A New Group in the Riordan Family of Matrix Groups: the Sprugnoli Group

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Dedicated to the memory of Renzo Sprugnoli

Abstract

We define a group of lower-triangular matrices whose columns are defined by power series. This group can be seen as a generalization of the (ordinary) Riordan group and the double Riordan group. Elements of this group are defined by three power series. Sequence bisections and vertically stretched Riordan arrays play an important role in the formulation of this group. We give a production matrix characterization of this new group. We have chosen to name this group in memory of Renzo Sprugnoli, who was a pioneer in the application of the Riordan group to combinatorial problems as well as contributing to an understanding of the rich structure of Riordan arrays. We indicate how higher order groups can be defined, based on n-tuples of power series and n-sections of sequences that they operate on.

Mathematics Subject Classifications: 15B36, 05A15, 11B83, 15A30, 20H25

1 Introduction

The Riordan group was first described by Shapiro, Getu, Woan, and Woodson [18] in 1991. Since then, it has played an important role at the interface of algebra, linear algebra and combinatorics. This connection across three domains occurs because the Riordan group has a linear representation by lower-triangular matrices. The operation of these matrices on vectors, whose elements arise from sequences, describe in linear fashion the operation of the group elements on power series, where these power series are the generating functions of the sequences in question. The group operation on power series is given by a weighted composition. Such weighted compositions play a role in all the generalizations of the Riordan group [6, 2, 17]. The weighted composition in the Riordan group has acquired the name of "the fundamental theorem" of Riordan arrays. Maintaining this tradition, for each generalization of the Riordan group, the relevant weighted composition that describes the group action on power series is called "the fundamental theorem" of the group.

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The goal of this paper is to introduce a new group whose group action on power series is again given by weighted compositions, but in this case this will involve a sum of weighted compositions involving the bisections of the power series acted upon. The group elements will be triples of power series, and each such triple will have a lower-triangular matrix representation. We note that the previously defined *double Riordan group* also consists of triples of power series, so we use a special notation (detailed below) to distinguish the double Riordan group from the new group introduced here.

In this paper, we shall use the usual notation (g, f) to denote an element of the (ordinary) Riordan group [1, 17, 18], the (non-standard) notation $(\!(g, f_1, f_2)\!)$ to denote an element of the double Riordan group [6], and (g, f_1, f_2) to denote an element of the set of Sprugnoli arrays (defined hereafter). Note that integer sequences that occur in this note, where known, will be referred to by their On-Line Encyclopedia of Integer Sequences (OEIS) reference [19, 20].

We recall that $\mathcal{F}_0 = \{\sum_{n=0}^{\infty} a_n x^n \,|\, a_0 \neq 0\}$ and that $\mathcal{F}_1 = \{\sum_{n=0}^{\infty} a_n x^n \,|\, a_0 = 0, a_1 \neq 0\}$ and in general

$$\mathcal{F}_r = \{\sum_{n=r}^{\infty} a_n x^n \,|\, a_r \neq 0\}.$$

Elements of $\mathcal{F}_0$ are multiplicatively invertible, and elements of $\mathcal{F}_1$ are compositionally invertible. For simplicity, we assume that the coefficients a_n are elements of a field $\mathbf{F}$, usually $\mathbb{R}$ or $\mathbb{C}$. Note that in our examples, coefficients will be drawn from the set of integers $\mathbb{Z}$.

The Riordan group is then the group of pairs $(g(x), f(x)) \in \mathcal{F}_0 \times \mathcal{F}_1$ with the following product rule

$$(g(x), f(x)) \cdot (u(x), v(x)) = (g(x)u(f(x)), v(f(x)) \tag{1}$$

and inverse

$$(g(x), f(x))^{-1} = \left(\frac{1}{g(\bar{f}(x))}, \bar{f}(x)\right),$$

where $\bar{f}(x)$ is the compositional inverse of $f(x) \in \mathcal{F}_1$ (that is, $\bar{f}$ is the solution u of the equation $f(u) = x$ for which $u(0) = 0$). The identity of this group is $(1, x)$. To each element of this group we can associate in a unique way a lower-triangular matrix $(t_{n,k})$ with entries in the ground ring R by means of

$$t_{n,k} = [x^n]g(x)f(x)^k,$$

where $[x^n]$ is the functional on $\mathbf{F}[[x]]$ that extracts the coefficient of x^n. Under this correspondence, the product (1) corresponds to ordinary matrix multiplication. Note that we have

$$t_{n,k} = [u^n]g(u)f(u)^k,$$

so we often write (g, f) for this array, and use an explicit argument for the power series involved when it helps comprehension.

The columns of (g, f) have their generating functions given by the geometric series of power series

$$g, gf, gf^2, gf^3, gf^4, \ldots.$$

The bi-variate generating function of the Riordan array $(g(x), f(x))$ is given by $\frac{g(x)}{1-yf(x)}$.

Example 1. The Riordan array $(g(x), f(x)) = \left(\frac{1}{1-x}, \frac{x}{(1-x)^2}\right)$ begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 & 0 & 0 \\ 1 & 6 & 5 & 1 & 0 & 0 & 0 \\ 1 & 10 & 15 & 7 & 1 & 0 & 0 \\ 1 & 15 & 35 & 28 & 9 & 1 & 0 \\ 1 & 21 & 70 & 84 & 45 & 11 & 1 \end{pmatrix}.$$

We have $t_{n,k} = \binom{n+k}{2k}$. The bi-variate generating function of this array is given by

$$\frac{\frac{1}{1-x}}{1 - y\frac{x}{(1-x)^2}} = \frac{1-x}{1-x(y+2)+x^2}.$$

When $y = 1$, we obtain the generating function $\frac{1-x}{1-3x+x^2}$ of the row sums of this matrix. Thus the row sums begin

$$1, 2, 5, 13, 34, 89, 233, 610, 1597, 4181, 10946, \ldots,$$

or F_{2n+1} A122367.

The product law follows from the following result, called the "fundamental theorem of Riordan arrays", which details how a Riordan array operates on a power series. We have

$$(g(x), f(x)) \cdot h(x) = g(x)h(f(x)).$$

This is sometimes called a weighted composition.

A similar law holds for so-called "vertically stretched" Riordan arrays [5]. These are matrices corresponding to pairs of power series of the form $(g, xf(x))$ where as usual, $f(x) \in \mathcal{F}_1$, and so $xf(x) \in \mathcal{F}_2$. Again, we have a weighted composition action on power series given by

$$(g(x), xf(x)) \cdot h(x) = g(x)h(xf(x)).$$

The bi-variate generating function of the stretched Riordan array $(g, xf(x))$ is given by

$$(g(x), xf(x)) \cdot \frac{1}{1-xy} = \frac{g(x)}{1-yxf(x)}.$$

In particular, the generating function of the row sums of the stretched Riordan array (g, xf) will have generating function $\frac{g(x)}{1-xf(x)}$.

Example 2. The stretched Riordan array $\left(\frac{1}{1-x}, \frac{x^2}{1-x-x^2}\right)$ begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 7 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 12 & 8 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 20 & 18 & 4 & 0 & 0 & 0 & 0 & 0 \\ 1 & 33 & 38 & 13 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We have, for instance,

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 7 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 12 & 8 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 20 & 18 & 4 & 0 & 0 & 0 & 0 & 0 \\ 1 & 33 & 38 & 13 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 2 \\ 3 \\ 5 \\ 8 \\ 13 \\ 21 \\ 34 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 3 \\ 7 \\ 14 \\ 32 \\ 69 \\ 159 \end{pmatrix},$$

where the last sequence has generating function

$$\left(\frac{1}{1-x}, \frac{x^2}{1-x-x^2}\right) \cdot \frac{1}{1-x-x^2} = \frac{\frac{1}{1-x}}{1 - \frac{x^2}{1-x-x^2} - (\frac{x^2}{1-x-x^2})^2},$$

which is equal to

$$\frac{(1-x-x^2)^2}{(1-x)(1-2x+3x^3+x^4)}.$$

The *double Riordan group* [6] is a generalization of the checkerboard subgroup of the Riordan group [1, 17, 18]. It has as its elements 3-tuples of power series $(\!(g, f_1, f_2)\!)$ where $g \in \mathbf{F}[[x^2]]$, $g_0 \neq 0$, and $f_1, f_2 \in x\mathbf{F}[[x^2]]$, $(f_i)_1 \neq 0$, where $\mathbf{F}$ is an appropriate field such as $\mathbb{R}$ or $\mathbb{C}$. The element $(\!(g, f_1, f_2)\!)$ of the double Riordan group is then represented by the lower-triangular matrix $(a_{n,k})$ whose (n,k)-th element is given by

$$a_{n,k} = [x^n]g(x)f_1(x)^{\lfloor \frac{k+1}{2} \rfloor} f_2(x)^{\lfloor \frac{k}{2} \rfloor}. \tag{2}$$

Thus the columns of this matrix have generating functions given by

$$g, gf_1, gf_1f_2, gf_1^2f_2, gf_1^2f_2^2, \ldots.$$

The identity of this group is $(\!(1, x, x)\!)$. The product rule for this group is given by first letting $h(x) = \sqrt{f_1 f_2}$, and then we have

$$(\!(g, f_1, f_2)\!) \cdot (\!(G, F_1, F_2)\!) = (\!(gG(h), \frac{f_1}{h}F_1(h), \frac{f_2}{h}F_2(h))\!). \tag{3}$$

Again, under the association of the triple $(\!(g, f_1, f_2)\!)$ with the matrix $(a_{n,k})$ given by (2), the product in the group coincides with ordinary matrix multiplication. With regard to inverses in the group, we have

$$(\!(g, f_1, f_2)\!)^{-1} = (\!(\frac{1}{g(\bar{h})}, \frac{x\bar{h}}{f_1(\bar{h})}, \frac{x\bar{h}}{f_2(\bar{h})})\!),$$

where $\bar{h}$ is the compositional inverse of h. Note that $h(x) \in \mathcal{F}_1$, since both $f_1, f_2 \in \mathcal{F}_1$ and so $f_1(x)f_2(x) \in \mathcal{F}_2 \Longrightarrow h(x) = \sqrt{f_1(x)f_2(x)} \in \mathcal{F}_1$.

2 The Sprugnoli set

Definition 3. The *Sprugnoli set* is the set with elements (g, f_1, f_2) with $g(x) \in \mathcal{F}_0$, $f_1(x) \in \mathcal{F}_1$, and $f_2(x) \in \mathcal{F}_1$ and $f_2 \in x\mathbf{F}[[x^2]]$ (thus f_2 is an odd power series). The element (g, f_1, f_2) of this set has the matrix representation

$$t_{n,k} = [x^n]g(x)f_1(x)^{k \bmod 2}(xf_2(x))^{\lfloor \frac{k}{2} \rfloor}.$$

Thus we have

$$t_{n,k} = \begin{cases} [x^n]g(x)(xf_2(x))^m, & k = 2m, \\ [x^n]g(x)f_1(x)(xf_2(x))^m, & k = 2m+1. \end{cases}$$

The columns of this matrix then have generating functions given by the following products of generating functions

$$g(x), g(x)f_1(x), g(x)(xf_2(x)), g(x)f_1(x)(xf_2(x)), g(x)(xf_2(x))^2,$$
$$g(x)f_1(x)(xf_2(x))^2, g(x)(xf_2(x))^3, g(x)f_1(x)(xf_2(x))^3, \ldots$$

We can represent this sequence by the following schema.

$$\begin{array}{cccccccc} g & g & g & g & g & g & g & g \\ 1 & f_1 & 1 & f_1 & 1 & f_1 & 1 & f_1 \\ 1 & 1 & f_2 & f_2 & f_2^2 & f_2^2 & f_2^3 & f_2^3 \\ 1 & 1 & x & x & x^2 & x^2 & x^3 & x^3 \\ - & - & - & - & - & - & - & - \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array}$$

Here, the final row gives the order of the product of the terms above it, showing that this associated matrix is lower-triangular. We see that f_1 occurs only for alternated columns, and only to the first power. The power series $g(x)$ is present for each column, while x appears to the same power as f_2 where these elements occur.

We can represent this array as the sum of two matrices as follows. The first matrix is the matrix whose columns are generated by the power series

$$g(x), 0, g(x)(xf_2(x)), 0, g(x)(xf_2(x))^2, \ldots,$$

which is a horizontal aeration of the stretched Riordan array $(g(x), xf_2(x))$, and the second matrix is the matrix whose columns are generated by the power series

$$0, g(x)f_1(x), 0, g(x)f_1(x)(xf_2(x)), 0, g(x)f_1(x)(xf_2(x))^2, 0, \ldots,$$

which is the horizontal aeration of the stretched Riordan array $(g(x)f_1(x), xf_2(x))$ (but note that this matrix starts with a zero row, due to the fact that $f_1 \in \mathcal{F}_1$).

Example 4. We consider the Sprugnoli array $\left(\frac{1}{1-x}, \frac{x(1+x)}{1-x}, \frac{x}{1-x^2}\right)$. This array begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 5 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 7 & 2 & 3 & 1 & 0 & 0 & 0 & 0 \\ 1 & 9 & 2 & 6 & 1 & 1 & 0 & 0 & 0 \\ 1 & 11 & 3 & 10 & 3 & 3 & 1 & 0 & 0 \\ 1 & 13 & 3 & 15 & 3 & 7 & 1 & 1 & 0 \\ 1 & 15 & 4 & 21 & 6 & 13 & 4 & 3 & 1 \end{pmatrix}.$$

This is the sum of the matrices

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 3 & 0 & 3 & 0 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 3 & 0 & 1 & 0 & 0 \\ 1 & 0 & 4 & 0 & 6 & 0 & 4 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 7 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 9 & 0 & 6 & 0 & 1 & 0 & 0 & 0 \\ 0 & 11 & 0 & 10 & 0 & 3 & 0 & 0 & 0 \\ 0 & 13 & 0 & 15 & 0 & 7 & 0 & 1 & 0 \\ 0 & 15 & 0 & 21 & 0 & 13 & 0 & 3 & 0 \end{pmatrix}.$$

The first is a horizontal aeration of the stretched Riordan array $\left(\frac{1}{1-x}, \frac{x}{1-x^2}\right)$, while the second one is a horizontal aeration of the stretched Riordan array $\left(\frac{1}{1-x}\frac{x(1+x)}{1-x}, \frac{x}{1-x^2}\right) = \left(\frac{x(1+x)}{(1-x)^2}, \frac{x}{1-x^2}\right)$ with an extra initial column of zeros.

A special element of this set is given by $(1, x, x)$. Its columns are generated by the sequence $1, x, x^2, x^3, \ldots$ and so its matrix representation is given by the usual identity matrix.

Elements of this set will be referred to as Sprugnoli arrays, in reference to their matrix representation.

Proposition 5. *The bi-variate generating function of the Sprugnoli array (g, f_1, f_2) is given by*

$$\frac{g(x)}{1 - y^2 x f_2(x)} + \frac{y g(x) f_1(x)}{1 - y^2 x f_2(x)} = \frac{g(x)(1 + y f_1(x))}{1 - y^2 x f_2(x)}.$$

Proof. The array is given by the sum of the horizontal aeration of the stretched Riordan array $(g(x), xf_2(x))$ and the horizontal aeration of the stretched Riordan array $(g(x)f_1(x), xf_2(x))$. Interpreting this in terms of bi-variate generating functions gives the result. □

Corollary 6. *The row sums of the Sprugnoli array (g, f_1, f_2) have generating function* $\frac{g(x)(1+f_1(x))}{1-xf_2(x)}$.

Proof. This results by setting $y = 1$ in the generating function of the array. □

Corollary 7. *The diagonal sums of the Sprugnoli array (g, f_1, f_2) have generating function $\frac{g(x)(1+xf_1(x))}{1-x^3f_2(x)}$.*

Proof. This results by setting $y = x$ in the generating function of the array. □

We wish to show that this set is a group. Thus we need first to state what the product in the group is. For this, we need to show how the elements of this set operate on power series. As with the Riordan group, this will then be translated in matrix terms into how the associated matrix operates (by matrix multiplication) on the vector whose elements are the coefficients of the given power series. This will give us the "fundamental theorem of Sprugnoli arrays".

3 The "fundamental theorem" of Sprugnoli arrays

The following example illustrates the result that follows.

Example 8. We consider the Sprugnoli array $\left(\frac{1}{1-x}, \frac{x(1+x)}{1-x}, \frac{x}{1-x^2}\right)$ acting on the positive Fibonacci numbers. We have

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 5 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 7 & 2 & 3 & 1 & 0 & 0 & 0 & 0 \\ 1 & 9 & 2 & 6 & 1 & 1 & 0 & 0 & 0 \\ 1 & 11 & 3 & 10 & 3 & 3 & 1 & 0 & 0 \\ 1 & 13 & 3 & 15 & 3 & 7 & 1 & 1 & 0 \\ 1 & 15 & 4 & 21 & 6 & 13 & 4 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 2 \\ 3 \\ 5 \\ 8 \\ 13 \\ 21 \\ 34 \end{pmatrix} =$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 3 & 0 & 3 & 0 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 3 & 0 & 1 & 0 & 0 \\ 1 & 0 & 4 & 0 & 6 & 0 & 4 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \\ 5 \\ 0 \\ 13 \\ 0 \\ 34 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 7 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 9 & 0 & 6 & 0 & 1 & 0 & 0 & 0 \\ 0 & 11 & 0 & 10 & 0 & 3 & 0 & 0 & 0 \\ 0 & 13 & 0 & 15 & 0 & 7 & 0 & 1 & 0 \\ 0 & 15 & 0 & 21 & 0 & 13 & 0 & 3 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 3 \\ 0 \\ 8 \\ 0 \\ 21 \\ 0 \end{pmatrix} =$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 4 & 6 & 4 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 5 \\ 13 \\ 34 \\ 89 \\ 233 \\ 610 \\ 1597 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 5 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 7 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 9 & 6 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 11 & 10 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 13 & 15 & 7 & 1 & 0 & 0 & 0 & 0 & 0 \\ 15 & 21 & 13 & 3 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ 8 \\ 21 \\ 55 \\ 144 \\ 377 \\ 987 \\ 2584 \end{pmatrix}.$$

Here, the sequences $1, 2, 5, \ldots$ and $1, 3, 8, \ldots$ are the even and odd bisections (see Appendix) of the positive Fibonacci numbers. The resulting sequence

$$1, 2, 6, 11, 26, 45, 100, 170, 370, \ldots$$

is the expansion of

$$\left(\frac{1}{1-x}, \frac{x(1+x)}{1-x}, \frac{x}{1-x^2}\right) \cdot \frac{1}{1-x-x^2} = \frac{(1+x)(1+x+x^3)}{1-5x^2+5x^4}.$$

The general result is as follows.

Proposition 9. *Let (g, f_1, f_2) be a Sprugnoli array, and let $h(x)$ be a power series $h(x) = \sum_{n=0}^{\infty} a_n x^n$. We let*

$$h^e(x) = \sum_{n=0}^{\infty} a_{2n} x^n \quad \text{and} \quad h^o(x) = \sum_{n=0}^{\infty} a_{2n+1} x^n$$

be the even and odd bisections of $h(x)$. Then we have

$$(g(x), f_1(x), f_2(x)) \cdot h(x) = g(x)h^e(xf_2(x)) + g(x)f_1(x)h^o(xf_2(x)).$$

Proof. We partition the matrix representing (g, f_1, f_2) as

$$(g, 0, g(xf_2), 0, g(xf_2)^2, 0, g(xf_2)^3, 0, \ldots) + (0, gf_1, 0, gf_1(xf_2), 0, gf_1(xf_2)0, \ldots).$$

These matrices operate, respectively, on $(a_0, 0, a_2, 0, a_4, 0, \ldots)$ and $(0, a_1, 0, a_3, 0, a_5, 0, \ldots)$. In terms of Riordan arrays, this is equivalent, after compression, to the stretched Riordan array (g, xf_2) operating on h^e, and the stretched Riordan array (gf_1, xf_2) operating on $h^o(x)$. The result follows from this. □

Note that we can write this operation in the symbolic notation

$$\begin{pmatrix} (g, xf_2), & (gf_1, xf_2) \end{pmatrix} \begin{pmatrix} h^e \\ h^o \end{pmatrix}$$

reminiscent of a scalar product.

Example 10. We have

$$\begin{aligned}(1, x, x) \cdot h(x) &= 1.h^e(x^2) + 1.x.h^o(x^2) \\ &= h^e(x^2) + xh^o(x^2) \\ &= h(x).\end{aligned}$$

4 The product rule

We are now in a position to say what we mean by the product of two Sprugnoli arrays. We note that the product rule in the Riordan group may be written

$$\begin{aligned}(g(x), f(x)) \cdot (u(x), v(x)) &= (g(x)u(f(x)), v(f(x)) \\ &= \left((g(x), f(x)) \cdot u, \frac{(g(x), f(x)) \cdot u(x)v(x)}{(g(x), f(x)) \cdot u(x)}\right).\end{aligned}$$

This is so because

$$\begin{aligned}(g(x), f(x)) \cdot u(x) &= g(x)u(f(x)), \\ \frac{(g(x), f(x)) \cdot u(x)v(x)}{(g(x), f(x)) \cdot u(x)} &= \frac{g(x)u(f(x))v(f(x))}{g(x)u(f(x))} = v(f(x)).\end{aligned}$$

Note that $u(x)$ and $u(x)v(x)$ are the generating functions of the first two columns of $(u(x), v(x))$. We can generalize this to give the following definition, using the first three columns of the second array.

Definition 11. We define the product of two Sprugnoli arrays (g, f_1, f_2) and (u, v_1, v_2) as follows.

$$(g, f_1, f_2) \cdot (u, v_1, v_2) = \left((g, f_1, f_2) \cdot u, \frac{(g, f_1, f_2) \cdot uv_1}{(g, f_1, f_2) \cdot u}, \frac{1}{x}.\frac{(g, f_1, f_2) \cdot uxv_2}{(g, f_1, f_2) \cdot u}\right). \tag{4}$$

Theorem 12. *The product (4) is well defined and for this product $(g, f_1, f_2) \cdot (u, v_1, v_2)$ is a Sprugnoli array.*

Proof. To show that this is well defined and again a Sprugnoli array, we must show that

- $(g, f_1, f_2) \cdot u \in \mathcal{F}_0$,
- $\frac{(g,f_1,f_2)\cdot uv_1}{(g,f_1,f_2)\cdot u} \in \mathcal{F}_1$,
- $\frac{1}{x}.\frac{(g,f_1,f_2)\cdot uxv_2}{(g,f_1,f_2)\cdot u} \in \mathcal{F}_1$ and is odd.

We have

$$(g, f_1, f_2) \cdot u = gu^e(xf_2) + gf_1u^o(xf_2).$$

Since $g(x) \in \mathcal{F}_0$ and $u^e(x) \in \mathcal{F}_0$, we obtain that $(g, f_1, f_2) \cdot u \in \mathcal{F}_0$. In particular, $(g, f_1, f_2) \cdot u$ is invertible so we have $\frac{1}{(g,f_1,f_2)\cdot u} \in \mathcal{F}_0$. Thus we must now show that $(g, f_1, f_2) \cdot uv_1 \in \mathcal{F}_1$. We have

$$(g, f_1, f_2) \cdot uv_1 = g(x)(uv_1)^e(xf_2(x)) + g(x)f_1(x)(uv_1)^o(xf_2(x)).$$

We let $u(x) = \sum_{n\geqslant 0} a_n x^n$ and $v_1(x) = \sum_{n\geqslant 0} b_n x^n$. Then uv_1 expands as

$$a_0b_0, a_0b_1 + a_1b_0, a_0b_2 + a_1b_1 + a_2b_0, \ldots.$$

Now $b_0 = 0$, so $(uv_1)^e$ expands as $0, a_0b_2+a_1b_1+a_2b_0, \ldots = 0, a_0b_2+a_1b_1, \ldots$. Thus $(uv_1)^e(x) \in \mathcal{F}_1$. Similarly, we have that $(uv_1)^o(x)$ expands as $a_0b_1+a_1b_0, \ldots = a_0b_1, \ldots$ and so $(uv_1)^o(x) \in$

$\mathcal{F}_0$. This implies that $g(x)(uv_1)^e(xf_2(x)) \in \mathcal{F}_1$, and that $g(x)f_1(x)(uv_1)^o(xf_2(x)) \in \mathcal{F}_1$ (since $f_1(x) \in \mathcal{F}_1$). We conclude that $(g, f_1, f_2) \cdot uv_1 \in \mathcal{F}_1$, and hence so is $\frac{(g,f_1,f_2)\cdot uv_1}{(g,f_1,f_2)\cdot u}$.

Applying similar reasoning to the previous paragraph, we can show that

$$\frac{1}{x}(g, f_1, f_2) \cdot uxv_2 \in \mathcal{F}_1,$$

and hence we have $\frac{1}{x}.\frac{(g,f_1,f_2)\cdot uxv_2}{(g,f_1,f_2)\cdot u} \in \mathcal{F}_1$.

Lemma 13. *Let $m \in \mathbb{Z}, m \geqslant 0$. We have*

$$(u(x)(xv_2(x))^m)^e = u^e(x)(\sqrt{x}v_2(\sqrt{x}))^m.$$

Proof. We have

$$\begin{aligned}
(u(x)(xv_2(x))^m)^e(x) &= \frac{u(\sqrt{x})(\sqrt{x}v_2(\sqrt{(x)}))^m + u(-\sqrt{x})(-\sqrt{x}v_2(-\sqrt{(x)}))^m}{2} \\
&= \frac{u(\sqrt{x})(\sqrt{x}v_2(\sqrt{(x)}))^m + u(-\sqrt{x})(\sqrt{x}v_2(\sqrt{(x)}))^m}{2} \quad (v_2 \text{ is odd}) \\
&= \frac{u(\sqrt{x}) + u(-\sqrt{x})}{2}(\sqrt{x}v_2(\sqrt{x}))^m \\
&= u^e(x)(\sqrt{x}v_2(\sqrt{x}))^m.
\end{aligned}$$

□

In a similar way we have the following.

Lemma 14. *We have*

$$(u(x)v_1(x)(xv_2(x))^m)^e = (uv_1)^e(x)(\sqrt{x}v_2(\sqrt{x}))^m.$$

Proposition 15. *We have*

$$(g, f_1, f_2) \cdot u(x)(xv_2(x))^m = ((g, f_1, f_2) \cdot u)(\sqrt{xf_2(x)}v_2(\sqrt{xf_2(x)}))^m.$$

Proof. We have

$$\begin{aligned}
(g, f_1, f_2) \cdot u(x)(xv_2(x))^m &= (g, xf_2) \cdot u^e(x)(\sqrt{x}v_2(\sqrt{x}))^m + (gf_1, xf_2) \cdot u^o(x)(\sqrt{x}v_2(\sqrt{x}))^m \\
&= (g(x)u^e(xf_2)(\sqrt{xf_2}v_2(\sqrt{xf_2}))^m + g(x)f_1(x)u^o(xf_2)(\sqrt{xf_2}v_2(\sqrt{xf_2}))^m \\
&= (g(x)u^e(xf_2) + g(x)f_1(x)u^o(xf_2))(\sqrt{xf_2}v_2(\sqrt{xf_2}))^m \\
&= ((g, f_1, f_2) \cdot u)(\sqrt{xf_2}v_2(\sqrt{xf_2}))^m.
\end{aligned}$$

□

Corollary 16. *We have*

$$\frac{1}{x}\frac{(g, f_1, f_2) \cdot uxv_2}{(g, f_1, f_2) \cdot u} = \frac{1}{x}\sqrt{xf_2}v_2(\sqrt{xf_2}).$$

Proof. We set $m = 1$ in the above result. □

Proposition 17. $\frac{1}{x}.\frac{(g,f_1,f_2)\cdot uxv_2}{(g,f_1,f_2)\cdot u}$ *is odd.*

Proof. We have

$$\frac{1}{x}\frac{(g,f_1,f_2)\cdot uxv_2}{(g,f_1,f_2)\cdot u}=\frac{1}{x}\sqrt{xf_2}v_2(\sqrt{xf_2}).$$

Changing x to $-x$, we get

$$\frac{1}{-x}\sqrt{-xf_2(-x)}v_2(\sqrt{-xf_2(-x)})=\frac{1}{-x}\sqrt{xf_2(x)}v_2(\sqrt{xf_2(x)})=-\frac{1}{x}\sqrt{xf_2(x)}v_2(\sqrt{xf_2(x)})$$

since $f_2(x)$ is odd. □

We are now in a position to conclude that the product (4) is well defined and returns a Sprugnoli array. □

Thus the set of Sprugnoli arrays is closed under multiplication. Note that we now have

$$(g,f_1,f_2)\cdot(u,v_1,v_2)=\left((g,f_1,f_2)\cdot u,\frac{(g,f_1,f_2)\cdot uv_1}{(g,f_1,f_2)\cdot u},\frac{1}{x}\sqrt{xf_2}v_2(\sqrt{xf_2})\right). \tag{5}$$

Furthermore, the product rule (5), when we look to the matrix representation of Sprugnoli arrays, coincides with matrix multiplication.

Proposition 18. *In the matrix representation of Sprugnoli arrays, the product rule (5) corresponds to ordinary matrix multiplication.*

Proof. To see this, we take the case of $k=2m$. In terms of matrix multiplication, the k-th column of the product of (g,f_1,f_2) and (u,v_1,v_2) will have generating function given by (g,f_1,f_2) acting on the generating function of the k-th column of (u,v_1,v_2). This is given by

$$(g,f_1,f_2)\cdot(u(x)(xv_2(x))^m=((g,f_1,f_2)\cdot u)(\sqrt{xf_2}v_2(\sqrt{xf_2}))^m.$$

Now the k-th column of the product (5) is given by $((g,f_1,f_2)\cdot u)(x\frac{1}{x}\sqrt{xf_2}v_2(\sqrt{xf_2}))^m$. Thus these coincide. A similar argument can be made for k odd. □

We conclude that in the matrix representation, the product (5) is given by ordinary matrix multiplication. Associativity of the product now follows from this.

5 The inverse of a Sprugnoli array

Our quest now is to define the inverse of a Sprugnoli array. For this, we first look at connections to the Riordan group and the double Riordan group.

Lemma 19. *If $f_1=\sqrt{xf_2}$, then the Sprugnoli array (g,f_1,f_2) is given by the Riordan array $(g,f_1)=(g,\sqrt{xf_2})$.*

Proof. The columns of $(g,\sqrt{xf_2},f_2)$ are generated by the sequence

$$g,g\sqrt{xf_2},g(xf_2),g\sqrt{xf_2}(xf_2),g(xf_2)^2,\ldots.$$

Clearly, this is

$$g,gf_1,gf_1^2,gf_1^3,gf_1^4,\ldots,$$

as required. □

Note that from $f_1 = \sqrt{xf_2}$, we get $xf_2 = f_1^2$ and $f_2 = \frac{1}{x}f_1^2$. Thus we can write $\left(g, \sqrt{xf_2}, f_2\right) = \left(g, \sqrt{xf_2}\right)$. Now we have the Riordan array inverse

$$\left(g, \sqrt{xf_2}\right)^{-1} = \left(\frac{1}{g(\overline{\sqrt{xf_2}})}, \overline{\sqrt{xf_2(x)}}\right).$$

Re-interpreting this in terms of Sprugnoli arrays, we then have

$$(g, \sqrt{xf_2}, f_2)^{-1} = \left(\frac{1}{g(\overline{\sqrt{xf_2}})}, \overline{\sqrt{xf_2}}, \frac{1}{x}\left(\overline{\sqrt{xf_2}}\right)^2\right).$$

We now look at a situation where there is a link between Sprugnoli arrays and double Riordan arrays.

Lemma 20. *If $g(x) \in \mathbf{F}[[x^2]], f_1 \in x\mathbf{F}[[x^2]]$, and we set $F_1 = f_1, F_2 = \frac{xf_2}{f_1}$, then*

$$(g, f_1, f_2) = (\!(g, F_1, F_2)\!).$$

Proof. The double Riordan array $(\!(g, F_1, F_2)\!)$ is defined by its column defining sequence

$$g, gF_1, gF_1F_2, gF_1^2F_2, gF_1^2F_2^2, \ldots.$$

But by definition, this is the same as

$$g, gf_1, g(xf_2), gf_1(xf_2), g(xf_2)^2, \ldots.$$

By matching terms we obtain the result. □

Thus under the conditions of the last lemma we have

$$(g, f_1, f_2)^{-1} = (\!(g, F_1, F_2)\!)^{-1} = \left(\!\left(\frac{1}{g(\bar{H})}, \frac{x\bar{H}}{F_1(\bar{H})}, \frac{x\bar{H}}{F_2(\bar{H})}\right)\!\right),$$

where $H = \sqrt{F_1F_2} = \sqrt{f_1\frac{xf_2}{f_1}} = \sqrt{xf_2}$, and so $\bar{H} = \overline{\sqrt{xf_2}}$. Then under the conditions of the lemma, we have

$$\begin{aligned}
(g, f_1, f_2)^{-1} &= \left(\!\left(\frac{1}{g(\bar{H})}, \frac{x\bar{H}}{F_1(\bar{H})}, \frac{x\bar{H}}{F_2(\bar{H})}\right)\!\right) \\
&= \left(\frac{1}{g(\bar{H})}, \frac{x\bar{H}}{f_1(\bar{H})}, \frac{x\bar{H}}{F_2(\bar{H})} \cdot \frac{x\bar{H}}{F_1(\bar{H})} \cdot \frac{1}{x}\right) \\
&= \left(\frac{1}{g(\bar{H})}, \frac{x\bar{H}}{f_1(\bar{H})}, \frac{x\bar{H}^2}{(xf_2)(\bar{H})}\right), \quad \text{where } F_1F_2 = xf_2 \\
&= \left(\frac{1}{g(\bar{H})}, \frac{x\bar{H}}{f_1(\bar{H})}, \frac{x\left(\overline{\sqrt{xf_2}}\right)^2}{(xf_2)(\overline{\sqrt{xf_2}})}\right)
\end{aligned}$$

$$= \left(\frac{1}{g(\bar{H})}, \frac{x\bar{H}}{f_1(\bar{H})}, \frac{1}{x}\left(\overline{\sqrt{xf_2}}\right)^2 \right).$$

We have now seen two special cases in which the inverse of (g, f_1, f_2) satisfies

$$(g, f_1, f_2)^{-1} = \left(w(x), s_1(x), \frac{1}{x}\left(\overline{\sqrt{xf_2}}\right)^2 \right),$$

where $w(x)$ and $s_1(x)$ are as yet undetermined elements of the inverse. To progress our search for an inverse in the general case, we have the following lemma concerning a canonical factorization of Sprugnoli arrays.

Lemma 21. *We have*

$$(g, x, x) \cdot (1, f_1, f_2) = (g, f_1, f_2).$$

Proof. We have

$$(g, x, x) \cdot 1 = g \cdot 1^e(x^2) + g \cdot 1^o(x^2) = g,$$

since $1^e = 1$ and $1^o = 0$. Now

$$(g, x, x) \cdot (1, f_1, f_2) = \left((g, x, x) \cdot 1, \frac{(g, x, x) \cdot f_1}{(g, x, x) \cdot 1}, \frac{1}{x}\frac{(g, x, x) \cdot xf_2}{(g, x, x) \cdot 1} \right).$$

We have

$$\begin{aligned}(g, x, x) \cdot f_1 &= g.f_1^e(x^2) + gxf_1^o(x^2) \\ &= g(f_1^e(x^2) + xf_1^o(x^2)) \\ &= gf_1.\end{aligned}$$

Then

$$\frac{(g, x, x) \cdot f_1}{(g, x, x) \cdot 1} = \frac{gf_1}{g} = f_1.$$

The third element of $(g, x, x) \cdot (1, f_1, f_2)$ is given by

$$\frac{1}{x}\frac{(g, x, x) \cdot xf_2}{(g, x, x) \cdot 1} = \frac{1}{x}\frac{(g, x, x) \cdot xf_2}{g}.$$

This is equal to

$$\begin{aligned}\frac{1}{x}\frac{g.(xf_2)^e(x^2) + xg.(xf_2)^o(x^2)}{g} &= \frac{1}{x}((xf_2)^e(x^2) + x(xf_2)^o(x^2)) \\ &= \frac{1}{x}(x(f_2)^e(x^2) + x^2(f_2)^o(x^2)) = (f_2)^e(x^2) + x(f_2)^o(x^2) \\ &= f_2(x).\end{aligned}$$

□

Corollary 22.

$$(g, f_1, f_2)^{-1} = (1, f_1, f_2)^{-1} \cdot (g, x, x)^{-1}.$$

Proof. This is so since the product in the set of Sprugnoli arrays is given by matrix multiplication in the matrix representation. Thus the inverse of a product is given in the usual way. □

Now $(g, x, x) = (g, x)$ so we have

$$(g, x, x)^{-1} = (g, x)^{-1} = \left(\frac{1}{g}, x\right) = \left(\frac{1}{g}, x, x\right).$$

Thus we have

$$(g, f_1, f_2)^{-1} = (1, f_1, f_2)^{-1} \cdot \left(\frac{1}{g}, x, x\right). \tag{6}$$

We now let $(1, r_1, r_2) = (1, f_1, f_2)^{-1}$. We have

$$(1, r_1, r_2) \cdot (1, f_1, f_2) = (1, x, x)$$

which gives us that

$$(1, r_1, r_2) \cdot f_1 = x,$$

or

$$1.f_1^e(xr_2) + 1.r_1 f_1^o(xr_2) = x.$$

Solving for r_1, we obtain

$$r_1 = \frac{x - f_1^e(xr_2)}{f_1^o(xr_2)}. \tag{7}$$

We are now in a position to say what the inverse of (g, f_1, f_2) in the Sprugnoli group is.

Definition 23. The inverse of the Sprugnoli group element (g, f_1, f_2) is given by

$$(g, f_1, f_2)^{-1} = \left(1, \frac{x - f_1^e(xr_2)}{f_1^o(xr_2)}, r_2\right) \cdot \left(\frac{1}{g}, x, x\right) \tag{8}$$

where

$$r_2 = \frac{1}{x}\left(\overline{\sqrt{xf_2}}\right)^2. \tag{9}$$

Letting $(g, f_1, f_2)^{-1} = (w(x), s_1(x), s_2(x))$ we then have

$$\begin{aligned} w(x) &= (1, r_1, r_2) \cdot \frac{1}{g} \\ &= 1.\left(\frac{1}{g}\right)^e (xr_2) + \frac{x - f_1^e(xr_2)}{f_1^o(xr_2)}\left(\frac{1}{g}\right)^o (xr_2). \end{aligned}$$

$$\begin{aligned} s_1(x) &= \frac{(1, r_1, r_2)(x/g)}{(1, r_1, r_2)(1/g)} \\ &= \frac{1.\,(x/g)^e\,(xr_2) + 1.r_1\,(x/g)^o\,(xr_2)}{1.\,(1/g)^e\,(xr_2) + 1.r_1\,(1/g)^o\,(xr_2)} \end{aligned}$$

$$
\begin{aligned}
&= \frac{(x\,(1/g)^o)(xr_2) + r_1\,(1/g)^e\,(xr_2)}{(1/g)^e\,(xr_2) + r_1\,(1/g)^o\,(xr_2)} \\
&= \frac{xr_2(x).\,(1/g)^o\,(xr_2) + r_1\,(1/g)^e\,(xr_2)}{w(x)}.
\end{aligned}
$$

$$
\begin{aligned}
s_2(x) &= \frac{1}{x}\frac{(1, r_1, r_2)(x^2/g)}{(1, r_1, r_2)(x^2/g)} \\
&= \frac{1}{x}\frac{1.\,(x^2/g)^e\,(xr_2) + 1.r_1\,(x^2/g)^o\,(xr_2)}{1.\,(1/g)^e\,(xr_2) + 1.r_1\,(1/g)^o\,(xr_2)} \\
&= \frac{1}{x}\frac{((x)(\frac{1}{g})^e(xr_2) + r_1((x)(\frac{1}{g})^o(xr_2))}{(\frac{1}{g})^e(xr_2) + r_1(\frac{1}{g})^o(xr_2)} \\
&= \frac{1}{x}xr_2(x) \\
&= r_2(x).
\end{aligned}
$$

6 The Sprugnoli matrix group

We have the following result.

Theorem 24. *The set of Sprugnoli arrays is a group.*

Proof. The set of Sprugnoli arrays is closed under the product (4) that we have defined. Moreover, this product, in the matrix representation, coincides with matrix multiplication. The product is therefore associative. We have also defined an identity element $(1, x, x)$ and have demonstrated that each element has an inverse. Thus the set is a group. □

To honour the memory of Renzo Sprugnoli, we call this group the *Sprugnoli group*.

We now study some simple subgroups of the Sprugnoli group.

Proposition 25. *The set of Sprugnoli arrays of the form (g, x, x) for $g \in \mathcal{F}_0$ is a subgroup of the Sprugnoli group.*

Proof. The identity for this subgroup is $(1, x, x)$. The inverse of (g, x, x) is $\left(\frac{1}{g}, x, x\right)$. We have

$$
(g, x, x) \cdot (h, x, x) = \left((g, x, x) \cdot h, \frac{(g, x, x) \cdot hx}{(g, x, x) \cdot h}, \frac{1}{x}\sqrt{x^2}x(\sqrt{x^2})\right).
$$

We have $\frac{1}{x}\sqrt{x^2}x(\sqrt{x^2}) = x$.

$$
\begin{aligned}
(g, x, x) \cdot hx &= (g, x^2)(hx)^e + (gx, x^2)(hx)^o(x) \\
&= (g, x^2)xh^o(x) + (gx, x^2)h^e(x) \quad \text{see Appendix} \\
&= g(x)x^2h^o(x^2) + gxh^e(x^2) \\
&= x(gh^e(x^2) + gxh^o(x^2)) \\
&= x(g, x, x) \cdot h.
\end{aligned}
$$

Thus $(g, x, x) \cdot (h, x, x) = ((g, x, x)h, x, x)$. □

Proposition 26. *The set of Sprugnoli arrays of the form* $(1, f_1, f_2)$ *is a subgroup of the Sprugnoli group.*

Proof. We have

$$\begin{aligned}(1, f_1, f_2) \cdot (1, v_1, v_2) &= \left((1, f_1, f_2) \cdot 1, \frac{(1, f_1, f_2) \cdot v_1}{(1, f_1, f_2) \cdot 1}, \frac{1}{x}\frac{(1, f_1, f_2) \cdot xv_2}{(1, f_1, f_2) \cdot 1}\right) \\ &= \left(1, (1, f_1, f_2) \cdot v_1, \frac{1}{x}(1, f_1, f_2) \cdot xv_2\right)\end{aligned}$$

Also, $(1, f_1, f_2)^{-1} = (1, r_1, r_2)$ as calculated above. □

Proposition 27. *The set of Sprugnoli arrays of the form* $(1, x, f_2)$ *is a subgroup of the Sprugnoli group.*

Proof. We have $(1, x, f_2)^{-1} = \left(1, x, \frac{1}{x}(\overline{\sqrt{xf_2}})^2\right)$. Also,

$$\begin{aligned}(1, x, f_2) \cdot (1, x, v_2) &= \left((1, x, f_2) \cdot 1, \frac{(1, x, f_2) \cdot 1.x}{(1, x, f_2) \cdot 1}, \dots\right) \\ &= (1, (1, xf_2) \cdot x^e(x) + (x, xf_2)x^o(x), \dots) \\ &= (1, 0 + x, \dots).\end{aligned}$$

□

Proposition 28. *The set of Sprugnoli arrays of the form* $(1, f_1, x)$ *is a subgroup of the Sprugnoli group.*

Proof. We have

$$(1, f_1, x) \cdot (1, v_1, x) = \left((1, f_1, x) \cdot 1, \frac{(1, f_1, x) \cdot v_1}{(1, f_1, x) \cdot 1}, \frac{1}{x}\frac{(1, f_1, x) \cdot x^2}{(1, f_1, x) \cdot 1}\right).$$

Now $(1, f_1, x) \cdot 1 = 1$, and

$$(1, f_1, x) \cdot x^2 = (1, x^2)(x^2)^e(x) + (f_1, x^2) \cdot (x^2)^o(x) = (1, x^2) \cdot x = x^2.$$

Hence

$$\frac{1}{x}\frac{(1, f_1, x) \cdot x^2}{(1, f_1, x) \cdot 1} = x.$$

We have

$$(1, f_1, x)^{-1} = \left(1, \frac{x - f_1^e(x^2)}{f_1^o(x^2)}, x\right),$$

which finishes the proof. □

7 Examples of inverse calculations

Example 29. We consider the Sprugnoli array A051159 which is $\left(\frac{1}{1-x}, \frac{x}{1+x}, \frac{x}{1-x^2}\right)$. We have $xf_2(x) = \frac{x^2}{1-x^2}$. We seek the reversion of $\sqrt{xf_2(x)}$. Thus we wish to solve the equation

$$\frac{u}{\sqrt{1-u^2}} = x$$

for the solution $u(x)$ with $u(0) = 0$. We have

$$\frac{u^2}{1-u^2} = x^2 \Longrightarrow u^2 = \frac{x^2}{1+x^2}.$$

Now for the inverse, we have $s_2 = r_2 = \frac{u^2}{x} = \frac{x}{1+x^2}$. We are now in a position to solve for r_1. We have

$$\begin{aligned} r_1 &= \frac{x - f_1^e(xr_2)}{f_1^o(xr_2)} \\ &= \frac{x - \left(\frac{-x}{1-x}\right)\left(\frac{x^2}{1+x}\right)}{\left(\frac{1}{1-x}\right)\left(\frac{x^2}{1+x^2}\right)} \\ &= \frac{x - \frac{-x^2/(1+x^2)}{1-x^2/(1+x^2)}}{\frac{1}{1-x^2/(1+x^2)}} \\ &= x\left(1 - \frac{x^2}{1+x^2}\right) + \frac{x^2}{1+x^2} \\ &= \frac{x(1+x)}{1+x^2}. \end{aligned}$$

We have $\frac{1}{g} = 1 - x$ and so we have

$$\left(\frac{1}{g}\right)^e = 1 \quad \text{and} \quad \left(\frac{1}{g}\right)^o = -1.$$

Then

$$w(x) = (1, r_1, r_2) \cdot \frac{1}{g} = 1 - \frac{x(1+x)}{1+x^2} \cdot 1 = 1 - \frac{x(1+x)}{1+x^2} = \frac{1-x}{1+x^2}.$$

Now $\frac{x}{g} = x(1-x) = x - x^2$, which gives us $\left(\frac{x}{g}\right)^e = -x$, and $\left(\frac{x}{g}\right)^o = 1$. Using these values and the equation

$$\frac{1.\,(x/g)^e\,(xr_2) + 1.r_1\,(x/g)^o\,(xr_2)}{1.\,(1/g)^e\,(xr_2) + 1.r_1\,(1/g)^o\,(xr_2)}$$

we find that

$$s_1(x) = \frac{x}{1-x}.$$

Thus we have

$$\left(\frac{1}{1-x}, \frac{x}{1+x}, \frac{x}{1-x^2}\right)^{-1} = \left(\frac{1-x}{1+x^2}, \frac{x}{1-x}, \frac{x}{1+x^2}\right).$$

We note that the rows of this matrix are palindromic (the matrix is "Pascal-like").

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 2 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 3 & 0 & 3 & 0 & 1 & 0 & 0 \\ 1 & 1 & 3 & 3 & 3 & 3 & 1 & 1 & 0 \\ 1 & 0 & 4 & 0 & 6 & 0 & 4 & 0 & 1 \end{pmatrix}.$$

Example 30. We take the example of the Sprugnoli array

$$S = \left(\frac{1+2x}{1-4x}, \frac{x(1+3x)}{1-2x}, \frac{x(1+x^2)}{1-x^2} \right).$$

We wish to find its inverse. In order to find $s_2 = r_2$, we have $s_2 = \frac{u^2}{x}$ were u satisfies the equation

$$u\sqrt{\frac{1+u^2}{1-u^2}} = x.$$

Squaring this equation, we get

$$u^4 + (1+x^2)u^2 - x^2 = 0$$

from which we deduce that

$$r_2 = s_2 = \frac{u^2}{x} = \frac{\sqrt{1+6x^2+x^4} - x^2 - 1}{2x}.$$

To calculate r_1, we must first get the even and odd bisections of $f_1(x) = \frac{x(1+3x)}{1-2x}$. For this, we have

$$\frac{f_1(x) + f_1(-x)}{2} = \frac{5x^2}{1-4x^2},$$

and so we have

$$f_1^e(x) = \frac{5x}{1-4x}.$$

Then

$$\frac{f_1(x) - f_1(-x)}{2x} = \frac{1+6x^2}{1-4x^2},$$

and so we have

$$f_1^o = \frac{1+6x}{1-4x}.$$

Thus we get

$$r_1 = \frac{x - \left(\frac{5x}{1-4x}\right)\left(\frac{\sqrt{1+6x^2+x^4}-x^2-1}{2}\right)}{\left(\frac{1+6x}{1-4x}\right)\left(\frac{\sqrt{1+6x^2+x^4}-x^2-1}{2}\right)}.$$

This give us

$$r_1 = \frac{5(1+2x)\sqrt{1+6x^2+x^4} - 46x^3 - 65x^2 - 5}{2(5+42x^2)}.$$

We now have the components of

$$(1, f_1, f_2)^{-1} = (1, r_1, r_2).$$

We must apply this array to

$$\left(\frac{1}{g(x)}, x, x\right) = \left(\frac{1-4x}{1+2x}, x, x\right).$$

We have

$$\left(\frac{1}{g}\right)^e = \frac{1+8x}{1-4x} \quad \text{and} \quad \left(\frac{1}{g}\right)^o = \frac{-6}{1-4x}.$$

We get

$$\begin{aligned} w(x) &= (1, r_1, r_2) \cdot \frac{1}{g} \\ &= 1.\left(\frac{1+8x}{1-4x}\right)(xr_2) + 1.r_1\left(\frac{-6}{1-4x}\right)(xr_2). \end{aligned}$$

Using the values for r_1 and r_2 already calculated, we find that

$$w(x) = \frac{3(5-30x+204x^2+72x^3)\sqrt{1+6x^2+x^4} + 216x^5 + 1620x^4 + 54x^3 + 543x^2 - 60x + 10}{(5-12x^2)(5+42x^2)}.$$

To calculate s_1, we first calculate the numerator of the required quotient, to obtain the generating function of the second column of the inverse. This is given by

$$\left(\frac{-6x}{1-4x}\right)(xr_2) + r_1\left(\frac{1+8x}{1-4x}\right)(xr_2).$$

Expanding this last expression, we obtain the generating function of the second column of the inverse to be

$$\frac{(24x^3 - 492x^2 - 10x - 65)\sqrt{1+6x^2+x^4} - 1320x^5 - 3684x^4 + 406x^4 + 137x^2 + 60x + 65}{2(5-12x^2)(5+42x^2)}.$$

Then s_1 is this generating function divided by $w(x)$. This gives us that s_1 is equal to

$$\frac{(24x^3 - 492x^2 - 10x - 65)\sqrt{1+6x^2+x^4} - 1320x^5 - 3684x^4 + 406x^3 + 137x^2 + 60x + 65}{2(3(5-30x+204x^2+72x^3)\sqrt{1+6x^2+x^4} + 216x^5 + 1620x^4 + 54x^2 - 60x + 10)}.$$

Finally we have $s_2(x) = r_2(x)$ to obtain the inverse $(w(x), s_1(x), s_2(x))$.

8 Production matrix and linear recurrences

A defining feature of Riordan arrays is that they have a sequence characterization. For instance, in a Riordan array $M = (g, f)$, any matrix element $t_{n,k}$ not in the first column will obey a linear recurrence equation involving the elements in the row above it, starting with the element $t_{n-1,k-1}$ above and to the left of it. The same recurrence coefficients are valid for each such element, irrespective of the row it is in. This means that the production matrix P of a Riordan array has a special simple structure, where $P = M^{-1}\overline{M}$, where $\overline{M}$ is the matrix M with its top row removed [7]. The elements in the first column of M normally follow a distinct recurrence. This results in two sequences of recurrence coefficients, the Z sequence (for the first column elements) and the A sequence, for all the other elements [8]. The production array will then have a structure as in the following matrix, where the obvious pattern continues.

$$\begin{pmatrix} z_0 & a_0 & 0 & 0 & 0 & 0 & \cdots \\ z_1 & a_1 & a_0 & 0 & 0 & 0 & \cdots \\ z_2 & a_2 & a_1 & a_0 & 0 & 0 & \cdots \\ z_3 & a_3 & a_2 & a_1 & a_0 & 0 & \cdots \\ z_4 & a_4 & a_3 & a_2 & a_1 & a_0 & \cdots \\ z_5 & a_5 & a_4 & a_3 & a_2 & a_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

For Sprugnoli arrays, a similar situation arises, but this time, we have two A-sequences (we designate them the A sequence and the B-sequence), one for elements in even columns, and one for elements in odd columns. Thus the form of the production array for a Sprugnoli array is given by

$$P = \begin{pmatrix} z_0 & a_0 & 0 & 0 & 0 & 0 & \cdots \\ z_1 & a_1 & b_0 & 0 & 0 & 0 & \cdots \\ z_2 & a_2 & b_1 & a_0 & 0 & 0 & \cdots \\ z_3 & a_3 & b_2 & a_1 & b_0 & 0 & \cdots \\ z_4 & a_4 & b_3 & a_2 & b_1 & a_0 & \cdots \\ z_5 & a_5 & b_4 & a_3 & b_2 & a_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{10}$$

giving it a "striped" appearance. We let $A(x)$ be the generating function of the second column of the production matrix (beginning $a_0, a_1, a_2, \ldots$). If the third column of the production matrix begins $0, b_0, b_1, b_2, \ldots$, the we let $B(x)$ be the generating function of the sequence $b_0, b_1, b_2, \ldots$. We have the following result.

Proposition 31. *We have*

$$A(x) = (1, r_1, r_2) \cdot \frac{f_1(x)}{x},$$
$$B(x) = \frac{1}{x}(1, r_1, r_2) \cdot f_2(x).$$

Proof. $A(x)$ is the generating function obtained by applying the inverse array $(g, f_1, f_2)^{-1} = (1, r_1, r_2) \cdot \left(\frac{1}{g}, x, x\right)$ to the second column of $\overline{(g, f_1, f_2)}$. This gives that

$$\begin{aligned} A(x) &= (1, r_1, r_2) \cdot \left(\frac{1}{g}, x, x\right) \cdot g(x) \frac{f_1(x)}{x} \\ &= (1, r_1, r_2) \cdot \frac{1}{g(x)} g(x) \frac{f_1(x)}{x} \\ &= (1, r_1, r_2) \cdot \frac{f_1(x)}{x}. \end{aligned}$$

Similarly, by applying the inverse matrix to the third column of $\overline{(g, f_1, f_2)}$, we get

$$\begin{aligned} xB(x) &= (1, r_1, r_2) \cdot \left(\frac{1}{g}, x, x\right) \cdot g f_2 \\ &= (1, r_1, r_2) \cdot \frac{1}{g(x)} g f_2 \\ &= (1, r_1, r_2) \cdot f_2. \end{aligned}$$

Thus $B(x) = \frac{1}{x}(1, r_1, r_2) \cdot f_2$. □

Corollary 32. *We have*

$$A(x) = f_1^o(xr_2) + \frac{r_1(x) f_1^e(xr_2)}{xr_2(x)}.$$

Proof. We have

$$\begin{aligned} (1, r_1, r_2) \cdot \frac{f_1(x)}{x} &= \left(\frac{f_1(x)}{x}\right)^e (xr_2(x)) + r_1(x) \left(\frac{f_1(x)}{x}\right)^o (xr_2(x)) \\ &= f_1^o(xr_2(x)) + r_1(x)(\frac{f_1^e}{x})(xr_2(x)) \\ &= f_1^o(xr_2(x)) + r_1(x) \frac{f_1^e(xr_2(x))}{xr_2(x)}. \end{aligned}$$

□

We note that $Z(x)$, the generating function of the first column of the production matrix, is given by

$$\begin{aligned} Z(x) &= \left(\frac{1 - g_0/g(x)}{x}\right)^e (xr_2(x)) + r_1(x) \left(\frac{1 - g_0/g(x)}{x}\right)^o (xr_2(x)) \\ &= \left(1 - \frac{g_0}{g(x)}\right)^o (xr_2(x)) + \frac{r_1(x)}{x} \left(1 - \frac{g_0}{g(x)}\right)^e (xr_2(x)). \end{aligned}$$

In the expression $B(x) = \frac{1}{x}(1, r_1, r_2) \cdot f_2$, we recall that $f_2(x)$ is an odd power series, and thus f_2^e is zero. Hence we have

$$xB(x) = r_1(x)(f_2)^o(xr_2(x))$$

$$= \frac{x - f_1^e(xr_2(x))}{f_1^o(xr_2(x))}(f_2)^o(xr_2(x))$$
$$= (x - f_1^e(xr_2(x)) \cdot \frac{f_2^o(xr_2(x))}{f_1^o(xr_2(x))}.$$

Proposition 33. *The fourth column of the production matrix has generating function* $x^2A(x)$.

Proof. The fourth column of (g, f_1, f_2) has generating function $gf_1(xf_2)$. Thus the generating function of the fourth column of $\overline{(g, f_1, f_2)}$ is given by gf_1f_2. It follows that the generating function of the fourth column of the production matrix is given by

$$(1, r_1, r_2) \cdot (\frac{1}{g}, x, x) \cdot gf_1f_2 = (1, r_1, r_2) \cdot f_1f_2 = (f_1f_2)^e(xr_2) + r_1(x)(f_1f_2)^o(xr_2).$$

Now

$$\begin{aligned}(f_1f_2)^e(x) &= (f_1(\sqrt{x})f_2(\sqrt{x}) + f_1(-\sqrt{x})f_2(-\sqrt{x}))/2\\ &= (f_1(\sqrt{x})f_2(\sqrt{x}) - f_1(-\sqrt{x})f_2(\sqrt{x}))/2\\ &= f_2(\sqrt{x})(f_1(\sqrt{x} - f_1(-\sqrt{x}))/2\\ &= f_2(\sqrt{x})\sqrt{x}f_1^o(x)\\ &= (xf_2)(\sqrt{x})f_1^o(x).\end{aligned}$$

Also we have

$$\begin{aligned}(f_1f_2)^o(x) &= (f_1(\sqrt{x})f_2(\sqrt{x}) - f_1(-\sqrt{x})f_2(-\sqrt{x}))/(2\sqrt{x})\\ &= (f_1(\sqrt{x})f_2(\sqrt{x}) + f_1(-\sqrt{x})f_2(\sqrt{x}))/(2\sqrt{x})\\ &= f_2(\sqrt{x})(f_1(\sqrt{x}) + f_1(-\sqrt{x}))/(2\sqrt{x})\\ &= \frac{f_2(\sqrt{x})}{\sqrt{x}}f_1^e(x)\\ &= \left(\frac{f_2}{x}\right)(\sqrt{x})f_1^e(x).\end{aligned}$$

Then

$$\begin{aligned}(1, r_1, r_2) \cdot f_1f_2 &= (f_1f_2)^e(xr_2) + r_1(x)(f_1f_2)^o(xr_2)\\ &= (xf_2)(\sqrt{xr_2})f_1^o(xr_2) + r_1(x)\left(\frac{f_2}{x}\right)(\sqrt{xr_2})f_1^e(xr_2)\\ &= (xf_2)(\overline{\sqrt{xf_2}})f_1^o(xr_2) + r_1(x)\left(\frac{f_2}{x}\right)(\overline{\sqrt{xf_2}})f_1^e(xr_2)\\ &= x^2f_1^o(xr_2) + r_1(x)\left(\frac{xf_2}{x^2}\right)(\overline{\sqrt{xf_2}})f_1^e(xr_2)\\ &= x^2f_1^o(xr_2) + r_1(x)\frac{x^2}{(\overline{\sqrt{xf_2}})^2}f_1^e(xr_2)\end{aligned}$$

$$
\begin{aligned}
&= x^2 f_1^o(xr_2) + r_1(x)\frac{x}{\frac{1}{x}(\overline{\sqrt{xf_2}})^2} f_1^e(xr_2) \\
&= x^2\left(f_1^o(xr_2) + \frac{r_1(x)f_1^e(xr_2)}{xr_2(x)}\right) \\
&= x^2 A(x).
\end{aligned}
$$

□

Subsequent odd indexed columns involve terms $f_1 f_2^{2m}$, and thus will yield $x^{2m}A(x)$ as their generating function. This explains the A-stripes. A similar argument holds for the B-stripes. In order to see what the relationship between $A(x)$ and $B(x)$ is, we shall need the following lemma.

Lemma 34. *We have*

$$r_2(x) = \frac{x}{f_2^o(xr_2(x))}.$$

Proof. We have

$$f_2^o(x) = \frac{f_2(\sqrt{x}) - f_2(-\sqrt{x})}{2\sqrt{x}} = \frac{f_2(\sqrt{x}) + f_2(\sqrt{x})}{2\sqrt{x}} = \frac{f_2(\sqrt{x})}{\sqrt{x}},$$

where we have used the fact that $f_2(x)$ is odd. Now

$$xr_2(x) = x.\frac{1}{x}\left(\overline{\sqrt{xf_2(x)}}\right)^2 = \left(\overline{\sqrt{xf_2(x)}}\right)^2,$$

and so

$$f_2^o(xr_2(x)) = f_2^o\left(\left(\overline{\sqrt{xf_2(x)}}\right)^2\right) = \frac{f_2(\overline{\sqrt{xf_2}})}{\overline{\sqrt{xf_2}}}.$$

Then

$$
\begin{aligned}
\frac{x}{f_2^o(xr_2(x))} &= \frac{x\overline{\sqrt{xf^2}}}{f_2(\overline{\sqrt{xf_2}})} \\
&= \frac{x(\overline{\sqrt{xf_2}})^2}{\overline{\sqrt{xf_2}}f_2(\overline{\sqrt{xf_2}})} \\
&= \frac{x(\overline{\sqrt{xf_2}})^2}{(xf_2)(\overline{\sqrt{xf_2}})} \\
&= \frac{x(\overline{\sqrt{xf_2}})^2}{x^2} \\
&= \frac{1}{x}(\overline{\sqrt{xf_2}})^2 \\
&= r_2(x).
\end{aligned}
$$

□

This leads us to the following characterization of $A(x)$ and $B(x)$ for Sprugnoli arrays.

Proposition 35. *Let $A(x)$ and $xB(x)$ correspond to the generating functions of the second and third columns of the production array of the Sprugnoli array (g, f_1, f_2). Then $A(x)+B(x)$ is an even power series.*

Proof. We have

$$A(x)+B(x) = \frac{\frac{1}{x}(x - f_1^e(xr_2(x)))f_2^o(xr_2(x))}{f_1^o(xr_2(x))} + \frac{x - f_1^e(xr_2(x))}{f_1^o(xr_2(x))} \left(\frac{f_1(x)}{x}\right)^o (xr_2(x)) + f_1^o(xr_2(x)).$$

Now

$$\left(\frac{f_1(x)}{x}\right)^o = \frac{f_1^e}{x}.$$

Thus

$$A(x) + B(x) = \frac{\frac{1}{x}(x - f_1^e(xr_2(x)))f_2^o(xr_2(x))}{f_1^o(xr_2(x))} + \frac{x - f_1^e(xr_2(x))}{f_1^o(xr_2(x))} \frac{f_1^e(xr_2(x))}{xr_2(x)} + f_1^o(xr_2(x)).$$

Gathering terms, we find that

$$\begin{aligned} A(x) + B(x) &= x\frac{f_1^e(xr_2(x))}{xr_2(x)} - \frac{1}{x} f_2^o(xr_2(x)) f_1^e(xr_2(x)) + \text{even terms in } xr_2(x) \\ &= f_1^e(xr_2(x)) \left(\frac{x}{xr_2(x)} - \frac{1}{x} f_2^o(xr_2(x))\right) + \text{even terms} \\ &= 0 + \text{even terms.} \end{aligned}$$

□

Example 36. We consider the Sprugnoli array $\left(\frac{1}{1-x-x^2}, \frac{x(1+x)}{1-x}, \frac{x}{1-x^2}\right)$. This matrix begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 6 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 5 & 11 & 3 & 3 & 1 & 0 & 0 & 0 & 0 \\ 8 & 19 & 4 & 7 & 1 & 1 & 0 & 0 & 0 \\ 13 & 32 & 8 & 14 & 4 & 3 & 1 & 0 & 0 \\ 21 & 53 & 12 & 26 & 5 & 8 & 1 & 1 & 0 \\ 34 & 87 & 21 & 46 & 12 & 17 & 5 & 3 & 1 \end{pmatrix}.$$

The production matrix for this array is as follows.

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & -2 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ -2 & -2 & -1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 4 & 4 & 2 & -2 & -2 & 1 & 0 & 0 & 0 \\ 4 & 4 & 2 & -2 & -1 & 2 & 1 & 0 & 0 \\ -8 & -8 & -4 & 4 & 2 & -2 & -2 & 1 & 0 \\ -8 & -8 & -4 & -4 & 2 & -2 & -1 & 2 & 1 \\ 16 & 16 & 8 & -8 & -4 & 4 & 2 & -2 & -2 \end{pmatrix}.$$

Thus we have

$$\begin{aligned} Z: &\quad 1,1,-2,-2,4,4,-8,-8,16,\ldots \\ A: &\quad 1,2,-2,-2,4,4,-8,-8,16,\ldots \\ B: &\quad 1,-2,-1,2,2,-4,-4,8,8,\ldots \end{aligned}$$

Taking the first column term $t_{6,0} = 13$, we have (using dot-product notation)

$$13 = (8,19,4,7,1,1,\ldots)\cdot(1,1,-2,-2,4,4,\ldots).$$

Taking the term $t_{7,1} = 53$ in the second column, we have

$$53 = (13,32,8,14,4,3,1,\ldots)\cdot(1,2,-2,-2,4,4,-8,\ldots).$$

Note that the index of the column is odd, so we use the A sequence as coefficients. Now taking the term $t_{6,2} = 8$, we have

$$8 = (19,4,7,1,1,\ldots)\cdot(1,-2,-1,2,2,\ldots).$$

In general, we have

$$\begin{aligned} t_{n,0} &= t_{n-1,0}z_0 + t_{n-1,1}z_1 + t_{n-1,2}z_2 + \cdots \\ t_{n,k} &= t_{n-1,k-1}a_0 + t_{n-1,k}a_1 + t_{n-1,k+1}a_2 + \cdots \qquad \text{for } k \text{ odd} \\ t_{n,k} &= t_{n-1,k-1}b_0 + t_{n-1,k}b_1 + t_{n-1,k+1}b_2 + \cdots \qquad \text{for } k \text{ even.} \end{aligned}$$

Example 37. We consider the family of polynomials $P_n(x)$ defined by

$$P_n(x) = \begin{cases} (x+1)P_{n-1}(x) - P_{n-2}(x), & \text{if } n \text{ is even,} \\ (x-1)P_{n-1}(x) - P_{n-2}(x), & \text{if } n \text{ is odd,} \end{cases}$$

with $P_n(x) = 0$ for $n < 0$ and $P_0(x) = 1$. These polynomials begin

$$1, x-1, x^2-2, x^3-x^2-3x+3, x^4-5x^2+5, \ldots.$$

The coefficient array of this sequence of polynomials is given by the Sprugnoli array

$$\left(\frac{1-x+x^2}{1+3x^2+x^4}, \frac{x}{1-x+x^2}, \frac{x}{1+3x^2+x^4}\right)$$

which begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & -3 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 5 & 0 & -5 & 0 & 1 & 0 & 0 & 0 & 0 \\ -8 & 8 & 6 & -6 & -1 & 1 & 0 & 0 & 0 \\ -13 & 0 & 19 & 0 & -8 & 0 & 1 & 0 & 0 \\ 21 & -21 & -25 & 25 & 9 & -9 & -1 & 1 & 0 \\ 34 & 0 & -65 & 0 & 42 & 0 & -11 & 0 & 1 \end{pmatrix}.$$

We recognize a signed version of the Fibonacci numbers A000045 in the first column. The inverse of this matrix begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 3 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 5 & 0 & 5 & 0 & 1 & 0 & 0 & 0 & 0 \\ 5 & 10 & 5 & 6 & 1 & 1 & 0 & 0 & 0 \\ 15 & 0 & 21 & 0 & 8 & 0 & 1 & 0 & 0 \\ 15 & 36 & 21 & 29 & 8 & 9 & 1 & 1 & 0 \\ 51 & 0 & 86 & 0 & 46 & 0 & 11 & 0 & 1 \end{pmatrix}.$$

and its production matrix begins

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

For this matrix, we have

$$g(x) = \frac{1+x}{2x^2}\left(1 - \sqrt{\frac{1-5x^2}{1-x^2}}\right) = \frac{1-x^2-\sqrt{(1-x^2)(1-5x^2)}}{2x^2(1-x)},$$

$$f_1(x) = \frac{x(1-x)}{2x^2}\left(1 - \sqrt{\frac{1-5x^2}{1-x^2}}\right) = \frac{1-x^2-\sqrt{(1-x^2)(1-5x^2)}}{2x(1+x)},$$

$$f_2(x) = \frac{1}{2x^3}(1-3x^2-\sqrt{(1-x^2)(1-5x^2)}).$$

The first column of this inverse matrix, which begins

$$1, 1, 2, 2, 5, 5, 15, 15, 51, 51, \ldots,$$

or A055879 can be regarded as moments of the family of (generalized) orthogonal polynomials $P_n(x)$. This sequence is a doubling of the binomial transform of the Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$ A000108. The generating function of this moment sequence can be expressed as the Jacobi continued fraction

$$\cfrac{1}{1-x-\cfrac{x^2}{1+x-\cfrac{x^2}{1-x-\cfrac{x^2}{1+x-\cfrac{x^2}{1+x-\cdots}}}}}.$$

We have shown that having a two-striped production matrix of the form (10) is a necessary condition for an array to be a Sprugnoli array. This author has not be able to demonstrate its sufficiency. Thus we leave this as a conjecture.

Conjecture 38. If a lower-triangular matrix has a two-striped production matrix of the form (10), with $A(x) + B(x)$ even, then it is a Sprugnoli matrix.

9 Further examples

The following are examples of some simple Sprugnoli arrays and their inverses.

$$\left(\frac{1}{1+x}, \frac{x}{1-x}, \frac{x}{1-x^2}\right)^{-1} = \left(\frac{1+x}{1+x^2}, \frac{x}{1+x}, \frac{x}{1+x^2}\right)$$
$$\left(\frac{1}{1-x}, x, x(1+x^2)\right)^{-1} = \left(1-x, \frac{x(1-xc(-x^2))}{1-x}, xc(-x^2)\right)$$

Here, we have $c(x) = \frac{1-\sqrt{1-4x}}{2x}$, the generating function of the Catalan numbers.

$$\left(\frac{1}{1-x}, x(1+x), x(1+x^2)\right)^{-1} = (1-x+x^2c(-x^2), x(-1+(2-x)c(-x^2)), xc(-x^2))$$
$$\left(\frac{1}{1-2x}, \frac{x(1+x)}{1-x}, \frac{x}{1-x^2}\right)^{-1} = \left(\frac{1-2x+6x^2}{1+2x^2}, \frac{x(1-4x-x^2-6x^3}{(1+x^2)(1-2x+6x^2)}, \frac{x}{1+x^2}\right)$$
$$\left(\frac{1}{1-x}, \frac{x(1+2x)}{1-x}, \frac{x}{1-x^2}\right)^{-1} = \left(\frac{1-x+6x^2}{1+3x^2}, \frac{x(1-4x+x^2-6x^3)}{(1+x^2)(1-x+6x^2)}, \frac{x}{1+x^2}\right)$$
$$\left(\frac{1}{1-x}, \frac{x(1+x)}{1-x}, \frac{x}{1-x^2}\right)^{-1} = \left(\frac{1-x+4x^2}{1+2x^2}, \frac{x(1-3x+x^2-4x^3)}{(1+x^2)(1-x+4x^2)}, \frac{x}{1+x^2}\right).$$

Example 39. An involution in the Sprugnoli group. We have

$$\left(\frac{1}{1-x}, \frac{-x}{1+x}, \frac{-x}{1-x^2}\right)^{-1} = \left(\frac{1}{1-x}, \frac{-x}{1+x}, \frac{-x}{1-x^2}\right).$$

Thus this matrix M satisfies $M^2 = I$. This is an example of an involution in the Sprugnoli group. We note that the production matrix of this Sprugnoli array has the simple form as follows.

$$\begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Example 40. Given $r, s, t \in \mathbb{Z}$, we call the sequence $s_n(r,s,t) = rs^{n \bmod 2} t^{\lfloor \frac{n}{2} \rfloor}$ the $(r,s,t)-$ *Sprugnoli* sequence. The generating function of the sequence $s_n(r,s,t)$ is given by

$$\frac{r(1+sx)}{1-tx^2}.$$

The sequence $s_n(1,2,3)$ (A038754) which begins

$$1, 2, 3, 6, 9, 18, 27, 54, 81, 162, 243, \ldots,$$

is then an eigensequence of the Sprugnoli array $\left(\frac{1}{1-x}, x(1-x), x\right)$ A070909. By this we mean that we have, for instance,

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 2 \\ 3 \\ 6 \\ 9 \\ 18 \\ 27 \\ 81 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 6 \\ 9 \\ 18 \\ 27 \\ 81 \\ 162 \end{pmatrix}$$

10 Generalizations of the Sprugnoli group

The double Riordan array has been generalized to suitable n-tuples of power series [17]. Thus it is natural to seek to do the same for the Sprugnoli group. For instance, a "next'" Sprugnoli group of elements (g, f_1, f_2, f_3) is based on $f_3 \in x\mathbf{F}[[x^3]]$, and involves sequence trisections in the corresponding fundamental theorem, and cubic roots in the construction of the inverses. We call this group the 3-Sprugnoli group. We can consider the Riordan group to be the 1-Sprugnoli group, the Sprugnoli group defined above to be the 2-Sprugnoli group, and this new proposed group to be the 3-Sprugnoli group.

To define the "fundamental theorem of the 3-Sprugnoli group", we need to calculate the three trisections of a power series. For a generating function $h(x)$, we can calculate its trisections $h^{(1)}(x)$, $h^{(2)}(x)$ and $h^{(3)}(x)$ using the 3×3 discrete Fourier transform [25] as follows. We let $\omega = \exp(2\pi i/3)$. Then

$$\frac{1}{3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & x^{-1/3} & 0 \\ 0 & 0 & x^{-2/3} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{pmatrix} \begin{pmatrix} h\left(x^{1/3}\right) \\ h\left(\omega x^{1/3}\right) \\ h\left(\omega^2 x^{1/3}\right) \end{pmatrix} = \begin{pmatrix} h^{(1)}(x) \\ h^{(2)}(x) \\ h^{(3)}(x) \end{pmatrix}.$$

The fundamental theorem for this group, describing how an element (g, f_1, f_2, f_3) acts on a power series $h(x)$, is given by

$$(g, f_1, f_2, f_3) \cdot h(x) = g(x)h^{(1)}(x^2 f_3(x)) + g(x)f_1(x)h^{(2)}(x^2 f_3(x)) + g(x)f_1(x)f_2(x)h^{(3)}(x^2 f_3(x)).$$

Note that we can write this in the symbolic form

$$(g, f_1, f_2, f_3) \cdot h(x) = \big((g, x^2 f_3), \quad (g f_1, x^2 f_3), \quad (g f_1 f_2, x^2 f_3)\big) \begin{pmatrix} h^{(1)} \\ h^{(2)} \\ h^{(3)} \end{pmatrix}.$$

This allows us to define the multiplication in this group. We have

$$(g, f_1, f_2, f_3) \cdot (u, v_1, v_2, v_3) = \left((g, f_1, f_2, f_3) \cdot u, \frac{(g, f_1, f_2, f_3) \cdot u v_1}{(g, f_1, f_2, f_3) \cdot u}, \frac{(g, f_1, f_2, f_3) \cdot u v_1 v_2}{(g, f_1, f_2, f_3) \cdot u v_1}, \frac{1}{x^2} \frac{(g, f_1, f_2, f_3) \cdot u x^2 v_3}{(g, f_1, f_2, f_3) \cdot u}\right).$$

The identity for this product is then given by $(1, x, x, x)$. In order to calculate the inverse of a group element, we look at the case of the element $(1, f_1, f_2, f_3)$ first. We let

$$(1, f_1, f_2, f_3)^{-1} = (1, r_1, r_2, r_3).$$

We have

$$r_3(x) = \frac{1}{x^2} \left(\overline{(x^2 f_3(x))^{1/3}}\right)^3.$$

In order to calculate r_1 and r_2, we use the following matrix formulation, where $f_{12}(x) = f_1(x) f_2(x)$.

$$\begin{pmatrix} f_1^{(2)}(x^2 r_3(x)) & f_1^{(3)}(x^2 r_3(x)) \\ f_{12}^{(2)}(x^2 r_3(x))/x & f_{12}^{(3)}(x^2 r_3(x))/x \end{pmatrix} \begin{pmatrix} r_1(x) \\ r_1(x) r_2(x) \end{pmatrix} = \begin{pmatrix} x - f_1^{(1)}(x^2 r_3(x)) \\ x - f_{12}^{(1)}(x^2 r_3(x))/x \end{pmatrix}.$$

This allows us to solve for $r_1(x)$ and $r_1(x) r_2(x)$. Thus we can find $r_2(x)$. Finally, in the general case, we use

$$(g, f_1, f_2, f_3)^{-1} = (1, f_1, f_2, f_3)^{-1} \cdot \left(\frac{1}{g(x)}, x, x, x\right).$$

In order to guarantee that this inverse exists for every element (g, f_1, f_2, f_3), we have the following proposition.

Proposition 41. *The determinant of the matrix*

$$\begin{pmatrix} f_1^{(2)}(x^2 r_3(x)) & f_1^{(3)}(x^2 r_3(x)) \\ f_{12}^{(2)}(x^2 r_3(x))/x & f_{12}^{(3)}(x^2 r_3(x))/x \end{pmatrix}$$

is non-zero.

Proof. We have $f_1, f_2 \in \mathcal{F}_1$, hence $f_1 f_2 \in \mathcal{F}_2$. Then $f_1^(2) \in \mathcal{F}_0$ and $(f_1 f_2)^{(3)} \in \mathcal{F}_0$. Also $(f_1 f_2)^{(2)} \in \mathcal{F}_1$. This now ensures that

$$f_1^{(2)}(x^2 r_3(x))(f_1 f_2)^{(3)}(x^2 r_3(x)) - f_1^{(3)}(x^2 r_3(x))(f_1 f_2)^{(2)}(x^2 r_3(x)) \neq 0.$$

Thus the determinant is non-zero. □

Example 42. We let

$$(g_1, f_1, f_2, f_3) = \left(\frac{1}{1-x}, x(1+x), \frac{x}{1-3x}, \frac{x}{1-x^3}\right).$$

Then the matrix whose $(n,k)-$th term is given by

$$t_{n,k} = [x^n]g(x)f_1(x)^{\frac{2}{3}\left(1-\cos\left(\frac{2\pi k}{3}\right)\right)}f_2(x)^{\frac{2}{3}\left(\frac{1}{2}+\cos\left(\frac{2\pi(k+1)}{3}\right)\right)}(x^2 f_3(x))^{\lfloor\frac{k}{3}\rfloor},$$

begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 5 & 1 & 0 & 0 & 0 \\ 1 & 2 & 17 & 1 & 1 & 0 & 0 \\ 1 & 2 & 53 & 1 & 2 & 1 & 0 \\ 1 & 2 & 161 & 2 & 2 & 5 & 1 \end{pmatrix}.$$

The production array of this matrix will then begin

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 3 & 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & -4 & 1 & 0 & 0 \\ 0 & 12 & 0 & 12 & 1 & 1 & 0 \\ 0 & 24 & 0 & -23 & -1 & 3 & 1 \\ 0 & 8 & 0 & -12 & 4 & 0 & -4 \end{pmatrix},$$

showing three vertical stripes after the first (Z) column. The production array of the inverse of this matrix begins

$$\begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ -3 & 5 & -3 & 1 & 0 & 0 & 0 \\ -31 & 61 & -35 & 4 & 1 & 0 & 0 \\ -103 & 205 & -119 & 17 & -1 & 1 & 0 \\ -279 & 557 & -330 & 49 & 5 & -3 & 1 \\ -779 & 1557 & -934 & 125 & 61 & -35 & 4 \end{pmatrix}.$$

The triple striped nature of these production matrices is clear.

In this case, we have

$$f_1^{(1)}(x) = 0, f_1^{(2)} = 1, f_1^{(3)} = 1,$$

and

$$f_{12}^{(1)}(x) = \frac{4}{1-27x}, f_{12}^{(2)}(x) = \frac{12x}{1-27x}, f_{12}^{(3)}(x) = \frac{1+9x}{1-27x}.$$

With these values, we obtain

$$r_1(x) = \frac{x(1-x+4x^2+10x^3+26x^4)}{1-2x^3},$$

$$r_2(x) = \frac{x(1-4x-12x^2-26x^3)}{1-x+4x^2+10x^3+26x^4},$$

and

$$r_3(x) = \frac{x}{1+x^3}.$$

This gives

$$(1, r_1(x), r_2(x), r_3(x)) = (1, f_1(x), f_2(x), f_3(x))^{-1},$$

and so

$$\begin{aligned}(g(x), f_1(x), f_2(x), f_3(x))^{-1} &= (1, r_1(x), r_2(x), r_3(x)) \cdot \left(\frac{1}{g(x)}, x, x, x\right) \\ &= \left((1, r_1, r_2, r_3) \cdot (1/g), \frac{(1, r_1, r_2, r_3) \cdot (x/g)}{(1, r_1, r_2, r_3) \cdot (1/g)}, \frac{(1, r_1, r_2, r_3) \cdot (x^2/g)}{(1, r_1, r_2, r_3) \cdot (x/g)}, r_3(x)\right) \\ &= (w(x), s_1(x), s_2(x), s_3(x))\end{aligned}$$

where

$$\begin{aligned}w(x) &= \frac{1-x+x^2-6x^3-10x^4-26x^5}{1-2x^3} \\ s_1(x) &= \frac{x(1-2x+8x^2+22x^3+52x^4)}{1-x+x^2-6x^3+10x^4+26x^5} \\ s_2(x) &= \frac{x(1-5x-12x^2-25x^3-2x^4-12x^5-26x^6)}{(1+x^3)(1-2x+8x^2+22x^3+52x^4)} \\ s_3(x) &= \frac{x}{1+x^3}.\end{aligned}$$

The general production matrix for this group on four elements will have the form

$$\begin{pmatrix} z_0 & a_0 & 0 & 0 & 0 & 0 & \cdots \\ z_1 & a_1 & b_0 & 0 & 0 & 0 & \cdots \\ z_2 & a_2 & b_1 & c_0 & 0 & 0 & \cdots \\ z_3 & a_3 & b_2 & c_1 & a_0 & 0 & \cdots \\ z_4 & a_4 & b_3 & c_2 & a_1 & b_0 & \cdots \\ z_5 & a_5 & b_4 & c_3 & a_2 & b_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The distinguishing character of these production matrices is illustrated by aligning the A, B, and C sequences of the above numeric example.

$$\left(\begin{array}{cccc|cccc} A & B & C & A+B+C & A^* & B^* & C^* & A^*+B^*+C^* \\ 1 & 1 & 1 & 3 & 1 & 1 & 1 & 3 \\ 1 & 3 & -4 & 0 & -1 & -3 & 4 & 0 \\ -1 & 0 & 12 & 11 & 5 & -35 & 17 & -13 \\ 4 & 0 & -23 & -19 & 61 & -119 & 49 & -9 \\ 12 & 0 & -12 & 0 & 205 & -350 & 125 & 0 \\ 24 & 0 & -36 & -12 & 557 & -934 & 365 & -12 \\ 8 & 0 & -72 & -64 & 1557 & -2710 & 1125 & -28 \\ 24 & 0 & -24 & 0 & 4485 & -7918 & 3433 & 0 \\ 48 & 0 & -72 & -24 & 13029 & -23458 & 10393 & -36 \end{array}\right).$$

We see an emerging pattern of zeros repeating after every third number. Here, A^*, B^* and C^* refer to the inverse matrix. Matrices from this Sprugnoli group on four elements conform to the following schema.

$$\begin{array}{ccccccccc}
g & g & g & g & g & g & g & g & g \\
1 & f_1 & f_1 & 1 & f_1 & f_1 & 1 & f_1 & f_1 \\
1 & 1 & f_2 & 1 & 1 & f_2 & 1 & 1 & f_2 \\
1 & 1 & 1 & f_3 & f_3 & f_3 & f_3^2 & f_3^2 & f_3^2 \\
1 & 1 & 1 & x^2 & x^2 & x^2 & x^4 & x^4 & x^4 \\
- & - & - & - & - & - & - & - & - \\
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8
\end{array}$$

In a similar way, elements (g, f_1, f_2, f_3, f_4) (with $f_4 \in x\mathbf{F}[[x^4]]$) in the Sprugnoli group on five elements conform to the following schema.

$$\begin{array}{ccccccccccc}
g & g & g & g & g & g & g & g & g & g & g \\
1 & f_1 & f_1 & f_1 & 1 & f_1 & f_1 & f_1 & 1 & f_1 & f_1 \\
1 & 1 & f_2 & f_2 & 1 & 1 & f_2 & f_2 & 1 & 1 & f_2 \\
1 & 1 & 1 & f_3 & 1 & 1 & 1 & f_3 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & f_4 & f_4 & f_4 & f_4 & f_4^2 & f_4^2 & f_4^2 \\
1 & 1 & 1 & 1 & x^3 & x^3 & x^3 & x^3 & x^6 & x^6 & x^6 \\
- & - & - & - & - & - & - & - & - & - & - \\
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10
\end{array}$$

In this case, the four quadrisections of a generating function $g(x)$ are given by

$$\frac{1}{4}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & x^{-1/4} & 0 & 0 \\ 0 & 0 & x^{-2/4} & 0 \\ 0 & 0 & 0 & x^{-3/4} \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix}\begin{pmatrix} g\left(x^{1/4}\right) \\ g\left(ix^{1/4}\right) \\ g\left(-x^{1/4}\right) \\ g\left(-ix^{1/4}\right) \end{pmatrix} = \begin{pmatrix} g^{(1)}(x) \\ g^{(2)}(x) \\ g^{(3)}(x) \\ g^{(4)}(x) \end{pmatrix}.$$

Setting $(1, r_1, r_2, r_3, r_4) = (1, f_1, f_2, f_3, f_4)^{-1}$, we solve for r_1, r_2 and r_3 using the matrix formulation

$$\begin{pmatrix} f_1^{(2)}(x^3 r_4(x)) & f_1^{(3)}(x^3 r_4(x)) & f_1^{(4)}(x^3 r_4(x)) \\ (f_1 f_2)^{(2)}(x^3 r_4(x))/x & (f_1 f_2)^{(3)}(x^3 r_4(x))/x & (f_1 f_2)^{(4)}(x^3 r_4(x))/x \\ (f_1 f_2 f_3)^{(2)}(x^3 r_4(x))/x^2 & (f_1 f_2 f_3)^{(3)}(x^3 r_4(x)) & (f_1 f_2 f_3)^{(4)}(x^3 r_4(x)) \end{pmatrix}\begin{pmatrix} r_1(x) \\ r_1(x) r_2(x) \\ r_1(x) r_2(x) r_3(x) \end{pmatrix}$$
$$= \begin{pmatrix} x - f_1^{(1)}(x^3 r_4(x)) \\ x - (f_1 f_2)^{(1)}(x^3 r_4(x))/x \\ x - (f_1 f_2 f_3)^{(1)}(x^3 r_4)/x^2 \end{pmatrix}.$$

Here, we have

$$r_4(x) = \frac{1}{x^3}\left(\overline{(x^3 f_4(x))^{1/4}}\right)^4.$$

Continuing in this fashion indicates that we can construct general n-Sprugnoli groups on elements $(g, f_1, f_2, \ldots, f_n)$. In particular, the product in the n-Sprugnoli group is given by

$$(g, f_1, \ldots, f_n) \cdot (h, r_1, \ldots, r_n) =$$
$$\left((g, f_1, \ldots, f_n) \cdot h, \frac{(g, f_1, \ldots, f_n) \cdot h r_1}{(g, f_1, \ldots, f_n) \cdot h}, \ldots, \frac{(g, f_1, \ldots, f_n) \cdot h r_1 \cdots r_{i+1}}{(g, f_1, \ldots, f_n) \cdot h r_1 \cdots r_i}, \ldots, \frac{1}{x^{n-1}} \frac{(g, f_1, \ldots, f_n) \cdot h x^{n-1} r_n}{(g, f_1, \ldots, f_n) \cdot h}\right).$$

11 Conclusions

Matrix groups based on the idea of using generating functions to prescribe the columns of the matrices in the group have grown in popularity since the first definition of the Riordan group [18]. The combination of generating functions and matrices has led to many applications of these groups in combinatorics and elsewhere, where the idea of generalized weighted compositions comes into play. This note adds to this context by defining a new group of matrices whose columns are defined by generating functions in an original way. We have indicated how these ideas define a hierarchy of Sprugnoli groups with elements $(g, f_1, f_2, \ldots, f_n)$ where there is a specific prescription on the form of f_n. In the definition of the Sprugnoli group on elements (g, f_1, f_2) above, there is a strong implicit role played by the (once-)stretched Riordan arrays (g, xf_2) and $(gf_1/x, xf_2)$. Embedded in the inverse $(g, f_1, f_2)^{-1}$ we can find left-inverses [5] of these stretched Riordan arrays. Higher order Sprugnoli groups are based on further stretchings of base Riordan arrays. In this way, the Riordan group is a fundamental building block for the Sprugnoli family of groups.

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Renzo Sprugnoli (1942-2024)

Following the publication of the seminal paper [18] on the Riordan group by Shapiro, Getu, Woan, and Woodson, Renzo Sprugnoli was one of the first people to exploit the expressive power of this new group. In a succession of papers [3, 4, 5, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 21, 22, 23, 24], and through his collaboration with others, he did much to show the power of these new methods, notably in the area of combinatorial sums and lattice paths. His fine analytical skills and his willingness to share with colleagues did much to further research in the area of Riordan arrays and their applications.

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Appendix: Even and odd bisections

We let $g(x)$ by a power series $g(x) = \sum_{n=0}^{\infty} a_n x^n$. The even bisection of $g(x)$ is the power series

$$g^e(x) = \sum_{n=0}^{\infty} a_{2n} x^n$$

while the odd bisection of $g(x)$ is the series

$$g^o(x) = \sum_{n=0}^{\infty} a_{2n+1} x^n.$$

We then have

$$g^e(x) = \frac{g(\sqrt{x}) + g(-\sqrt{x})}{2},$$

and

$$g^o(x) = \frac{g(\sqrt{x}) - g(-\sqrt{x})}{2\sqrt{x}}.$$

We can write this in matrix form as

$$\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & x^{-1/2} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} g\left(x^{1/2}\right) \\ g\left(-x^{1/2}\right) \end{pmatrix} = \begin{pmatrix} g^e(x) \\ g^o(x) \end{pmatrix}.$$

Example 43. Given a power series $g(x)$, we calculate the even and odd bisections of $xg(x)$. We have

$$\begin{aligned}
(xg)^e(x) &= \frac{(xg)(\sqrt{x}) + (xg)(-\sqrt{x})}{2} \\
&= \frac{\sqrt{x}g(\sqrt{x}) - \sqrt{x}g(-\sqrt{x})}{2} \\
&= \frac{\sqrt{x}(g(\sqrt{x}) - g(-\sqrt{x}))}{2} \\
&= \frac{x(g(\sqrt{x}) - g(-\sqrt{x}))}{2\sqrt{x}} \\
&= xg^o(x).
\end{aligned}$$

Thus

$$(xg)^e(x) = xg^o(x).$$

Similarly, we have

$$\begin{aligned}
(xg)^o(x) &= \frac{(xg)(\sqrt{x}) - (xg)(-\sqrt{x})}{2\sqrt{x}} \\
&= \frac{\sqrt{x}g(\sqrt{x}) + \sqrt{x}g(-\sqrt{x})}{2\sqrt{x}} \\
&= \frac{g(\sqrt{x}) + g(-\sqrt{x})}{2}
\end{aligned}$$

$$= g^e(x).$$

Thus

$$(xg)^o(x) = g^e(x).$$

(Concerned with sequence A000045, A000108, A038754, A051159, A055879, A070909, and A122367).