathrm{D}_{10} : \mathrm{C}_2$. Checking the list of the maximal subgroups of S_7 (see [5, Page 22] for the list), we yield that A has no maximal subgroup isomorphic to S_4 or $\mathrm{D}_{10} : \mathrm{C}_2$, a contradiction. Assume that $\mathrm{A}_{q-2} = \mathrm{A}_6 \cong \mathrm{PSL}_2(9)$. Then $k = 6$ and $A_{\alpha\beta} \cong \mathrm{A}_5$, or $k = 10$ and $A_{\alpha\beta} \cong \mathrm{C}_3^2 : \mathrm{C}_4$, in particular, $q = 8$ and $A \cong \mathrm{S}_9$. By the structure of the subgroups of A (see [5, page 37] for example), we can deduce that $A_{\{\alpha,\beta\}} = A_{\alpha\beta} : \mathrm{C}_2$ is not a maximal subgroup of A in those cases, a contradiction. Suppose that $q - 2 = 7$. Then $A_\alpha \cong \mathrm{A}_7$ and $A \cong \mathrm{S}_{10}$. Derived from [4, Table 7.4], one yields that either $k = 7$ and $A_{\alpha\beta} \cong \mathrm{A}_6$, or $k = 15$ and $A_{\alpha\beta} \cong \mathrm{PSL}_2(7)$. By the structure of the subgroups of $A \cong \mathrm{S}_{10}$ (see [5, page 48] for example), we can deduce that $A_{\{\alpha,\beta\}} = A_{\alpha\beta} : \mathrm{C}_2$ is not a maximal subgroup of A in those cases, a contradiction. Suppose that $q - 2 = 8$. Then

$A \cong \mathrm{S}_{11}$ and $A_\alpha \cong \mathrm{A}_8 \cong \mathrm{PSL}_4(2)$. It follows that $k = 8$ and $A_{\alpha\beta} \cong \mathrm{A}_7$, or $k = 15$ and $A_{\alpha\beta} \cong \mathrm{C}_2^3 : \mathrm{PSL}_3(2)$. By the structure of the subgroups of $A \cong \mathrm{S}_{11}$ (see [5, page 75] for example), we can deduce that $A_{\{\alpha,\beta\}} = A_{\alpha\beta} : \mathrm{C}_2$ is not a maximal subgroup of A in those cases, a contradiction. Now that $q - 2 \geqslant 9$. Then $k = q - 2$ and $A_{\alpha\beta} \cong \mathrm{A}_{q-3}$ (see [4, Table 7.4] for example) and so $A_{\{\alpha,\beta\}} \cong \mathrm{S}_{q-3}$ is insolvable. It is clear that $A_{\{\alpha,\beta\}}$ is not a maximal subgroup of $A \cong \mathrm{S}_q$, a contradiction. □

Lemma 24. *If Γ satisfies Condition 5.1 and G is insolvable, then A_α is insolvable.*

Proof. Assume that G is insolvable and A_α is solvable. Derived from Lemma 22, one yields that A^+ is an almost simple group with $T = \mathrm{soc}(A^+) = \mathrm{soc}(A)$ and G is core-free in A^+. On the other hand, by Lemma 6, one yields that A is biquasiprimitive on $V\Gamma$. Since $A^+ = GA_\alpha$ is an exact factorization with A_α being solvable, one can get all the candidates of (A^+, G, A_α) in the Lemma 5. However, since A_α, $A_{\alpha\beta}$ and $A_{\{\alpha,\beta\}}$ are solvable, the tuple $(A^+, A_\alpha, A_{\{\alpha,\beta\}})$ can be get from [11, Theorem 1.1]. Noting that A acts primitively on the edges of Γ, one yields that $A_{\{\alpha,\beta\}}$ is a solvable maximal subgroup of A. Followed from [18, Theorem 1.1 (ii)], one yields that there exist a normal subgroup of A^+ denoted by L, which is minimal such that $L_{\{\alpha,\beta\}} = L \cap A_{\{\alpha,\beta\}}$ is a maximal subgroup of L for $T = \mathrm{soc}(A) = \mathrm{soc}(L)$ and $(L, L_{\{\alpha,\beta\}})$ is given in [18, Tables 14-20]. Note that A_α has a 2-transitive permutation representation of degree k and we can get the structure of $(A_\alpha, A_{\alpha\beta})$ from [4, Tables 7.3 and 7.4] and formulas (1) and (2), in particular, both of L_α and T_α are transitive on $\Gamma(\alpha)$ as $T_\alpha \trianglelefteq L_\alpha \trianglelefteq A_\alpha$ and $T_\alpha \trianglelefteq A_\alpha$.

(I). Assume that the part (i) or part (ii) of the Lemma 5 holds, that is $T \cong \mathrm{A}_n$. For part (i), $G \cong \mathrm{A}_{n-1}$ or S_{n-1}, and A_α is a transitive subgroup of A on $\{1, \cdots, n\}$ of order n, and for part (ii), $n = p^a$ is a prime power, $A_\alpha \leqslant \mathrm{A\Gamma L}_1(p^a) \cong \mathrm{C}_p^a : \mathrm{C}_{p^a-1}$ is 2-homogeneous on $\{1, \cdots, n\}$ and $\mathrm{A}_{n-2} \trianglelefteq G \leqslant \mathrm{S}_{n-2} \times \mathrm{S}_2$. Since $T \cong \mathrm{A}_n$ is a nonabelian simple group, $n \geqslant 5$. Proved that part (i) occurs. If $n = 6$, then $(A^+, A) \cong (\mathrm{A}_6, \mathrm{A}_6.\mathrm{C}_2)$ or $(\mathrm{A}_6.\mathrm{C}_2, \mathrm{A}_6.\mathrm{C}_2^2)$, and $G \cong \mathrm{A}_5$ or S_5, in particular, A_α is a solvable transitive subgroup of A on $\{1, 2, 3, 4, 5, 6\}$. Noting that $|A_\alpha| = |A^+ : G|$, we deduce that $|A_\alpha| = 6$ or 12. It follows that $|A_{\{\alpha,\beta\}}| \leqslant 4$ or 8 respectively, as $|A_{\{\alpha,\beta\}}| = 2|A_\alpha|/k$ for $k \geqslant 3$. By a detailed inspection on the list of the maximal subgroups of A (see [5, Page 4] for the list), we can deduce that A has no maximal subgroup with order being not greater than 8 and so this case dose not occur. Now that $n \neq 6$. Then $(A^+, A) \cong (\mathrm{A}_n, \mathrm{S}_n)$ and $G \cong \mathrm{A}_{n-1}$, and so $|A_\alpha| = |A^+ : G| = n$. Since $|A_{\{\alpha,\beta\}}| = 2|A_\alpha|/k$ for $k \geqslant 3$, one yields that $|A_{\{\alpha,\beta\}}| < |A_\alpha| = n$ contradicting to the result of Lemma 12. Suppose that the part (ii) holds. Then $n = p^a \neq 6$, and so $(A^+, A) \cong (\mathrm{A}_n, \mathrm{S}_n)$, in particular, $\mathrm{O}_p(A_\alpha) = \mathrm{C}_p^a$ and the valency of Γ is p^a. On the other hand, derived from the 2-transitivity of A_α of degree p^a, one deduces that $p^a \cdot (p^a - 1) \mid |A_\alpha|$ and $A_\alpha \cong \mathrm{C}_p^a : \mathrm{C}_{p^a-1}$, in particular, $G = A_{n-2}$ as $|G| = |A^+ : A_\alpha|$. However, by [18, Table 14], $(A, A_{\{\alpha,\beta\}}) \cong (\mathrm{A}_p, \mathrm{C}_p : \mathrm{C}_{p-1/2})$ for $p \neq 7, 11, 17, 23$, leading to $|A_{\alpha\beta}| = \frac{p(p-1)}{4}$ and $k = |A_\alpha : A_{\alpha\beta}| = 4$, a contradiction occurs as $n = k = p^a = 4$ and $T \cong \mathrm{A}_4$ is solvable. Hence these cases do not occur.

(II). Assume that Lemma 5.(iii) holds, that is $T = \mathrm{soc}(A) \cong \mathrm{PSL}_n(q)$, $T_\alpha = A_\alpha \cap T \leqslant \mathrm{C}_{\frac{q^n-1}{(q-1)d}} : \mathrm{C}_n$ and $\mathrm{C}_p^{(n-1)f} : \mathrm{SL}_{n-1}(q) \trianglelefteq G \cap T$ with $d = (n, q-1)$ and $q = p^f$. Note that the tuple $(L, L_{\{\alpha,\beta\}})$ is given in [18, Table 16] and $\frac{1}{2}|L_{\{\alpha,\beta\}}| = \frac{1}{k}|L_\alpha|$, i.e., $k = 2|L_\alpha|/|L_{\{\alpha,\beta\}}|$

for $k \geqslant 3$. Then $k = |L_\alpha|/|L_{\alpha\beta}| = |T_\alpha|/|T_{\alpha\beta}|$ is a factor of the integer $|\mathrm{C}_{\frac{q^n-1}{(q-1)d}} : \mathrm{C}_n|/|T_{\alpha\beta}|$ as $T_{\alpha\beta} \leqslant T_\alpha \leqslant \mathrm{C}_{\frac{q^n-1}{(q-1)d}} : \mathrm{C}_n$. Let τ be the transpose-inverse automorphism, σ be a field automorphism and δ be the diagonal automorphism.

Suppose that $L \cong \mathrm{PSL}_3(2)$, just as the line 5 or 6 of [18, Table 16]. Then $L = T$ and $(A^+, A) \cong (\mathrm{PSL}_3(2), \mathrm{PSL}_3(2).\mathrm{C}_2)$ with $T_{\{\alpha,\beta\}} \cong \mathrm{C}_7 : \mathrm{C}_3$ or S_4. Note that $T_\alpha \leqslant \mathrm{C}_7 : \mathrm{C}_3$. Since $|T_{\{\alpha,\beta\}}| = 2|T_{\alpha\beta}|$ is even, one deduces that $T_{\{\alpha,\beta\}} \ncong \mathrm{C}_7 : \mathrm{C}_3$. It follows that $|T_{\alpha\beta}| = \frac{1}{2}|T_{\{\alpha,\beta\}}| = 12$, contradicting to the assumption that $T_\alpha \leqslant \mathrm{C}_7 : \mathrm{C}_3$.

For the other lines of [18, Table 16], we can get contradiction by replacing A by A^+ in the corresponding proof of the part (II) of Lemma 20.

(III). Assume that Lemma 5.(iv) holds, that is $T = \mathrm{soc}(A) \cong \mathrm{PSp}_{2m}(q)$ with q even and $m \geqslant 3$, $T_\alpha = A_\alpha \cap T \leqslant \mathrm{C}_p^{mf} : \mathrm{C}_{q^m-1}.\mathrm{C}_m$ and $G \cap T = \Omega^-_{2m}(q)$. Then $q = 2^f$ for some positive f and the tuple $(L, L_{\{\alpha,\beta\}})$ is given in [18, Table 17]. Then we have $L \cong \mathrm{PSp}_4(q).\langle\mu\rangle$, $\mathrm{PSp}_4(3)$, $\mathrm{PSp}_4(4).\mathrm{C}_4$, $\mathrm{PSp}_6(2)$, $\mathrm{PSp}_6(3)$ or $\mathrm{PSp}_8(3)$, where μ is a graph automorphism defined in [18, Remark 10.4]. Note that $2m \geqslant 6$ and q is even, and one yields that $L \cong \mathrm{PSp}_6(2)$ and $L_{\{\alpha,\beta\}} \cong \mathrm{C}_2.[2^6] : \mathrm{S}_3^2$, just as the line 4 of [18, Table 17]. Then $T = L \cong \mathrm{PSp}_6(2)$ and $A^+ = L = T$ as $\mathrm{Out}(\mathrm{PSp}_6(2)) = 1$. However, $L_\alpha \leqslant \mathrm{C}_p^{mf} : \mathrm{C}_{q^m-1}.\mathrm{C}_m = \mathrm{C}_2^3 : \mathrm{C}_7.\mathrm{C}_3$, and $|L_\alpha| \leqslant 168 < 2304 = |L_{\alpha\beta}| = |L_{\{\alpha,\beta\}}|/2$, contradicting the fact that $L_{\alpha\beta} \leqslant L_\alpha$. Hence this case is excluded.

(IV). Suppose that Lemma 5.(v) holds, that is (A^+, A_α, G) is one of the cases listed in [2, Table 4]. On the other hand, the edge-primitive graphs with soluble edge-stabilizers are determined in Lemmas 10, [11, Theorem 1.1], [9, Theorem 1.1], [10, Theorem 1.1] and [18, Theorem 1.3]. By comparing the above results, we can conclude that there does not exist any graph satisfying our condition and so the this case does not occur.

Finally, we complete the proof of Lemma 24. □

Lemma 25. *Assume that* Γ *satisfies Condition 5.1. Then* Γ *is isomorphic to the Heawood graph, the point-line non-incidence graph and a graph given in Examples 14-18.*

Proof. By Lemma 22 and 24, we have that G is solvable. Moreover, by the Lemma 22, A^+ is an almost simple group with $\mathrm{soc}(A^+) = \mathrm{soc}(A) = T$, in particular, $A^+ = GA_\alpha$ is an exact factorization with G being core-free in A^+. Since G is solvable, the triples (A^+, G, A_α) can be get from the Lemma 5. Note that $A_\alpha^+ = A_\alpha$, A_α has a 2-transitive permutation representation of degree k, $A_{\{\alpha,\beta\}}$ is a maximal subgroup of A, $A = A^+ : \mathrm{C}_2$, $k = |A : A_{\alpha\beta}|$ and $A_{\{\alpha,\beta\}} = A_{\alpha\beta} : \mathrm{C}_2$.

(I). Assume that the part (i) of the Lemma 5 holds, namely, $T \cong \mathrm{A}_n$, H is transitive on $\{1, \cdots, n\}$ and $K \cong \mathrm{A}_{n-1}$ or S_{n-1} for $\{H, K\} = \{G, A_\alpha\}$. Suppose that $n = 5$. Then $A^+ \cong \mathrm{A}_5$ and $A \cong \mathrm{S}_5$. If $A_\alpha = K$, then $A_\alpha \cong \mathrm{A}_4$ as $A_\alpha \leqslant A^+ \cong \mathrm{A}_5$. Consequently, $k = 4$ and $(A_{\alpha\beta}, A_{\{\alpha,\beta\}}) \cong (\mathrm{C}_3, \mathrm{S}_3)$, a contradiction arisen as $A_{\{\alpha,\beta\}} \cong \mathrm{S}_3$ is not a maximal subgroup of $A \cong \mathrm{S}_5$. Assume that $A_\alpha = H$ and $G = K$. Then $G \cong \mathrm{A}_4$ and $A_\alpha \cong \mathrm{C}_5$ contradicting the assumption that A_α has a 2-transitive permutation representation. Suppose that $n = 6$, i.e., $T \cong \mathrm{A}_6$. Then either $(A^+, A) \cong (\mathrm{A}_6, \mathrm{A}_6.\mathrm{C}_2)$ with $A_\alpha \cong \mathrm{A}_5$, or $(\mathrm{A}_6.\mathrm{C}_2, \mathrm{A}_6.\mathrm{C}_2^2)$ with $A_\alpha \cong \mathrm{A}_5$ or S_5. For the former, $(A_{\alpha\beta}, k) = (\mathrm{A}_4, 5)$ or $(\mathrm{D}_{10}, 6)$, and $A_{\{\alpha,\beta\}} \cong \mathrm{S}_4$ or $\mathrm{C}_5 : \mathrm{C}_4$ respectively. Followed from the maximal subgroups of $A \cong \mathrm{S}_6$ (see [5, Page 4] for example), neither S_4 nor $\mathrm{C}_5 : \mathrm{C}_4$ can be isomorphic to a maximal subgroup

of $A \cong \mathrm{S}_6$, a contradiction. Now that $(A^+, A) \cong (\mathrm{A}_6.\mathrm{C}_2, \mathrm{A}_6.\mathrm{C}_2^2)$ with $A_\alpha \cong \mathrm{A}_5$ or S_5. Then $(A_{\alpha\beta}, k) = (\mathrm{A}_4, 5)$, $(\mathrm{D}_{10}, 6)$, $(\mathrm{S}_4, 5)$ or $(\mathrm{C}_5 : \mathrm{C}_4, 6)$. It follows that $A_{\{\alpha,\beta\}} \cong \mathrm{S}_4$, $\mathrm{D}_{10} : \mathrm{C}_2$, $\mathrm{S}_4 : \mathrm{C}_2$ or $(\mathrm{C}_5 : \mathrm{C}_4) : \mathrm{C}_2$ respectively. However, the maximal subgroups structure of $A \cong \mathrm{A}_6.\mathrm{C}_2^2$ (see [5, Page 4] for example) shows that all above candidates of $A_{\{\alpha,\beta\}}$ are not maximal in A, a contradiction. Assume that $n = 7$, i.e., $(A^+, A, A_\alpha) \cong (\mathrm{A}_7, \mathrm{S}_7, \mathrm{A}_6)$. Then either $k = 6$ and $(A_{\alpha\beta}, A_{\{\alpha,\beta\}}) \cong (\mathrm{A}_5, \mathrm{S}_5)$, or $k = 10$ and $(A_{\alpha\beta}, A_{\{\alpha,\beta\}}) \cong (\mathrm{C}_3^2 : \mathrm{C}_4, (\mathrm{C}_3^2 : \mathrm{C}_4) : \mathrm{C}_2)$. However, the maximal subgroups structure of S_7 (see [5, Page 10] shows that all above candidates of $A_{\{\alpha,\beta\}}$ are not maximal in A, a contradiction. Suppose that $n = 8$, i.e., $(A^+, A, A_\alpha) \cong (\mathrm{A}_8, \mathrm{S}_8, \mathrm{A}_7)$. Then either $k = 7$ and $(A_{\alpha\beta}, A\{\alpha, \beta\}) \cong (\mathrm{A}_6, \mathrm{S}_6)$, or $k = 15$ and $(A_{\alpha\beta}, A\{\alpha, \beta\}) \cong (\mathrm{PSL}_2(7), \mathrm{PSL}_2(7) : \mathrm{C}_2)$ (see [3, Table 7.4] for example). Checking the maximal subgroups structure of S_8 (see [5, Page 22] for example), we can deduce that $A_{\{\alpha,\beta\}} \cong \mathrm{PSL}_2(7) : \mathrm{C}_2$, $k = 15$ and $(A^+, A, A_\alpha) \cong (\mathrm{A}_8, \mathrm{S}_8, \mathrm{A}_7)$. It follows that $|V\Gamma| = 2|G| = 16$ and Γ is isomorphic to the complete graph K_{16}. Clearly, it is impossible as by the assumption, Γ is a bipartite graph. Assume that $n = 9$, i.e., $(A^+, A, A_\alpha) \cong (\mathrm{A}_9, \mathrm{S}_9, \mathrm{A}_8)$. Then either $k = 8$ and $(A_{\alpha\beta}, A_{\{\alpha,\beta\}}) \cong (\mathrm{A}_7, \mathrm{S}_7)$, or $k = 15$ and $(A_{\alpha\beta}, A_{\{\alpha,\beta\}}) \cong (\mathrm{C}_2^3 : \mathrm{PSL}_3(2), (\mathrm{C}_2^3 : \mathrm{PSL}_3(2)) : \mathrm{C}_2)$ (see [3, Table 7.4] and [5, Page 22] for example). By the maximal subgroups structure of S_9 (see [5, Page 37] for example), one yields that all above candidates of $A_{\{\alpha,\beta\}}$ are not maximal in $A \cong \mathrm{S}_9$, a contradiction. Now suppose that $n \geqslant 10$. Then $(A^+, A, A_\alpha) \cong (\mathrm{A}_n, \mathrm{S}_n, \mathrm{A}_{n-1})$. It follows that $k = n - 1$ and $(A_{\alpha\beta}, A_{\{\alpha,\beta\}}) \cong (A_{n-2}, \mathrm{S}_{n-2})$. It is clear that S_{n-2} is not a maximal subgroup of S_n for $n \geqslant 10$, a contradiction.

(II). Assume that the part (ii) of Lemma 5 holds, that is $T \cong \mathrm{A}_n$, $n = p^a$, $H \leqslant \mathrm{A\Gamma L}_1(p^a)$ is 2-homogeneous on $\{1, \ldots, n\}$ and $A_{n-2} \trianglelefteq K \leqslant \mathrm{S}_{n-2} \times \mathrm{S}_2$ for $\{G, A_\alpha\} = \{H, K\}$, where p is some prime and a is some positive integer. Since $n = p^a$, one yields that $n \neq 6$ and so $(A^+, A) \cong (\mathrm{A}_n, \mathrm{S}_n)$. Assume that $n = 5$. Then $(A^+, A) \cong (\mathrm{A}_5, \mathrm{S}_5)$. If $A_\alpha = K$, i.e., $A_\alpha = A_\alpha^+ \cong \mathrm{A}_3$ or S_3, and so $k = 3$ and $(A_{\alpha\beta}, A_{\{\alpha,\beta\}}) \cong (1, \mathrm{C}_2)$ or $(\mathrm{C}_2, \mathrm{C}_2^2)$, respectively. It clear that neither C_2 nor C_2^2 is a maximal subgroup of S_5, a contradiction. Now that $G = K$ and $A_\alpha = H$. Then $G \cong \mathrm{A}_3$ or S_3 as $G \leqslant A^+ \cong \mathrm{A}_5$. It follows that $|A_\alpha| = 20$ or 10. Due to the 2-transitivity of A_α on $\Gamma(\alpha)$, on yields that $k = 5$ and $A_\alpha \cong \mathrm{C}_5 : \mathrm{C}_4$. Consequently, $A_{\alpha\beta} \cong \mathrm{C}_4$ and $A_{\{\alpha,\beta\}} \cong \mathrm{C}_4 : \mathrm{C}_2$. Derived from the subgroups construction of $A \cong \mathrm{S}_5$ (see [5, Page 2] for example), one yields that $A_{\{\alpha,\beta\}}$ is not a maximal subgroup of A. Now suppose that $n = 7$. Then $k = 5$ and $(A_\alpha, A_{\alpha\beta}, A_{\{\alpha,\beta\}}) \cong (\mathrm{A}_5, \mathrm{A}_4, \mathrm{S}_4)$, $k = 6$ and $(A_\alpha, A_{\alpha\beta}, A_{\{\alpha,\beta\}}) \cong (\mathrm{PSL}_2(4), \mathrm{D}_{10}, \mathrm{C}_5 : \mathrm{C}_4)$, $k = 5$ and $(A_\alpha, A_{\alpha\beta}, A_{\{\alpha,\beta\}}) \cong (\mathrm{S}_5, \mathrm{S}_4, \mathrm{S}_4 : \mathrm{C}_2)$ or $k = 6$ and $(A_\alpha, A_{\alpha\beta}, A_{\{\alpha,\beta\}}) \cong (\mathrm{PGL}_2(4), \mathrm{C}_5 : \mathrm{C}_4, (\mathrm{C}_5 : \mathrm{C}_4) : \mathrm{C}_2)$. The maximal subgroups structure of S_7 (see [5, Page 10] for example) shows that all above candidates of $A_{\{\alpha,\beta\}}$ are not maximal in A, a contradiction. Assume that $n = 8$. Then $(k, A_\alpha, A_{\alpha\beta}, A_{\{\alpha,\beta\}}) = (6, \mathrm{A}_6, \mathrm{A}_5, \mathrm{S}_5)$, $(10, \mathrm{PSL}_2(9), \mathrm{C}_3^2 : \mathrm{C}_4)$, $(6, \mathrm{S}_6, \mathrm{S}_5, \mathrm{S}_5 : \mathrm{C}_2)$, $(10, \mathrm{PSL}_2(9).\mathrm{C}_2, \mathrm{C}_3^2 : \mathrm{D}_8, (\mathrm{C}_3^2 : \mathrm{D}_8) : \mathrm{C}_2)$, $(10, \mathrm{PSL}_2(9).\mathrm{C}_2, \mathrm{C}_3^2 : \mathrm{C}_8, (\mathrm{C}_3^2 : \mathrm{C}_8) : \mathrm{C}_2)$ or $(10, \mathrm{PSL}_2(9).\mathrm{C}_2, \mathrm{C}_3^2 : \mathrm{Q}_8, (\mathrm{C}_3^2 : \mathrm{Q}_8) : \mathrm{C}_2)$ (see [5, Page 4] for example). Checking the maximal subgroups structure of S_8 (see [5, Page 22] for example), we can deduce that all above candidates of $A_{\{\alpha,\beta\}}$ are not maximal in A, a contradiction. Suppose that $n = 9$, i.e., $(A^+, A) \cong (\mathrm{A}_9, \mathrm{S}_9)$ and $A_\alpha \cong \mathrm{A}_7$ or S_7. Then $(k, A_{\alpha\beta}, A_{\{\alpha,\beta\}}) = (7, \mathrm{A}_6, \mathrm{S}_6)$, $(15, \mathrm{PSL}_2(7), \mathrm{PSL}_3(2) : \mathrm{C}_2)$ or $(7, \mathrm{S}_6, \mathrm{S}_6 : \mathrm{C}_2)$ (see [3, Table 7.4] and [5, Page 10] for

example). By the maximal subgroups structure of S_9 (see [5, Page 37] for example), one yields that all above candidates of $A_{\{\alpha,\beta\}}$ are not maximal in A, a contradiction. Now suppose that $n = p^a \geqslant 11$. Then $(A^+, A, A_\alpha) \cong (\mathrm{A}_n, \mathrm{S}_n)$ and $A_\alpha \cong \mathrm{A}_{n-2}$ or S_{n-2}. It follows that $k = n-2$ and $(A_{\alpha\beta}, A_{\{\alpha,\beta\}}) \cong (\mathrm{A}_{n-3}, \mathrm{S}_{n-3})$ or $(\mathrm{S}_{n-3}, \mathrm{S}_{n-3} : \mathrm{C}_2)$ respectively. It is clear that neither S_{n-3} nor $\mathrm{S}_{n-3} : \mathrm{C}_2$ is a maximal subgroup of S_n for $n = p^a \geqslant 11$ (see [22, Theorem] for example), a contradiction.

(III). Assume that the part (iii) of the Lemma 5 holds, that is $T \cong \mathrm{PSL}_n(q)$ and either $G \cap T \leqslant \mathrm{C}_{\frac{(q^n-1)}{(q-1)d}} : \mathrm{C}_n$ and $\mathrm{C}_p^{(n-1)f} : \mathrm{SL}_{n-1}(q) \trianglelefteq T_\alpha$, or $T_\alpha \leqslant \mathrm{C}_{\frac{(q^n-1)}{(q-1)d}} : \mathrm{C}_n$ and $\mathrm{C}_p^{(n-1)f} : \mathrm{SL}_{n-1}(q) \trianglelefteq G \cap T$, where $d = (n, q-1)$. For the latter case, since G is solvable, one yields that $\mathrm{SL}_{n-1}(q)$ is solvable and then either $n-1 = 1$, or $(n-1, q) = (2, 2)$ or $(2, 3)$. It leads to that $T \cong \mathrm{PSL}_2(q)$, $\mathrm{PSL}_3(2)$ or $\mathrm{PSL}_3(3)$ respectively. Consequently, $T_\alpha \leqslant \mathrm{C}_{\frac{q+1}{d}} : \mathrm{C}_2$, S_3 or C_2^2 respectively. On the other hand, since $A^+ = TA_\alpha$, $A^+/T = (TA_\alpha)/T \cong A_\alpha/(T \cap A_\alpha) = A_\alpha/T_\alpha$ and so $A_\alpha = T_\alpha.o$ for $o \cong A^+/T \leqslant \mathrm{Out}(T)$. Consequently, A_α is solvable and $A_{\alpha\beta}$ is solvable, so is $A_{\{\alpha,\beta\}}$. Thus the tuple $(A, A_\alpha, A_{\{\alpha,\beta\}})$ can be get from [11, Theorem 1.1]. Noting that A acts primitively on the edges of Γ, one deduces that $A_{\{\alpha,\beta\}}$ is a solvable maximal subgroup of A. Followed from [18, Theorem 1.1 (ii)], one yields that there exist a normal subgroup of A denoted by L, which is minimal such that $L_{\{\alpha,\beta\}} = L \cap A_{\{\alpha,\beta\}}$ is a maximal subgroup of L for $T = \mathrm{soc}(A) = \mathrm{soc}(L)$, and $(L, L_{\{\alpha,\beta\}})$ is given in [18, Tables 14-20]. Since A_α acts 2-transitively on $\Gamma(\alpha)$ and $T_\alpha \trianglelefteq A_\alpha$, T_α is transitive on $\Gamma(\alpha)$ and $k = |T_\alpha : T_{\alpha\beta}|$. Due to $|T_{\{\alpha,\beta\}}| = 2|T_{\alpha\beta}| = \frac{2|T_\alpha|}{k}$, one concludes that $|T_{\{\alpha,\beta\}}|$ is a factor of $\frac{4(q+1)}{kd}$, $\frac{12}{k}$ or $\frac{8}{k}$ respectively. On the other hand, detail checking the [18, Table 16], one yields that $T \cong \mathrm{PSL}_2(q)$ and $T_{\{\alpha,\beta\}}$ is a maximal subgroup of T such that $T_{\{\alpha,\beta\}} \cong \mathrm{D}_{\frac{2(q+1)}{d}}$, $\mathrm{D}_{\frac{2(q-1)}{d}}$, $\mathrm{C}_p^f : \mathrm{C}_{\frac{q-1}{d}}$, A_4 or S_4. However, non of them has order as a factor of $\frac{4(q+1)}{kd}$, $\frac{12}{k}$ or $\frac{8}{k}$, a contradiction arisen.

Now that $G \cap T \leqslant \mathrm{C}_{\frac{(q^n-1)}{(q-1)d}} : \mathrm{C}_n$ and $\mathrm{C}_p^{(n-1)f} : \mathrm{SL}_{n-1}(q) \trianglelefteq T_\alpha$. If $\mathrm{C}_p^{(n-1)f} : \mathrm{SL}_{n-1}(q)$ is solvable, then either $n-1 = 1$, or $(n-1, q) = (2, 2)$ or $(2, 3)$, and so $T \cong \mathrm{PSL}_2(q)$ with $\mathrm{C}_p^f \trianglelefteq T_\alpha$, $T \cong \mathrm{PSL}_3(2)$ with $\mathrm{C}_2^2 : \mathrm{SL}_2(2) \trianglelefteq T_\alpha$ or $T \cong \mathrm{PSL}_3(3)$ with $\mathrm{C}_3^2 : \mathrm{SL}_2(3) \trianglelefteq T_\alpha$. For the first case, that is $T \cong \mathrm{PSL}_2(q)$ with $\mathrm{C}_p^f \trianglelefteq T_\alpha$, due to the [8, Theorem 1.3], we observe that this case does not occur. Now that $T \cong \mathrm{PSL}_3(2)$ with $\mathrm{C}_2^2 : \mathrm{SL}_2(2) \trianglelefteq T_\alpha$. Then $A \cong \mathrm{PSL}_3(2).\mathrm{C}_2$ and $A^+ = T \cong \mathrm{PSL}_3(2)$. By the maximal subgroups of $T \cong \mathrm{PSL}_3(2)$ (see [5, Page 3] for example), we have $A_\alpha = T_\alpha \cong \mathrm{S}_4$ and either $k = 3$ and $A_{\alpha\beta} \cong \mathrm{D}_8$, or $k = 4$ and $A_{\alpha\beta} \cong \mathrm{S}_3$. Further, we can find that $A_{\{\alpha,\beta\}} \cong \mathrm{D}_8 : \mathrm{C}_2$ or $\mathrm{S}_3 : \mathrm{C}_2$ respectively. Checking the list of the maximal subgroup of $\mathrm{PSL}_3(2).\mathrm{C}_2$ (see [5, Page 3] for the list), we can get that $k = 3$ and $A_\alpha \cong \mathrm{S}_4$ with $A_{\{\alpha,\beta\}} \cong \mathrm{D}_{16}$ and $A \cong \mathrm{PSL}_3(2).\mathrm{C}_2$, in particular, $|V\Gamma| = |A : A_\alpha| = 14$. Due to the Lemma 10, Γ is the Heawood graph of order 14. Now we consider the case that $T \cong \mathrm{PSL}_3(3)$ and $\mathrm{C}_3^2 : \mathrm{SL}_2(3) \trianglelefteq T_\alpha$. Then $A \cong \mathrm{PSL}_3(3).\mathrm{C}_2$ and $A^+ = T$. Due to the maximal subgroups of $T \cong \mathrm{PSL}_3(3)$ (see [5, Page 13] for example), $A_\alpha = T_\alpha \cong \mathrm{C}_3^2 : 2\mathrm{A}_4$ or $\mathrm{C}_3^2 : \mathrm{S}_4$. Since A_α has 2-transitive permutation representation, one yields that $k = 9$ and $(A_\alpha, A_{\alpha\beta}, A_{\{\alpha,\beta\}}) \cong (\mathrm{C}_3^2 : 2\mathrm{A}_4, 2\mathrm{A}_4, 2\mathrm{A}_4 : \mathrm{C}_2)$ or $(\mathrm{C}_3^2 : 2\mathrm{A}_4, 2\mathrm{S}_4, 2\mathrm{S}_4 : \mathrm{C}_2)$. Detail checking the maximal subgroups of $A \cong \mathrm{PSL}_3(3).\mathrm{C}_2$ (see [5, Page 13] for example), we yield that $A_\alpha \cong \mathrm{C}_3^2 : 2\mathrm{S}_4$ and $|V\Gamma| = |A : A_\alpha| = 26$. Further,

under the help of the Magma, we find that Γ is isomorphic to a graph of the Example 14.

Now we suppose that $\mathrm{C}_p^{(n-1)f} : \mathrm{SL}_{n-1}(q)$ is insolvable. Then T_α is insolvable, so is A_α. It follows that $n-1 \geqslant 3$ or $n-1=2$ with $q \neq 2$ or 3. Followed from $T_\alpha \trianglelefteq A_\alpha$ and A_α acts 2-transitive on $\Gamma(\alpha)$, on yields that T_α is transitive on $\Gamma(\alpha)$. The irreducibility of the action of $\mathrm{SL}_{n-1}(q)$ on $\mathrm{C}_p^{(n-1)f}$ leads to $\mathrm{C}_p^{(n-1)f} \trianglelefteq T_\alpha$. Combing with the formulas (1) and (2), we can find that $\mathrm{C}_p^{(n-1)f}$ acts regularly on $\Gamma(\alpha)$ and so $k = q^{n-1}$ and $\mathrm{SL}_{n-1}(q) \trianglelefteq T_{\alpha\beta}$. It is clear that T_α acts 2-transitively on $\Gamma(\alpha)$. Due to the [3, Table 7.3], we observe that $\mathrm{SL}_{n-1}(q) \trianglelefteq T_{\alpha\beta} \leqslant \Gamma\mathrm{L}_{n-1}(q)$ and so $\mathrm{PSL}_{n-1}(q)$ is a unique insolvable composition factor of $T_{\alpha\beta}$ up to isomorphism, so is $T_{\{\alpha,\beta\}} = T_{\alpha\beta} : \mathrm{C}_2$. Further, we can deduce that $2|\mathrm{SL}_{n-1}(q)| \leqslant |T_{\{\alpha,\beta\}}| \leqslant 2|\Gamma\mathrm{L}_{n-1}(q)|$. On the other hand, since $A = T.\mathcal{O}$ and $A_{\{\alpha,\beta\}}$ is a maximal subgroup of A, we can deduce that $T_{\{\alpha,\beta\}} = T \cap A_{\{\alpha,\beta\}}$ can be get from the [13, Table 3.5.A]. However, since $T_{\{\alpha,\beta\}} = T_{\alpha\beta} : \mathrm{C}_2$ and $\mathrm{SL}_{n-1}(q) \trianglelefteq T_{\alpha\beta}$, one yields that $T_{\{\alpha,\beta\}}$ is not a parabolic subgroup of T. Checking the subgroups structure of $T \cong \mathrm{PSL}_n(q)$ (see [13, Table 3.5.A] for the list), we can find that $T_{\{\alpha,\beta\}}$ can be get from the [13, Proposition 1.4], that is $T_{\{\alpha,\beta\}} \cong [a^+_{1,n-1}/(q-1,n)].\mathrm{PSL}_{n-1}(q).[b^+_{1,n-1}]$, where $a^+_{1,n-1}$ and $b^+_{1,n-1}$ are given in [13, Page 86]. It is clear that Γ is isomorphic to the point-line non-incidence graph.

(IV). Assume that the part (iv) of Lemma 5 holds, that is $T = \mathrm{soc}(A) \cong \mathrm{PSp}_{2m}(q)$ with q even and $m \geqslant 3$. Derived from $m \geqslant 3$, one yields that $\Omega^-_{2m}(q)$ is insolvable and then $G \cap T \leqslant \mathrm{C}_q^m : \mathrm{C}_{q^m-1}.\mathrm{C}_m$ and $T_\alpha = \Omega^-_{2m}(q)$. Note that $A_\alpha = T_\alpha.O \cong \Omega^-_{2m}(q).O$ for $O \leqslant \mathrm{Out}(T)$. Then [3, Table 7.4] shows that $\Omega^-_{2m}(q).O$ has no 2-transitive permutation representation and so this case does not occur.

(V). Assume that the part (v) of Lemma 5 holds, that is (A^+, G, A_α) is one of the cases listed in [2, Table 4]. Since $|A : A^+| = 2$, we can exclude the lines 4, 7, 9, 11, 21, 29, 33, 34, 36, 37 and 40-47.

Suppose that $(A^+, A_\alpha) \cong (\mathrm{A}_8.\mathcal{O}, (\mathrm{A}_5 \times \mathrm{C}_3).\mathrm{C}_2.\mathcal{O})$, just as the line 1 of the [2, Table 4]. Since $|A : A^+| = 2$, one yields that $A^+ \cong \mathrm{A}_8$ and $A \cong \mathrm{S}_8$, in particular, $A_\alpha = A^+_\alpha \cong (\mathrm{A}_5 \times \mathrm{C}_3).\mathrm{C}_2$. Followed from [4, Tables 7.4], we have $A_{\alpha\beta} \cong (\mathrm{A}_4 \times \mathrm{C}_3).\mathrm{C}_2$ with $k = 5$ or $(\mathrm{D}_{10} \times \mathrm{C}_3).\mathrm{C}_2$ with $k = 6$. However, the maximal subgroups structure of A (see [5, Page 22] for example) shows that $A_{\{\alpha,\beta\}} = A_{\alpha\beta} : \mathrm{C}_2$ is not a maximal subgroup of A, a contradiction. By the complete similar method with above case, we can rule out the line 2 of [2, Table 4].

Assume that $(A^+, A_\alpha) \cong (\mathrm{A}_{32}.\mathcal{O}, (\mathrm{A}_{29} \times \mathrm{C}_3).\mathrm{C}_2.\mathcal{O})$, just as the line 5 of the [2, Table 4]. Then $A^+ \cong \mathrm{A}_{32}$ and $A \cong \mathrm{S}_{32}$. Followed from [4, Tables 7.3 and 7.4], we have $A_{\alpha\beta} \cong (\mathrm{A}_{28} \times \mathrm{C}_3).\mathrm{C}_2$ and $k = 29$. However, the maximal subgroups structure of $A \cong \mathrm{S}_{32}$ (see [22, Theorem]) shows that $A_{\{\alpha,\beta\}} = A_{\alpha\beta} : \mathrm{C}_2$ is isomorphic to a subgroup of $\mathrm{S}_{28} \times \mathrm{S}_4 < \mathrm{S}_{32}$ contradicting to that $A_{\{\alpha,\beta\}}$ is a maximal subgroup of A. On the same line with above case, the lines 3, 6, 8 and 10 of [2, Table 4] are excluded.

Assume that $(A^+, A_\alpha) \cong (\mathrm{PSL}_2(11).\mathcal{O}, \mathrm{A}_5)$, just as the case 12 of [2, Table 4]. Then $A^+ \cong \mathrm{PSL}_2(11)$ and $A \cong \mathrm{PSL}_2(11) : \mathrm{C}_2$. Followed from [4, Table 7.4], we have $A_{\alpha\beta} \cong \mathrm{A}_4$ with $k = 5$ or D_{10} with $k = 6$. Note that S_4 and D_{20} are maximal subgroups of $\mathrm{PSL}_2(11) : \mathrm{C}_2$ (see [5, Page 7] for example). With the help of Magma, we can deduce that if $A_{\alpha\beta} \cong \mathrm{A}_4$, then $A_{\{\alpha,\beta\}} = A_{\alpha\beta} : \mathrm{C}_2 \cong \mathrm{S}_4$ and the graph Γ is isomorphic to the graph given in Example 15, and if $A_{\alpha\beta} \cong \mathrm{D}_{10}$, then $A_{\{\alpha,\beta\}} = A_{\alpha\beta} : \mathrm{C}_2 \cong \mathrm{D}_{10} \times \mathrm{C}_2 \cong \mathrm{D}_{20}$ and

the graph Γ is isomorphic to the graph given in Example 15.

Suppose that $(A^+, A_\alpha) \cong (\mathrm{PSL}_2(11).\mathcal{O}, \mathrm{A}_4.\mathcal{O}_2)$, just as the line 13 of [2, Table 4]. Then $(A^+, A) \cong (\mathrm{PSL}_2(11), \mathrm{PSL}_2(11) : \mathrm{C}_2)$ and $A_\alpha \cong \mathrm{A}_4$. The 2-transitivity of A_α shows that $k = 4$ and $A_{\alpha\beta} \cong \mathrm{C}_3$ and so $A_{\{\alpha,\beta\}} = A_{\alpha\beta} : \mathrm{C}_2 \cong \mathrm{S}_3$. It is clear that S_3 is not a maximal subgroup of $\mathrm{PSL}_2(11) : \mathrm{C}_2$ (see [5, Page 7] for example), a contradiction. By the complete similar method with this case, we can rule out the lines 14, 17 and 18 of [2, Table 4].

Suppose that $(A^+, A_\alpha) \cong (\mathrm{PSL}_2(29), \mathrm{A}_5)$, just as the line 15 of [2, Table 4]. Then $A \cong \mathrm{PSL}_2(29) : \mathrm{C}_2$. Followed from [4, Table 7.4], one yields that $A_{\alpha\beta} \cong \mathrm{A}_4$ with $k = 5$ or D_{10} with $k = 6$. It follows that $A_{\{\alpha,\beta\}} = A_{\alpha\beta} : \mathrm{C}_2 \cong \mathrm{S}_4$ or $\mathrm{D}_{20} : \mathrm{C}_2$ respectively. Derived from the maximal subgroups structure of $A \cong \mathrm{PSL}_2(29) : \mathrm{C}_2$ (see [5, Page 20] for example), one yields that $A_{\{\alpha,\beta\}} \cong \mathrm{S}_4$ with $k = 4$ and by Magma, we find the graph Γ is given in Example 16.

Assume that $(A^+, A_\alpha) \cong (\mathrm{PSL}_2(59).\mathcal{O}, \mathrm{A}_5)$, just as the line 16 of [2, Table 4]. Then $A^+ \cong \mathrm{PSL}_2(59)$ and $A \cong \mathrm{PSL}_2(59) : \mathrm{C}_2$. Followed from [4, Table 7.4], we have $A_{\alpha\beta} \cong \mathrm{A}_4$ with $k = 5$ or D_{10} with $k = 6$. It follows that $A_{\{\alpha,\beta\}} = A_{\alpha\beta} : \mathrm{C}_2 \cong \mathrm{S}_4$ or $\mathrm{D}_{10} : \mathrm{C}_2$ respectively. With the help of Magma, we deduce that only S_4 is a maximal subgroup of $\mathrm{PSL}_2(59) : \mathrm{C}_2$ and the graph Γ is isomorphic to the graph given in Example 17 in this case.

Assume that $(A^+, A_\alpha) \cong (\mathrm{PGL}_4(3), ((\mathrm{C}_4 \times \mathrm{PSL}_2(9)) : \mathrm{C}_2).\mathrm{C}_2)$, just as the line 20 of [2, Table 4]. Then $A^+ \cong \mathrm{PGL}_4(3)$ and $A \cong \mathrm{PGL}_4(3).\mathrm{C}_2 \cong \mathrm{PSL}_4(3).\mathrm{C}_2^2$. Followed from [4, Table 7.4], we have $(k, A_{\alpha\beta}) \cong (6, ((\mathrm{C}_4 \times \mathrm{A}_5) : \mathrm{C}_2).\mathrm{C}_2)$ or $(10, ((\mathrm{C}_4 \times (\mathrm{C}_9 : \mathrm{C}_4)) : \mathrm{C}_2).\mathrm{C}_2)$. However, the maximal subgroups structure of A (see [5, Page 68] for example) shows that $A_{\{\alpha,\beta\}} = A_{\alpha\beta} : \mathrm{C}_2$ is not a maximal subgroup of A, a contradiction. On the same line with above case, the lines 22, 25, 26, 27, 28, 30, 31, 32 and 35 of [2, Table 4] are excluded.

Assume that $(A^+, A_\alpha) \cong (\mathrm{PSL}_5(2).\mathcal{O}, \mathrm{C}_2^6 : (\mathrm{S}_3 \times \mathrm{PSL}_3(2)))$, just as the line 23 of [2, Table 4]. Then $A^+ \cong \mathrm{PSL}_5(2)$ and $A \cong \mathrm{PSL}_5(2).\mathrm{C}_2$. Combing with [4, Tables 7.3 and 7.4] and formulas (1) and (2), we have $k = 2^6 = 64$ and $A_{\alpha\beta} \cong \mathrm{S}_3 \times \mathrm{PSL}_3(2)$, in particular, $A_{\{\alpha,\beta\}} \cong (\mathrm{S}_3 \times \mathrm{PSL}_3(2)) : \mathrm{C}_2$. By [5, Page 70], we can deduce that $A_{\{\alpha,\beta\}} \cong \mathrm{S}_3 \times \mathrm{PSL}_3(2) : \mathrm{C}_2$ is a maximal subgroup of $A \cong \mathrm{PSL}_5(2).\mathrm{C}_2$. Further, under the help of Magma, we can deduce that Γ is isomorphic to the graph given in Example 18. By the complete similar method with this case, we can rule out the lines 19 and 24 of [2, Table 4].

Suppose that $(A^+, A_\alpha) \cong (\mathrm{M}_{12}.\mathcal{O}, \mathrm{PSL}_2(11).\mathcal{O})$, just as the line 38 of [2, Table 4]. Then $(A^+, A, A_\alpha) \cong (\mathrm{M}_{12}, \mathrm{M}_{12} : \mathrm{C}_2, \mathrm{PSL}_2(11))$. Followed from [4, Table 7.4], we have $A_{\alpha\beta} \cong \mathrm{C}_{11} : \mathrm{C}_5$ with $k = 12$ or A_5 with $k = 11$. Then $A_{\{\alpha,\beta\}} = A_{\alpha\beta} : \mathrm{C}_2 \cong (\mathrm{C}_{11} : \mathrm{C}_5) : \mathrm{C}_2$ or S_5 respectively. Derived from the maximal subgroups structure of $A \cong \mathrm{M}_{12} : \mathrm{C}_2$ (see [5, Page 33] for example), one yields that $A_{\alpha\beta} \cong \mathrm{A}_5$ and $A_{\{\alpha,\beta\}} = A_{\alpha\beta} : \mathrm{C}_2 \cong \mathrm{S}_5$ with $k = 11$. However, computation by Magma shows that there dose not exist any 2-element x in $\mathbf{N}_G(G_{uv}) \setminus \mathbf{N}_G(G_u)$ such that $\langle x, G_u \rangle = G$, contradicting to Lemma 4.

Assume that $(A^+, A_\alpha) \cong (\mathrm{M}_{12}, \mathrm{M}_{11})$, just as the line 39 of [2, Table 4]. Then $A \cong \mathrm{M}_{12} : \mathrm{C}_2$ and $|V\Gamma| = 2|A^+ : A_\alpha| = 24$. Followed from [4, Table 7.4], we have $A_{\alpha\beta} \cong \mathrm{PSL}_2(11)$ with $k = 12$, or $A_{\alpha\beta} \cong \mathrm{M}_{10}$ with $k = 11$. It follows that $A_{\{\alpha,\beta\}} = A_{\alpha\beta} : \mathrm{C}_2 \cong \mathrm{PSL}_2(11) : \mathrm{C}_2$ or $\mathrm{M}_{10} : \mathrm{C}_2$, respectively. The maximal subgroups of A (see [5, Page 33] for example) shows that $k = 12$ and $A_{\{\alpha,\beta\}} \cong \mathrm{PSL}_2(11) : \mathrm{C}_2$ and the graph Γ is isomorphic to the complete

bipartite graph $\mathsf{K}_{12,12}$. This shows that $A \cong \mathrm{S}_{12} \wr \mathrm{S}_2$, a contradiction to $A \cong \mathrm{M}_{12} : \mathrm{C}_2$.

Finally, we complete the proof of Lemma 25. □

6 Proof of Theorem 1

Proof. Let Γ be an edge-primitive 2-arc-transitive Cayley graph or bi-Cayley graph with valency no less than three. Let $A = \mathrm{Aut}\Gamma$. First, we assume that Γ is a non-bipartite Cayley graph over a finite group G. By Lemma 9, one yields that A is an almost simple group. Let $T = \mathrm{soc}(A)$. By Lemmas 19, 20 and 21, the first half of Theorem 1 holds.

Next, suppose that Γ is a bi-Cayley graph over a finite group G. By Lemma 9, we can get that either Γ is a complete bipartite graph $\mathsf{K}_{m,m}$, or A is an almost simple group. For the former, since the valency of Γ is no less than 3, $m \geqslant 3$ and one part of the Theorem 1 holds. Now that A is an almost simple group. Then Lemma 22 shows that A^+ is almost simple with $\mathrm{soc}(A^+) = \mathrm{soc}(A)$, $A^+ = GA_\alpha$ is an exact factorization and G is core-free in A^+. Further, combing with Lemmas 23, 24 and 25, we can get the last part of the Theorem 1. □

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