Proof of a Conjecture of Matherne, Morales, and Selover on Encodings of Unit Interval Orders

Félix Gélinas Adrien Segovia Hugh Thomas

Submitted: Jan 27, 2023; Accepted: Sep 1, 2026; Published: Nov 6, 2026
© The authors. Released under the CC BY license (International 4.0).

Abstract

There are two bijections from unit interval orders on n elements to Dyck paths from $(0,0)$ to (n,n). One is to consider the pairs of incomparable elements as forming the set of boxes between a Dyck path and the diagonal. Another is to find a particular part listing (in the sense of Guay-Paquet, Rowland and Morales) which yields an isomorphic poset, and to interpret the part listing as the area sequence of a Dyck path. Matherne, Morales, and Selover conjectured that, for any unit interval order, these two Dyck paths are related by Haglund's well-known zeta bijection. In this paper we prove their conjecture.

Mathematics Subject Classifications: 06A07

1 Introduction

Let $\mathcal{I} = \{I_1, \ldots, I_n\}$ be a set of n intervals of unit length, numbered from left to right. $U(\mathcal{I})$ is the poset on the elements 1 to n which is defined by $i \prec j$ if and only if interval I_i is strictly to the left of interval I_j.

The posets that can be described in this way are called *unit interval orders*. We write $\mathcal{U}_n$ for the collection of unit interval orders on $\{1, \ldots, n\}$.

A *Dyck path* of length n is a path which proceeds by steps of $(1,0)$ and $(0,1)$ from $(0,0)$ to (n,n), while not passing below the diagonal line from $(0,0)$ to (n,n). Dyck paths are counted by the well-known Catalan numbers. We write $\mathcal{D}_n$ for the collection of Dyck paths of length n.

A Dyck path can be specified by identifying the collection of complete 1×1 boxes with vertices at lattice points which lie between the Dyck path and the diagonal line. We refer to the collection of these boxes as the *area set* of the Dyck path. The boxes which may appear in the area set of a Dyck path in $\mathcal{D}_n$ can be indexed by pairs (i,j) with $1 \leqslant i < j \leqslant n$, where the pair (i,j) corresponds to the box $i-1 \leqslant x \leqslant i$, $j-1 \leqslant y \leqslant j$.

Université du Québec à Montréal (felixgel@yorku.ca, adrien.segovia@gmail.com, thomas.hugh_r@uqam.ca).

 https://doi.org/10.37236/11825

The *area sequence* of a Dyck path $D \in \mathcal{D}_n$ is the sequence $(a_1, \ldots, a_n)$ consisting of the number of boxes in the area set on each horizontal line, from bottom to top. The possible area sequences of Dyck paths are characterized by the fact that $a_1 = 0$ and $a_i \leqslant a_{i-1} + 1$.

For U a unit interval order, let $a(U)$ be the Dyck path whose area set is given by the boxes (i, j) such that $i < j$ and $i \not\prec j$. In Section 2, we will recall the proof that this defines a bijection from $\mathcal{U}_n$ to $\mathcal{D}_n$.

We now turn to the definition of a second map from unit interval orders to Dyck paths, as described in Section 3. Let $w = (w_1, \ldots, w_n)$ be a sequence of non-negative integers. Associated to w is a poset $P(w)$ defined as follows [3, Section 2]. (This encoding is implicit in [4, Remark 3.2 and Proposition 3.11].)

For $1 \leqslant i, j \leqslant n$, we set $i \prec j$ if either of the following is satisfied:

- $w_j - w_i \geqslant 2$,
- or $w_j - w_i = 1$ and $i < j$.

For any w, the poset $P(w)$ is isomorphic to a unique unit interval order. The word w is referred to as a *part listing* for $P(w)$. Note that the labelling of the elements of $P(w)$ and the labelling of the elements of the isomorphic unit interval order may not be the same (see Figure 2).

For a unit interval order $U \in \mathcal{U}_n$, there is a unique part listing w such that $P(w)$ is isomorphic to U and w is the area sequence of a Dyck path. Define $\tilde{p}(U)$ to be this part listing. Define $p(U)$ to be the Dyck path whose area sequence is $\tilde{p}(U)$.

We have now recalled two ways to associate a Dyck path to a unit interval order, namely the maps a and p. Considering these two maps, Matherne, Morales and Selover raised this question of how they are related. (In fact, the definition of p used in [6] is slightly different from what we have taken as the definition, and seems to include a small inaccuracy. The authors of [6] have confirmed to us that the map p which we consider is the one which they intended.)

On the basis of computer evidence, Matherne, Morales, and Selover made the following conjecture:

Conjecture 1. [6] For $U \in \mathcal{U}_n$, we have that $a(U) = \zeta(p(U))$.

The map ζ which relates $a(U)$ and $p(U)$ in the conjecture of Matherne, Morales, and Selover is the famous zeta map of Haglund [5, Proof of Theorem 3.15]. It is a bijection from $\mathcal{D}_n$ to $\mathcal{D}_n$ which plays an important role in (q, t)-Catalan combinatorics. We recall its definition in Section 4 below. This unexpected connection between ζ and unit interval orders seems worthy of further investigation.

The main result of this paper is to prove Conjecture 1 (Theorem 17) which we carry out in Section 5.

Additionally, in Section 6, we establish that the part listing $\tilde{p}(U)$ can be characterized among those part listings w with $P(w)$ isomorphic to U as the one which is minimal with respect to graded reverse lexicographic order. This is not needed for our argument, but provides an interesting alternative description of the map $\tilde{p}$ (and thus also of p).

Subsequent to the appearance of our preprint, Fang provided a different proof of Conjecture 1 in [2]. While our proof is elementary and use an inductive argument, his proof is bijective and uses bijections on trees recently considered in the study of parabolic Tamari lattices.

2 The map a

Let us give an equivalent definition of the unit interval orders.

Lemma 2. *An order $\prec$ on $\{1,\dots,n\}$ is a unit interval order if and only if for all $x, y \in \{1,\dots,n\}$, we have the two properties*

(i) $x \prec y$ implies $x < y$,

(ii) $x \prec y$ implies that for all $x' \leqslant x$ and all $y' \geqslant y$, we have $x' \prec y'$.

Proof. It is obvious from the definition that a unit interval order satisfies the two properties. We prove the converse direction by contradiction. Since it is clearly true for $n = 1$, suppose, seeking a contradiction, that $\prec$ is an order on $\{1,\dots,n\}$, with $n \geqslant 2$ minimal, that satisfies the two properties but such that it is not a unit interval order. The two properties are such that when $\prec$ is restricted to $\{1,\dots,n-1\}$, it still satisfies the two properties. By minimality of n, the order $\prec$ restricted to $\{1,\dots,n-1\}$ is a unit interval order. Let $\{I_1,\dots,I_{n-1}\}$ be a set of intervals of unit length realizing this unit interval order.

If the integer n is not comparable, with respect to $\prec$, to any i with $i < n$, then property (ii) implies that there are no relations for $\prec$. The poset $\prec$ can therefore be realized as the unit interval order corresponding to n intervals all of which intersect. This contradicts our assumption.

Otherwise let k be the greatest integer such that $k \prec n$. By property (ii), this means that the intervals I_i for $k < i < n$ (if any) all intersect. We can therefore choose I_n so that it intersects exactly these interval and no others. Using property (ii), the unit interval order defined by $\{I_1,\dots,I_n\}$ is precisely $\prec$, which is a contradiction and finishes the proof of the lemma. □

Given $U \in \mathcal{U}_n$, define

$$\tilde{a}(U) = \{(x, y) \mid x \not\prec y, 1 \leqslant x < y \leqslant n\}$$

Lemma 3. *For $U \in \mathcal{U}_n$, we have that $\tilde{a}(U)$ is the area set of a Dyck path in $\mathcal{D}_n$.*

Proof. Since $(x, y) \in \tilde{a}(U)$ implies $x < y$, we only have boxes above the diagonal. Let $(x, y) \in \tilde{a}(U)$. Then, by the converse of property (ii) from Lemma 2, for any $x \leqslant x' < y' \leqslant y$, we have that $(x', y') \in \tilde{a}(U)$. We just proved that if there is a box (x, y) in $\tilde{a}(U)$, all the boxes above the diagonal and weakly south east of (x, y) are also in $\tilde{a}(U)$. This implies that $\tilde{a}(U)$ is the area set of a Dyck path. □

Thanks to Lemma 3, for U a unit interval order, we can define $a(U)$ to be the Dyck path whose area set is given by $\tilde{a}(U)$.

Lemma 4. *The map a is a bijection from $\mathcal{U}_n$ to $\mathcal{D}_n$.*

Proof. We give the inverse map. Let $D \in \mathcal{D}_n$ be a Dyck path with area set A. Let $\prec$ be the order on $\{1, \ldots, n\}$ defined by $x \prec y$ if $x < y$ and $(x, y) \notin A$. Proving that $\prec$ is a unit interval order would finish the proof, as the map sending D to $\prec$ would be the inverse of a. Since A is the area set of a Dyck path we know that if $(x, y) \notin A$, then for any $x' \leqslant x$ and any $y' \geqslant y$, we have that $(x', y') \notin A$. The order $\prec$ is therefore a unit interval order by Lemma 2. □

3 Part listings

In this section, we study the way that unit interval orders can be defined via part listings, as already described in Section 1.

We give an algorithm which, starting from a unit interval order $U \in \mathcal{U}_n$, gives the part listing $\tilde{p}(U)$ defined in the introduction as the unique part listing whose associated poset is isomorphic to U and such that it is the area sequence of a Dyck path. This is in contrast to the approaches of [3, 4]. In [3], different part listings can yield isomorphic posets, while in [4], the non-uniqueness of the part listing is avoided by not choosing a total order on the "parts".

Given U a unit interval order in $\mathcal{U}_n$, we inductively define a function ℓ from $\{1, \ldots, n\}$ to $\mathbb{Z}_{\geqslant 0}$, as follows. We fix that $\ell(1) = 0$. We suppose that $\ell(i)$ has been defined for all $1 \leqslant i \leqslant j - 1$. Now define $\ell(j)$ by

$$\ell(j) = \begin{cases} \max_{i \prec j} \ell(i) + 1 \text{ if } j \text{ is not a minimal element of } U, \\ 0 \text{ otherwise.} \end{cases}$$

We call $\ell(i)$ the level of i (or of the interval I_i).

Algorithm 5. *Let $U \in \mathcal{U}_n$. We will successively define words q_1, q_2, …, q_n. The word q_i is of length i, and is obtained by inserting a copy of $\ell(i)$ into q_{i-1}.*

- *We begin by defining $q_1 = 0$. Now suppose that q_{i-1} has already been constructed.*
- *Let C_i be the number of elements of level $\ell(i) - 1$ comparable to i.(Note that they are necessarily to the left of i.) The letter $\ell(i)$ is added into q_{i-1} directly after the occurrences of the letter $\ell(i)$ (if any) immediately following the C_i-th letter $\ell(i) - 1$.*

Finally, define $q(U) = q_n$.

See Figure 1 for an example of this algorithm.

Lemma 6. *For $U \in \mathcal{U}_n$, the part listing $q(U)$ is the area sequence of a Dyck path in $\mathcal{D}_n$.*

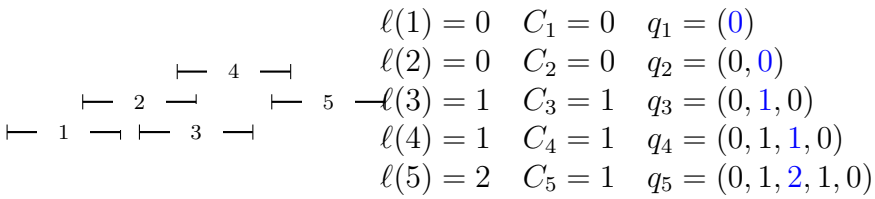


Figure 1: Example of Algorithm 5



Figure 2: On the left is the Hasse diagram of the unit interval order of Figure 1, on the right is the one obtained from the part listing $(0, 1, 2, 1, 0)$.

Proof. Recall that the area sequences of Dyck paths in $\mathcal{D}_n$ are the sequences of non-negative integers $(a_1, \ldots, a_n)$ characterized by the two properties that $a_1 = 0$ and $a_i \leqslant a_{i-1} + 1$.

Let $U \in \mathcal{U}_n$ and $q(U)$ be the sequence constructed by the above algorithm. By definition of the algorithm, 0 is the first element added to the list and will remain at the first position in $q(U)$ since every level $\ell(i)$ will be added after this 0. Thus, we have that $q(U)$ starts with a 0. Also by construction, we have that $\ell(i)$ is inserted directly after a letter which is either $\ell(i) - 1$ or $\ell(i)$. Thus, when $\ell(i)$ is inserted, it satisfies the condition that its value is at most one more than the value of its preceding letter. And since at each step we add letters greater or equal to the maximal letter of the previous word, the condition will remain true as we carry out all subsequent insertions. □

Given a part listing w of length n, we define a partial order on $\{1, \ldots, n\}$ as described in the introduction. We denote it by $P(w)$, and we write $\prec_{P(w)}$ for its order relation.

Definition 7. Given a unit interval order $U \in \mathcal{U}_n$, and its corresponding part listing $w = q(U)$, we define a permutation, called standardization, $f_U : \{1, \ldots, n\} \to \{1, \ldots, n\}$ as follows. We first consider the positions i in the part listing with $w_i = 0$, and we number them starting with 1 from left to right. We then number the positions i with $w_i = 1$ from left to right, and continue in the same way. The number that is eventually assigned to position i is $f(i)$.

We now define a new order $\prec_f$ on $\{1, \ldots, n\}$ by saying that $i \prec_f j$ if and only if $f^{-1}(i) \prec_{P(w)} f^{-1}(j)$. By definition, this poset is isomorphic to $P(w)$.

Proposition 8. *For U a unit interval order, $w = q(U)$ the corresponding part listing, and f the standardization, $\prec_f$ agrees with the original order on U.*

Proof. Note that ℓ is a weakly increasing function of $\{1, \ldots, n\}$. Let $1 \leqslant i \leqslant n$. Because the copies of ℓ in w are inserted from left to right, $f^{-1}(i)$ is the position in w where $\ell(i)$ wound up, that is to say, $w_{f^{-1}(i)} = \ell(i)$.

Let $1 \leqslant i < j \leqslant n$. Suppose I_i is strictly to the left of I_j in U. Thus, $\ell(j) > \ell(i)$. Suppose first that $\ell(j) \geqslant \ell(i)+2$. In this case, $w_{f^{-1}(j)} \geqslant w_{f^{-1}(i)}+2$, so $j \succ_f i$, as desired.

Suppose now that $\ell(j) = \ell(i)+1$. By assumption, I_j is strictly to the right of I_i. Thus, when $\ell(j)$ was inserted into w, it was inserted to the right of the letter $\ell(i)$; this persists as other letters are inserted. It follows that also in this case, $j \succ_f i$, as desired.

Suppose now that I_i and I_j overlap. In this case, $\ell(j) \leqslant \ell(i)+1$, because the I_k strictly to the left of I_j are weakly to the left of I_i, so have level at most $\ell(i)$. We must therefore consider two cases, when $\ell(j) = \ell(i)$ and when $\ell(j) = \ell(i)+1$.

Consider first the case where $\ell(j) = \ell(i)$. In this case, $w_{f^{-1}(j)} = w_{f^{-1}(i)}$, so i and j are incomparable with respect to $\prec_f$, as desired.

Suppose now that $\ell(j) = \ell(i)+1$. We know that in w the letter $\ell(j)$ is inserted directly to the right of the letters of value $\ell(j)-1$ that correspond to the intervals I_k that are strictly to the left of I_j and are of level $\ell(j)-1$. The interval I_i has level $\ell(j)-1$, but by assumption it overlaps with I_j, thus at the moment we insert the letter $\ell(j)$ it is inserted to the left of the letter $\ell(i)$. As other letters are inserted to obtain w, the letter $\ell(j)$ will remain to the left of the letter $\ell(i)$. Thus $f^{-1}(j) < f^{-1}(i)$, and since $w_{f^{-1}(j)} = \ell(j)$ and $w_{f^{-1}(i)} = \ell(j)-1$, this proves that i and j are incomparable with respect to $\prec_f$, as desired. □

In the introduction, for a unit interval order U, we defined $\tilde{p}(U)$ to be the unique part listing w such that $P(w)$ is isomorphic to U and w is also an area sequence of a Dyck path. We are now in a position to establish that the map $\tilde{p}$ is well-defined.

Proposition 9. *For a unit interval order U, we have that $\tilde{p}(U)$ is well-defined and $\tilde{p}(U) = q(U)$.*

Proof. For each unit interval order $U \in \mathcal{U}_n$, the part listing $q(U)$ is the area sequence of a Dyck path by Lemma 6. By Proposition 8, the poset $P(q(U))$ is isomorphic to U. This shows in particular that the map q must be injective. Since there are the same number of unit interval orders in $\mathcal{U}_n$ as of Dyck paths in $\mathcal{D}_n$, q must be a bijection. Thus, Proposition 8 tells us that if w and w' are two different area sequences of Dyck paths, then $P(w)$ and $P(w')$ cannot be isomorphic. It follows that, for any U, there is exactly one Dyck path w such that $P(w)$ is isomorphic to U, namely, $q(U)$. Thus, $\tilde{p}(U)$ is well-defined and equals $q(U)$. □

4 The zeta map

In this section we describe the map $\zeta : \mathcal{D}_n \to \mathcal{D}_n$ (see [5, proof of Theorem 3.15]). While it is often associated with Haglund's name, in fact, the inverse of the map appeared earlier in [1].

Algorithm 10. *Start with $D \in \mathcal{D}_n$.*

- *We begin by labelling the lattice points that make up the path D (except the very first): we label the top end-point of an up step with the letter a, and we label the right endpoint of a right step with the letter b.*

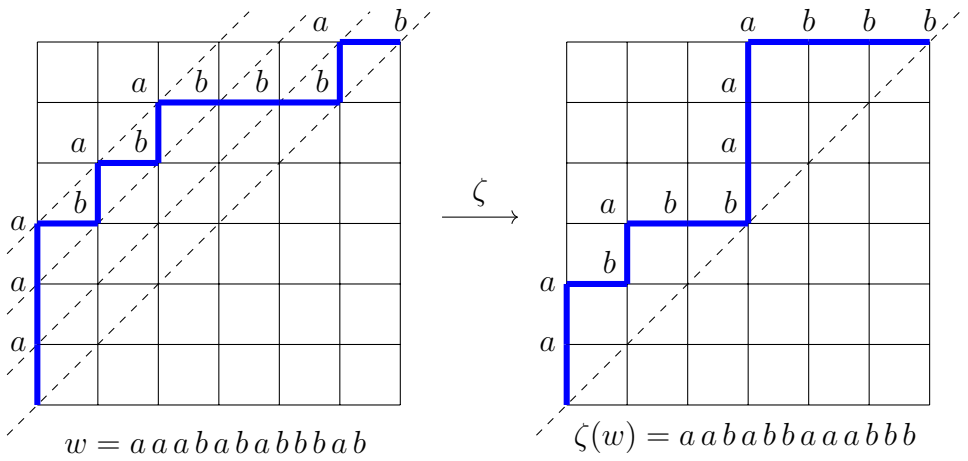


Figure 3: Example of the zeta map

- *Read the labels: first on the line $y = x$, from bottom left to top right, then on the line $y = x + 1$, again in the same direction, then on the line $y = x + 2$, etc.*
- *Interpret b as designating an up step, and a as designating a right step. (Note that this means that up steps in the Dyck path contribute right steps in the result, and vice versa.)*

This defines a lattice path from $(0, 0)$ to (n, n). Define this to be $\zeta(D)$.

See Figure 3 for an example of this map. The following lemma is well-known, but we include a proof for completeness.

Lemma 11. *Starting from $D \in \mathcal{D}_n$, the path $\zeta(D)$ is a Dyck path.*

Proof. Let $D \in \mathcal{D}_n$. We define a matching between the up steps and the right steps of D as follows. For all non-negative integers t, look at the part of D between the lines $y = x + t$ and $y = x + (t + 1)$. This necessarily consists of an alternating sequence of the same number of up steps and right steps. We define our pairing by matching the i-th up step with the i-th right step. By definition of ζ, these two matched edges contribute an up step and a right step to $\zeta(D)$, and the up step comes before the right step. Thus $\zeta(D)$ will always stay above the diagonal. □

Remark 12. Let $D \in \mathcal{D}_n$. By Proposition 9 there exists a unit interval order U such that $D = \tilde{p}(U)$. The intervals of U are obtained by pairing of the up steps and right steps of D as explained in the proof of Lemma 11.

5 Proof of the conjecture

The proof of the conjecture (Theorem 17) will proceed by induction. We suppose that for a unit interval order U, we know that $\zeta(p(U))$ and $a(U)$ coincide. We then consider

what happens when we add a new rightmost interval to U. Proving that this changes the result of applying each of the maps in the same way, we conclude that the two maps also coincide on the larger poset. This proves the conjecture by induction.

Definition 13. A peak in a Dyck path consists of an up step followed by a right step. In the Dyck word, this amounts to an occurrence of the consecutive pair of letters 'ab' and in the corresponding area sequence $(a_1, \dots, a_n)$ this amounts to having $a_i \leqslant a_{i-1}$ for a given i.

By adding a peak to a Dyck path, we mean the insertion, at some position, of 'ab' into the Dyck path. The result of adding a peak is again a Dyck path. Note that adding a peak does not necessarily increase the number of peaks: if the peak is added after a letter 'a' and before a letter 'b', the number of peaks does not change.

We say that a peak of a Dyck path is a maximal peak if the top of the up step lies on the highest line of slope 1 that touches the Dyck path. We will refer to the last peak of the Dyck path as the final peak, and to the last maximal peak as the final maximal peak.

We say that a peak is of height i if the top of the up step is on the line $y = x + i$.

In the sequel, let $U \in \mathcal{U}_n$ and U' be obtained from U by adding an $(n+1)$-st interval of unit length to the right of those of U (which possibly overlap with some intervals of U). We note ℓ the level of this added interval.

For an illustration of the following lemmas, see Figure 4.

Lemma 14. *Let U be a unit interval order, and let U' be obtained from it by adding an interval to the right of the intervals of U. The Dyck path $p(U')$ is obtained from $p(U)$ by adding a final maximal peak.*

Proof. By Algorithm 5, in going from $p(U)$ to $p(U')$, we insert a letter ℓ into $p(U)$ such that the letter before is either ℓ or $\ell - 1$. The letters before are at most ℓ and the letters after are at most $\ell - 1$. Adding a letter ℓ in this way adds a final maximal peak, since we add the rightmost maximal letter ℓ to the area sequence $p(U)$. □

Lemma 15. *Let $U \in \mathcal{U}_n$, and let U' be obtained from it by adding an interval to the right of the intervals of U. The Dyck path $\zeta(p(U'))$ is obtained from $\zeta(p(U))$ by adding a final peak in position $(n - r, n + 1)$, where r is the sum of the number of occurrences of the letter ℓ in $p(U)$ and of the number of occurrences of the letter $\ell - 1$ appearing after the position of the added letter ℓ in $p(U')$.*

Proof. By Lemma 14, we know that $p(U')$ is obtained from $p(U)$ by adding a final maximal peak. The height of this peak is $\ell+1$ since we added ℓ in the area sequence $p(U)$ to obtain this peak. The fact that this is the final maximal peak means that the peaks after are of heights smaller than $\ell + 1$.

Let us recall that the map ζ builds a Dyck path by scanning along the lines of slope 1 from bottom left to top right, by order of increasing height. Adding the peak of height $\ell + 1$ does not change what we read on any height below ℓ, nor what we read on height ℓ before we reach the right endpoint of the right step of the added peak. When we reach

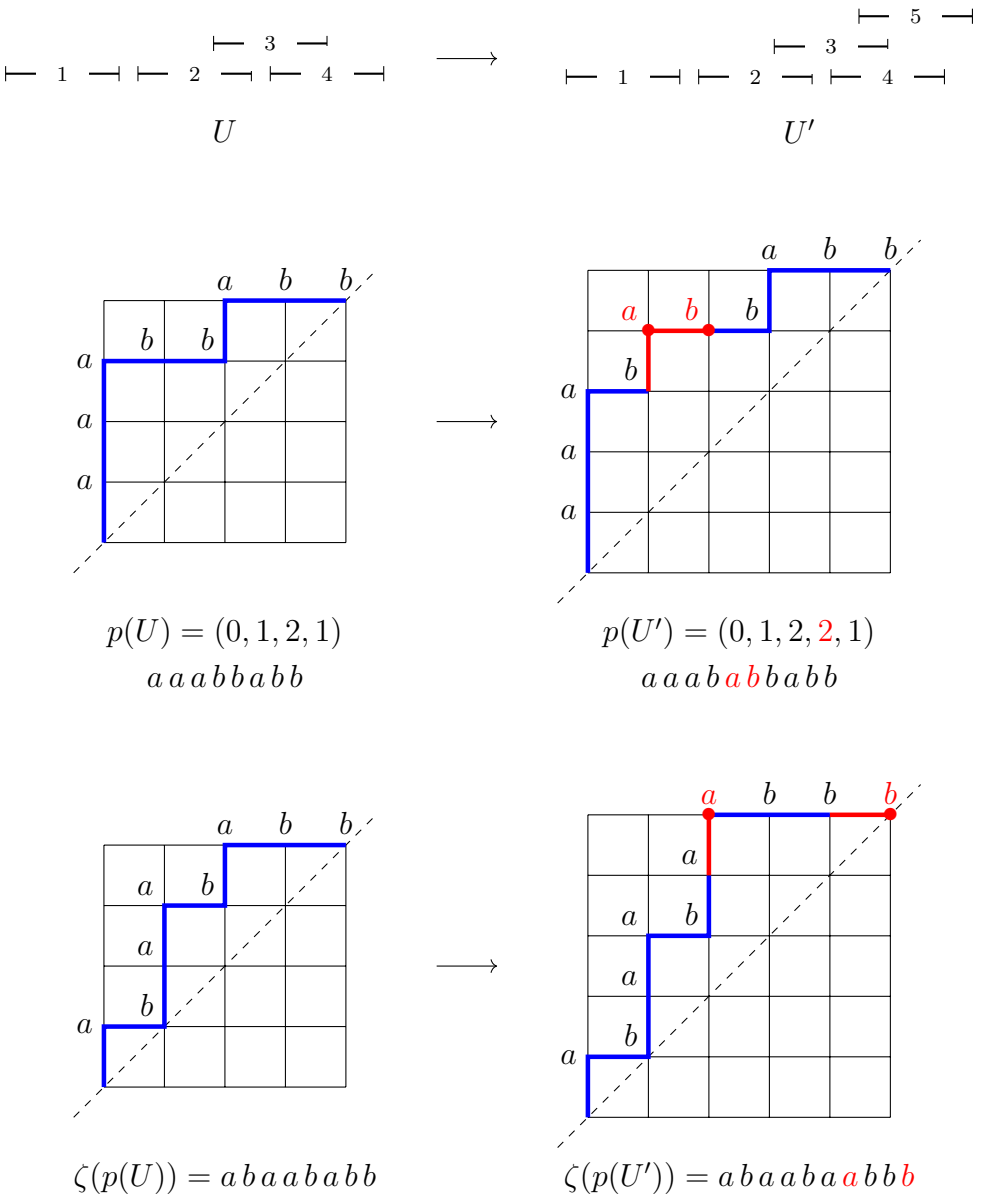


Figure 4: Adding of an interval of level 2 that creates a new peak in p.

this right endpoint, in $\zeta(p(U'))$ we put an up step 'a'. Then on the same height ℓ we read top endpoints of peaks of height ℓ appearing after the added peak (if any). Such peaks correspond to the letters $\ell-1$ appearing after the position of the added letter ℓ in $p(U')$, which will give right steps 'b' in $\zeta(p(U'))$. Finally, we read the line of height $\ell+1$, putting a right step 'b' in $\zeta(p(U'))$ for all maximal peaks of $p(U')$, which correspond to the occurrences of the letter ℓ in $p(U')$. Thus after the last up step 'a' of $\zeta(p(U'))$ we have exactly $r+1$ right steps 'b', so the added peak is in position $(n-r, n+1)$. □

Lemma 16. *The Dyck path $a(U')$ is obtained from $a(U)$ by adding a final peak in position $(n-s, n+1)$, where s is the number of intervals in U' not comparable to the rightmost interval I_{n+1}.*

Proof. Let us recall that by definition $a(U)$ is the Dyck path whose area set is given by the boxes (i, j) such that $i < j$ and $i \not\prec j$. We then add I_{n+1}, which is not comparable to the last s intervals in U (those of level ℓ together with a subset of those of level $\ell-1$). Thus in U' we have only the new non comparable relations $i \not\prec n+1$ for $i \in \{n-s+1, \ldots, n\}$, giving the corresponding boxes $(i, n+1)$ in the area set. This proves the lemma. □

We can now prove the main result of the paper.

Theorem 17. *The maps $\zeta \circ p$ and a coincide.*

Proof. We proceed by induction. We know that $a(U) = \zeta \circ p(U)$ is true for the unique $U \in \mathcal{U}_1$, thus the initial case is verified.

Let $n \geqslant 1$. Suppose that $a(U) = \zeta \circ p(U)$ is true for all $U \in \mathcal{U}_n$. Let $U' \in \mathcal{U}_{n+1}$. There exists a unit interval order poset $U \in \mathcal{U}_n$ such that U' is obtained from U by adding an $(n+1)$-st interval I_{n+1} of level ℓ to the right of those of U.

We now establish that the number r in Lemma 15 is equal to the number s in Lemma 16. Indeed, in U' the intervals not comparable to I_{n+1} are all the intervals of level ℓ in U and a subset of those of level $\ell-1$. The latter are exactly those which correspond to the occurrences of the letter $\ell-1$ appearing after the position of the added ℓ in $p(U')$.

Then using Lemma 15 and Lemma 16, and the induction hypothesis that gives $a(U) = \zeta(p(U))$, we obtain that $a(U') = \zeta(p(U'))$, thereby finishing the induction and the proof of the theorem. □

6 Graded reverse lexicographic minimality of $\tilde{p}(U)$

In this section, we establish that the part listing $\tilde{p}(U)$ can be characterized among those part listings w with $P(w)$ isomorphic to U as the one which is minimal with respect to graded reverse lexicographic order. It provides an alternative description of the map $\tilde{p}$ (and thus also of p).

We define the graded reverse lexicographic order on finite sequences of n non-negative integers as follows. We say that $(a_1, \ldots, a_n) < (b_1, \ldots, b_n)$ if $\sum_i a_i < \sum_i b_i$, or in the case that the sums are equal if $a_j > b_j$, where j is the index of the first position where the two strings differ. (Note that the inequality $a_j > b_j$ is reversed from what one might expect! The expected inequality, $a_j < b_j$, defines graded lexicographic order.)

Lemma 18. *Let U be a unit interval order. The graded reverse lexicographically minimal part listing for U is the area sequence of a Dyck path.*

Proof. Let $(a_1, \ldots, a_n)$ be the graded reverse lexicographically minimal part listing for U. Suppose, seeking a contradiction, that $a_i > a_{i-1} + 1$. Consider the part listing in which a_i and a_{i-1} have swapped positions. This part listing defines an isomorphic poset, and the new part listing is lower in graded reverse lexicographic order, so we would have preferred it. This is a contradiction.

Now suppose that $a_1 > 0$. Consider the part listing in which a_1 is removed and $a_1 - 1$ is inserted at the end. This part listing produces an isomorphic poset, and since its sum is lower, it is lower in graded reverse lexicographic order, so we would have preferred it. This, too, is a contradiction.

It follows that the graded reverse lexicographically minimal part listing for U is the area sequence of a Dyck path. □

Corollary 19. *For U a unit interval order, the part listing $\tilde{p}(U)$ is the graded reverse lexicographically minimal part listing for U.*

Proof. By Lemma 18, the graded reverse lexicographically minimal part listing for U is the area sequence of some Dyck path. We know that $\tilde{p}$ defines a bijection from unit interval orders to area sequences of Dyck paths. Thus, there is at most one area sequence of a Dyck path which, when interpreted as a part listing, yields a poset isomorphic to U. The graded reverse lexicographically minimal part listing for U is therefore $\tilde{p}(U)$. □

Acknowledgements

The authors would like to thank Mathieu Guay-Paquet, Alejandro Morales, and Viviane Pons for helpful discussions. We thank the referee for their valuable suggestions.

The authors benefitted from the support of the NSERC Discovery Grants program (RGPIN-2022-03960) and the Canada Research Chairs program (CRC-2021-00120). F.G. was supported by an NSERC USRA.

References

[1] G. Andrews, C. Krattenthaler, L. Orsina, and P. Papi, ad-nilpotent $\mathfrak{b}$-ideals in $\mathrm{sl}(n)$ having a fixed class of nilpotence: combinatorics and enumeration. *Trans. Amer. Math. Soc.* **354** (10), 3835–3853, 2002.

[2] W. Fang, Bijective proof of a conjecture on unit interval posets. *Discrete Mathematics & Theoretical Computer Science* **26** (2), #1, 2024.

[3] M. Guay-Paquet, A modular relation for the chromatic symmetric functions of (3+1)-free posets. arXiv:1306.2400.

[4] M. Guay-Paquet, A. Morales, and A.H. Rowland, Structure and enumeration of $(3+1)$-free posets. *Annals of Combinatorics* **18** (4), 645–674, 2014.

[5] J. Haglund, The q,t-Catalan numbers and the space of diagonal harmonics, with an appendix on the combinatorics of Macdonald polynomials. University Lecture Series **41**. American Mathematical Society, Providence, RI. 2008.

[6] J. Matherne, A. Morales, and J. Selover, The Newton polytope and Lorentzian property of chromatic symmetric functions. *Selecta Mathematica* **30** (3), #42, 2024.