An Extended Generalization of RSK Correspondence via A Type Quiver Representations

Benjamin Dequêne[a]

Submitted: Feb 20, 2026; Accepted: Aug 28, 2026; Published: Nov 6, 2026
©The author. Released under the CC BY-ND license (International 4.0).

Abstract

Let $\lambda = (\lambda_1 \geqslant \ldots \geqslant \lambda_k > 0)$. For any c Coxeter element of $\mathfrak{S}_{\lambda_1+k-1}$, we construct a bijection from fillings of λ to reverse plane partitions. We recover two previous generalizations of the Robinson–Schensted–Knuth correspondence for particular choices of Coxeter element depending on λ: one based on the work of, among others, Burge, Hillman, Grassl, Knuth, and uniformly presented by Gansner; the other developed by Garver, Partrias, and Thomas, and independently by Dauvergne, called Scrambled RSK.

Our results in this paper develop the combinatorial consequences of our previous work of type A_{λ_1+k} quivers.

Mathematics Subject Classifications: 05E10, 05A17, 16G20

1 Introduction

This article has a short version in the proceedings of the 36th edition of the Formal Power Series and Algebraic Combinatorics (FPSAC) Conference [12].

1.1 RSK and its generalizations

Let $n \in \mathbb{N}^*$. The ***Robinson–Schensted correspondence*** is a famous one-to-one correspondence from elements of the symmetric group $\mathfrak{S}_n$ to pairs of standard Young tableaux of the same shape and of size n. It first arose from the representation theory of the symmetric group, thanks to the work of Robinson [34], before receiving a combinatorial realization via Schensted row insertions [36]. This correspondence was studied for numerous combinatorial consequences, as a combinatorial proof of a representation-theoretic identity involving the dimension of the irreducible representations of $\mathfrak{S}_n$ (see Theorem 8), Viennot's geometric construction [41], plactic monoids [30, 38], or the Erdős–Szekeres theorem [14]. We refer the reader to [15, 35] for more details.

[a]School of Mathematics, University of Leeds, Leeds, UK (b.d.dequene@leeds.ac.uk).

The ***Robinson–Schensted–Knuth (RSK) correspondence***, denoted by **RSK**, is a generalization of the Robinson–Schensted correspondence, introduced by Knuth [27], and presented as a bijection from nonnegative integer matrices to pairs of semi-standard Young tableaux of the same shape. We recover the Robinson-Schensted correspondence by restricting **RSK** to permutation matrices. The RSK correspondence extends many properties of the previous correspondence; for instance, its symmetry under transposition of the matrix results in interchanging the tableaux. As a remarkable consequence, we cite the Cauchy identity for symmetric functions (see [40, 15] for more details), which generalizes the representation-theoretic identity mentioned above. It also has many interpretations in different settings, using deformations and generalizations of this correspondence. We refer the reader to [31, 28, 1, 24].

In this paper, we focus on two of those generalizations. Gansner introduced the first [17, 18], based on observations from various works of Burge [6], Hillman–Grassl [25] and Knuth [27]. Given a fixed nonzero integer partition λ, via Greene–Kleitman invariants [23], he defines a map, denoted by $\boldsymbol{\mathcal{RSK}}_\lambda$, which realizes a bijection from arbitrary fillings of λ to reverse plane partitions of λ. Garver, Patrias, and Thomas give the second one [21], in terms of quiver representation theory. Independently, Dauvergne [9], in a combinatorial setting, introduced it as *"Scrambled RSK"*. Here, we focus on the quiver representation theory point of view. This variant can be introduced as a family of one-to-one correspondences $(\mathbf{RSK}_{m,c})_{m,c}$, parametrized by orientations of an A_n type quiver (seen here as a Coxeter element $c \in \mathfrak{S}_{n+1}$) — see Section 3.1), and $m \in \{1, \dots, n\}$, from $m \times (n-m+1)$ integer matrices to reverse plane partitions of $(n-m+1)^m$ (seen as n-tuples of integer partitions satisfying storability conditions – see Section 4).

Our main goal is to exhibit a construction of an extended generalization of $\boldsymbol{\mathcal{RSK}}_\lambda$, for any nonzero integer partitions λ, based on a combinatorial extraction of results from [13], involving any Coxeter element $c \in \mathfrak{S}_{n+1}$, where n is the hook-length of the box $(1,1)$ in λ, using the combinatorics of the A_n type quivers. We denote those maps by $\boldsymbol{\mathcal{RSK}}_{\lambda,c}$. In Figure 1, we pictured how those maps $\boldsymbol{\mathcal{RSK}}_{\lambda,c}$ can be seen as an extended generalization that contains the previously mentioned correspondence. We state the precise results in Section 1.3.

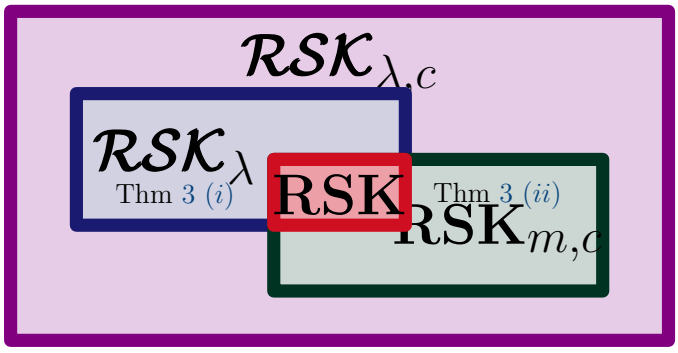


Figure 1: Illustration showing the purpose of the extended generalization $\boldsymbol{\mathcal{RSK}}_{\lambda,c}$.

1.2 Quiver representation theory

In this section, we recall the setting of [21] and state the main result of [13], which motivates our work.

Fix $\mathbb{K}$ an algebraically closed field, and $n \geqslant 1$. Consider an A_n type quiver Q: this is a directed graph whose underlying graph is a line with n vertices. We label the vertices from 1 to n, from left to right. A *representation* E is an assignment of a vector space E_q at each vertex q of Q, and an assignment of a linear transformation E_α to each arrow α of Q. We say that E is *finite-dimensional* whenever, for all $q \in Q_0$, E_q is finite-dimensional. Given two representations E and F, a *morphism* $\phi : E \longrightarrow F$ is a collection of linear maps $(\phi_q : E_q \longrightarrow F_q)_q$ assigned to each vertex of Q, such that it satisfies some commutativity properties (see Section 5). Denote by $\mathrm{rep}_{\mathbb{K}}(Q)$ the category of (finite-dimensional) representations of Q over $\mathbb{K}$. One can view this category as a set of representations of Q equipped with morphisms between them.

A representation is said to be ***indecomposable*** whenever it is not isomorphic to a direct sum of two nonzero representations. Write $\mathrm{ind}_{\mathbb{K}}(Q)$ for the set of isomorphism classes of indecomposable representations of Q. We can encode the data of $\mathrm{ind}_{\mathbb{K}}(Q)$, and the morphisms between indecomposable representations, by the ***Auslander–Reiten quiver***, denoted by $\mathrm{AR}_{\mathbb{K}}(Q)$. It is a directed graph whose vertices are elements of $\mathrm{ind}_{\mathbb{K}}(Q)$, and arrows are irreducible morphisms between them. We recall that any $E \in \mathrm{rep}_{\mathbb{K}}(Q)$ is characterized, up to isomorphisms, by the multiplicities of its indecomposable summands. Write $\mathrm{Mult}(E) : \mathrm{ind}_{\mathbb{K}} \longrightarrow \mathbb{N}$ for the map which associates indecomposable representations to their multiplicities. It could be seen as a filling of $\mathrm{AR}_{\mathbb{K}}(Q)$.

An endomorphism $N : E \longrightarrow E$ is said to be *nilpotent* whenever, for every vertex q of Q, N_q is nilpotent. Write $\mathsf{NEnd}(E)$ for the set of nilpotent endomorphisms of a given representation E.

We define an invariant on isomorphism classes of $\mathrm{rep}_{\mathbb{K}}(Q)$, called the ***generic Jordan form data*** as follows. Given $E \in \mathrm{rep}_{\mathbb{K}}(Q)$, we study the set of its nilpotent endomorphisms, denoted by $\mathsf{NEnd}(E)$, by determining their Jordan form: it is displayed as an n-tuple of integer partitions. Garver, Patrias, and Thomas [21] proved that a (Zariski) dense open set $\Omega \subset \mathsf{NEnd}(E)$ exists in which all the nilpotent endomorphisms have the same Jordan form. This common Jordan form data is called the generic Jordan form data of X, denoted by $\mathrm{GenJF}(E)$ — see Theorem 47 for the precise statement. Note that this invariant can be computed combinatorially (see Section 5.3).

Note that if $n > 1$, GenJF is not a complete invariant. However, we can still be interested in determining the full subcategories of $\mathrm{rep}_{\mathbb{K}}(Q)$ (closed under direct sums and summands) in which GenJF becomes complete. We call these subcategories ***Jordan recoverable***.

Determining all the Jordan recoverable subcategories of $\mathrm{rep}_{\mathbb{K}}(Q)$ is still a difficult task. A conjecture is stated in [13] and is recalled in Section 7. Another question raised is how to recover the representation, up to isomorphisms, from its generic Jordan form data. Garver, Patrias, and Thomas described an algebraic way to do so, and they called ***canonically Jordan recoverable*** any subcategory in which their algebraic procedure succeeds. Note that any canonically Jordan recoverable subcategory is Jordan recoverable,

but the converse is false, which explains the refined notion.

They prove that, for any vertex m in Q, the subcategories additively generated by indecomposable representations X such that $X_m \neq 0$, denoted by $\mathscr{C}_{Q,m}$, are canonically Jordan recoverable. Moreover, they show that GenJF can be seen as a generalization of the RSK correspondence, as they recover **RSK** if Q is oriented such that m is the only sink (respectively, only source) of Q. They also showed that they recover the Hillman–Grassl correspondence when Q is linearly oriented (i.e., Q has a unique source and only one sink). We refer the reader to [21, Section 6] for more details.

Recall that a filling of $\mathrm{AR}_{\mathbb{K}}(Q)$ corresponds to a representation $E \in \mathrm{rep}_{\mathbb{K}}(Q)$ up to isomorphism. Now, see GenJF as a map from fillings of $\mathrm{AR}_{\mathbb{K}}(Q)$ (which define, up to isomorphism, representations of Q) to n-tuples of integer partitions. This map becomes a bijection if we restrict its domain to fillings f which vanish on indecomposable representations X such that $X_m = 0$, and its codomain to n-tuples of integer partitions that satisfy some storability conditions (see Section 4 and [13]). In this case, GenJF coincides with the Dauvergne's Scrambled RSK $\mathbf{RSK}_{m,c}$ mentioned earlier, where $c \in \mathfrak{S}_{n+1}$ is the Coxeter element corresponding to Q (see Section 1.3).

The main result of [13] generalizes one of the results of [21] by describing all the canonically Jordan recoverable subcategories of $\mathrm{rep}_{\mathbb{K}}(Q)$.

Recall that, for any A_n type quiver Q, the isomorphism classes of indecomposable representations are in bijection with intervals $[\![i,j]\!] = \{i, i+1, \ldots, j\}$ in $\{1, \ldots, n\}$. Moreover, their indecomposable representations characterize subcategories closed under direct sums and summands. Thus, for any subcategory $\mathscr{C}$ of $\mathrm{rep}_{\mathbb{K}}(Q)$, we write $\mathrm{Int}(\mathscr{C})$ for the set of intervals corresponding to the indecomposable representations in $\mathscr{C}$.

Two intervals $[\![i,j]\!]$ and $[\![k,\ell]\!]$ are ***adjacent*** whenever either $j+1=k$ or $\ell+1=i$. An interval set $\mathscr{J}$ is said to be ***adjacency-avoiding*** if no pair of intervals in $\mathscr{J}$ are adjacent.

Theorem 1 ([13]). *Let $n \geqslant 1$ and Q be an A_n type quiver. A subcategory $\mathscr{C}$ is canonically Jordan recoverable if and only if $\mathrm{Int}(\mathscr{C})$ is adjacency-avoiding.*

Note that this result shows that canonical Jordan recoverability does not depend on the orientation of Q.

A ***bipartition*** of any given set $\mathbf{A}$ is a pair $(\mathbf{L}, \mathbf{R})$ of subsets of $\mathbf{A}$ such that $\mathbf{L} \cap \mathbf{R} = \varnothing$ and $\mathbf{L} \cup \mathbf{R} = \mathbf{A}$.

We highlight another remarkable fact from [13]. As any subset of an adjacency-avoiding interval set is adjacency-avoiding, we can focus on maximal ones. Those maximal adjacency-avoiding interval sets are parametrized by bipartitions $(\mathbf{L}, \mathbf{R})$ of $\{2, \ldots, n\}$. Precisely, for any maximal adjacency-avoiding interval set $\mathscr{J}$, there exists a unique bipartition $(\mathbf{L}, \mathbf{R})$ of $\{2, \ldots, n\}$ such that:

$$\mathscr{J} = \{[\![\ell, r-1]\!] \mid \ell \in \mathbf{L} \cup \{1\} \text{ and } r \in \mathbf{R} \cup \{n+1\}\}$$

1.3 Main results

We proceed to a combinatorial extraction of the results in [13]. We summarize this extraction in Table 1. The bijective link between integer partitions with $h_\lambda(1,1) = n$ and

Combinatorial tools	Representation-theoretic tools
Coxeter element of $\mathfrak{S}_{n+1}$	Orientation of an A_n type quiver Q
Transposition in $\mathfrak{S}_{n+1}$	Indecomposable representation in $\mathrm{rep}_{\mathbb{K}}(Q)$
AR quiver of c	AR quiver of $\mathrm{rep}_{\mathbb{K}}(Q)$
Integer partition λ with hook-length of $(1,1)$ in λ equals to n	maximal CJR subcategory $\mathscr{C}$ of $\mathrm{rep}_{\mathbb{K}}(Q)$
Filling of λ	$\mathrm{Mult}(E)$ for some $E \in \mathscr{C}$
Reverse plane partition of λ	$\mathrm{GenJF}(E)$ for some $E \in \mathscr{C}$.

Table 1: Identifications between tools from quiver representation theory and combinatorics.

maximal canonically Jordan recoverable (CJR) subcategories of $\mathrm{rep}_{\mathbb{K}}(Q)$ is given by the parametrization with bipartitions of $\{2, \dots, n\}$. Note also that Reading's bijection [33, Lemma 1.7] allows us to define a Coxeter element $\mathbf{c}(\lambda) \in \mathfrak{S}_{n+1}$ from such a λ.

Given such a λ, we build a one-to-one correspondence from generic Jordan form data of a representation in the category coming from λ to reverse plane partitions of shape λ, thanks to the notion of $\mathbf{c}(\lambda)$-storability for n-tuples of partitions (see Section 4).

See Section 6 to get the combinatorial construction of $\boldsymbol{\mathcal{RSK}}_{\lambda,c}$. Our main result is the following.

Theorem 2. *Let $n \geqslant 1$, λ be an integer partition such that $h_\lambda(1,1) = n$, and $c \in \mathfrak{S}_{n+1}$ be a Coxeter element. The map $\boldsymbol{\mathcal{RSK}}_{\lambda,c}$ realizes a one-to-one correspondence from fillings of shape λ to reverse plane partitions of shape λ.*

We also show some secondary results that justify the name of "extended generalization" of RSK.

Theorem 3. *Let $n \geqslant 1$, λ be an integer partition such that $h_\lambda(1,1) = n$, and $c \in \mathfrak{S}_{n+1}$ be a Coxeter element.*

(i) If $c = \mathbf{c}(\lambda)^{\pm 1}$, then $\boldsymbol{\mathcal{RSK}}_{\lambda,c} = \boldsymbol{\mathcal{RSK}}_{\lambda}$.

(ii) If $\lambda = (n-m+1)^m$ for some $m \in \{1, \dots, n\}$, then $\boldsymbol{\mathcal{RSK}}_{\lambda,c} = \mathbf{RSK}_{m,c}$.

(iii) If $c = (1, \dots, n+1)$, then $\boldsymbol{\mathcal{RSK}}_{\lambda,c}$ coincides with the Hillman–Grassl correspondence.

We refer the reader to Section 6 for the proofs of the main theorems, and more details about the local transformation mentioned above.

1.4 Outline of the paper

First, in Section 2, while introducing some essential notations and vocabulary, we recall the classical story of the Robinson–Schensted–Knuth correspondence and discuss a first generalization described by Gansner [18] allowing us to obtain, for any integer partition λ, a one-to-one correspondence from fillings to reverse plane partitions of shape λ.

In Section 3, we go through relevant reminders on Coxeter elements of $\mathfrak{S}_{n+1}$. In particular, we highlight their bijective links with bipartitions of $\{2, \dots, n\}$, integer partitions such that the hook-length of the box $(1, 1)$ is n, and representations of A_n type quivers.

In Section 4, given a Coxeter element $c \in \mathfrak{S}_{n+1}$, we focus on some behavior for n-tuples of integer partitions, called c-storability. We enumerate basic combinatorial results, and we introduce diagonal transformations which, for instance, can change c-storability into c'-storability for some other Coxeter element $c' \in \mathfrak{S}_{n+1}$.

Then, in Section 5, we started to gather the pieces of the puzzle with the representation-theoretic notion of Jordan recoverability. After giving some motivations, among others, we recall operations preserving a refinement of this notion, called canonical Jordan recoverability, and a characterization of the maximal canonically Jordan recoverable subcategories for A_n type quivers. This also relates to the c-storability properties via the n-tuple of integer partitions obtained by computing the generic Jordan form of the nilpotent endomorphism of any given representation in such a category.

In Section 6, we prove the main result by extracting the results from the previous section into a setting allowing us to see the correspondence highlighted in Table 1.

Finally, in Section 7, we gave further directions to extend this work: especially a realization of $\boldsymbol{\mathcal{RSK}}_{\lambda,c}$ via local transformations motivated by [26, 21], and the elaboration of variations of $\boldsymbol{\mathcal{RSK}}_{\lambda,c}$ for other Weyl group, or for string algebras.

2 The Robinson–Schensted–Knuth correspondence

2.1 Notations and vocabulary

This section introduces the basic objects we need throughout this paper.

Quivers and directed graphs

A ***quiver*** is a quadruplet $Q = (Q_0, Q_1, s, t)$ where Q_0 is a set called the *vertex set*, Q_1 is another set called the *arrow set*, and $s, t : Q_1 \longrightarrow Q_0$ are functions called *source* and *target functions*. Given a quiver Q, we denote by Q^{op} its ***opposite quiver*** defined from Q by reversing all its arrows.

Let $Q = (Q_0, Q_1, s, t)$ and $\Xi = (\Xi_0, \Xi_1, \sigma, \tau)$ be two quivers. A ***morphism of quivers*** Ψ is a pair of maps $(\Psi_0 : Q_0 \longrightarrow \Xi_0, \Psi_1 : Q_1 \longrightarrow \Xi_1)$ such that, for all $\alpha \in Q_1$, $\sigma(\Xi_1(\alpha)) = \Xi_0(s(\alpha))$ and $\tau(\Xi_1(\alpha)) = \Xi_0(t(\alpha))$. Such a morphism Ψ is an ***isomorphism*** whenever Ψ_0 and Ψ_1 are bijective. We say that Q and Ξ are isomorphic in such a case.

A quiver Q is said to be ***finite*** whenever Q_0 and Q_1 are finite. We say that Q ***has no multi-arrows*** whenever

$$\#\{\alpha \in Q_1 \mid s(\alpha) = q_1 \text{ and } t(\alpha) = q_2\} \leqslant 1$$

for all pairs $(q_1, q_2) \in (Q_0)^2$. In the combinatorial setting, we call ***directed graph*** any finite quiver without multi-arrows. Since the arrows in any directed graph are uniquely determined by their source and target, we denote directed graphs by pairs $G = (G_0, G_1)$, where we view the arrow set G_1 as a subset of $(G_0)^2$.

Let $G = (G_0, G_1)$ be a directed graph. A ***path*** γ in G as a finite sequence of vertices $(v_0, \dots, v_k)$ such that $(v_i, v_{i+1}) \in G_1$. A *lazy path at* $v \in G_0$ is the path (v). In the following, we denote by $\Pi(G)$ the set of paths in G. For any $\gamma = (v_0, \dots, v_k) \in \Pi(G)$, we denote by $s(\gamma) = v_0$ its source and by $t(\gamma) = v_k$ its target. We also write $\mathrm{Supp}(\gamma) = \{v_0, \dots, v_k\}$ for the ***support of*** $\boldsymbol{\gamma}$. For $\ell \geqslant 1$, we extend the notion of support to ℓ-tuples of paths $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_\ell) \in \Pi(G)^\ell$ as $\mathrm{Supp}(\boldsymbol{\gamma}) = \bigcup_{i=1}^{\ell} \mathrm{Supp}(\gamma_i)$.

A directed graph G is said to be ***connected*** whenever for any pair $(v, w) \in G_0^2$, there exist $\ell \in \mathbb{N}^*$ and $(\gamma_1, \dots, \gamma_\ell) \in \Pi(G)^\ell$ such that:

- $v = s(\gamma_1)$,
- for any $i \in \{1, \dots, \ell - 1\}$ odd, $t(\gamma_i) = t(\gamma_{i+1})$, and if ℓ is odd, then $t(\gamma_\ell) = w$;
- for any $i \in \{1, \dots, \ell - 1\}$ even, $s(\gamma_i) = s(\gamma_{i+1})$, and if ℓ is even, then $s(\gamma_\ell) = w$.

We say that G is ***acyclic*** if the only paths γ in G such that $s(\gamma) = t(\gamma)$ are the lazy ones. Call ***antichain*** of G any subset of vertices $\{w_1, \dots, w_r\} \subset G_0$ such that there is no $\gamma \in \Pi(G)$ with $s(\gamma) = w_i$ and $t(\gamma) = w_j$ for all $1 \leqslant i, j \leqslant r$ with $i \neq j$.

Integer partitions

An ***integer partition*** is a finite weakly decreasing sequence $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_p)$ of positive integers. Define its *size* as $|\lambda| = \lambda_1 + \ldots + \lambda_k$ and its *length* by $\ell(\lambda) = p$. If needed, we can extend the definition of an integer partition to an infinite weakly decreasing sequence of nonnegative integers with finitely many nonzero entries.

Given $a, b \in \mathbb{N}^*$, we denote by a^b the integer partition λ such that $\ell(\lambda) = b$, and $\lambda_i = a$ for $1 \leqslant i \leqslant b$.

We endow $(\mathbb{N}^*)^2$ with the *cartesian product order* $\trianglelefteq$ defined by

$$(i, j) \trianglelefteq (i', j') \iff \begin{cases} i \leqslant i' \\ j \leqslant j' \end{cases} .$$

A ***Ferrers diagram*** is a finite ideal of $((\mathbb{N}^*)^2, \trianglelefteq)$. Recall that we have a one-to-one correspondence between Ferrers diagrams and integer partitions. For a given integer partition λ, we define the Ferrers diagram of shape λ to be

$$\mathrm{Fer}(\lambda) = \{(i, j) \in (\mathbb{N}^*)^2 \mid j \leqslant \lambda_i\}.$$

Call *box of* λ any element of $\mathrm{Fer}(\lambda)$. We use matrix coordinates for the boxes of any partition, meaning that we use English conventions to draw Ferrers diagrams.

Given a box $b \in \mathrm{Fer}(\lambda)$, we write $h_\lambda(b)$ for the ***hook-length of*** b ***in*** $\boldsymbol{\lambda}$, which is defined, if $b = (i, j)$, as

$$h_\lambda(b) = \#\{(u, v) \in \mathrm{Fer}(\lambda) \mid u \geqslant i, v \geqslant j, \text{ and } u = i \text{ or } v = j\}.$$

Explicitly, one can show that $h_\lambda(b) = \lambda_i - i + \lambda'_j - j + 1$ where λ' is the *conjugate* of λ; meaning λ' is the unique integer partition such that

$$\mathrm{Fer}(\lambda') = \{(j, i) \mid (i, j) \in \mathrm{Fer}(\lambda)\}.$$

In particular, we have $h_\lambda(1,1) = \lambda_1 + \ell(\lambda) - 1$. In the following, for any $n \in \mathbb{N}$, we write Hk_n for the set of integer partitions such that $h_\lambda(1,1) = n$.

Given a nonzero integer partition λ, we consider the λ-*diagonal coordinates* for elements in $\mathrm{Fer}(\lambda)$ as follows. For $k \in \mathbb{Z}$, we define the ***kth diagonal of λ*** as the set $D_k(\lambda)$ of boxes such that $\lambda_1 + i - j = k$. Note that $D_k(\lambda) \neq \varnothing$ if and only if $k \in \{1, \dots, h_\lambda(1,1)\}$. Assume that $\lambda \in \mathrm{Hk}_n$. For $k \in \{1, \dots, n\}$, we set $\delta_k = \max(\{\min(i,j) \mid (i,j) \in D_k(\lambda)\})$. We define the ***$\lambda$-diagonal coordinates*** of a box $(i,j) \in \mathrm{Fer}(\lambda)$ to be the pair $\langle k, \delta \rangle_\lambda$ where $k \in \{1, \dots, n\}$ and $\delta \in \{1, \dots, \delta_k\}$ such that $(i,j) \in D_k(\lambda)$ and $\delta = \delta_k - \min(i,j) + 1$. See Figure 2 for an example with $\lambda = (5,3,2)$. Given $k \in \{1, \dots, n\}$, we define $\square_k(\lambda)$ the

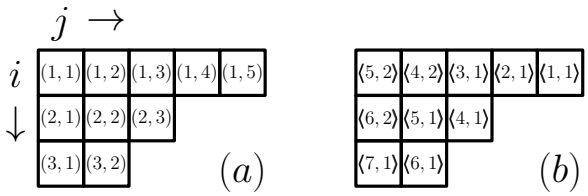


Figure 2: With $\lambda = (5,3,2)$, (a): the Ferrers diagram of λ with the usual coordinates for English convention; (b) : the λ-diagonal coordinates of the boxes in $\mathrm{Fer}(\lambda)$

kth square of λ as the order ideal in $(\mathrm{Fer}(\lambda), \trianglelefteq)$ generated by $D_k(\lambda)$. Note that $D_{\lambda_1}(\lambda)$ corresponds to the *Durfee square of* λ.

A ***filling*** of shape λ is an function $f : \mathrm{Fer}(\lambda) \longrightarrow \mathbb{N}$. Such a filling f is a ***(weak) reverse plane partition (of shape λ)*** whenever f is weakly increasing with respect to $\trianglelefteq$. We call these reverse plane partitions "weak" because we allow 0 as the value of a box, but we drop this adjective from now on. We denote by $\mathrm{RPP}(\lambda)$ the set of reverse plane partitions of shape λ.

A reverse plane partition f (of shape λ) is a ***(weak) semi-standard Young tableau (of shape λ)*** if $f(i,j) > f(i',j) \geqslant 0$ for any $(i,j), (i',j) \in \mathrm{Fer}(\lambda)$ such that $i' < i$. Write $\mathrm{SSYT}(\lambda)$ for the set of such semi-standard Young tableau of shape λ, and, given an integer $m \in \mathbb{N}$, $\mathrm{SSYT}(\lambda, m)$ for those with values in $\{0, \dots, m\}$.

A filling f is a ***(weak) increasing tableau (of shape λ)*** whenever f is a reverse plane partition which is strictly increasing with respect to $\trianglelefteq$. We denote by $\mathrm{INC}(\lambda)$ the set of increasing tableaux of shape λ.

We define ***standard Young tableaux (of shape λ)*** as a bijective increasing tableau $f : \mathrm{Fer}(\lambda) \longrightarrow \{1, \dots, |\lambda|\}$. We write $\mathrm{SYT}(\lambda)$ for the set of standard Young tableaux of shape λ.

See Figure 3 for an example of each of the previous notions for $\lambda = (5,3,2)$.

2.2 The classical story

We recall the classical way to present the *Robinson–Schensted–Knuth correspondence*. For more details, we invite the reader to look at the following references: [40], [15].

Let us first recall the Schensted row-insertion.

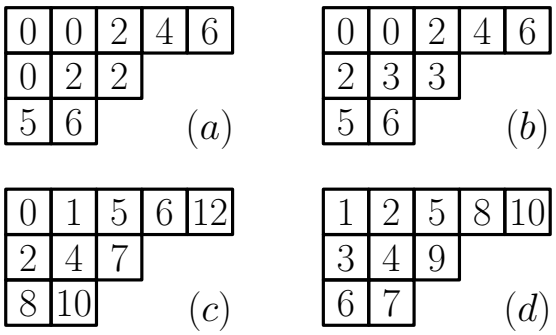


Figure 3: With $\lambda = (5, 3, 2)$, (a): a (weak) reverse plane partition of shape λ; (b) : a (weak) semi-standard Young tableau of shape λ; (c) : an increasing tableaux of shape λ; (d) a standard Young tableaux of shape λ.

Let $k \in \mathbb{N}$. The ***Schensted row-insertion of k***, denoted $k \xrightarrow{\text{Sch.}} -$, is a function on semi-standard Young tableaux defined as follows. Let f be a semi-standard Young tableau of shape λ. Then the filling $\left(k \xrightarrow{\text{Sch.}} f\right) = g$ is obtained thanks to the following procedure:

1) Put $i' = 1$ and $x = k$;

2) If it exists, let j' be the smallest index such that $x < f(i', j')$, otherwise, we put $j' = \lambda_{i'} + 1$;

3) If $(i', j') \notin \text{Fer}(\lambda)$, then we put $g(i', j') = x$ and for $(i, j) \in \text{Fer}(\lambda)$ such that $i \geqslant i'$ we put $g(i, j) = f(i, j)$, and we are done;

4) Otherwise, we put $g(i', j') = x$ and for $j \neq j'$ such that $(i', j) \in \text{Fer}(\lambda)$, we put $g(i', j) = f(i', j)$; put $x = f(i', j')$, we increase i' by 1, and we come back to step 2).

We end with a filling g of the integer partition obtained from λ by adding the box (i', j') from step 3) of the algorithm. We illustrate how Schensted row insertion works with the following example.

Example 4. Consider f to be the semi-standard Young tableau of shape $\lambda = (5, 2)$ below (Figure 4).

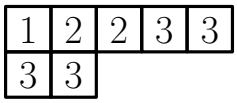


Figure 4: Example of a semi-standard Young tableau f.

We obtain $1 \xrightarrow{\text{Sch.}} f$ by replacing the value in the box $(1, 2)$ by 1, the value in the box $(2, 1)$ by 2, by adding a box at $(3, 1)$ and giving it the value 3. The filling obtained

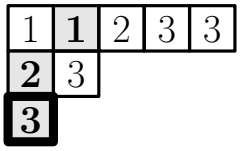


Figure 5: The filling $1 \xrightarrow{\text{Sch.}} f$ obtained from f in Figure 4

following the Schensted row-insertion algorithm is illustrated below (Figure 5). We greyed the modified boxes and framed the added box. Remark that this new filling is a semi-standard Young tableau of shape $\mu = (5, 2, 1)$.

We can now present the RSK correspondence.

The ***RSK correspondence*** is a map from nonnegative integer matrices and pairs of semi-standard Young tableaux of the same shape, described as follows:

1) From $A = (a_{i,j})$, an $n \times m$ nonnegative integer matrix, consider the associated two-line array,
$$w_A = \begin{pmatrix} i_1 & i_2 & \dots & i_s \\ j_1 & j_2 & \dots & j_s \end{pmatrix}$$
such that, for any $(i, j) \in \{1, \dots, n\} \times \{1, \dots, m\}$, there are $a_{i,j}$ copies of the column $\begin{pmatrix} i \\ j \end{pmatrix}$, and all the columns are in lexicographic order, meaning:
   - $i_1 \leqslant \dots \leqslant i_s$, and;
   - if $i_p = i_{p+1}$ then $j_p \leqslant j_{p+1}$.

   They are usually called *biwords*.

2) We construct two sequences of semi-standard Young tableaux, $(P(k))_{0 \leqslant k \leqslant s}$ and $(Q(k))_{0 \leqslant k \leqslant s}$, as it follows:
   - we begin with $P(0) = Q(0) = \varnothing$;
   - for all $k \in \{1, \dots, s\}$, we put $P(k) = j_k \xrightarrow{\text{Sch.}} P(k-1)$.
   - for all $k \in \{1, \dots, s\}$, we get $Q(k)$ from $Q(k-1)$ by recording i_k in box created when passing from $P(k-1)$ to $P(k)$.

3) We define $\mathbf{RSK}(A) = (P(s), Q(s))$.

Example 5. Let us take
$$A = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Then we get

$$w_A = \begin{pmatrix} 1 & 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 \\ 1 & 3 & 3 & 3 & 2 & 2 & 3 & 1 & 2 \end{pmatrix}.$$

In Figure 6, we make explicit the step-by-step calculations of $(P(k), Q(k))_{1 \leqslant k \leqslant 9}$. Here $\mathbf{RSK}(A) = (P(9), Q(9))$.

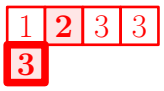
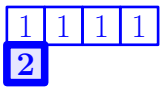
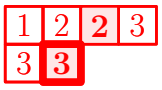
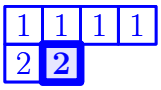
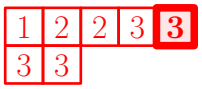
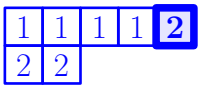
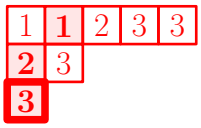
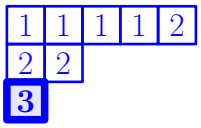
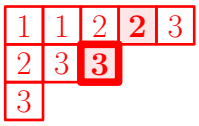
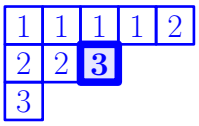
k	(i_k, j_k)	$P(k)$	$Q(k)$
1	$(1,1)$		
2	$(1,3)$		
3	$(1,3)$		
4	$(1,3)$		
5	$(2,2)$		
6	$(2,2)$		
7	$(2,3)$		
8	$(3,1)$		
9	$(3,2)$		

Figure 6: Illustration of the calculations for $\mathbf{RSK}(A)$. The framed box at the row k is the one added by Schensted row-insertion from $P(k-1)$ to $P(k)$ and the colored boxes are the modified ones from $P(k-1)$ to $P(k)$ following the row-insertion process.

Theorem 6. *Let $n, m \in \mathbb{N}^*$. The RSK correspondence gives a bijection from $n \times m$ nonnegative integer matrices to pairs (P, Q) of semi-standard Young tableaux of the same shape such that their entries are from 1 to m for P, and from 1 to n for Q.*

A well-known combinatorial consequence is the Cauchy identity. Before stating it, we recall Schur polynomials.

Let $n \in \mathbb{N}^*$, and $x_1, \ldots, x_n$ be n formal variables. Consider λ to be a nonzero integer partition. We define the ***Schur polynomial of λ*** as follows:

$$s_\lambda(x_1, \ldots, x_n) = \sum_{f \in \mathrm{SSYT}(\lambda, n-1)} \prod_{b \in \mathrm{Fer}(\lambda)} x_{f(b)+1}.$$

We write $x_{f(b)+1}$ instead of $x_{f(b)}$ because we are considering weak semi-standard Young tableaux. It is well-known that this is a homogeneous symmetric polynomial of degree $|\lambda|$. Moreover, for any $1 \leqslant m \leqslant n$, $(s_\lambda)_{\lambda \vdash m}$ gives a basis of the vector space of the homogeneous symmetric polynomials of degree m.

For $n = 2$ and $\lambda = (2, 1)$ we get $s_\lambda(x_1, x_2) = x_1^2 x_2 + x_1 x_2^2$.

Corollary 7 (**Cauchy identity** (see [40], [15]))**.** *For any $n, m \in \mathbb{N}^*$, and for any $x_1, \ldots, x_n$ and $y_1, \ldots, y_m$ sets of formal variables, we have*

$$\sum_\lambda s_\lambda(x_1, \ldots, x_n) s_\lambda(y_1, \ldots, y_m) = \prod_{i=1}^{n} \prod_{j=1}^{m} \frac{1}{1 - x_i y_j}$$

where the sum is over all the integer partitions λ.

Remark 8. The RSK correspondence induces a bijection from permutations of $\mathfrak{S}_n$ to pairs of standard Young tableaux (P, Q) of size n. This recovers the Robinson–Schensted correspondence. It allows us to establish the following representation-theoretic identity combinatorially:

$$n! = \sum_{\lambda \vdash n} t_\lambda^2,$$

where t_λ is both the number of standard Young tableaux of shape λ, with values in $\{1, \ldots, n\}$, and the dimension of the irreducible representation of $\mathfrak{S}_n$ corresponding to the partition λ.

2.3 The Greene–Kleitman invariant

Let $G = (G_0, G_1)$ be a directed graph. Assume that G is acyclic. Consider a filling $f : G_0 \longrightarrow \mathbb{N}$ of G. We assign to any ℓ-tuple of paths $\boldsymbol{\gamma} \in \Pi(G)^\ell$ in G a ***f-weight*** defined by

$$\mathsf{wt}_f(\boldsymbol{\gamma}) = \sum_{v \in \mathrm{Supp}(\boldsymbol{\gamma})} f(v).$$

Set $M_0^G(f) = 0$, and for all integers $\ell \geqslant 1$, $M_\ell^G(f) = \max\left(\{\mathsf{wt}_f(\boldsymbol{\gamma}) \mid \boldsymbol{\gamma} \in \Pi(G)^\ell\}\right)$. We define the ***Greene–Kleitman invariant*** of f in G as

$$\mathsf{GK}_G(f) = \left(M_\ell^G(f) - M_{\ell-1}^G(f)\right)_{\ell \geqslant 1}.$$

See Figure 7 for an explicit computation example.

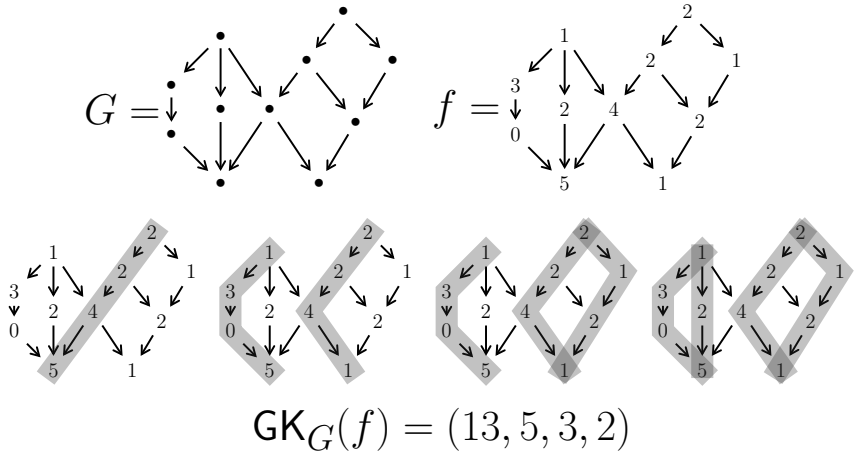


Figure 7: An example of the computation of GK_G.

Proposition 9 (Greene–Kleitman [23]). *Let G be an acyclic directed graph and f be a filling of G. The integer sequence $\mathsf{GK}_G(f)$ is an integer partition of length equal to the maximal cardinality of an antichain in G. Moreover,*

$$|\,\mathsf{GK}_G(f)| = \sum_{v \in G_0} f(v).$$

2.4 The Gansner story

In the following, we present another way to realize the RSK correspondence. We refer the reader to [18] for more details.

Let $A = (a_{i,j})_{1 \leqslant i,j \leqslant n}$ be a $n \times n$ integer matrix. We construct a directed graph G_A where the vertices are labelled by (i, j) for $1 \leqslant i, j \leqslant n$, and the arrows are given by $(i, j) \longrightarrow (i + 1, j)$ and $(i, j) \longrightarrow (i, j + 1)$.

For $1 \leqslant i \leqslant n$, consider $G_A^{[i,-]}$ the full subgraph of G_A whose vertices are (k, j) for $1 \leqslant k \leqslant i$ and $1 \leqslant j \leqslant n$. The entries of A endows the graph $G_A^{[i,-]}$ with a filling $f_A^{[i,-]}$. We define a sequence of integer partitions $(\nu^i)_{1 \leqslant i \leqslant n}$ by

$$\forall i \in \{1, \ldots, n\},\ \nu^i = \mathsf{GK}_{G_A^{[i,-]}}\left(f_A^{[i,-]}\right).$$

Analogously, by considering, for $1 \leqslant j \leqslant n$, $G_A^{[-j]}$ the full subgraph of G_A whose vertices are (i, k) for $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant k$, we define a sequence of integer partitions $(\mu^j)_{1 \leqslant j \leqslant n}$ by

$$\forall j \in \{1, \ldots, n\},\ \mu^j = \mathsf{GK}_{G_A^{[-,j]}}\left(f_A^{[-,j]}\right).$$

Note $\nu^n = \mu^n$. Moreover, we can show that $\nu^i \subseteq \nu^{i+1}$ for all $1 \leqslant i < n$, and $\mu^j \subseteq \mu^{j+1}$ for all $1 \leqslant j < n$.

The pairs of integer partitions sequences $((\nu_i)_{1 \leqslant i \leqslant n}, (\mu_j)_{1 \leqslant j \leqslant n})$ are called the ***Greene–Kleitman invariants of A***. They allow us to recover $\mathbf{RSK}(A)$. The semi-standard

Young tableau $P(s)$ is the filling of μ^n obtained by labelling j the boxes in $\mu^j \setminus \mu^{j-1}$. The other one, $Q(s)$, is constructed similarly but with the sequence (ν^i).

Example 10. In Figure 8, we give the explicit calculations of the sequences (μ^j) and (ν^i) for the matrix A of Theorem 5.

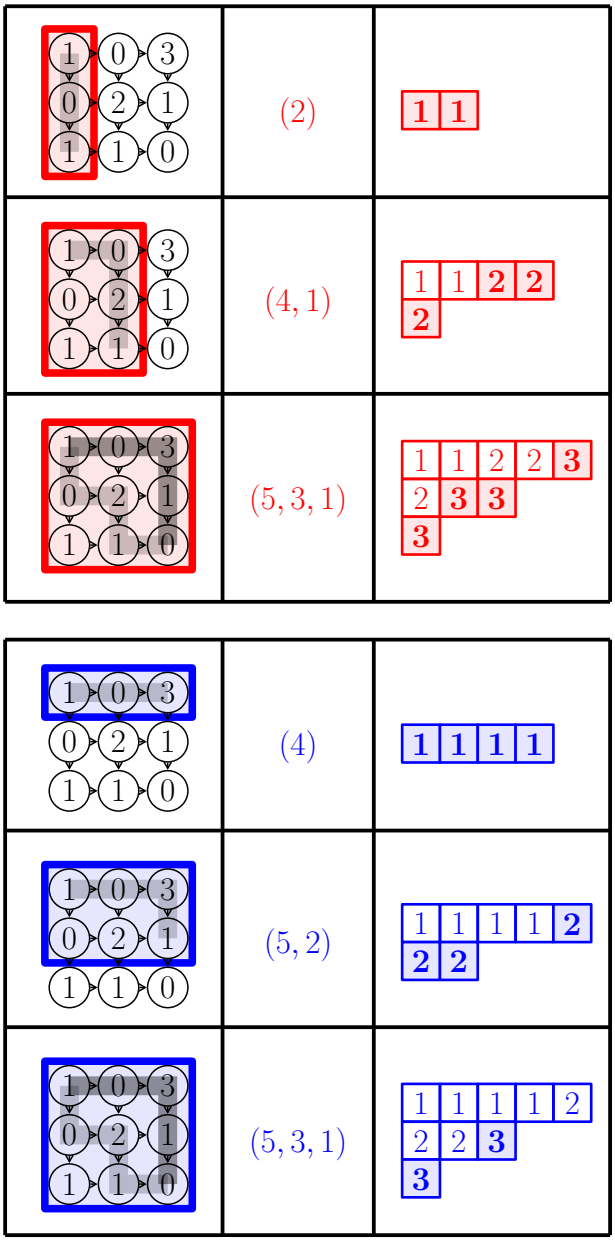


Figure 8: Illustration of the calculations for **RSK**(A) in the "Greene-Kleitman" way. (Top): The sequence (μ^j); (Bottom): The sequence (ν^i).

2.5 The generalized Gansner story

We can display these sequences $(\mu^j)_{1\leqslant j\leqslant n}$ and $(\nu^i)_{1\leqslant j\leqslant n}$ as a reverse plane partition of shape the $n\times n$ box partition. We first locate the sink of the subgraph used to calculate ν^j or μ^i. This sink corresponds to a box in the Young tableau we want. We fill the boxes in the same diagonal with the parts of ν^j or μ^i from the bottom left to the top right. We can look at Figure 9 to see how it goes with the results of Theorem 10.

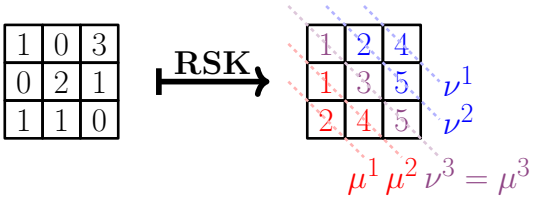


Figure 9: Displaying the results from the application of **RSK** seen in Figure 8 in the "Greene-Kleitman" way.

We can generalize this way of applying **RSK** to a correspondence from fillings of a Ferrers diagram of any fixed integer partition λ to reverse plane partitions of shape λ.

Let $f : \mathrm{Fer}(\lambda) \longrightarrow \mathbb{N}$ be a filling of $\mathrm{Fer}(\lambda)$ We consider the directed graph G_λ associated to λ whose vertex set is $\mathrm{Fer}(\lambda)$ and the arrows are given by $(i,j) \longrightarrow (i,j+1)$ and $(i,j) \longrightarrow (i+1,j)$. Indeed G_λ corresponds to the Hasse diagram of $(\mathrm{Fer}(\lambda), \trianglelefteq)$.

Assume that $\lambda \in \mathrm{Hk}_n$. For each $k \in \{1,\dots,n\}$, we consider $G_\lambda^{[k]}$ the full subgraph of G_λ whose vertices are boxes in $\square_k(\lambda)$. Write $f^{[k]}$ for the induced filling of $G_\lambda^{[k]}$ from f of G_λ. We define an integer partition π^k by:

$$\pi^k = \mathsf{GK}_{G_\lambda^{[k]}}\left(f^{[k]}\right).$$

We can show that $\ell(\pi^k) \leqslant \#D_k(\lambda)$, as there exists $\boldsymbol{\gamma} \in \Pi(G_\lambda^{[k]})^{\#D_k(\lambda)}$ such that $\mathrm{Supp}(\boldsymbol{\gamma}) = (G_\lambda^{[k]})_0$. So we can place the values of π^k on the diagonal $D_k(\lambda)$. We define a filling $\boldsymbol{\mathcal{RSK}}_\lambda(f) : \mathrm{Fer}(\lambda) \longrightarrow \mathbb{N}$ as follows. Using the λ-diagonal coordinates for the boxes of λ, we compute $\boldsymbol{\mathcal{RSK}}_\lambda(f)$ by:

$$\forall \langle k,\delta\rangle_\lambda \in \mathrm{Fer}(\lambda),\ \boldsymbol{\mathcal{RSK}}_\lambda(f)(\langle k,\delta\rangle_\lambda) = \pi^k_\delta.$$

See Figure 10 for a detailed example.

Theorem 11 ([18, 17])**.** *For any nonempty integer partition λ, The map $\boldsymbol{\mathcal{RSK}}_\lambda$ is a one-to-one correspondence from fillings of* $\mathrm{Fer}(\lambda)$ *to* $\mathrm{RPP}(\lambda)$.

Remark 12. By reversing the arrows $(i,j) \longrightarrow (i+1,j)$ in G_λ, and by proceeding to the same calculations that defined $\boldsymbol{\mathcal{RSK}}_\lambda$, it realizes the Hillman–Grassl correspondence. See [18] for more details.

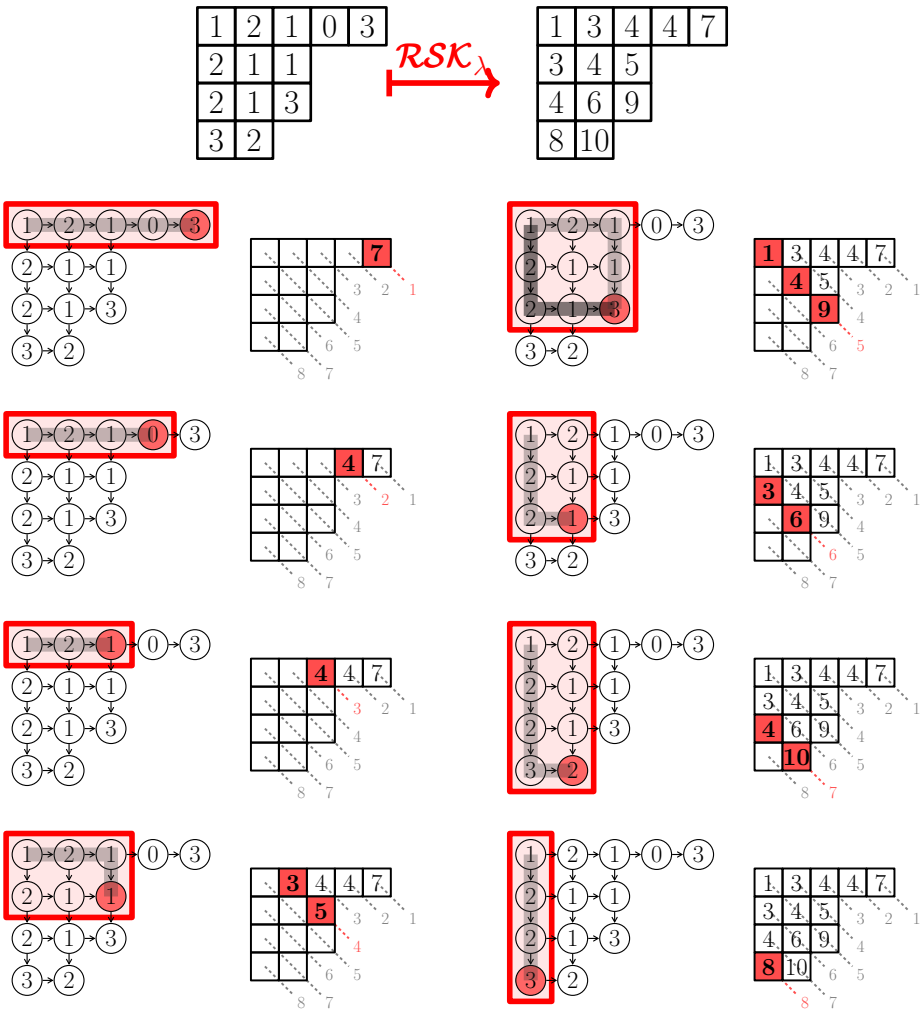


Figure 10: Explicit calculations of $\boldsymbol{\mathcal{RSK}}_\lambda(f)$ for a given filling f of shape $\lambda = (5,3,3,2)$. For $1 \leqslant m \leqslant 8$, each framed subgraph corresponds to the subgraph $G_\lambda(m)$, and each filled diagonal colored in red correspond to $\mathsf{GK}_{G_\lambda(m)}(f)$.

We introduce some notations before stating a combinatorial identity, consequence of Theorem 11. Fix $n \in \mathbb{N}^*$. Consider $\lambda \in \mathrm{Hk}_n$. We assign a weight to the boxes of λ, using the λ-diagonal coordinates, as follows:

$$\forall b = \langle k, \delta \rangle_\lambda \in \mathrm{Fer}(\lambda),\ w_{\lambda,b}(x_1, \ldots, x_n) = \prod_{1 \leqslant \ell \leqslant n,\ b \in \square_\ell(\lambda)} x_\ell.$$

We define the ***trace generating function for*** $\mathbf{RPP}(\boldsymbol{\lambda})$:

$$\rho_\lambda(x_1, \ldots, x_n) = \sum_{f \in \mathrm{RPP}(\lambda)} \prod_{\langle \ell, \varepsilon \rangle_\lambda \in \mathrm{Fer}(\lambda)} x_\ell^{f(\langle \ell, \varepsilon \rangle_\lambda)}.$$

Corollary 13 ([18]). *Let $n \geqslant 1$ and $\lambda \in \mathrm{Hk}_n$. Let $x_1, \ldots, x_n$ be n formal variables. We have*

$$\rho_\lambda(x_1, \ldots, x_n) = \prod_{b \in \mathrm{Fer}(\lambda)} \frac{1}{1 - w_{\lambda,b}(x_1, \ldots, x_n)}.$$

This result induces a well-known equality involving the norm-generating function of $\mathrm{RPP}(\lambda)$, previously proved by R. Stanley [39] using $\mathcal{P}$-partition theory. Here, by setting $\sigma(f) = \sum_{b \in \mathrm{Fer}(\lambda)} f(b)$ and mapping all the x_i to x, we have

$$\sum_{f \in \mathrm{RPP}(\lambda)} x^{\sigma(f)} = \prod_{b \in \mathrm{Fer}(\lambda)} \frac{1}{1 - x^{h_\lambda(b)}}.$$

3 Tools from Coxeter elements

In this section, we define combinatorial objects related to Coxeter elements that help present and study our extended version of Gansner's RSK correspondence.

3.1 (Type A) Coxeter elements

For any $n \geqslant 1$, let $\mathfrak{S}_{n+1}$ be the symmetric group on $n+1$ letters. For $1 \leqslant i < j \leqslant n+1$, write (i, j) for the transposition exchanging i and j. For $1 \leqslant i \leqslant n$, let s_i be the adjacent transposition $(i, i+1)$. Let Σ_n be the set of adjacent transpositions of $\mathfrak{S}_{n+1}$. Recall that $\mathfrak{S}_{n+1}$ admits a presentation in terms of generators and relations using Σ_n as follows:

$$\mathfrak{S}_{n+1} = \left\langle \Sigma_n \;\middle|\; \begin{array}{ll} s_i^2 = 1 & \text{for } i \in \{1, \ldots, n\} \\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} & \text{for } i \in \{1, \ldots, n-1\} \\ s_i s_j = s_j s_i & \text{for } i, j \in \{1, \ldots, n\} \text{ such that } |i-j| > 1 \end{array} \right\rangle$$

For any $w \in \mathfrak{S}_{n+1}$, call an *expression of* w a way to write w as a product of transpositions in Σ_n. The *length of* w, denoted by $\ell(w)$, is the minimal number of transpositions in Σ_n needed to express w. Whenever $\ell(sw) < \ell(w)$ for some $s \in \Sigma_n$, we say that s is *initial in* w. Similarly, we call $s \in \Sigma_n$ *final in* w whenever $\ell(ws) < \ell(w)$.

A ***Coxeter element (of $\mathfrak{S}_{n+1}$)*** is an element $c \in \mathfrak{S}_{n+1}$ which can be written as a product of all the transpositions of Σ_n, in some order, where each of them appears precisely once.

Example 14. The permutation $c = s_2 s_1 s_3 s_6 s_5 s_4 s_8 s_7 = (1, 3, 4, 7, 9, 8, 6, 5, 2)$ is a Coxeter element of $\mathfrak{S}_9$. Note that s_1, s_3, s_6 and s_8 are initial in c, and s_7 and s_4 are final in c.

First, we state the result on conjugating Coxeter elements with their initial or final adjacent transpositions.

Lemma 15.

(*i*) *Let $c \in \mathfrak{S}_{n+1}$ be a Coxeter element. For any $s \in \Sigma_n$, either initial or final in c, the permutation scs is a Coxeter element in $\mathfrak{S}_{n+1}$.*

(ii) *[33, Lemma 1.7] Given two Coxeter elements $c, c' \in \mathfrak{S}_{n+1}$, there exist $k \in \mathbb{N}$, and a sequence $c = c_0, c_1, \ldots, c_k = c'$ of Coxeter elements such that for any $i \in \{0, \ldots, k-1\}$ there exists an initial adjacent transposition t_i such that $c_{i+1} = t_i c_i t_i$.*

Then, as observed in Theorem 14, the following lemma allows us to write any Coxeter element of $\mathfrak{S}_{n+1}$ as a long cycle of a precise form. It is a direct consequence of Theorem 15.

Lemma 16. *An element $c \in \mathfrak{S}_{n+1}$ is a Coxeter element if and only if c is a long cycle which can be written as follows*

$$c = (c_1, c_2, \ldots, c_m, c_{m+1}, \ldots, c_{n+1})$$

where $c_1 = 1 < c_2 < \ldots < c_m = n+1 > c_{m+1} > \ldots > c_{n+1} > c_1 = 1$.

Consider a Coxeter element $c \in \mathfrak{S}_{n+1}$. Write it $c = (c_1, \ldots, c_{n+1})$ as said in the previous lemma. We define the ***left part of c*** as $\mathbf{L}_c = \{c_2, \ldots, c_{m-1}\}$ and the ***right part of c*** as $\mathbf{R}_c = \{c_{m+1}, \ldots, c_{n+1}\}$.

The following lemma characterizes initial and final adjacent transpositions thanks to $\mathbf{L}_c$ and $\mathbf{R}_c$.

Lemma 17. *Let $c \in \mathfrak{S}_{n+1}$ be a Coxeter element. For any $k \in \{2, \ldots, n-1\}$,*

- *s_k is final in c if and only if $k \in \mathbf{L}_c$ and $k+1 \in \mathbf{R}_c$; and,*
- *s_k is initial in c if and only if $k \in \mathbf{R}_c$ and $k+1 \in \mathbf{L}_c$.*

In special cases,

- *s_1 is initial in c if and only if $2 \in \mathbf{L}_c$; and,*
- *s_n is final in c if and only if then $n \in \mathbf{L}_c$.*

We recall, via the following definition, that we can associate to any Coxeter element $c \in \mathfrak{S}_{n+1}$ a unique A_n type quiver.

Definition 18. Let $c \in \mathfrak{S}_{n+1}$ be a Coxeter element. We define the quiver ***Q(c)*** as follows:

- its set of vertices is $Q(c)_0 = \{1, \ldots, n\}$;
- its set of arrows is given by arrows between i and $i+1$, for all $i \in \{1, \ldots, n\}$:
  - we have $i \longrightarrow i+1$ if s_i precedes s_{i+1} in a reduced expression of c;
  - we have $i \longleftarrow i+1$ otherwise.

Example 19. For $c = (1, 3, 4, 7, 9, 8, 6, 5, 2)$, we obtain

$$Q(c) = \quad 1 \leftarrow 2 \rightarrow 3 \rightarrow 4 \leftarrow 5 \leftarrow 6 \rightarrow 7 \leftarrow 8$$

Proposition 20. *Let $n \in \mathbb{N}^*$. The map $c \longmapsto Q(c)$ realizes a one-to-one correspondence from Coxeter elements of $\mathfrak{S}_{n+1}$ to A_n type quivers. Moreover:*

- *v is a source of $Q(c)$ if and only if s_v is initial in c;*
- *v is a sink of $Q(c)$ if and only if s_v is final in c;*

This map is crucial for connecting to representation-theoretic results.

Finally, we give a tiny result that links the inverse operation on Coxeter elements in $\mathfrak{S}_{n+1}$ and the opposite action on A_n type quivers.

Lemma 21. *Let $c \in \mathfrak{S}_{n+1}$ be a Coxeter element. Then $Q(c^{-1}) = Q(c)^{\mathrm{op}}$.*

3.2 Interval bipartitions

Let $\mathbf{A} \subset \mathbb{N}^*$. A ***bipartition*** of $\mathbf{A}$ is a pair $(\mathbf{L}, \mathbf{R})$ sucht that $\mathbf{L} \cup \mathbf{R} = \mathbf{A}$ and $\mathbf{L} \cap \mathbf{R} = \varnothing$. We do not identify the pair $(\mathbf{L}, \mathbf{R})$ with the pair $(\mathbf{R}, \mathbf{L})$. The following result is a direct consequence of Theorem 16.

Corollary 22. *Let $n \geqslant 1$. The map*

$$\Psi_{n+1} : \begin{cases} \{\textit{Coxeter elements of } \mathfrak{S}_{n+1}\} & \longrightarrow & \{\textit{Bipartitions of } \{2, \dots, n\}\} \\ c & \longmapsto & (\mathbf{L}_c, \mathbf{R}_c) \end{cases}$$

is bijective.

In the following, we focus on bipartitions of intervals in $\mathbb{N}^*$.

An *interval (in $\mathbb{N}^*$)* is a set $[\![i, j]\!] = \{i, i+1, \dots, j\}$ for some $i, j \in \mathbb{N}^*$ with $i \leqslant j$. For all $i \in \mathbb{N}^*$, we set $[\![i, i]\!] = [\![i]\!]$. We denote by $\mathcal{I}$ the set of all the intervals in $\mathbb{N}^*$, and by $\mathcal{I}_n$ the subset of those included in $\{1, \dots, n+1\}$. An ***interval bipartition*** is a bipartition $(\mathbf{L}, \mathbf{R})$ of an interval in $\mathbb{N}^*$. Call it ***elementary*** whenever either $\mathbf{L} = \mathbf{R} = \varnothing$, or both $1 \in \mathbf{L}$ and $\max(\mathbf{L} \cup \mathbf{R}) \in \mathbf{R}$.

Fix $(\mathbf{L}, \mathbf{R})$ as a bipartition of some finite set $\mathbf{A} \subset \mathbb{N}^*$. If $\mathbf{L}$ is nonempty, write $\mathbf{L} = \{\ell_1 < \ell_2 < \ldots < \ell_p\}$. We define the integer partition $\boldsymbol{\lambda}(\mathbf{L}, \mathbf{R})$, for all $i \in \{1, \dots, p\}$, by $\boldsymbol{\lambda}(\mathbf{L}, \mathbf{R})_i = \#\{r \in \mathbf{R} \mid \ell_i < r\}$ if $\mathbf{L}$ is not empty, and $\boldsymbol{\lambda}(\mathbf{L}, \mathbf{R}) = (0)$ otherwise.

Lemma 23. *For any bipartition $(\mathbf{L}, \mathbf{R})$ of some finite set $\mathbf{A} \subset \mathbb{N}^*$, there exists an elementary interval bipartition $(\mathbf{L}', \mathbf{R}')$ such that $\boldsymbol{\lambda}(\mathbf{L}, \mathbf{R}) = \boldsymbol{\lambda}(\mathbf{L}', \mathbf{R}')$*

Proof. If $\boldsymbol{\lambda}(\mathbf{L}, \mathbf{R}) = (0)$, then we set $\mathbf{L}' = \mathbf{R}' = \varnothing$ and we are done.

Otherwise consider $\mathbf{M} = \{\ell \in \mathbf{L} \mid \ell < \max(\mathbf{R})\}$ and $\mathbf{S} = \{r \in \mathbf{R} \mid r > \min(\mathbf{L})\}$. By construction, we easily check that $\boldsymbol{\lambda}(\mathbf{L}, \mathbf{R}) = \boldsymbol{\lambda}(\mathbf{M}, \mathbf{S})$. Let $p = \#(\mathbf{A} \cup \mathbf{B})$, and consider the stricly increasing map from $\varphi : \{1, \dots, p\} \longrightarrow \mathbf{M} \cup \mathbf{S}$. By setting $\mathbf{L}' = \varphi^{-1}(\mathbf{M})$ and $\mathbf{R}' = \varphi^{-1}(\mathbf{S})$, we can check that $(\mathbf{L}', \mathbf{R}')$ is an elementary interval bipartition of $[\![1, p]\!]$, and $\boldsymbol{\lambda}(\mathbf{L}', \mathbf{R}') = \boldsymbol{\lambda}(\mathbf{M}, \mathbf{S}) = \boldsymbol{\lambda}(\mathbf{L}, \mathbf{R})$. □

From now on, we assume that $(\mathbf{L}, \mathbf{R})$ is an elementary interval bipartition. By also writing $\mathbf{R} = \{r_1 < \ldots < r_q\}$, we can picture $\boldsymbol{\lambda}(\mathbf{L}, \mathbf{R})$ by its Ferrers diagram: we have $(i, j) \in \mathrm{Fer}(\boldsymbol{\lambda}(\mathbf{L}, \mathbf{R}))$ whenever $\ell_i < r_{q-j+1}$. It allows us to label the ith row of $\mathrm{Fer}(\boldsymbol{\lambda}(\mathbf{L}, \mathbf{R}))$ by ℓ_i and the jth column by r_{q-j+1}.

Given a Coxeter element $c \in \mathfrak{S}_{n+1}$, we write $\boldsymbol{\lambda}(c)$ for the integer partition $\boldsymbol{\lambda}(\{1\} \cup \mathbf{L}_c, \mathbf{R}_c \cup \{n+1\})$. Thanks to the observation above, we introduce the ***c-coordinates*** of any box in $\mathrm{Fer}(\boldsymbol{\lambda}(c))$ as it follows. By setting $\mathbf{L} = \mathbf{L}_c \cup \{1\} = \{\ell_1 < \ldots < \ell_p\}$ and $\mathbf{R} = \mathbf{R}_c \cup \{n+1\} = \{r_1 < \ldots < r_q\}$, we write $[\![\ell_i, r_{q-j+1}]\!]_c$ for the box $(i,j) \in \mathrm{Fer}(\boldsymbol{\lambda}(c))$ whenever $\ell_i < r_{q-j+1}$. See Figure 11 for an example of such an object.

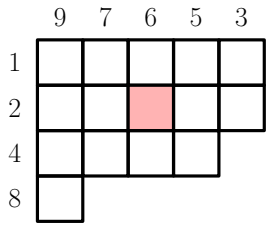


Figure 11: The (labelled) integer partition $\boldsymbol{\lambda}(\mathbf{L}, \mathbf{R})$ with $\mathbf{L} = \{1, 2, 4, 8\}$ and $\mathbf{R} = \{3, 5, 6, 7, 9\}$. It also corresponds to $\lambda(c)$ with $c = (1, 2, 4, 8, 9, 7, 6, 5, 3)$. The red box has c-coordinates $[\![2, 6]\!]_c$.

Proposition 24. *For any integer partition λ, there exists a unique elementary interval bipartition $(\mathbf{L}, \mathbf{R})$ such that $\lambda = \boldsymbol{\lambda}(\mathbf{L}, \mathbf{R})$.*

Proof. If $\lambda = (0)$, then we set $\mathbf{L} = \mathbf{R} = \varnothing$ and we are done. Otherwise, we label the segments of the southeast border of the shape of $\mathrm{Fer}(\lambda)$ from 1 to its length, going from the top-right to the bottom-left. This defines a label for each row and each column of $\mathrm{Fer}(\lambda)$. We set $\mathbf{L}$, the set of labels assigned to the rows, and $\mathbf{R}$, the set of labels assigned to the columns. We can easily check that $(\mathbf{L}, \mathbf{R})$ is an elementary interval bipartition (of the interval $[\![1, h_\lambda(1,1) + 1]\!]$). By construction, it is unique. □

Corollary 25. *Let $n \in \mathbb{N}^*$. For any $\lambda \in \mathrm{Hk}_n$, there exists a unique Coxeter element c of $\mathfrak{S}_{n+1}$ such that $\boldsymbol{\lambda}(c) = \lambda$.*

Proof. It follows automatically from Theorem 22 and Theorem 24. □

Given a $\lambda \in \mathrm{Hk}_n$, we denote by $\mathbf{c}(\lambda)$ the unique Coxeter element of $\mathfrak{S}_n$ such that $\boldsymbol{\lambda}(\mathbf{c}(\lambda)) = \lambda$.

Lemma 26. *Let $c \in \mathfrak{S}_{n+1}$. For $k \in \{1, \ldots, n\}$, we have:*

$$\#D_k(\boldsymbol{\lambda}(c)) = \min(\#\{\ell \in \mathbf{L}_c \cup \{1\} \mid \ell \leqslant k\}, \#\{r \in \mathbf{R}_c \cup \{n+1\} \mid r > k\})$$

Proof. This result follows by interpreting $D_k(\lambda)$ as "the diagonal" of the rectangle made of boxes (i,j) such that $\ell_i \leqslant k < r_{\lambda_1 - j + 1}$. □

3.3 Auslander–Reiten quiver

Let $c \in \mathfrak{S}_{n+1}$ be a Coxeter element. The ***Auslander–Reiten quiver of c***, denoted by $\mathrm{AR}(c)$, is the oriented graph satisfying the following conditions:

- The vertices of $\mathrm{AR}(c)$ are the transpositions (i, j), with $i < j$, in $\mathfrak{S}_{n+1}$;
- The arrows of $\mathrm{AR}(c)$ are given, for all $i < j$, by
  - $(i, j) \longrightarrow (i, c(j))$ whenever $i < c(j)$;
  - $(i, j) \longrightarrow (c(i), j)$ whenever $c(i) < j$.

Let us state an obvious and useful proposition about these quivers.

Proposition 27. *For any Coxeter element $c \in \mathfrak{S}_{n+1}$, the Auslander–Reiten quiver $\mathrm{AR}(c)$ is an acyclic connected directed graph. Moreover:*

- *its sources are the initial adjacent transpositions in c, and*
- *its sinks are the final adjacent transpositions in c.*

To construct such a graph recursively, we can first find the initial adjacent transpositions of c, which are all the sources, and step by step, using the second rule, construct the arrows and the vertices of $\mathrm{AR}(c)$ until we reach all the transpositions of $\mathfrak{S}_{n+1}$. Note that the sinks of $\mathrm{AR}(c)$ are given by the final adjacent transpositions of c. See Figure 12 for an explicit example.

Moreover, one can notice that we can construct $\mathrm{AR}(c)$ from any transposition $(i, j) \in \mathfrak{S}_{n+1}$ using the second rule.

Proposition 28. *Let $c \in \mathfrak{S}_{n+1}$ be a Coxeter element. Then $\mathrm{AR}(c^{-1}) = \mathrm{AR}(c)^{\mathrm{op}}$.*

Remark 29. The Auslander–Reiten quiver of a Coxeter element has a representation-theoretic meaning for the quiver $Q(c)^{\mathrm{op}}$ (see Section 5.1).

4 Storability

In this section, we first recall the notion of storability, introduced in [13], and we enumerate a few primary results. Then, given a positive integer $n \in \mathbb{N}^*$ and a Coxeter element $c \in \mathfrak{S}_{n+1}$, we introduce the notion of c-storabilty for n-tuples of integer partitions. We highlight their bijective link with the reverse plane partitions of $\boldsymbol{\lambda}(c)$.

4.1 Storable pairs and storable triplets

Definition 30. Let λ and μ be two integer partitions. The pair (λ, μ) is ***storable*** whenever for all $i \in \mathbb{N}^*$, $\lambda_i \geqslant \mu_i \geqslant \lambda_{i+1}$ (we can add zero parts if needed). Such a pair is ***strongly storable*** if we have $\lambda_1 = \mu_1$.

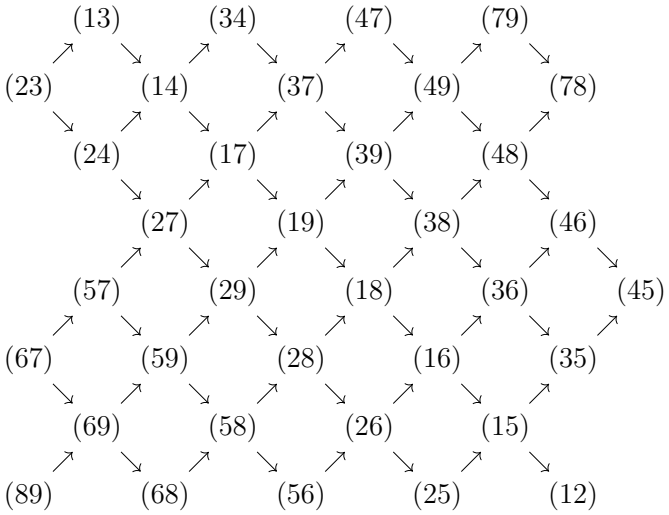


Figure 12: The Auslander–Reiten quiver of $c = (1, 3, 4, 7, 9, 8, 6, 5, 2) = s_2 s_1 s_3 s_6 s_5 s_4 s_8 s_7$.

We can picture storable pairs as follows. See a partition λ as a right-infinite row of forty-five-degree rotated squares filled with the parts of λ from left to right. We can add infinitely many zeros to the right. See μ in the same way. We say that two such rows of squares are *intertwining* if, for all $i \geqslant 1$, the ith square of one row is placed between the ith and the $(i+1)$th squares of the other row.

Then (λ, μ) is a storable pair if and only if we can intertwine the two rows of filled squares such that, when we read the two rows together from left to right, the values remain decreasing (Figure 13). In other words, the square μ_i intertwines the squares λ_i and λ_{i+1} whenever $\lambda_i \geqslant \mu_i \geqslant \lambda_{i+1}$.

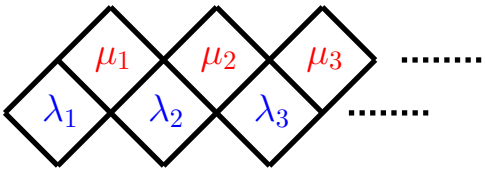


Figure 13: Illustration of storability of (λ, μ).

We give two results that arise from the definition.

Lemma 31. *Let λ and μ be two integer partitions.*

1) If (λ, μ) and (μ, λ) are both storable, then $\lambda = \mu$;

2) If (λ, μ) is storable, then $\ell(\lambda) \in \{\ell(\mu), \ell(\mu) + 1\}$.

Definition 32. Let λ, μ and ν be three integer partitions. The triplet (λ, μ, ν) is ***storable*** if the two following conditions are satisfied:

- either (λ, μ) or (μ, λ) is a storable pair;
- either (μ, ν) or (ν, μ) is a storable pair.

More precisely, we say that (λ, μ, ν) is:

(⊞⊞) $(\boxplus, \boxplus)$-storable if (λ, μ) and (ν, μ) are storable pairs;

(⊞⊟) $(\boxplus, \boxminus)$-storable if (λ, μ) and (μ, ν) are storable pairs;

(⊟⊞) $(\boxminus, \boxplus)$-storable if (μ, λ) and (ν, μ) are storable pairs;

(⊟⊟) $(\boxminus, \boxminus)$-storable if (μ, λ) and (μ, ν) are storable pairs.

Such a triplet is ***strongly storable*** whenever $\lambda_1 = \mu_1$ or $\mu_1 = \nu_1$.

We illustrate the four storability configurations in Figure 14.

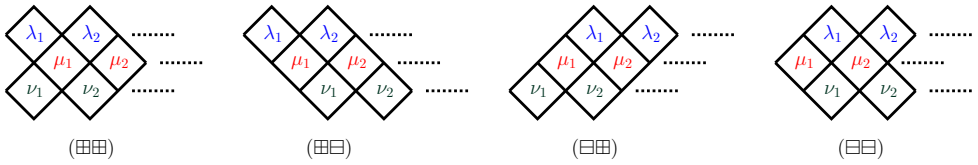


Figure 14: Illustration of the four storability configurations of (λ, μ, ν).

We introduce the following notion via the representation-theoretic interpretation introduced in Section 5.2.

Definition 33. Let λ, μ, and ν be three integer partitions. Assume that (λ, μ, ν) is a storable triplet. We define the ***diagonal transformation of μ in (λ, μ, ν)***, denoted $\mathrm{diag}(\lambda, \mu, \nu)$, to be the integer partition $\theta = (\theta_1, \theta_2, \ldots)$ such that:

- if (λ, μ, ν) is $(\boxplus, \boxplus)$-storable, then we define, for all $i \geqslant 1$,

$$\theta_i = \begin{cases} \max(\lambda_1, \nu_1) & \text{if } i = 1 \\ \min(\lambda_{i-1}, \nu_{i-1}) + \max(\lambda_i, \nu_i) - \mu_{i-1} & \text{otherwise;} \end{cases}$$

- if (λ, μ, ν) is $(\boxplus, \boxminus)$-storable, then we define, for all $i \geqslant 1$,

$$\theta_i = \begin{cases} \lambda_1 + \max(\lambda_2, \nu_1) - \mu_1 & \text{if } i = 1 \\ \min(\lambda_i, \nu_{i-1}) + \max(\lambda_{i+1}, \nu_i) - \mu_i & \text{otherwise;} \end{cases}$$

- if (λ, μ, ν) is $(\boxminus, \boxplus)$-storable, then we define, for all $i \geqslant 1$,

$$\theta_i = \begin{cases} \nu_1 + \max(\lambda_1, \nu_2) - \mu_1 & \text{if } i = 1 \\ \min(\lambda_{i-1}, \nu_i) + \max(\lambda_i, \nu_{i+1}) - \mu_i & \text{otherwise;} \end{cases}$$

- if (λ, μ, ν) is $(\boxminus, \boxminus)$-storable, then we define, for all $i \geqslant 1$,

$$\theta_i = \min(\lambda_i, \nu_i) + \max(\lambda_{i+1}, \nu_{i+1}) - \mu_{i+1}.$$

We can picture the diagonal operation as doing local operations for each square of μ in the diagram representing the storable triple (λ, μ, ν) (Figure 15).

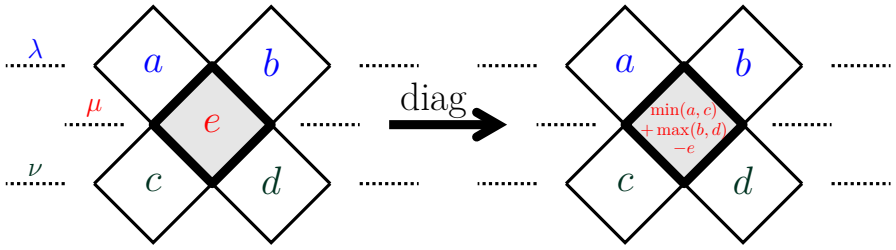


Figure 15: Illustration of the local operations to calculate $\mathrm{diag}(\lambda, \mu, \nu)$.

Remark that λ and ν play symmetric roles: $\mathrm{diag}(\lambda, \mu, \nu) = \mathrm{diag}(\nu, \mu, \lambda)$. Here are some elementary statements about the diagonal transformation.

Lemma 34. *Let λ, μ and ν be three integer partitions. When it is well-defined, consider $\theta = \mathrm{diag}(\lambda, \nu, \mu)$.*

1) If (λ, μ) is a storable pair, then $\mathrm{diag}(\lambda, \mu, \mu) = \lambda$.

2) If (λ, μ, ν) is $(\boxplus, \boxplus)$-storable, then (λ, θ, ν) is strongly $(\boxminus, \boxminus)$-storable.

3) If (λ, μ, ν) is $(\boxplus, \boxminus)$-storable, then (λ, θ, ν) is $(\boxplus, \boxminus)$-storable.

4) If (λ, μ, ν) is $(\boxminus, \boxplus)$-storable, then (λ, θ, ν) is $(\boxminus, \boxplus)$-storable.

5) If (λ, μ, ν) is $(\boxminus, \boxminus)$-storable, then (λ, θ, ν) is $(\boxplus, \boxplus)$-storable.

6) If (λ, μ, ν) is $(\boxplus, \boxplus)$-storable, $(\boxplus, \boxminus)$-storable, $(\boxminus, \boxplus)$-storable or strongly $(\boxminus, \boxminus)$-storable, then $\mathrm{diag}(\lambda, \theta, \nu) = \mu$.

4.2 *c*-storability

For $\mathbf{A} \subset \mathbb{Z}$ and $j \in \mathbb{Z}$, we set $\mathbf{A}[j] = \{a + j \mid a \in A\}$.

Let $n \in \mathbb{N}^*$. Let $\boldsymbol{\pi} = (\pi^k)_{1 \leqslant k \leqslant n}$ a $n+2$-tuple of integer partitions. In the following, we set $\pi^0 = \pi^{n+1} = (0)$ to make things more convenient with the following definitions. We say that $\boldsymbol{\pi}$ is ***(strongly) $(\boxminus, \boxminus)$-storable at k*** if $(\pi^{k-1}, \pi^k, \pi^{k+1})$ is a (strongly) $(\boxminus, \boxminus)$-storable triplet. We use the same formulation for the three other storability configurations.

Definition 35. Let $(\mathbf{L}, \mathbf{R})$ be a pair of disjoint subsets of $\{1, \ldots, n+1\}$ such that $\min(\mathbf{L}, \mathbf{R}) \in \mathbf{L}$ and $\max(\mathbf{L}, \mathbf{R}) \in \mathbf{R}$. A n-tuple of integer partitions $\boldsymbol{\pi} = (\pi^k)_{1 \leqslant k \leqslant n}$ is said to be $(\mathbf{L}, \mathbf{R})$-storable whenever the following assertions hold:

(a) $\boldsymbol{\pi}$ is $(\boxplus, \boxplus)$-storable at i for $i \notin \mathbf{L} \cup \mathbf{R}[-1]$;

(*b*) $\boldsymbol{\pi}$ is $(\boxplus, \boxminus)$-storable at i for $i \in \mathbf{R}[-1] \setminus \mathbf{L}$;

(*c*) $\boldsymbol{\pi}$ is $(\boxminus, \boxplus)$-storable at i if $i \in \mathbf{L} \setminus \mathbf{R}[-1]$;

(*d*) $\boldsymbol{\pi}$ is $(\boxminus, \boxminus)$-storable at i for $i \in \mathbf{L} \cap \mathbf{R}[-1]$.

(*e*) $\pi^k = (0)$ for $k \notin \{\min(\mathbf{L}), \ldots, \max(\mathbf{R})\}$.

Let $c \in \mathfrak{S}_{n+1}$ be a Coxeter element. A n-tuple of integer partitions $\boldsymbol{\pi}$ is ***c-storable*** whenever $\boldsymbol{\pi}$ is $(\mathbf{L}_c \cup \{1\}, \mathbf{R}_c \cup \{n+1\})$-storable.

Note that (*a*) and (*e*) are not exclusive.

Example 36. Fix $n = 5$ and the Coxeter element $c = (1, 2, 4, 6, 5, 3) \in \mathfrak{S}_n$. The tuple $\boldsymbol{\pi} = ((2), (4, 2), (2, 1), (3, 2), (3))$ is c-storable. From the drawing (Figure 16), we can determine the c-storability property.

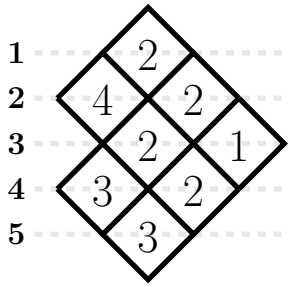


Figure 16: Illustration of the c-storability of $\boldsymbol{\pi}$ in Theorem 36.

Given a Coxeter element $c \in \mathfrak{S}_{n+1}$, we denote by $\boldsymbol{\mathcal{ST}}(c)$ the set of n-tuples of integer partitions that are c-storable.

For $k \in \{1, \ldots, n\}$, we define the ***diagonal transformation of $\boldsymbol{\pi}$ at k***, denoted by $\mathbf{diag}_k(\boldsymbol{\pi})$, to be the n-tuple of integer partitions obtained from $\boldsymbol{\pi}$ by replacing π^k with $\mathrm{diag}(\pi^{k-1}, \pi^k, \pi^{k+1})$. The following lemma follows from Theorem 34.

Lemma 37. *Let $c \in \mathfrak{S}_{n+1}$ and $\boldsymbol{\pi} \in \boldsymbol{\mathcal{ST}}(c)$. For any $k \in \{1, \ldots, n\}$:*

1) if s_k is initial in c, then then $\mathbf{diag}_k(\boldsymbol{\pi}) \in \boldsymbol{\mathcal{ST}}(s_k c s_k)$;

2) if s_k is final in c, then $\mathbf{diag}_k(\boldsymbol{\pi}) \in \boldsymbol{\mathcal{ST}}(s_k c s_k)$, and $\mathbf{diag}_k(\boldsymbol{\pi})$ is strongly $(\boxminus, \boxminus)$-storable at k;

3) otherwise $\mathbf{diag}_k(\boldsymbol{\pi}) \in \boldsymbol{\mathcal{ST}}(c)$.

This result will help explore ways to realize the extended generalization of RSK we present in Section 6 via local transformations. See 7 for additional details. In the remainder of this section, we show that we have a bijection from $\boldsymbol{\mathcal{ST}}(c)$ to $\mathrm{RPP}(\boldsymbol{\lambda}(c))$.

We first exhibit a symmetric operation. We define the *reverse map* rev_n on $\{1, \ldots, n\}$ by $\mathrm{rev}_n(k) = n + 1 - k$ for all $k \in \{1, \ldots, n\}$. It induces an action on:

- subsets of $\{1,\dots,n\}$: for any $\mathbf{A}\subseteq\{1,\dots,n\}$, we set $\mathrm{rev}_n(\mathbf{A})=\{\mathrm{rev}_n(a)\mid a\in\mathbf{A}\}$;
- Coxeter elements of $\mathfrak{S}_{n+1}$: for any Coxeter element $c\in\mathfrak{S}_{n+1}$, write $\mathrm{rev}_{n+1}(c)$ the Coxeter element obtained by conjugating c with rev_{n+1} seen as a permutation;
- n-tuples of integer partitions: given $\boldsymbol{\pi}=(\pi^k)_{1\leqslant k\leqslant n}$, we set $\mathrm{rev}_n(\boldsymbol{\pi})=(\pi^{\mathrm{rev}_n(k)})_{1\leqslant k\leqslant n}$.

Lemma 38. *Consider $c\in\mathfrak{S}_{n+1}$ a Coxeter element. For any $\boldsymbol{\pi}\in\boldsymbol{\mathcal{ST}}(c)$, we have $\mathrm{rev}_n(\boldsymbol{\pi})\in\boldsymbol{\mathcal{ST}}(\mathrm{rev}_{n+1}(c))$.*

The following result allows us to control the length of the integer partitions that constitute any c-storable n-tuple of integer partitions.

Proposition 39. *Let $c\in\mathfrak{S}_{n+1}$ be a Coxeter element. For all $\boldsymbol{\pi}=(\pi^k)_{1\leqslant k\leqslant n}\in\boldsymbol{\mathcal{ST}}(c)$, and for all $1\leqslant k\leqslant n$,*

$$\ell(\pi^k)\leqslant\min\left(\#\{i\in\mathbf{L}_c\cup\{1\}\mid i\leqslant k\},\#\{j\in\mathbf{R}_c\cup\{n+1\}\mid j>k\}\right).$$

Proof. Let $c\in\mathfrak{S}_{n+1}$ and $\boldsymbol{\pi}\in\boldsymbol{\mathcal{ST}}(c)$. We set $\mathbf{L}=\mathbf{L}_c\cup\{1\}$ and $\mathbf{R}=\mathbf{R}_c\cup\{n+1\}$.

We will prove that $\ell(\pi^k)\leqslant\#\{i\in\mathbf{L},i\leqslant k\}$ by induction over $1\leqslant k\leqslant n$. By hypothesis, as $1\in\mathbf{L}$, we have that $(\pi^1,\pi^0=(0))$ is a storable pair. Via Theorem 31, we get that $\ell(\pi^1)\leqslant\ell(\pi^0)+1=1$. In addition, $\#\{i\in\mathbf{L}\mid i\leqslant 1\}=1$.

Assume that, for a fixed $1\leqslant k<n$, $\ell(\pi^k)\leqslant\#\{i\in\mathbf{L},i\leqslant k\}$. We can distinguish two cases.

- If $k+1\notin\mathbf{L}$, then $\boldsymbol{\pi}$ is $(\boxplus,\boxplus)$-storable or $(\boxplus,\boxminus)$-storable at $k+1$. Thus (π^k,π^{k+1}) is a storable pair. By Theorem 31, either $\ell(\pi^k)=\ell(\pi^{k+1})+1$ or $\ell(\pi^k)=\ell(\pi^{k+1})$. In either way,
$$\ell(\pi^{i+1})\leqslant\ell(\pi^k)\leqslant\#\{i\in\mathbf{L}\mid i\leqslant k\}=\#\{i\in\mathbf{L}\mid i\leqslant k+1\}.$$
- If $k+1\in\mathbf{L}$, then $\boldsymbol{\pi}$ is $(\boxminus,\boxplus)$-storable or $(\boxminus,\boxminus)$-storable at $k+1$. Thus (π^{k+1},π^k) is a storable pair. By Theorem 31, either $\ell(\pi^{k+1})=\ell(\pi^i)+1$ or $\ell(\pi^{k+1})=\ell(\pi^i)$. Either way,
$$\ell(\pi^{k+1})\leqslant\ell(\pi^k)+1\leqslant\#\{i\in\mathbf{L}\mid i\leqslant k\}+1=\#\{i\in\mathbf{L}\mid i\leqslant k+1\}.$$

This completes the proof of the first desired inequality.

To get the other one, we play with the symmetry the reverse map gives. By Theorem 38, we know that $\mathrm{rev}_n(\boldsymbol{\pi})\in\boldsymbol{\mathcal{ST}}(\mathrm{rev}_{n+1}(c))$. Therefore we have, for $1\leqslant k\leqslant n$,

$$\begin{aligned}\ell(\pi^k)=\ell(\mathrm{rev}(\boldsymbol{\pi})^{n+1-k})&\leqslant\{j'\in\mathrm{rev}_{n+1}(\mathbf{R}_c)\mid j'\leqslant n+2-k\}\\&=\{j\in\mathbf{R}_c\mid j>k\}.\end{aligned}$$

We can, therefore, conclude the desired result. □

Now, thanks to Theorem 26, we can define a map from $\boldsymbol{\mathcal{ST}}(c)$ to fillings of $\boldsymbol{\lambda}(c)$ as follows:

$$\Phi_c : \begin{cases} \boldsymbol{\mathcal{ST}}(c) & \longrightarrow & \{\text{fillings of } \boldsymbol{\lambda}(c)\} \\ \boldsymbol{\pi} = (\pi_k)_{1 \leqslant k \leqslant n} & \longmapsto & (\Phi_c(\boldsymbol{\pi}) : \langle k, r \rangle_{\boldsymbol{\lambda}(c)} \longmapsto \pi_r^k) \end{cases}$$

The map Φ_c consists on, for any $\boldsymbol{\pi} \in \boldsymbol{\mathcal{ST}}(c)$, and $k \in \{1, \ldots, n\}$, placing the kth integer partition π^k of $\boldsymbol{\pi}$ in the boxes of $D_k(\boldsymbol{\lambda}(c))$ in the decreasing order, from right to left. We complete the diagonal with zeros if necessary. Therefore, one can check that the map Φ_c consists of a 135°-degree counterclockwise rotation of the picture of the c-storable n-tuples of integer partitions. Then, the c-storability conditions allow us to check that $\Phi_c(\boldsymbol{\pi})$ is a reverse plane partition of shape $\boldsymbol{\lambda}(c)$.

Theorem 40. *For any $n \in \mathbb{N}^*$, and for any Coxeter element $c \in \mathfrak{S}_{n+1}$, The map Φ_c induces a bijection from $\boldsymbol{\mathcal{ST}}(c)$ to* $\mathrm{RPP}(\boldsymbol{\lambda}(c))$.

5 Overview of Jordan recoverability

5.1 Quiver representations

Let $Q = (Q_0, Q_1, s, t)$ be a finite connected quiver. Consider $\mathbb{K}$ an algebraically closed field. A ***(finite-dimensionnal) representation of Q over*** $\mathbb{K}$ is a pair

$$E = ((E_q)_{q \in Q_0}, (E_\alpha)_{\alpha \in Q_1})$$

seen as an assignement of:

- a finite-dimensional $\mathbb{K}$ -vector space E_q to each $q \in Q_0$;
- a linear transformation $E_\alpha : E_{s(\alpha)} \longrightarrow E_{t(\alpha)}$ to each $\alpha \in Q_1$.

Write $\mathrm{rep}_{\mathbb{K}}(Q)$ for the set of the representations of Q over $\mathbb{K}$. For $E \in \mathrm{rep}_{\mathbb{K}}(Q)$, we denote by $\mathbf{dim}(E) = (\dim(E_q))_{q \in Q_0}$ the *dimension vector* of E.

Given $E, F \in \mathrm{rep}_{\mathbb{K}}(Q)$, a ***morphism of representations*** $\phi : E \longrightarrow F$ is a collection $(\phi_q)_{q \in Q_0}$ seen as an assignement of a linear transformation $\phi_q : E_q \longrightarrow F_q$ to each $q \in Q_0$ such that for any $\alpha \in Q_1$, $F_\alpha \phi_{s(\alpha)} = \phi_{t(\alpha)} E_\alpha$. Such a morphism of representations ϕ is an isomorphism whenever, for all $q \in Q_0$, ϕ_q is an isomorphism of vector spaces. We say that two representations, E and F, are isomorphic. We denote it by $E \cong F$ whenever an isomorphism exists from E to F Write $\mathrm{Hom}_{\mathbb{K}}(E, F)$ for the $\mathbb{K}$-vector space of morphisms of representations from E to F. In particular, we set $\mathrm{End}_{\mathbb{K}}(E) = \mathrm{Hom}_{\mathbb{K}}(E, E)$ to refer to the endomorphisms of a representation E.

We define the *path algebra* $\mathbb{K}Q$ to be the $\mathbb{K}$-vector space generated as a basis by the paths over Q endowed with multiplication acting on the basis as a concatenation of paths Recall that $\mathrm{rep}_{\mathbb{K}}(Q)$ endowed with the morphisms of representations between them is a category. This category is equivalent to the category of finitely generated (right) $\mathbb{K}Q$-modules. See [2] for more details.

Given $F, G \in \mathrm{rep}_{\mathbb{K}}(Q)$, we write $F \oplus G$ the direct sum of F and G. A representation E is said to be ***indecomposable*** whenever, if $E \cong F \oplus G$, then $F = 0$ or $G = 0$. By $\mathrm{ind}_{\mathbb{K}}(Q)$, we denote the isomorphism classes of indecomposable representations of Q.

In our setting, we can describe indecomposable representations thanks to intervals in $\{1, \ldots, n\}$. Let $n \geqslant 1$ be an integer Assume that Q is an A_n type quiver Given an interval $K \subset \{1, \ldots, n\}$, we denote by X_K the indecomposable representation such that:

- $(X_K)_q = \mathbb{K}$ if $q \in K$, and $(X_K)_q = 0$ otherwise;
- $(X_K)_\alpha = \mathrm{Id}_{\mathbb{K}}$ if $s(\alpha), t(\alpha) \in K$, and $(X_K)_\alpha = 0$ otherwise.

Proposition 41. *Let Q be an A_n type quiver Any indecomposable representation $E \in \mathrm{rep}_{\mathbb{K}}(Q)$ is isomorphic to X_K for some interval $K \subset \{1, \ldots, n\}$.*

Note that knowing the indecomposable representations of $\mathrm{rep}_{\mathbb{K}}(Q)$ and knowing the morphisms between them is enough to describe the entire category $\mathrm{rep}_{\mathbb{K}}(Q)$ This data is contained in the so-called ***Auslander–Reiten quiver of Q (over $\mathbb{K}$)***, denoted by $\mathrm{AR}_{\mathbb{K}}(Q)$. The vertices of $\mathrm{AR}_{\mathbb{K}}(Q)$ are the isomorphism classes of the indecomposable representations of $\mathrm{rep}_{\mathbb{K}}(Q)$, and the arrows correspond to the irreducible representations between them.

Proposition 42. *Let $c \in \mathfrak{S}_{n+1}$ be a Coxeter element. Then the quivers $\mathrm{AR}(c)$ and $\mathrm{AR}_{\mathbb{K}}(Q(c))^{\mathrm{op}}$ are isomorphic. More precisely, the map $\Psi : \mathrm{AR}(c) \longrightarrow \mathrm{AR}_{\mathbb{K}}(Q(c))^{\mathrm{op}}$ defined by:*

- $\Psi_0((i,j)) = X_{[\![i,j-1]\!]}$ *for all* $1 \leqslant i < j \leqslant n+1$, *and,*
- $\Psi_1(((i,j),(k,\ell))) = (X_{[\![i,j-1]\!]}, X_{[\![k,\ell-1]\!]})$ *for all arrow* $((i,j),(k,\ell))$ *in* $\mathrm{AR}(c)$,

is a quiver (directed graph) isomorphism.

Remark 43. To see further details about Auslander-Reiten quivers of A_n type quivers, we refer the reader to [37, Section 3.1]. To learn more about quiver representation theory and for more in-depth knowledge on the notion of Auslander–Reiten quivers, we invite the reader to look at [2].

In the following, we focus on full subcategories of $\mathrm{rep}_{\mathbb{K}}(Q)$ that are closed under sums and summands. Thus, those categories are additively generated by indecomposable representations, up to isomorphism, and are characterized by the sets of indecomposable representations.

From now one, we write $\mathcal{I}_n$ the set of intervals in $\{1, \ldots, n\}$. Given such a category $\mathscr{C}$, we write $\mathrm{Int}(\mathscr{C})$ for the interval set $\mathscr{J} \subseteq \mathcal{I}_n$ such that $\mathscr{C}$ is additively generated by the indecomposable representations X_K for $K \in \mathscr{J}$. Given an interval set $\mathscr{J} \subseteq \mathcal{I}_n$, we denote by $\mathrm{Cat}_Q(\mathscr{J})$ for the category additively generated by the X_K for $K \in \mathscr{J}$.

5.2 Reflection functors

In this subsection, we recall the definition of *reflection functors* for any quiver Q. For our purposes in this paper, defining these functors only on objects is sufficient. We refer the reader to [3] and [2, Section VII.5] for more details.

Let Q be an arbitrary quiver and v be a vertex Q Denote $\sigma_v(Q)$ the quiver obtained from Q by reversing the directions of the arrows incident to v If $\alpha \in Q_1$ such that $v \in \{s(\alpha), t(\alpha)\}$, denote $\tilde{\alpha}$ the reversed arrow of α in $\sigma_v(Q)$.

Now assume that v is a sink of Q Consider $\Xi = \sigma_v(Q)$. The ***reflection functor***

$$\mathcal{R}_v^+ : \mathrm{rep}(Q) \longrightarrow \mathrm{rep}(\Xi)$$

is defined as follows. Let $E = ((E_q)_{q\in Q_0}, (E_\beta)_{\beta\in Q_1}) \in \mathrm{rep}(Q)$. We set $\mathcal{R}_v^+(E) = ((F_q)_{q\in \Xi_0}, (F_\beta)_{\beta\in \Xi_1}) \in \mathrm{rep}(\Xi)$ where

- $F_q = E_q$ for $q \neq v$ and

$$F_v = \mathrm{Ker}\left(\bigoplus_{\alpha\in Q_1,\ t(\alpha)=v} E_\alpha : \bigoplus_{\alpha\in Q_1,\ t(\alpha)=v} E_{s(\alpha)} \longrightarrow E_v\right);$$

- $F_\beta = E_\beta$ if $\beta \in Q_1$ such that $t(\beta) \neq v$, otherwise $F_{\tilde{\beta}} : Y_v \longrightarrow E_{s(\beta)}$ is the composition of the kernel inclusion of F_v to $\bigoplus_{\alpha\in Q_1,\ t(\alpha)=v} E_{s(\alpha)}$ with the projection onto the direct summand $E_{s(\beta)}$.

If v is a source of Q, the ***reflection functor***

$$\mathcal{R}_v^- : \mathrm{rep}(Q) \longrightarrow \mathrm{rep}(\sigma_v(Q))$$

is defined dually. We refer the reader to [13] for an explicit example of the calculation of $\mathcal{R}_v^\pm(E)$ given an A type quiver Q, and a representation $E \in \mathrm{rep}_{\mathbb{K}}(Q)$.

The reflection functors are additive, so we can understand their actions on objects by knowing their actions on indecomposable objects.

By the following propositions, we recall the interpretation of reflection functors for Coxeter elements of $\mathfrak{S}_{n+1}$, and their action on $\mathrm{ind}_{\mathbb{K}}(Q)$, for Q an A_n type quiver.

Proposition 44. *Let $c \in \mathfrak{S}_{n+1}$ be a Coxeter element. Then if v is either a sink or a source of $Q(c)$, then $\sigma_v(Q(c)) = Q(s_v c s_v)$.*

Proposition 45. *Let Q be an A_n type quiver, $v \in Q_0$ and $[\![v]\!] \neq K \in \mathcal{I}_n$. Write $\Xi = \sigma_v(Q)$. If v is a sink of Q, then $\mathcal{R}_v^+(X_K) \cong X_{K'} \in \mathrm{rep}(\Xi)$ where*

$$K' = \begin{cases} K \cup \{v\} & \text{if either } e(K) = v-1 \text{ or } b(k) = v+1; \\ K \setminus \{v\} & \text{if either } e(K) = v \text{ or } b(K) = v; \\ K & \text{otherwise.} \end{cases}.$$

If v is a source of Q, then $\mathcal{R}_v^-(X_K) = X_{K'}$ where K' is defined as above.

Note that, if v is a sink of Q, $\mathcal{R}_v^+(X_{[\![v]\!]}) = 0$, and if v is a source, $\mathcal{R}_v^-(X_{[\![v]\!]}) = 0$. We also recall the following result, which highlights one of the main algebraic uses of those functors.

Theorem 46. *Let Q be a quiver and v be one of its sinks. Write $\Xi = \sigma_v(Q)$. The reflection functor $\mathcal{R}_v^+ : \mathrm{rep}(Q) \longrightarrow \mathrm{rep}(\Xi)$ induces a category equivalence between the full subcategory of $\mathrm{rep}(Q)$ additively generated by the indecomposable representations of Q except the simple projective representation at v and the full subcategory of $\mathrm{rep}(\Xi)$ additively generated by indecomposable representations of Ξ except the simple injective representation at v The quasi-inverse is induced by the reflection functor $\mathcal{R}_v^- : \mathrm{rep}(\Xi) \longrightarrow \mathrm{rep}(Q)$.*

For more details, see [2, Theorem VII.5.3].

5.3 Jordan recoverability and canonical Jordan recoverability

Consider $E \in \mathrm{rep}_{\mathbb{K}}(Q)$. An endomorphism $N : E \longrightarrow E$ is ***nilpotent*** if $N^k = 0$ for some integer $k \geqslant 1$. One can see a nilpotent endomorphism as a collection of nilpotent transformations $(N_q)_{q\in Q_0}$ with some additional compatibility relations. Write $\mathsf{NEnd}(E)$ for the set of nilpotent endomorphisms.

Assume that $\mathbf{dim}(E) = \boldsymbol{d} = (d_q)_{q\in Q_0}$. Given $N \in \mathsf{NEnd}(E)$, we consider the Jordan form of N_q at each $q \in Q_0$ It gives us a sequence of integer partitions $\lambda^q \vdash d_q$ We refer to $\boldsymbol{\lambda} = (\lambda^q)_{q\in Q_0}$ as the ***Jordan form of N***, and we set $\mathrm{JF}(N) = \boldsymbol{\lambda}$. On integer partitions, we consider the ***dominance order***, denote by $\trianglelefteq$, defined as follows: for λ and μ two integer partitions of $d \in \mathbb{N}$, $\lambda \trianglelefteq \mu$ whenever, for all $k \geqslant 1$, $\lambda_1 + \ldots + \lambda_k \leqslant \mu_1 + \ldots + \mu_k$. We extend this order to n-tuples of partitions First, let us introduce a notation Given $\boldsymbol{d} = (d_k)_{1\leqslant k\leqslant n} \in \mathbb{N}^n$ and $\boldsymbol{\lambda} = (\lambda^k)_{1\leqslant k\leqslant n}$ a n-tuple of integer, we write $\boldsymbol{\lambda} \vdash \boldsymbol{d}$ whenever $\lambda^k \vdash d_k$ for all $k \in \{1, \ldots, n\}$. Now fix $\boldsymbol{d} \in \mathbb{N}^n$, and $\boldsymbol{\lambda}, \boldsymbol{\mu} \vdash \boldsymbol{d}$. We write $\boldsymbol{\lambda} \trianglelefteq \boldsymbol{\mu}$ whenever for all $k \in \{1, \ldots, n\}$, $\lambda^k \trianglelefteq \mu^k$.

Theorem 47 ([21],[13])**.** *Let Q be an A_n type quiver, and $\mathbb{K}$ be an algebraically closed field Consider $E \in \mathrm{rep}_{\mathbb{K}}(Q)$.*

(i) The set $\mathsf{NEnd}(E)$ is an irreducible algebraic variety.

(ii) There exists a maximal value of JF in $\mathsf{NEnd}(E)$, with respect to $\trianglelefteq$, and it is attained in a dense open set (in the Zariski topology) of $\mathsf{NEnd}(E)$.

Note that the previous result can be generalized to finitely generated modules over an arbitrary $\mathbb{K}$-algebra (see [21]).

Definition 48. Let Q be an A_n type quiver, and $\mathbb{K}$ be an algebraically closed field For all $E \in \mathrm{rep}_{\mathbb{K}}(Q)$, we call the ***generic Jordan form data of E***, denoted by $\mathrm{GenJF}(E)$, the maximal value of JF in $\mathsf{NEnd}(E)$

This definition is entirely algebraic. We introduce a combinatorial method using Greene–Kleitman invariant calculations to compute $\mathrm{GenJF}(E)$ effectively.

Fix an A_n type quiver Q Let $E \in \mathrm{rep}_{\mathbb{K}}(Q)$. We decompose E as below

$$E \cong \bigoplus_{K \in \mathcal{I}_n} X_K^{m_K},$$

with $m_K \in \mathbb{N}$. Using this decomposition above allows us to see a representation E, up to isomorphism, as a filling $\mathbf{f}_Q(E)$ of $\mathrm{AR}_{\mathbb{K}}(Q)$ as follows:

$$\forall K \in \mathcal{I}_n,\ \mathbf{f}_Q(E)(K) = m_K.$$

In the following, for any filling f of $\mathrm{AR}_{\mathbb{K}}(Q)$, we denote by $\mathbf{E}_Q(f)$ its associated representation in $\mathrm{rep}_{\mathbb{K}}(Q)$, defined up to isomorphism.

For each $k \in \{1, \ldots, n\}$, write $\mathrm{AR}_{\mathbb{K}}^{[k]}(Q)$ for the complete subquiver of $\mathrm{AR}_{\mathbb{K}}(Q)$ whose vertices are given by X_K such that $k \in K$. It induces a filling $\mathbf{f}_Q(E)^{[k]}$ of $\mathrm{AR}_{\mathbb{K}}(Q)$. Using Theorem 27 and Theorem 42, the direct graph $\mathrm{AR}_{\mathbb{K}}(Q)$ is acyclic, which allows us to calculate the Greene–Kleitman invariant of $\mathbf{f}_Q(E)^{[k]}$.

Theorem 49 ([13, Theorem 3.6])**.** *Let Q be an A_n type quiver For any $X \in \mathrm{rep}_{\mathbb{K}}(Q)$, we have*

$$\mathrm{GenJF}(E) = \left(\mathsf{GK}_{\mathrm{AR}_{\mathbb{K}}^{(k)}(Q)}(\mathbf{f}_Q(E)^{[k]})\right)_{k \in Q_0}.$$

Remark 50. For any A_n type quiver Q, the acyclicity of $\mathrm{AR}_{\mathbb{K}}(Q)$, needed to calculate the Greene–Kleitman invariants, can be justified using Theorem 42 and Theorem 27. There exists a representation-theoretic explanation: Q is a *representation-directed quiver*. We refer the reader to [2, Section IX.3.] for more details.

The generic Jordan form data is an invariant on $\mathrm{rep}_{\mathbb{K}}(Q)$, but not complete. Nevertheless, we remain interested in determining the subcategories of $\mathrm{rep}_{\mathbb{K}}(Q)$ such that GenJF is complete, which brings us to the following definition.

Definition 51. Let Q be an A_n quiver A subcategory $\mathscr{C} \subset \mathrm{rep}_{\mathbb{K}}(Q)$ is ***Jordan recoverable*** if, for any tuple of integer partitions $\boldsymbol{\lambda}$, there exists at most one $E \in \mathscr{C}$ such that $\mathrm{GenJF}(E) \cong \boldsymbol{\lambda}$.

[13] and [21].

It is still difficult to characterize all the Jordan recoverable subcategories of $\mathrm{rep}_{\mathbb{K}}(Q)$. In Section 7, we recall and restate a conjecture originally stated in [13]. However, we can focus on subcategories in which we can make explicit an algebraic inverse to GenJF.

Fix Q an A_n type quiver Consider $\boldsymbol{\lambda}$ to be an n-tuple of integer partitions We can consider the set $\mathrm{rep}_{\mathbb{K}}(Q, \boldsymbol{\lambda})$ of representations $F \in \mathrm{rep}_{\mathbb{K}}(Q)$ such that it admits a nilpotent endomorphism N with Jordan form $\mathrm{JF}(N) = \boldsymbol{\lambda}$. We can ask for the existence of a dense open set (for Zariski's topology) $\Omega \subset \mathrm{rep}_{\mathbb{K}}(Q, \boldsymbol{\lambda})$ such that all the representations in Ω are isomorphic to each other. Via [21, Corollary 2.5], such a set Ω exists.

Definition 52. Given Q and A_n type quiver, and any n-tuple of integer partitions $\boldsymbol{\lambda}$, we define the ***generic representation of $\boldsymbol{\lambda}$***, denoted by $\mathrm{GenRep}(\boldsymbol{\lambda})$, to be the representation such that there exists a dense open set Ω in $\mathrm{rep}_{\mathbb{K}}(Q, \boldsymbol{\lambda})$ such that, for all $F \in \Omega$, $F \cong \mathrm{GenRep}(\boldsymbol{\lambda})$.

Note that, in general, for any $E \in \mathrm{rep}_{\mathbb{K}}(Q)$, $\mathrm{GenRep}(\mathrm{GenJF}(E)) \ncong E$, and for any n-tuple of partitions $\boldsymbol{\lambda}$, $\mathrm{GenJF}(\mathrm{GenRep}(\boldsymbol{\lambda})) \neq \boldsymbol{\lambda}$.

It makes sense that whenever $\boldsymbol{\lambda}$ does not correspond to a generic Jordan form data, applying $\mathrm{GenJF} \circ \mathrm{GenRep}$ does not give us back $\boldsymbol{\lambda}$ Similarly, whenever we have two representations $E, F \in \mathrm{rep}_{\mathbb{K}}(Q)$ such that $\boldsymbol{\lambda} = \mathrm{GenJF}(E) = \mathrm{GenJF}(F)$ but $E \ncong F$, then $\mathrm{GenRep}(\boldsymbol{\lambda}) \ncong E$ or $\mathrm{GenRep}(\boldsymbol{\lambda}) \ncong F$. Thus, we can focus on categories in which, at least, we can recover any representation from its generic Jordan form data by applying GenRep.

Definition 53. A subcategory $\mathscr{C} \subset \mathrm{rep}_{\mathbb{K}}(Q)$ is said to be ***canonically Jordan recoverable*** whenever for all $E \in \mathscr{C}$, $\mathrm{GenRep}(\mathrm{GenJF}(E)) \cong E$.

Note that any canonically Jordan recoverable category is Jordan recoverable, but the converse is false. We refer the reader to [13, 21] for examples.

5.4 Operations preserving canonical Jordan recoverability

In this section, we restate some results about operations that preserve the canonical Jordan recoverability property from [13].

First, we recall results involving the reflection functors.

Proposition 54 ([13, Section 4.3])**.** *Let v be a vertex of a quiver of A_n type. Let $\boldsymbol{\pi}$ be a n-tuple of integer partitions such that $\boldsymbol{\pi}$ is either $(\boxplus, \boxplus)$-storable, $(\boxplus, \boxminus)$-storable, $(\boxminus, \boxplus)$ storable or strongly $(\boxminus, \boxminus)$-storable at v. Consider $E = \mathrm{GenRep}(\boldsymbol{\pi})$ and assume that $\mathrm{GenJF}(E) = \boldsymbol{\pi}$. The following assertions hold:*

(i) If v is a source, $\mathrm{GenJF}(\mathcal{R}_v^-(E)) = \mathbf{diag}_v(\boldsymbol{\pi})$, and $\mathcal{R}_v^-(E) \cong \mathrm{GenRep}(\mathbf{diag}_v(\boldsymbol{\pi}))$.

(ii) If v is a sink, $\mathrm{GenJF}(\mathcal{R}_v^+(E)) = \mathbf{diag}_v(\boldsymbol{\pi})$, and $\mathcal{R}_v^+(E) \cong \mathrm{GenRep}(\mathbf{diag}_v(\boldsymbol{\pi}))$.

Theorem 55 ([13, Section 5.2])**.** *Let v be a source of an A_n-type quiver Q. Let $\mathscr{C} \subset \mathrm{rep}_{\mathbb{K}}(Q)$ be a canonically Jordan recoverable category such that*

(∇) For all $E \in \mathscr{C}$, $\mathrm{GenJF}(E)$ is either $(\boxplus, \boxplus)$-storable, $(\boxminus, \boxplus)$-storable, $(\boxplus, \boxminus)$-storable or strongly $(\boxminus, \boxminus)$-storable at v.

Then $R_v^-(\mathscr{C})$ is a canonically Jordan recoverable subcategory of $\mathrm{rep}_{\mathbb{K}}(\sigma_v(Q))$. Similarly, if v is a sink, $R_v^+(\mathscr{C})$ is a canonically Jordan recoverable subcategory of $\mathrm{rep}_{\mathbb{K}}(\sigma_v(Q))$.

Then, we recall some results about the *adding simple operations*. Let Q be an A_n type quiver. For any $v \in Q_0$, we define the ***adding simple at v operation***, denoted by AddS_v, on every subcategory $\mathscr{C}$ of $\mathrm{rep}_{\mathbb{K}}(Q)$ by $\mathrm{AddS}_v(\mathscr{C}) = \mathrm{add}(\mathscr{C}, X_{[\![v]\!]})$.

Lemma 56 ([13, Section 5.1])**.** *Let v be a source or a sink of an A_n type quiver Q. Consider $a \in \mathbb{N}$ and $E \in rep_{\mathbb{K}}(Q)$. Write $\boldsymbol{\pi} = \mathrm{GenJF}(E)$. Then $\mathrm{GenJF}(X_{[\![v]\!]}^a \oplus E) = \boldsymbol{\xi}$ where $\xi^q = \pi^q$ if $q \neq v$ and $\xi^v = (\pi_1^v + a, \pi_2^v, \pi_3^v, \ldots)$.*

Theorem 57 ([13, Section 5.1])**.** *Let Q be a quiver of A_n type, and v be a source or a sink of Q. Let $\mathscr{C} \subset \mathrm{rep}_{\mathbb{K}}(Q)$ be a canonically Jordan recoverable category such that*

($*$) For any $E \in \mathscr{C}$, $\mathrm{GenJF}(E)$ is strongly $(\boxminus, \boxminus)$-storable at v.

Then $\mathrm{AddS}_v(\mathscr{C})$ is a canonically Jordan recoverable subcategory of $\mathrm{rep}_{\mathbb{K}}(Q)$.

5.5 Results from previous work

In this section, we resume some results from [13], and by going through the main outlines of the proof, we establish some valuable consequences in the next section.

First, we state the main result of [13]. Let Q be an A_n type quiver. We recall that the subcategories we consider are characterized by their indecomposable representations and, therefore, by intervals of $\{1, \dots, n\}$.

For $K = [\![i; j]\!] \in \mathcal{I}_n$, we set $b(K) = i$ and $e(K) = j$. Two intervals $K, L \in \mathcal{I}_n$ are ***adjacent*** whenever either $b(L) = e(K) + 1$ or $b(K) = e(L) + 1$. Hence, K and L are not adjacent if either $K \cap L \neq \varnothing$, or $b(K) > e(L) + 1$, or $b(L) > e(K) + 1$.

Definition 58. An interval set $\mathscr{J} \subset \mathcal{I}_n$ is said to be ***adjacency-avoiding*** whenever there is no pair of intervals $K, L \in \mathscr{J}$ that are adjacent.

Example 59. Below are given all the maximal adjacency-avoiding interval sets of $\mathcal{I}_3$:

- $\{[\![1]\!], [\![1, 2]\!], [\![1, 3]\!]\}$;
- $\{[\![2]\!], [\![1, 2]\!], [\![2, 3]\!], [\![1, 3]\!]\}$;
- $\{[\![3]\!], [\![2, 3]\!], [\![1, 3]\!]\}$;
- $\{[\![1]\!], [\![3]\!], [\![1, 3]\!]\}$.

Below is the main result of [13], which gives a combinatorial characterization of all the subcategories of $\mathrm{rep}_{\mathbb{K}}(Q)$.

Theorem 60. *Let Q be an A_n type quiver A subcategory $\mathscr{C} \subset \mathrm{rep}_{\mathbb{K}}(Q)$ is canonically Jordan recoverable if, and only if, $\mathrm{Int}(\mathscr{C})$ is adjacency-avoiding.*

Remark 61.

- This result generalizes a previous result of [21] for A_n type quivers.
- For any interval set $\mathscr{J}$, the canonical Jordan recoverability of $\mathrm{Cat}_Q(\mathscr{J})$ does not depend of Q.

The first step of the proof was to highlight that the adjacency-avoiding property of $\mathrm{Int}(\mathscr{C})$ is a necessary condition for some category $\mathscr{C}$ to be canonically Jordan recoverable.

Then, we study the maximal adjacency-avoiding interval sets of $\{1, \dots, n\}$. Before stating the following result, let us introduce a relevant family of interval sets. For any pair of subsets $(\mathbf{B}, \mathbf{E})$ of $\{1, \dots, n\}$, we define the interval set $\mathscr{J}(\mathbf{B}, \mathbf{E})$ as follows:

$$\mathscr{J}(\mathbf{B}, \mathbf{E}) = \{[\![b, e]\!] \mid b \in \mathbf{B},\ e \in \mathbf{E}\}.$$

Theorem 62 ([13])**.** *Any adjacency-avoiding interval set is a subset of some $\mathscr{J}(\mathbf{B}, \mathbf{E})$ such that $(\mathbf{B}, \mathbf{E}[1])$ is an elementary interval bipartition of $\{1, \dots, n + 1\}$.*

By Theorem 22, we can parametrize maximal adjacency-avoiding interval subsets of $\mathcal{I}_n$ via a Coxeter element $c \in \mathfrak{S}_{n+1}$. Write $\mathscr{J}(c) = \mathscr{J}(\mathbf{L}_c \cup \{1\}, \mathbf{R}_c[-1] \cup \{n\})$.

We set $\mathrm{Cat}_Q(c) = \mathrm{Cat}_Q(\mathscr{J}(c))$. Finally, we show that, given an A_n quiver, all the subcategories $\mathrm{Cat}_Q(c) \subset \mathrm{rep}_{\mathbb{K}}(Q)$ arising from maximal adjacency-avoiding interval subsets of $\mathcal{I}_n$ are canonically Jordan recoverable.

Theorem 63 ([13, Theorem 7.5])**.** *Fix $n \in \mathbb{N}^*$. Let $\mathbb{K}$ be an algebraically closed field, and Q be an A_n type quiver. Consider $c \in \mathfrak{S}_{n+1}$ a Coxeter element. Then $\mathrm{Cat}_Q(c)$ is canonically Jordan recoverable. Moreover,* GenJF *induces a one-to-one correspondence from isomorphism classes of representations of $\mathrm{Cat}_Q(c)$ to $\boldsymbol{\mathcal{ST}}(c)$.*

Remark 64. In [13, Theorem 7.5], we state that GenJF induces an injective map from isomorphism classes of representations of $\mathrm{Cat}_Q(c)$ to $\boldsymbol{\mathcal{ST}}(c)$. The steps used in the proofs of [13, Theorem 7.4 and 7.5] can be reversed. Thus GenJF indeed induces a one-to-one correspondence.

The proof uses technical tools from combinatorics and quiver representation theory, particularly results presented in Section 5.4. The proof of [13, Theorem 7.5] uses induction on the orientations of the quiver Q, which can be seen in terms of Coxeter elements. We prove it for $Q(c_1)$ for $c_1 = (1, \dots, n+1)$, and then we prove that if the statement holds for $Q(c_2)$ given by some Coxeter element $c_2 \in \mathfrak{S}_{n+1}$, then it holds for $Q(sc_2s)$ for any $s \in \Sigma_n$ that is either initial or final in c_2.

Let $c \in \mathfrak{S}_{n+1}$ be a Coxeter element, and $k \in \{1, \dots, n\}$. We define $\sigma_k(c)$ by:

- $\sigma_k(c) = s_k c s_k$ whenever s_k is either initial or final in c, and
- $\sigma_k(c) = c$ otherwise.

Note that we use the same notation for mutations on quivers, and we saw that in our setting, they match (see Proposition 44). We also consider a tiny deformation of this mutation, denoted by $\widetilde{\sigma_k}$, defined by:

- $\widetilde{\sigma_k}(c) = c$ whenever $k = 1$ and s_1 initial in c, or $k = n$ and s_n final in c;
- $\widetilde{\sigma_k}(c) = \sigma_k(c)$ otherwise.

Proposition 65. *Let $c_1, c_2 \in \mathfrak{S}_{n+1}$ be two Coxeter elements. Fix $k \in \{1, \dots, n\}$ such that s_k is either initial or final in c_1. Then $\mathcal{R}_k^{\pm}(\mathrm{Cat}_{Q(c_1)}(c_2))$ is equivalent to a subcategory of $\mathrm{Cat}_{Q(c_1)}(\widetilde{\sigma_k}(c_2))$, as subcategory of $\mathrm{rep}_{\mathbb{K}}(Q(\sigma_k(c_1)))$.*

Proof. This is a consequence of Theorem 44 for c_1, and Theorem 45 for c_2. □

In the following proposition, we rephrase the precise result hidden in the proof of [13, Theorem 7.5].

Theorem 66 ([13])**.** *Let $\mathbb{K}$ be an algebraically closed field. Fix $n \in \mathbb{N}^*$. Let $c_1, c_2 \in \mathfrak{S}_{n+1}$ be two Coxeter elements. Let $\mathrm{Cat}_{Q(c_1)}(c_2) \subset \mathrm{rep}_{\mathbb{K}}(Q(c_1))$. Fix $k \in \{1, \dots, n\}$ such that $s_k \in \Sigma_n$ is either initial or final in c_1. For all $E \in \mathrm{Cat}_{Q(c_1)}(c_2)$,*

$$\mathrm{GenJF}(\mathcal{R}_k^{\pm}(E)) = \mathbf{diag}_k(\mathrm{GenJF}(E)).$$

Moreover, $\mathrm{GenJF}(\mathcal{R}_k^{\pm}(E)) \in \boldsymbol{\mathcal{ST}}(\widetilde{\sigma_k}(c_2))$, and if s_k is final in c_2, then $\mathrm{GenJF}(\mathcal{R}_k^{\pm}(E))$ is strongly $(\boxminus, \boxminus)$-storable at k.

6 Extended RSK via type A quiver representation

6.1 Construction of the extended generalization

Next, we describe an extended version of RSK using (type A) Coxeter elements and state the main result.

Let $n \geqslant 1$ and $\lambda \in \mathrm{Hk}_n$. Let $c \in \mathfrak{S}_{n+1}$, and consider $\mathrm{AR}(c)$ its Auslander–Reiten quiver. Recall that $\mathbf{c}(\lambda) \in \mathfrak{S}_{n+1}$ is the unique Coxeter element such that $\boldsymbol{\lambda}(\mathbf{c}(\lambda)) = \lambda$ by Theorem 22. We also recall that $(\mathbf{L} = \mathbf{L}_c \cup \{1\}, \mathbf{R} = \mathbf{R}_c \cup \{n+1\})$ is the unique elementary bipartition of $\{1, \dots, n+1\}$ such that $\boldsymbol{\lambda}(\mathbf{L}, \mathbf{R}) = \lambda$ by Theorem 24.

Using $\mathbf{c}(\lambda)$-coordinates on $\mathrm{Fer}(\lambda)$, any box labelled $[\ell, r]_{\mathbf{c}(\lambda)}$ can be associated to the transposition $(\ell, r) \in \mathfrak{S}_{n+1}$. We construct a one-to-one correspondence $\boldsymbol{rep}_{\lambda,c}$ from fillings of shape λ to those of the Auslander–Reiten quiver $\mathrm{AR}(c)$ which are supported on vertices $(\ell, r) \in \mathbf{L} \times \mathbf{R}$ such that $\ell < r$. Precisely, for any filling f of shape λ, $\boldsymbol{rep}_{\lambda,c}(f)(\ell, r) = f([\ell, r]_{\mathbf{c}(\lambda)})$ whenever $(\ell, r) \in \mathbf{L} \times \mathbf{R}$ such that $\ell < r$, and $\boldsymbol{rep}_{\lambda,c}(\ell, r) = 0$ otherwise.

As in Section 2.4, we also use the λ-diagonal coordinates. For each $k \in \{1, \dots, n\}$, we consider the subgraph $\mathrm{AR}^{[k]}(c)$ of $\mathrm{AR}(c)$ where the vertices are the transpositions (ℓ, r) with $\ell \leqslant k < r$. This subgraph has only one source and only one sink. We denote by $\boldsymbol{rep}_{\lambda,c}(f)^{[k]}$ the filling or $\mathrm{AR}^{[k]}(c)$ induced by $\boldsymbol{rep}_{\lambda,c}(f)$.

We define $\boldsymbol{\mathcal{RSK}}_{\lambda,c}(f)$ to be the fillings of shape λ defined, for $k \in \{1, \dots, n\}$, and for $\delta \in \{1, \dots, \delta_k\}$, by:

$$\boldsymbol{\mathcal{RSK}}_{\lambda,c}(f)(\langle k, \delta \rangle_\lambda) = \mathsf{GK}_{\mathrm{AR}^{[k]}(c)} \left(\boldsymbol{rep}_{\lambda,c}(f)^{[k]} \right)_\delta .$$

See Figure 17 for an explicit example.

Theorem 67. *Let $n \geqslant 1$ and $\lambda \in \mathrm{Hk}_n$. Let $c \in \mathfrak{S}_{n+1}$ be a Coxeter element. The map $\boldsymbol{\mathcal{RSK}}_{\lambda,c}$ realizes a one-to-one correspondence from fillings of shape λ to reverse plane partitions of shape λ.*

Proof. See the Coxeter element $c = c_1 \in \mathfrak{S}_{n+1}$, as the A_n type quiver $Q(c_1)$. See λ as the choice of maximal canonically Jordan recoverable subcategory $\mathrm{Cat}_{Q(c_1)}(c_2)$ of $\mathrm{rep}_{\mathbb{K}}(Q(c_1))$ where $c_2 \in \mathfrak{S}_{n+1}$ such that $\boldsymbol{\lambda}(c_2) = \lambda$.

Then the map $\boldsymbol{rep}_{\lambda,c}$ gives a one-to-one correspondence from fillings of λ to representations in $\mathrm{Cat}_{Q(c_1)}(c_2)$, up to isomorphism. Set $E = \mathsf{E}_{Q(c_1)}(\boldsymbol{rep}_{\lambda,c_1}(f))$ to be the representation of $\mathrm{Cat}_{Q(c_1)}(c_2)$ whose multiplicities are given by $\boldsymbol{rep}_{\lambda,c_1}(f)$. The calculations of $\boldsymbol{\mathcal{RSK}}_{\lambda,c_1}(f)$ correspond exactly to the calculations of $\mathrm{GenJF}(E)$ by Theorem 49 and 40. As stated by Theorem 63, we are done. □

Remark 68. One can notice that we could apply slightly broadened generalization by allowing to take any bipartition $(\mathbf{L}, \mathbf{R})$ such that $\min(\mathbf{L} \cup \mathbf{R}) \in \mathbf{L}$ and $\max(\mathbf{L} \cup \mathbf{R}) \in \mathbf{R}$, and an Coxeter element $c \in \mathfrak{S}_{n+1}$ with $n \geqslant \max(R)$. In fact, this slightly deformed correspondence corresponds to the application of $\boldsymbol{\mathcal{RSK}}_{\lambda,c}$, for $\lambda \in \mathrm{Hk}_n$ such that $\mathbf{L} \subseteq \mathbf{L}_{\mathbf{c}(\lambda)} \cup \{1\}$ and $\mathbf{R} \subseteq \mathbf{R}_{\mathbf{c}(\lambda)} \cup \{n+1\}$, with its domain restricted to fillings that are vanishing on boxes $[\ell, r]_{\mathbf{c}(\lambda)}$ where $\ell \notin \mathbf{L}$ or $r \notin \mathbf{R}$.

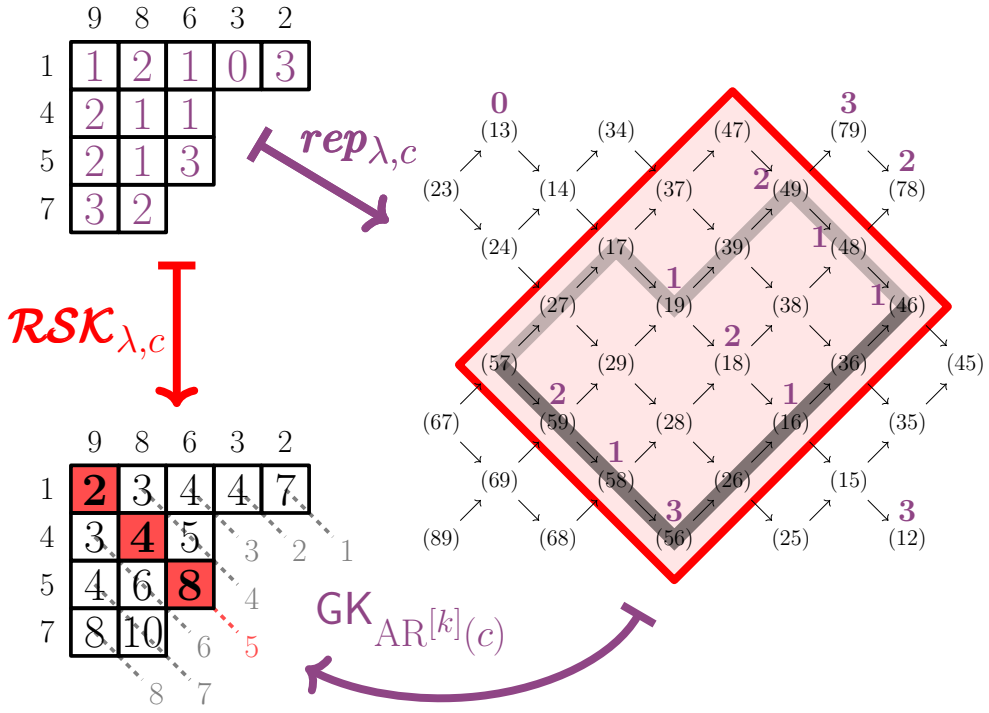


Figure 17: Explicit calculation of $\boldsymbol{\mathcal{RSK}}_{\lambda,c}(f)$ for the boxes in $D_k(\lambda)$, with $k=5$, from a filling of $\lambda=(5,3,3,2)$, with $c=(1,3,4,7,9,8,6,5,2)$.

6.2 Link to previous well-known combinatorial bijections

Let us start this section with an obvious proposition from the construction of $\boldsymbol{\mathcal{RSK}}_{\lambda,c}$.

Proposition 69. *Let $n\geqslant 1$. Consider a Coxeter element $c\in\mathfrak{S}_{n+1}$, and $\lambda\in\mathrm{Hk}_n$. Then $\boldsymbol{\mathcal{RSK}}_{\lambda,c^{-1}}=\boldsymbol{\mathcal{RSK}}_{\lambda,c}$ and $\boldsymbol{\mathcal{RSK}}_{\lambda',c}=(\boldsymbol{\mathcal{RSK}}_{\lambda,c})^{\top}$ where $f^{\top}$ is the filling of λ' obtained by transposing the filling f of λ.*

The following result shows that we extended Gansner's RSK correspondence.

Proposition 70. *Let $n\geqslant 1$ and $c\in\mathfrak{S}_{n+1}$ be a Coxeter element. Set $\lambda=\boldsymbol{\lambda}(c)$. Then $\boldsymbol{\mathcal{RSK}}_{\lambda,c}=\boldsymbol{\mathcal{RSK}}_{\lambda}$.*

Proof. In this configuration, by setting $\mathbf{L}=\{1\}\cup\mathbf{L}_c=\{\ell_1<\ldots\ell_p\}$ and $\mathbf{R}=\mathbf{R}_c\cup\{n+1\}=\{r_1<\ldots<r_q\}$, we can check that:

- for $i\in\{1,\ldots,p-1\}$, $c(\ell_i)=\ell_{i+1}$;
- for $j\in\{1,\ldots,q-1\}$, $c(r_{j+1})=r_j$.

Thus, the orientation of the Auslander–Reiten quiver $\mathrm{AR}(c)$ corresponds precisely to the reverse of the one used to calculate the $\boldsymbol{\mathcal{RSK}}_{\boldsymbol{\lambda}(c)}$. It induces the desired result. □

Proposition 71. *Let $n \geqslant 1$ and $\lambda \in \mathrm{Hk}_n$. Consider $c = (1, 2, \ldots, n+1)$. Then $\boldsymbol{\mathcal{RSK}}_{\lambda,c}$ corresponds exactly to the Hillman-Grassl correspondence.*

Proof. Set $\mathbf{L} = \{\ell_1 < \ldots \ell_p\}$ and $\mathbf{R} = \{r_1 < \ldots < r_q\}$ such that $(\mathbf{L}, \mathbf{R})$ is the elementary bipartition of $\{1, \ldots, n+1\}$ such that $\lambda = \boldsymbol{\lambda}(\mathbf{L}, \mathbf{R})$. We can check that, in $\mathrm{AR}(c)$, (ℓ_{i+1}, r_j) follows directly (ℓ_i, r_j), and (ℓ_i, r_{j+1}) follows directly (ℓ_i, r_j). Then, the orientation we must choose to realize the Hillman–Grassl correspondence coincides with the one given by $\mathrm{AR}(c)$. Therefore, we get the desired result. □

Now we highlight the link with the work of [21]

Proposition 72. *Let $n \geqslant 1$ and $\lambda \in \mathrm{Hk}_n$. Consider a Coxeter element $c \in \mathfrak{S}_{n+1}$. Then, for any filling f of λ,*

$$\Phi^{-1}_{\mathbf{c}(\lambda)} \circ \boldsymbol{\mathcal{RSK}}_{\lambda,c}(f) = \mathrm{GenJF}(\mathsf{E}_{Q(c)}(f)).$$

Then, the following corollary holds.

Corollary 73. *Let $n \geqslant 1$ and $\lambda \in \mathrm{Hk}_n$. Consider a Coxeter element $c \in \mathfrak{S}_{n+1}$. If $\mathbf{c}(\lambda) = (1, 2, \ldots, m, n+1, n, \ldots, m+1)$ for some $m \in \{1, \ldots, n\}$, then*

$$\boldsymbol{\mathcal{RSK}}_{\lambda,c} = \mathbf{RSK}_{m,c}$$

where $\mathbf{RSK}_{m,c}$ corresponds to Scrambled RSKs.

Note that choosing a filling of such a λ corresponds to a representation, up to isomorphism, of $\mathscr{C}_{Q(c),m}$.

6.3 Enumerative properties

Lemma 74. *Let $n \geqslant 1$, and $\lambda \in \mathrm{Hk}_n$. Consider a Coxeter elements $c \in \mathfrak{S}_{n+1}$. Then, for any filling f of λ, and for any $k \in \{1, \ldots, n\}$, we have*

$$\sum_{\varepsilon=1}^{\delta_k} \boldsymbol{\mathcal{RSK}}_{\lambda,c}(f)(\langle k, \varepsilon \rangle_\lambda) = \sum_{b \in \square_k(\lambda)} f(b).$$

Proof. This is a direct consequence of Theorem 9 and the definition of the extended generalization. Precisely, by setting $\mathbf{L} = \mathbf{L}_c \cup \{1\}$ and $\mathbf{R} = \mathbf{R}_c \cup \{n+1\}$,

$$\sum_{\varepsilon=1}^{\delta_k} \boldsymbol{\mathcal{RSK}}_{\lambda,c}(f)(\langle k, \epsilon \rangle_\lambda) = \sum_{(\ell,r) \in \mathbf{L} \times \mathbf{R}, \ell \leqslant k < r} \mathrm{rep}_{\lambda,c}(f)(\ell, r) = \sum_{b \in \square_k(\lambda)} f(b)$$

□

Remark 75. From the previous proof, one can notice that, given $\lambda \in \mathrm{Hk}_n$ and a Coxeter element $c \in \mathfrak{S}_{n+1}$, we have

$$\dim(\mathsf{E}_{Q(c)}(f)_k) = \sum_{b \in \square_k(\lambda)} f(b).$$

This value does not depend on c.

The following corollary extends the product formula of the trace generating function of reverse plane partitions of minuscule posets. Proctor originally established this version [32], and Garver, Patrias, and Thomas proved it differently [21].

Corollary 76. *Let $n \geqslant 1$, and $\lambda \in \mathrm{Hk}_n$. For any A_n type quiver Q, we have*

$$\rho_\lambda(x_1, \ldots, x_n) = \sum_{E \in \mathrm{Cat}_Q(\mathbf{c}(\lambda))} \prod_{\ell=1}^{n} x_\ell^{\dim(E_\ell)} = \prod_{M \in \mathrm{Cat}_Q(\mathbf{c}(\lambda)),\ \textit{indec.}} \frac{1}{1 - \prod_{\ell=1}^{n} x_\ell^{\dim(M_\ell)}},$$

where the sum is over the isomorphism classes of representations in $\mathrm{Cat}_Q(\mathbf{c}(\lambda))$, and the product is over isomorphism classes of indecomposable representations in $\mathrm{Cat}_Q(\mathbf{c}(\lambda))$

Proof. Let Q be an A_n type quiver. Consider the Coxeter element $c \in \mathfrak{S}_{n+1}$ such that $Q(c) = Q$. By Theorem 74 and Theorem 75, we have

$$\rho_\lambda(x_1, \ldots, x_n) = \sum_{f \in \mathrm{RPP}(\lambda)} \prod_{\ell=1}^{n} x_\ell^{\sum_{\varepsilon=1}^{\delta_l} f(\langle \ell, \varepsilon \rangle_\lambda)} = \sum_{f \in \mathrm{RPP}(\lambda)} \prod_{\ell=1}^{n} x_\ell^{\dim(E_{Q(c)}(f)_\ell))}.$$

So

$$\rho_\lambda(x_1, \ldots, x_n) = \sum_{E \in \mathrm{Cat}_Q(\mathbf{c}(\lambda))} \prod_{\ell=1}^{n} x_\ell^{\dim(E_\ell)}.$$

Moreover, each box $b \in \mathrm{Fer}(\lambda)$ corresponds bijectively to an indecomposable representation, using the $\mathbf{c}(\lambda)-$coordinates and $\mathbf{E}_Q \circ \boldsymbol{rep}_{\lambda,c}$. Set, for any $b \in \mathrm{Fer}(\lambda)$, M_b its associated indecomposable representation in $\mathrm{rep}_{\mathbb{K}}(Q)$, up to isomorphism. We have that:

- $b \in \square_\ell(\lambda)$ if and only if $\dim((M_b)_\ell) = 1$, and,
- $b \notin \square_\ell(\lambda)$ if and only if $\dim((M_b)_\ell) = 0$.

Therefore,

$$\omega_{\lambda,b}(x_1, \ldots, x_n) = \prod_{\ell=1}^{n} x_\ell^{\dim((M_b)_\ell)}.$$

We get the desired result by Theorem 13. □

Remark 77. Note that this identity (the second equality) is a consequence of our extended generalization $\boldsymbol{\mathcal{RSK}}_{\lambda,c}$ where the Coxeter element $c \in \mathfrak{S}_{n+1}$ corresponds to the choice of an A_n type quiver. The product formula comes from the fact that any representation $E \in \mathrm{rep}_{\mathbb{K}}(Q)$ can be decomposed, in a unique way, as a sum of indecomposable representations (See Section 5.3).

7 To go further

In this section, we suggest some research directions that could follow this work.

A realization of $\boldsymbol{\mathcal{RSK}_{\lambda,c}}$ via local transformations

[26] gives a complete realization of $\boldsymbol{\mathcal{RSK}_{\lambda}}$ via local transformations, for any integer partitions. In quiver representation settings, [21] give a realization of $\mathbf{RSK}_{m,c}$ via local transformations ("toggles"), for any $n \in \mathbb{N}^*$, and any Coxeter element $c \in \mathfrak{S}_{n+1}$. In parallel, [10] give a realization of $\mathbf{RSK}_{m,c}$ (mentioned as *Scrambled RSK*) via the same "toggles" but described using combinatorial terms.

Thinking about the crucial link with quiver representations and the local description given by [21, Section 4.3], one could think about considering a sequence of "toggles", translated from operations preserving canonical Jordan recoverability (see Section 5.4), based on the linear order on transpositions of $\mathfrak{S}_{n+1}$, compatible with (the opposite of) the Auslander–Reiten quiver.

We hope to elaborate such a realization of $\boldsymbol{\mathcal{RSK}_{\lambda,c}}$ in the near future.

Dynkin type (or other) variations of $\boldsymbol{\mathcal{RSK}_{\lambda,c}}$

Instead of considering Coxeter elements of the symmetric group, one could ask if it is possible to consider Coxeter elements of any Weyl group. We see at least two ways to think about that.

The first way is to adapt the setting to the Weyl group we are considering. For instance, if we work with the signed symmetric group, a B type Weyl group, we could think about type B RSK, seen as the domino correspondence. We refer the reader to the work of Garfinkle [19, 20] Bonafé, Geck, Iancu, and Lam [4] for more details. In the type D case, Gern did a few studies in his PhD thesis [22] in terms of Kazhdan–Lusztig polynomials.

The second way could be to work in the quiver representation setting and to determine all the canonically Jordan recoverable subcategories of any type D and type E quivers. [21, Theorem 1.3] gives us a type D and a type E versions of the RSK correspondence.

From the perspective of quiver representation theory, we can go deeper into it and work with well-behaved and well-known bounded quivers. For instance, for *string quivers*, we have a complete classification of the isomorphism classes of indecomposable representations [7] and of the morphisms between them [29, 8]. Some work was already done, for a subfamily of string quivers, called *gentle quivers*, on canonically Jordan recoverable subcategories in this domain [11], and we hope to characterize all of them. One interesting combinatorial outcome could be the construction of an "RSK correspondence via string algebras".

Acknowledgements

The author acknowledges the *CHARMS program grant (ANR-19-CE40-0017-02)* and the *Engineering and Physical Sciences Research Council (EP/W007509/1)* for their partial funding support. He thanks the selection committee of the 36th edition of the FPSAC Conference (Bochum, 2024) for its insightful comments on my extended abstract [12], accepted for the poster session.

He is grateful to Ben Adenbaum, Emily Gunawan, Florent Hivert, Yann Palu, GaYee Park, and Michael Schoonheere for their interest and discussions on this project. He especially thanks Phillippe Nadeau for his advice, which allowed him to simplify some combinatorial proofs. Finally, he acknowledges Hugh Thomas for his advice and comments on this work.

References

[1] F. Aigner and G. Frieden, qRSt: A Probabilistic Robinson–Schensted Correspondence for Macdonald Polynomials, *Int. Math. Res. Not.* **2022**, 13505–13568. DOI: https://doi.org/10.1093/imrn/rnab083

[2] I. Assem, A. Skowroński, and D. Simson, *Elements of the Representation Theory of Associative Algebras. Vol. 1: Techniques of Representation Theory*, London Mathematical Society Student Texts, Cambridge University Press, Cambridge, 2006. DOI: https://doi.org/10.1017/CBO9780511614309

[3] I. N. Bernstein, I. M. Gel'fand, and V. A. Ponomarev, Coxeter Functors and Gabriel's Theorem, *Russian Math. Surveys* **28** (1973), 17–32. DOI: https://doi.org/10.1070/RM1973v028n02ABEH001526

[4] C. Bonnafé, M. Geck, L. Iancu, and T. Lam, On Domino Insertion and Kazhdan–Lusztig Cells in Type B_n, in *Representation Theory of Algebraic Groups and Quantum Groups*, 2010, pp. 33–54. DOI: https://doi.org/10.1007/978-0-8176-4697-4_3

[5] T. Britz and S. Fomin, Finite Posets and Ferrers Shapes, *Adv. Math.* **158** (2001), 86–127. DOI: https://doi.org/10.1006/aima.2000.1966

[6] W. H. Burge, Four Correspondences Between Graphs and Generalized Young Tableaux, *J. Combin. Theory Ser. A* **17** (1974), 12–30. DOI: https://doi.org/10.1016/0097-3165(74)90024-7

[7] M. C. R. Butler and C. M. Ringel, Auslander–Reiten sequences with few middle terms and applications to string algebras, *Comm. Algebra* **15** (1987), no. 1–2, 145–179. DOI: https://doi.org/10.1080/00927878708823416

[8] W. W. Crawley-Boevey, Maps between representations of zero-relation algebras, *J. Algebra* **126** (1989), 259–263. DOI: https://doi.org/10.1016/0021-8693(89)90304-9

[9] D. Dauvergne, Hidden invariance of last passage percolation and directed polymers, *Ann. Probab.* **50** (2022), no. 1, 18–60. DOI: https://doi.org/10.1214/21-AOP1527

[10] D. Dauvergne, M. Nica, and B. Virág, RSK in last passage percolation: a unified approach, *Probab. Surveys* **19** (2022), 65–112. DOI: https://doi.org/10.1214/22-PS4

[11] B. Dequêne, Jordan recoverability of some subcategories of modules over gentle algebras, *J. Pure Appl. Algebra* **228** (2024), 107446. DOI: https://doi.org/10.1016/j.jpaa.2023.107446

[12] B. Dequêne, An extended generalization of RSK via the combinatorics of type A quiver representations, *Séminaire Lotharingien de Combinatoire*, Proceedings of the 36th Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2024), **91B** (2024), Art. 76. URL: https://www.mat.univie.ac.at/~slc/wpapers/FPSAC2024/76.html

[13] B. Dequêne, Canonically Jordan recoverable categories for modules over the path algebra of A_n type quivers, *Ann. Represent. Theory* **3** (2026), no. 3, 335–369. DOI: https://doi.org/10.5802/art.43

[14] P. Erdős and G. Szekeres, A Combinatorial Problem in Geometry, in *Classic Papers in Combinatorics*, 1987, pp. 49–56. DOI: https://doi.org/10.1007/978-0-8176-4842-8_3

[15] W. Fulton, *Young Tableaux: With Applications to Representation Theory and Geometry*, Cambridge University Press, Cambridge, 1996. DOI: https://doi.org/10.1017/CBO9780511626241

[16] E. R. Gansner, Acyclic Digraphs, Young Tableaux and Nilpotent Matrices, *SIAM J. Algebraic Discrete Methods* **2** (1981), 429–440. DOI: https://doi.org/10.1137/0602046

[17] E. Gansner, Matrix correspondences of plane partitions, *Pacific J. Math.* **92** (1981), 295–315. DOI: https://doi.org/10.2140/pjm.1981.92.295

[18] E. R. Gansner, The Hillman–Grassl correspondence and the enumeration of reverse plane partitions, *J. Combin. Theory Ser. A* **30** (1981), 71–89. DOI: https://doi.org/10.1016/0097-3165(81)90041-8

[19] D. Garfinkle, On the classification of primitive ideals for complex classical Lie algebras, I, *Compos. Math.* **75** (1990), no. 2, 135–169. URL: http://www.numdam.org/item/CM_1990__75_2_135_0/

[20] D. Garfinkle, On the classification of primitive ideals for complex classical Lie algebras, II, *Compos. Math.* **81** (1992), no. 3, 307–336. URL: http://www.numdam.org/item/CM_1992__81_3_307_0/

[21] A. Garver, R. Patrias, and H. Thomas, Minuscule reverse plane partitions via quiver representations, *Selecta Math.* **29** (2023), no. 3, Article 37. DOI: https://doi.org/10.1007/s00029-023-00831-4

[22] T. C. Gern, *Leading Coefficients of Kazhdan–Lusztig Polynomials in Type D*, PhD thesis, University of Colorado Boulder, 2013.

[23] C. Greene and D. J. Kleitman, The structure of Sperner k-families, *J. Combin. Theory Ser. A* **20** (1976), 41–68. DOI: https://doi.org/10.1016/0097-3165(76)90077-7

[24] M. Gillespie and A. Reimer-Berg, A Generalized RSK for Enumerating Linear Series on n-pointed Curves, *Algebraic Combinatorics* **6** (2023), no. 1. DOI: https://doi.org/10.5802/alco.250

[25] A. P. Hillman and R. M. Grassl, Reverse plane partitions and tableau hook numbers, *J. Combin. Theory Ser. A* **21** (1976), 216–221. DOI: https://doi.org/10.1016/0097-3165(76)90065-0

[26] S. Hopkins, *RSK via local transformations*, 2014. URL: https://www.samuelfhopkins.com/docs/rsk.pdf

[27] D. E. Knuth, Permutations, Matrices, and Generalized Young Tableaux, *Pacific J. Math.* **34** (1970), 709–727. DOI: https://doi.org/10.2140/pjm.1970.34.709

[28] C. Krattenthaler, Growth diagrams, and increasing and decreasing chains in fillings of Ferrers shapes, *Adv. Appl. Math.* **37** (2006), 404–431. DOI: https://doi.org/10.1016/j.aam.2005.12.006

[29] H. Krause, Maps Between Tree and Band Modules, *J. Algebra* **137** (1991), 186–194. DOI: https://doi.org/10.1016/0021-8693(91)90088-P

[30] A. Lascoux and M.-P. Schützenberger, Le monoïde plaxique, in *Noncommutative Structures in Algebra and Geometric Combinatorics* (Naples, 1978), **109**, Quad. "Ricerca Scientifica" CNR, Rome, 1981, pp. 129–156.

[31] I. Pak, Hook length formula and geometric combinatorics, *Sém. Lothar. Combin.* **46** (2001), Article B46f. URL: http://eudml.org/doc/121696

[32] R. A. Proctor, Bruhat lattices, plane partition generating functions, and minuscule representations, *European J. Combin.* **5** (1984), no. 4, 331–350. DOI: https://doi.org/10.1016/S0195-6698(84)80037-2

[33] N. Reading, Clusters, Coxeter-Sortable Elements and Noncrossing Partitions, *Trans. Amer. Math. Soc.* **359** (2007), 5931–5958. DOI: https://doi.org/10.1090/S0002-9947-07-04264-0

[34] G. de B. Robinson, On the Representations of the Symmetric Group, *Amer. J. Math.* **60** (1938), 745–760. DOI: https://doi.org/10.2307/2371609

[35] B. E. Sagan, *The Symmetric Group: Representations, Combinatorial Algorithms, and Symmetric Functions*, 2nd ed., Graduate Texts in Mathematics 203, Springer, New York, 2013. URL: https://books.google.com/books?id=Y6vTBwAAQBAJ

[36] C. Schensted, Longest Increasing and Decreasing Subsequences, *Canad. J. Math.* **13** (1961), 179–191. DOI: https://doi.org/10.4153/CJM-1961-015-3

[37] R. Schiffler, *Quiver Representations*, Springer, 2014. DOI: https://doi.org/10.1007/978-3-319-09204-9

[38] M.-P. Schützenberger, Pour le monoïde plaxique, *Math. Inform. Sci. Humaines* **140** (1997), 5–10. URL: http://www.numdam.org/item/MSH_1997__140__5_0/

[39] R. P. Stanley, *Ordered Structures and Partitions*, Memoirs of the American Mathematical Society, No. 119, American Mathematical Society, Providence, RI, 1972.

[40] R. P. Stanley, *Enumerative Combinatorics*, Vol. 2, Cambridge University Press, 1999. URL: http://math.mit.edu/~rstan/ec/

[41] G. Viennot, Une forme géométrique de la correspondance de Robinson–Schensted, in *Combinatoire et Représentation du Groupe Symétrique*, 1977, pp. 29–58. DOI: https://doi.org/10.1007/BFb0069178