Stability for hyperplane covers

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Abstract

An almost k-cover of the hypercube $Q^n = \{0,1\}^n$ is a collection of hyperplanes that avoids the origin and covers every other vertex at least k times. When k is large with respect to the dimension n, Clifton and Huang asymptotically determined the minimum possible size of an almost k-cover. Central to their proof was an extension of the LYM inequality, concerning a weighting of hyperplanes.

In this paper we completely characterise the hyperplanes of maximum weight, showing that there are $\binom{2n-1}{n}$ such hyperplanes. We further provide stability, bounding the weight of all hyperplanes that are not of maximum weight. These results allow us to effectively shrink the search space when using integer linear programming to construct small covers, and as a result we are able to determine the exact minimum size of an almost k-cover of Q^6 for most values of k. We further use the stability result to improve the Clifton–Huang lower bound for infinitely many choices of k in every sufficiently large dimension n.

Mathematics Subject Classifications: 52C17, 05B40

1 Introduction

While it is clear that the hypercube $Q^n = \{0,1\}^n$ can be covered by two hyperplanes, a classic and surprising result shows that if we have to avoid the origin, then n hyperplanes are needed to cover the remaining points. Lying at the intersection of finite geometry and extremal combinatorics, this problem and its variations have been studied by several researchers over the decades. In the finite geometry setting, such a hyperplane cover is closely related to the notion of blocking sets, and research in this direction was pioneered in the late 1970s by Jamison [7]. Meanwhile in the extremal context, this problem was first studied by Alon and Füredi [1], who resolved a problem of Komjáth [8] from Ramsey

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Theory. The Alon–Füredi Theorem was a precursor to the hugely influential Combinatorial Nullstellensatz, and indeed, this problem has proven a valuable test case in the development of the algebraic method. For a more thorough survey of the history of this problem, we refer the reader to [3].

1.1 Covering with multiplicities

In recent years, renewed interest in this hyperplane covering problem was sparked by the work of Clifton and Huang [6]. In this paper, they studied the multiplicity version of the problem. Given $k \in \mathbb{N}$, they defined an *almost k-cover* of Q^n to be a set of hyperplanes that avoids the origin while covering all other points of Q^n at least k times. They were interested in the minimum size of an almost k-cover, a quantity they denoted by $f(n,k)$. Note that the Alon–Füredi Theorem shows $f(n,1) = n$, and folklore has that $f(n,2) = n+1$ (also see [2]).

Clifton and Huang showed that the extremal function $f(n,k)$ exhibits different behaviour depending on the relative sizes of the two parameters. When n is large compared to k, they used algebraic methods to obtain lower bounds on $f(n,k)$, showing that $f(n,3) = n+3$ for all $n \geqslant 2$, and $f(n,k) \geqslant n+k+1$ whenever $k \geqslant 4$ and $n \geqslant 3$. In a subsequent paper, Sauermann and Wigderson [11] solved the algebraic version of this problem, showing any polynomial that vanishes to degree k at all points of Q^n but the origin must have degree at least $n+2k-3$ for $k \geqslant 2$ and $n \geqslant 2k-3$. This in particular improves the lower bound to $f(n,k) \geqslant n+2k-3$, but falls short of Clifton and Huang's upper bound of $f(n,k) \leqslant n + \binom{k}{2}$, which they conjecture to be the truth for all $k \in \mathbb{N}$ and n sufficiently large with respect to k.

In this paper, however, we will be interested in the other regime, where k is large with respect to n. In this range, Clifton and Huang [6] determined asymptotically the size of the smallest almost k-covers, showing $f(n,k) = (H_n + o(1))\,k$, where $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ is the nth Harmonic number. To prove this result, they solved the linear programming relaxation of the integer linear program that represents the hyperplane covering problem. In particular, to prove the lower bound, they first defined a weighting of the points of Q^n.

Definition 1. Let x be a nonzero point in Q^n with exactly t coordinates equal to 1. The *weight* of x is defined as

$$w(x) = \frac{1}{t\binom{n}{t}}.$$

More generally, given any set $S \subseteq \mathbb{R}^n$, we define the *weight* of S to be the sum of the weights of all nonzero points in $S \cap Q^n$; that is,

$$w(S) = \sum_{p \in S \cap Q^n \setminus \{\vec{0}\}} w(p). \tag{1}$$

Given this definition, the crucial step was the following theorem.

Theorem 2 ([6], Theorem 1.3)**.** *If h is a hyperplane avoiding the origin, then $w(h) \leqslant 1$.*

To see how this implies the lower bound, observe that the total weight of the hypercube minus the origin is H_n. Since every point is covered at least k times, it follows that the total weight of the hyperplanes in an almost k-cover must be at least $H_n k$. Since, by Theorem 2, each hyperplane can have weight at most 1, we must have

$$f(n,k) \geqslant \lceil H_n k \rceil. \tag{2}$$

There are a couple of key remarks to be made concerning this theorem. First, Clifton and Huang observed that the bound is tight; any hyperplane $h = \{x : \sum_{i=1}^n c_i x_i = 1\}$ where the coefficients c_i are all either 0 or 1 satisfies $w(h) = 1$. In particular, this gives exponentially many hyperplanes of maximum weight.

Next, Clifton and Huang observed that if the coefficients c_i are all positive, then the points covered by the hyperplane are the characteristic vectors of the sets in an antichain. The bound then follows immediately from the famous Lubell–Yamamoto–Meshalkin inequality [4, 9, 10, 12]. Thus, Theorem 2 can be viewed as a generalisation of the LYM inequality.

1.2 Our results

In this paper, we prove stability for Theorem 2, characterising all hyperplanes of maximum weight, and improving the bound on the weight of all other hyperplanes.

Theorem 3. *Let $h = \{x : \sum_{i=1}^n c_i x_i = 1\}$ be an affine hyperplane in $\mathbb{R}^n$ that does not pass through the origin. If the coefficients satisfy*

(i) $c_i \leqslant 1$ *for all* i,

(ii) $\sum_{i=1}^n c_i \geqslant 1$, *and*

(iii) $c_i \in \mathbb{Z}$ *for all* i,

then $w(h) = 1$. *Otherwise,* $w(h) \leqslant 1 - \frac{1}{n}$.

Note that the upper bound for the non-maximum-weight hyperplanes is best possible. Indeed, the hyperplane $h_\alpha = \{x : \alpha x_1 + \sum_{i=2}^n x_i = 1\}$ has $w(h_\alpha) = 1 - \frac{1}{n}$ whenever $\alpha \notin \{-(n-2), -(n-3), \ldots, -1, 0, 1\}$.

Our characterisation allows us to enumerate the hyperplanes of maximum weight.

Corollary 4. *The number of weight-*1 *hyperplanes in $\mathbb{R}^n$ is $\binom{2n-1}{n}$.*

Proof. We enumerate the weight-1 hyperplanes by establishing a bijection from the set $B_n \subseteq \{-1, +1\}^{2n-1}$ of all sequences $(b_1, \ldots, b_{2n-1})$ with exactly n positive entries. For a sequence $(b_1, \ldots, b_{2n-1})$ in B_n, let i_ℓ be the indices such that $b_{i_\ell} = +1$ for $\ell = 1, 2, \ldots, n$; we arrange these indices so that $i_1 < i_2 < \cdots < i_n$, and we set $i_0 = 0$ for later usage. Now we define a mapping φ from B_n to $\mathbb{Z}^n$ as follows: for $b = (b_1, \ldots, b_{2n-1}) \in B_n$, let $\varphi(b) = (c_1, \ldots, c_n)$, where the coefficients c_ℓ are given by

$$c_\ell = \sum_{j=i_{\ell-1}+1}^{i_\ell} b_j, \qquad \ell = 1, 2, \ldots, n.$$

For instance, $\varphi(+1,+1,-1,-1,+1,-1,+1,+1,-1) = (1,1,-1,0,1)$.

It is clear that the image of φ is exactly those vectors $(c_1,\ldots,c_n)$ that satisfy the conditions (i), (ii), and (iii) in Theorem 3, and the inverse mapping φ^{-1} can be easily defined: simply expand every coefficient c_ℓ into a sequence of $1-c_\ell$ terms of -1 followed by a $+1$, and append s terms of -1 to the end, where $s = \sum_{i=1}^{n} c_i - 1$. This shows that φ is a bijection, and the number of weight-1 hyperplanes in $\mathbb{R}^n$ is the cardinality of B_n. □

The characterisation of maximum weight hyperplanes is also useful for determining precise values of the extremal function $f(n,k)$. As an example of an application of our result, we will prove the following, extending results of Clifton and Huang (who determined $f(5,k)$ for all $k \geqslant 15$).

Theorem 5. *For $k \geqslant 16$, we have*

$$f(6,k) = \left\lceil \frac{49k}{20} \right\rceil$$

whenever $k \not\equiv 11 \pmod{20}$ and $k \notin \{22,42,62,82,102\}$.

Notation and organisation. Throughout our proofs, we will be dealing with sequences $(c_1,c_2,\ldots,c_n)$ of coefficients that define the hyperplane given by the equation $c_1x_1 + c_2x_2 + \ldots + c_nx_n = 1$, together with permutations $\pi \in S_n$ of the n indices. For convenience, we will often identify the indices $i \in [n]$ with the corresponding coefficients c_i. In particular, when we have a set $S = \{\pi(1),\pi(2),\ldots,\pi(|S|)\} \subseteq [n]$ given by an initial segment of π, by the *sum* of S we mean $\sum_{i=1}^{|S|} c_{\pi(i)}$. Furthermore, for some $1 \leqslant t \leqslant |S|$, we call $(\pi(1),\pi(2),\ldots,\pi(t))$ and $(\pi(t),\pi(t+1),\ldots,\pi(|S|))$ an initial interval and a tail interval of S respectively, and refer to their sums as an initial sum and a tail sum.

In Section 2, we will review Clifton and Huang's proof of Theorem 2, introducing terminology that we will need in our own proofs. In Section 3, we use this framework to prove Theorem 3, and prove some further stability. In Section 4, we combine our stability result with the linear programming approach to prove Theorem 5, and also show that the lower bound (2) is not tight infinitely often. Finally, in Section 5, we provide some concluding remarks and outline possible directions for further research.

2 The good, bad, and the redundant

In this section we will review the proof of Theorem 2, primarily with the aim of establishing some terminology and notation that we shall use in our own proofs in the next section.

Let h be the affine hyperplane in $\mathbb{R}^n$ defined by the equation $\sum_{i=1}^{n} c_ix_i = 1$. We wish to show that $w(h) \leqslant 1$; that is, the sum of the weights of the points of Q^n contained in h is at most 1. For this, it is useful to identify a point $x \in Q^n$ with the subset $S \subseteq [n]$ whose characteristic vector it is; that is, $\{i : x_i = 1\}$. The idea behind the proof is to associate each set S covered by the hyperplane h with a disjoint set of permutations in S_n, whose size is proportional to the weight of x. Since the total number of permutations is bounded by $n!$, this will in turn bound the weight of the hyperplane h.

Definition 6. Given some set $S \subseteq [n]$ such that $\sum_{i\in S} c_i = 1$, we say a permutation $\pi \in S_n$ *yields* S provided that

(i) $\{\pi(1), \pi(2), \ldots, \pi(|S|)\} = S$, and

(ii) $\displaystyle\sum_{i=1}^{\ell} c_{\pi(i)} \begin{cases} < 1 & \text{if } \ell < |S|, \\ = 1 & \text{if } \ell = |S|, \end{cases}$ where $1 \leqslant \ell \leqslant |S|$.

From the definition, it is clear that each $\pi \in S_n$ can yield at most one subset $S \subseteq [n]$. We call a permutation $\pi \in S_n$ *bad* if it does not yield any set $S \subseteq [n]$, and define $\mathcal{B}$ to be the collection of all bad permutations in S_n.

Clifton and Huang [6] proved that, for every set $S \subseteq [n]$ such that $\sum_{i\in S} c_i = 1$, and for every cyclic permutation σ of S, there is at least one starting point in S such that if we unfold σ to a linear permutation π_σ of S, then any permutation $\pi \in S_n$ that has π_σ as an initial segment yields S. In the case where σ admits two or more such starting points, we call σ *turnable*, and from the available options, we choose π_σ such that the initial entry $\pi_\sigma(1)$ is the largest. Finally, we say a linear permutation $\pi \in S_n$ is *switchable* if it yields a set S, and the cyclic permutation σ of S that it induces is turnable. That is to say, we can rotate the first $|S|$ elements of π to obtain another permutation that also yields S.

Now, consider a permutation $\pi \in S_n$. If π is not bad, there is some set S that π yields. Let $\sigma = \langle \pi(1), \pi(2), \ldots, \pi(|S|)\rangle$ be the cyclic permutation of S induced by $\pi|_{[|S|]}$. If $\pi|_{[|S|]} = \pi_\sigma$, we say that π is *good*, and otherwise we say that π is *redundant*. We let $\mathcal{G} \subseteq S_n$ be the set of good permutations, and $\mathcal{R} \subseteq S_n$ the set of redundant permutations. Note that this gives a partition $S_n = \mathcal{G} \cup \mathcal{B} \cup \mathcal{R}$ of the set of the permutations of $[n]$ into the subsets of good, bad, and redundant permutations, whence

$$n! = |\mathcal{G}| + |\mathcal{B}| + |\mathcal{R}|. \tag{3}$$

Given a set S covered by h, note that there are $(|S|-1)!$ cyclic permutations σ of S, each of which gives $(n-|S|)!$ good permutations (which must start with π_σ). Hence, abusing notation to write $S \in h$ to indicate that S is covered by h,

$$|\mathcal{G}| = \sum_{S\in h} (|S|-1)!(n-|S|)!,$$

or

$$\frac{|\mathcal{G}|}{n!} = \sum_{S\in h} \frac{(|S|-1)!(n-|S|)!}{n!} = \sum_{S\in h} \frac{1}{|S|\binom{n}{|S|}} = \sum_{x\in h} w(x) = w(h).$$

Dividing (3) by $n!$ and rearranging then yields

$$w(h) = \frac{|\mathcal{G}|}{n!} = 1 - \frac{|\mathcal{B}|}{n!} - \frac{|\mathcal{R}|}{n!} \leqslant 1. \tag{4}$$

This proves Theorem 2. Moreover, it outlines how one can establish stability — to prove that non-maximum-weight hyperplanes have weight bounded away from 1, we need to show that they admit many bad or redundant permutations.

3 Weights of hyperplanes

We will now use the framework set up in the previous section to prove our main result, Theorem 3. Following that, we shall prove some further stability, providing a partial characterisation of all hyperplanes of large weight.

3.1 The proof of Theorem 3

We first prove that if $h = \{x : \sum_{i=1}^n c_i x_i = 1\}$, where the coefficients c_i satisfy the three conditions in Theorem 3, then $w(h) = 1$. Per the discussion in Section 2, this amounts to showing that every permutation in S_n is good, i.e., $\mathcal{G} = S_n$.

For any permutation $\pi \in S_n$, let t be the smallest index such that

$$\sum_{j=1}^{t} c_{\pi(j)} = 1.$$

Note that the conditions (i), (ii) and (iii) guarantee that such an index exists, and moreover that $\sum_{j=1}^{\ell} c_{\pi(j)} < 1$ for all $1 \leqslant \ell \leqslant t-1$. Hence, π yields the subset $S = \{\pi(1), \ldots, \pi(t)\}$.

Now consider the cyclic permutation $\sigma = \langle \pi(1), \ldots, \pi(t) \rangle$ of S. We argue that $\pi(1)$ is the unique starting point for σ, that is, $\pi|_{[t]} = \pi_\sigma$. There is nothing to prove if $t = 1$, so we may assume $t > 1$ and $\pi(s)$ is another starting point of σ with $1 < s \leqslant t$. Then $\sum_{i=s}^{t} c_{\pi(i)} < 1$. Since all the coefficients c_j's are integers, this implies that $\sum_{i=s}^{t} c_{\pi(i)} \leqslant 0$. But this leads to

$$\sum_{i=1}^{s-1} c_{\pi(i)} = \sum_{i=1}^{t} c_{\pi(i)} - \sum_{i=s}^{t} c_{\pi(i)} \geqslant 1 - 0 = 1,$$

which contradicts our choice of t. Hence we have $\pi|_{[t]} = \pi_\sigma$, and so π is good. As π was an arbitrary permutation, it follows that $\mathcal{G} = S_n$, as required.

To complete the proof of Theorem 3, we need to show that if $w(h) < 1$, then $w(h) \leqslant 1 - \frac{1}{n}$. Note that by what we have just shown, we know at least one of the following three cases must occur:

(1) $c_i > 1$ for some $i \in [n]$,

(2) $\sum_{i=1}^n c_i < 1$, or

(3) $c_i \notin \mathbb{Z}$ for some $i \in [n]$.

We treat each of these cases in turn.

Case (1) $c_i > 1$ for some $i \in [n]$.

Observe that any permutation π with $\pi(1) = i$ is bad, since if a permutation yields a set $S \subseteq [n]$, its first $|S|$ partial sums are at most 1. Thus, $|\mathcal{B}| \geqslant (n-1)!$, and so by (4) we have $w(h) \leqslant 1 - \frac{1}{n}$.

Case (2) $\sum_{i=1}^{n} c_i < 1$.

We will show that each cyclic permutation σ of $[n]$ admits a starting point for which the resulting permutation $\pi \in S_n$ is bad. Then $|\mathcal{B}| \geqslant (n-1)!$, which once again implies $w(h) \leqslant 1 - \frac{1}{n}$.

Let π' be the permutation we obtain from σ by starting at 1. For $1 \leqslant t \leqslant n$, define $a_t = \sum_{i=1}^{t} c_{\pi'(i)}$. Let $t_0 \in [n]$ be any index such that $a_{t_0} \geqslant a_t$ for all $t \in [n]$. In particular,

$$a_{t_0} \geqslant a_n = \sum_{i=1}^{n} c_{\pi'(i)} = \sum_{i=1}^{n} c_i.$$

If $t_0 = n$, then $a_t \leqslant a_{t_0} = a_n < 1$ for each $t \in [n]$, which means that π' is already bad. Otherwise, take π to be the permutation obtained by starting σ at $t_0 + 1$. We shall show that π is bad. If not, then it yields some set $S \subseteq [n]$, and so there is some j with $\sum_{i=1}^{j} c_{\pi(i)} = 1$. If $j \leqslant n - t_0$ then

$$\sum_{i=1}^{j} c_{\pi(i)} = \sum_{i=t_0+1}^{t_0+j} c_{\pi'(i)} = a_{t_0+j} - a_{t_0} \leqslant 0,$$

by the choice of t_0. Otherwise, if $n - t_0 + 1 \leqslant j \leqslant n$, then

$$\sum_{i=1}^{j} c_{\pi(i)} = \sum_{i=t_0+1}^{n} c_{\pi'(i)} + \sum_{i=1}^{j-n+t_0} c_{\pi'(i)} = a_n - a_{t_0} + a_{j-n+t_0} \leqslant a_n < 1$$

where the penultimate inequality holds by $a_{t_0} \geqslant a_{j-n+t_0}$ and the last inequality follows by assumption. Hence, there is no such j, and π must be bad, completing this case.

Case (3) $c_i \notin \mathbb{Z}$ for some $i \in [n]$.

If c_i is the unique non-integral coefficient, then any permutation with $\pi(1) = i$ is bad, as all its partial sums will be non-integral. Thus, as in Case (1), we will have $w(h) \leqslant 1 - \frac{1}{n}$. Hence we may assume there are $r \geqslant 2$ non-integral coefficients, which we may assume (without loss of generality) to be $c_1, c_2, \ldots, c_r$. In this case, we shall show that for every cyclic permutation of $[n]$ there is a starting point for which the corresponding permutation $\pi \in S_n$ is either bad or redundant. Then we have $|\mathcal{B}| + |\mathcal{R}| \geqslant (n-1)!$, and so it again follows from (4) that $w(h) \leqslant 1 - \frac{1}{n}$.

Let σ' be a cyclic permutation of $[n]$, and partition it into the cyclic intervals I_1, I_2, $\ldots$, I_r starting at $1, 2, \ldots, r$, respectively. If one of these intervals has a partial sum at least 1, then note that it is in fact strictly greater than 1: as c_i is the only non-integral coefficient in I_i, all partial sums are non-integral. Thus, if we start π at i, we obtain a bad permutation. Otherwise, for every interval I_i, all the partial sums are strictly less than 1. In this case, consider the permutation π starting at 1. If π is not bad, it must yield a set $S \subseteq [n]$; let $\sigma = \langle \pi(1), \pi(2), \ldots, \pi(|S|) \rangle$ be the cyclic permutation of S induced by π.

Claim 7. *σ is turnable.*

To see why Claim 7 completes the proof, observe that since σ is turnable, it must have some other starting point that, when unfolded into a linear permutation, also yields S. As we chose π_σ to be the S-yielding permutation with the largest starting point, it follows that our original permutation π, which starts with 1, cannot have π_σ as an initial segment, and thus must be redundant. This shows that every cyclic permutation of $[n]$ admits a starting point giving a bad or redundant permutation, which resolves Case (3).

Proof of Claim 7. For $1 \leqslant t \leqslant |S|$, define $a_t = \sum_{i=t}^{|S|} c_{\pi(i)}$, and let t_0 be the smallest index that minimizes a_t. We first argue that $2 \leqslant t_0 \leqslant |S|$. Indeed, in order to have an integral sum, S must contain the index of at least one other non-integral coefficient apart from c_1. Let t be such that $\pi(t)$ is the last index in S of a non-integral coefficient. Then a_t is a partial sum of the interval $I_{\pi(t)}$, and hence $a_t < 1 = a_1$.

Let π' be a permutation obtained by rotating σ to start from $\pi(t_0)$, unfolding it to a linear permutation of S, and then extending it arbitrarily to a permutation of $[n]$. We claim that π' also yields S, which in particular implies σ is turnable.

Since $\pi'([|S|]) = S$, the only way that π' could fail to yield S is if there is some index $j \in [|S| - 1]$ such that $\sum_{i=1}^{j} c_{\pi'(i)} \geqslant 1$. If $j \leqslant |S| - t_0$, then $\pi'([j]) = \pi([t_0, t_0 + j - 1])$, and $t_0 + j \leqslant |S|$. If this partial sum is positive, let alone greater than or equal to 1, then

$$a_{t_0+j} = \sum_{i=t_0+j}^{|S|} c_{\pi(i)} = \sum_{i=t_0}^{|S|} c_{\pi(i)} - \sum_{i=t_0}^{t_0+j-1} c_{\pi(i)} = a_{t_0} - \sum_{i=1}^{j} c_{\pi'(i)} < a_{t_0},$$

which contradicts the choice of t_0. If $j = |S| - t_0 + 1$, then $\pi'([j]) = \pi([t_0, |S|])$, and so $\sum_{i=1}^{j} c_{\pi'(i)} = a_{t_0} \leqslant a_t < 1$, where t is as above. Thus, $j \geqslant |S| - t_0 + 2$, in which case $\pi'([j]) = \pi([t_0, |S|]) \cup \pi([1, j - |S| + t_0 - 1])$ and

$$\sum_{i=1}^{j} c_{\pi'(i)} = \sum_{i=1}^{j-|S|+t_0-1} c_{\pi(i)} + \sum_{i=t_0}^{|S|} c_{\pi(i)} \geqslant 1.$$

Recall that π yields S, and so we have $\sum_{i=1}^{|S|} c_{\pi(i)} = 1$. We can therefore deduce that $\sum_{i=t_0+j-|S|}^{t_0-1} c_{\pi(i)} \leqslant 0$, and so

$$a_{t_0+j-|S|} = \sum_{i=t_0+j-|S|}^{|S|} c_{\pi(i)} = \sum_{i=t_0+j-|S|}^{t_0-1} c_{\pi(i)} + \sum_{i=t_0}^{|S|} c_{\pi(i)} = \sum_{i=t_0+j-|S|}^{t_0-1} c_{\pi(i)} + a_{t_0} \leqslant a_{t_0},$$

which again contradicts our choice of t_0 (recall that $j \leqslant |S|-1$, which implies $t_0+j-|S| < t_0$). Hence, there is no such index j, and π' also yields S. □

This completes the proof of Theorem 3.

3.2 Further stability

Theorem 3 characterises the hyperplanes of maximum possible weight. It is then natural to ask what we can say about hyperplanes of large weight, and indeed, our methods allow us to at least partially describe their coefficients.

Theorem 8. *Let r be a positive integer, and let $h = \{x : \sum_{i=1}^n c_i x_i = 1\}$ be a hyperplane in $\mathbb{R}^n$ of weight $w(h) > 1 - \frac{r}{n}$. Then the following hold:*

(a) $|\{i : c_i > 1\}| \leqslant r - 1$, *and*

(b) $|\{i : c_i \notin \mathbb{Z}\}| \leqslant 2r^2 - 1$.

Before we embark on the proof, we first note that the bound in part (a) is best possible. Indeed, if $c_1 = \ldots = c_{r-1} = 2, c_r = \ldots = c_n = 1$, then it is easy to see that a permutation π is bad if $\pi(1) \in [r-1]$, and good otherwise. This implies $w(h) = 1 - \frac{r-1}{n} > 1 - \frac{r}{n}$, and we have $r - 1$ coefficients larger than 1.

On the other hand, we do not expect the bound in part (b) to be tight, and rather expect that there can be at most $O(r)$ non-integral coefficients. However, note that a bound of r is not true in general. For instance, if $r \geqslant 2$ and $n = 2r - 1$, then we can take $c_i = \frac{1}{2}$ for all i; this has weight $\frac{1}{2}$ but $n = 2r - 1$ non-integral coefficients. Perhaps, though, if n is sufficiently large with respect to r, then it could be true that one must have fewer than r non-integral coefficients.

Finally, we note that Theorem 8 only has two parts in its characterisation, as compared to the three parts in Theorem 3. One might also expect that, if the hyperplane has weight close to one, then the sum of the coefficients should not be too small. However, this is not true — one could have $c_1 = c_2 = \ldots = c_{n-1} = 1$, with c_n tending to $-\infty$. Then the sum of the coefficients is very small, but all permutations except those starting with n are good, meaning $w(h) = 1 - \frac{1}{n}$. Hence, it appears to be somewhat complicated to formulate an appropriate condition on the sums of the coordinates.

Proof. For (a), note that any permutation starting with a coefficient larger than 1 is bad. Thus, if there are at least r coefficients larger than 1, we have $|\mathcal{B}| \geqslant r(n-1)!$, and then (4) implies $w(h) \leqslant 1 - \frac{r}{n}$.

For (b), we make the following claim.

Claim 9. *If h has s non-integral coefficients, then for any cyclic permutation of $[n]$ and $b \in \mathbb{N}$, there are either at least b starting points that give bad permutations, or at least $\frac{s}{b}$ that give bad or switchable permutations.*

Let us first see how the claim gives the result. Suppose for the sake of contradiction that h has at least $s = 2r^2$ non-integral coefficients, and set $b = r$. Then we know that the $(n-1)!$ cyclic permutations of $[n]$ are of two types — the first give rise to at least r bad permutations, and the second give rise to at least $2r$ permutations that are either bad or switchable. Suppose there are $\alpha(n-1)!$ cyclic permutations of the first kind, and thus $(1-\alpha)(n-1)!$ cyclic permutations of the second kind. If we let $\mathcal{S}$ denote the set

of switchable permutations, we have $|\mathcal{B}| + |\mathcal{S}| \geqslant r\alpha(n-1)! + 2r(1-\alpha)(n-1)!$, with $|\mathcal{B}| \geqslant r\alpha(n-1)!$.

Now notice that at least half of all switchable permutations are redundant, and so it follows from the above inequalities that $|\mathcal{B}| + |\mathcal{R}| \geqslant r(n-1)!$, and so the weight of the hyperplane is at most $1 - \frac{r}{n}$. □

To finish, we prove the claim.

Proof of Claim 9. Fix a cyclic permutation σ of $[n]$, and let $I_1, I_2, \ldots, I_s$ be the (cyclic) intervals that start with non-integral coefficients, labelled in cyclic order. For $1 \leqslant i \leqslant s$, let π_i be the linear permutation of $[n]$ obtained from σ by starting at the ith non-integral coefficient (so I_i is an initial segment of π_i).

Now since each I_i starts with a non-integral coefficient, and has no other, all of its initial sums are non-integral. Thus, if I_i has any initial sum that is at least 1, it is strictly larger than 1, and so π_i will be a bad permutation. If there are at least b such intervals, then we obtain b bad permutations, and we are done.

If not, our goal will be to show that in every sequence of b consecutive permutations $\pi_j, \pi_{j+1}, \ldots, \pi_{j+b-1}$ (indices modulo s), at least one must be bad or switchable. Then, by averaging over the s such intervals, it follows that we have at least $\frac{s}{b}$ bad or switchable permutations, as required.

If this is not the case, then each of these permutations π_i must be good, yielding some set S_i. If one of the I_i has an initial sum in the interval $(0, 1)$, consider the longest tail interval of S_i that minimises the sum. This must be a proper subinterval of S_i, since all of S_i has sum 1, but dropping the positive initial sum in I_i gives a tail interval with smaller sum. We can then follow the proof of Claim 7 to show that the induced cyclic permutation of S_i is turnable, and π_i is therefore switchable.

Hence, we may assume that all initial sums of I_i, including I_i itself, are strictly negative. In this case, observe first that S_i and S_{i+1} cannot end in the same interval. Indeed, if this were the case, then $S_i \triangle S_{i+1}$ would be $I_i \cup J$, where J is the difference of two initial sums of the interval in which S_i and S_{i+1} end. However, the sum of I_i is non-integral, while the sum of J must be an integer, and thus we cannot have both S_i and S_{i+1} having sum equal to 1.

Next, observe that S_i must end after S_{i+1}. If not, then S_{i+1} contains $S_i \setminus I_i$ as an initial sum. However, the sum of S_i is 1, while the sum of I_i is negative, so this means that S_{i+1} has an initial sum that is strictly larger than 1, which is a contradiction.

Hence, it follows that the sets S_i, $j \leqslant i \leqslant j + b - 1$, all end in distinct intervals. If S_i ends in the interval I_ℓ, then notice that $I_\ell \cap S_i$ is a tail sum of S_i, and it must be positive. If this sum is in $(0, 1)$, then as before, by considering a longest tail interval of S_i with smallest sum, we can follow Claim 7 to deduce π_i is switchable. Otherwise, I_ℓ contains an initial sum that is larger than 1, but we only have at most $b - 1$ such intervals. Thus, one of $\pi_j, \ldots, \pi_{j+b-1}$ must be switchable, as required. □

4 Almost k-covers

In this section, we show how our stability result can be used in the determination of $f(n,k)$, the minimum number of hyperplanes needed for an almost k-cover of the n-dimensional hypercube Q^n.

4.1 Constructing small covers

Our extremal function $f(n,k)$ is the solution to the integer linear program where we have a variable for every hyperplane not through the origin, representing the multiplicity with which it appears in the cover, and a constraint for every nonzero point in $\{0,1\}^n$, ensuring the point is covered at least k times.

Unfortunately, integer linear programming is notoriously difficult to solve, and this is especially true in this setting, as the number of variables involved grows incredibly fast. Indeed, there are infinitely many hyperplanes in $\mathbb{R}^n$ that could, in principle, be included in our cover. However, we can finitise our problem by observing that we need only consider hyperplanes that intersect $\{0,1\}^n \setminus \{\vec{0}\}$ *maximally* — that is, if H_1 and H_2 are hyperplanes, both avoiding the origin, and $H_1 \cap \{0,1\}^n \subseteq H_2 \cap \{0,1\}^n$, then we can replace any occurrence of H_1 in a cover with a copy of H_2.

Moreover, any maximally intersecting hyperplane H must contain n affinely independent points of the hypercube. Indeed, write $H = \{x : h(x) = 1\}$ for some linear functional h, let A be the affine hull of $H \cap \{0,1\}^n$, and suppose $\dim A < n-1$. Since $\vec{0} \notin A$, the linear span L of A has dimension $\dim A + 1 < n$, and so some standard basis vector e_i lies outside L. We can therefore extend the restriction of h to L to a linear functional g on $\mathbb{R}^n$ with $g(e_i) = 1$. The hyperplane $H' = \{x : g(x) = 1\}$ then avoids the origin and satisfies $H \cap \{0,1\}^n \subsetneq (H \cap \{0,1\}^n) \cup \{e_i\} \subseteq H' \cap \{0,1\}^n$, contradicting the maximality of H. As any n affinely independent points in $\mathbb{R}^n$ determine a unique hyperplane, every maximally intersecting hyperplane is determined by some n nonzero points of the hypercube. In particular, this implies that there are at most $\binom{2^n-1}{n}$ hyperplanes we need to consider, making the integer linear program finite.

Unfortunately, there is a considerable difference between *finite* and *computationally feasible*, and this upper bound of $\binom{2^n-1}{n}$ grows far too fast to leave us with any hope of computing exact answers even when n is as small as 6. While it should be pointed out that this is indeed just an upper bound, and some hyperplanes are significantly overcounted, brute-force enumeration of the maximally intersecting hyperplanes (which we could only carry out for $n \leqslant 5$) shows that the true number also exhibits rapid growth.

n	1	2	3	4	5	6
Upper bound	1	3	35	1365	169911	67945521
Actual count	1	3	11	95	2629	??
Weight-1 hyperplanes	1	3	10	35	126	462

Table 1: The number of maximally intersecting and weight-1 hyperplanes in $\mathbb{R}^n$.

In order to be able to proceed computationally, then, it is necessary to restrict the

search space. A natural place to start is by only considering hyperplanes of maximum weight. Indeed, the Clifton–Huang lower bound (2) shows that in any almost k-cover, the total weight of the hyperplanes must be at least $H_n k$, where H_n is the nth Harmonic number. To minimise the size of the cover, then, we would want each hyperplane to have as much weight as possible.

As we saw in Corollary 4, there are far fewer hyperplanes of weight 1; the number of these is the much more modest $\binom{2n-1}{n}$. Moreover, Theorem 3 characterises these maximum-weight hyperplanes, so we are able to set up the corresponding integer linear program and efficiently search for small covers when $n = 6$. In many cases we are able to find covers matching the Clifton–Huang lower bound, as shown in Theorem 5, which we first restate.

Theorem 5. *For $k \geqslant 16$, we have*

$$f(6,k) = \left\lceil \frac{49k}{20} \right\rceil$$

whenever $k \not\equiv 11 \pmod{20}$ and $k \notin \{22, 42, 62, 82, 102\}$.

Proof. The Clifton–Huang lower bound (2) gives $f(6,k) \geqslant \lceil H_6 k \rceil = \lceil \frac{49k}{20} \rceil$, and so we only need to prove the upper bound. To this end, we solved the integer linear program corresponding to $f(6,k)$ for various values of k, restricting ourselves to using the 462 hyperplanes of weight 1.

We noted that $f(6,60) = 147$, which comes from the general construction given by Clifton and Huang [6]. We next considered the case $k = 20$. The lower bound implies $f(6,20) \geqslant 49$, with equality only possible if all hyperplanes involved have weight 1. Our integer linear program solver found such an almost-k cover, which we have provided in Appendix A. Note that this cover includes hyperplanes with coefficients other than 0 or 1; that is, it uses some of the new weight-1 hyperplanes described in Theorem 3.

Having thus established that $f(6,20) = 49$, we turn to other values of k. Now, since adding an almost ℓ-cover to an almost k-cover results in an almost $(k+\ell)$-cover, we have $f(n, k+\ell) \leqslant f(n,k) + f(n,\ell)$, which implies that $f(6, k+20m) \leqslant f(6,k) + 49m$ for all integers $m \geqslant 1$. We then ran the integer linear program through a solver for all $16 \leqslant k \leqslant 35$, and, except for $k = 22$ and $k = 31$, found that $f(6,k) = \lceil \frac{49k}{20} \rceil$. While these values are excluded from our result, we were able to resolve the case $k \equiv 2 \pmod{20}$ by finding $f(6,122) = 299 = \lceil \frac{49 \cdot 122}{20} \rceil$.

Using this finite set of computational results, together with the recursive upper bound, it follows that $f(6,k) \leqslant \lceil \frac{49k}{20} \rceil$ for all $k \geqslant 16$ with $k \not\equiv 11 \pmod{20}$ and $k \notin \{22, 42, 62, 82, 102\}$. □

It is natural to ask what happens when $k \equiv 11 \pmod{20}$. Our computational results were restricted to hyperplanes of weight 1. However, by Theorem 3, we know that all other hyperplanes have weight at most $\frac{5}{6}$. Thus, if we have a cover of m hyperplanes containing at least one hyperplane that is not of weight 1, the total weight of the hyperplanes in the cover is at most $m - \frac{1}{6}$.

For $k \equiv 11 \pmod{20}$, we have $\frac{49k}{20} \geqslant \lceil \frac{49k}{20} \rceil - \frac{1}{20}$. Hence, in light of the previous remark, if there is a cover of size $\lceil \frac{49k}{20} \rceil$, it must consist entirely of weight-1 hyperplanes. Our integer linear program solver neither found nor ruled out any such covers for $k \in \{11, 31, 51\}$; in all these cases, though, it provided covers showing $f(6,k) \leqslant \lceil \frac{49k}{20} \rceil + 1$.

However, it is not inconceivable that further along these sequences, the Clifton–Huang lower bound is realised; that is, $f(6, 11 + 20m_0) = 27 + 49m_0$ for some suitably large m_0. If this does happen, then using $f(6,20) = 49$, it again follows that we would have equality for all $m \geqslant m_0$. This is what happened for $k \equiv 2 \pmod{20}$: while we could not show $f(6,k) = \lceil \frac{49k}{20} \rceil$ for $k \in \{22, 42, 62, 82, 102\}$, by establishing this for $k = 122$, the result follows for all larger terms in this residue class.

4.2 Improving the lower bound

In some cases, though, one can show that the lower bound $f(n,k) \geqslant \lceil H_n k \rceil$ is not tight for arbitrarily large k. Using numerical data about the hyperplanes in $\mathbb{R}^5$, Clifton and Huang were able to show that $f(5,k) = \lceil \frac{137k}{60} \rceil + 1$ for all $k = 67 + 60m$ with $m \geqslant 1$. Using our stability result, we can extend this to all sufficiently large n, showing that the lower bound is not tight infinitely often.

Theorem 10. *Let $k \geqslant 2$ and $n \in \mathbb{N}$ be such that $0 < \lceil kH_n \rceil - kH_n < \frac{1}{n\binom{n-1}{\lfloor n/2 \rfloor}}$. Then $f(n,k) \geqslant \lceil kH_n \rceil + 1$.*

In particular, if n is sufficiently large, there are infinitely many choices of k for which $f(n,k) \geqslant \lceil kH_n \rceil + 1$.

Proof. Consider a smallest almost k-cover $\mathcal{H}$ of Q^n, and for each hyperplane h, let x_h denote the number of copies of h in $\mathcal{H}$. We then have

$$|\mathcal{H}| = \sum_h x_h = \sum_h \left(\sum_{S : S \in h} w(S) + 1 - \sum_{S : S \in h} w(S) \right) x_h,$$

where here, and below, we are summing over all nonempty subsets S of $[n]$.

We obtain

$$|\mathcal{H}| = \sum_S \left(\sum_{h : h \ni S} x_h \right) w(S) + \sum_h \left(1 - \sum_{S : S \in h} w(S) \right) x_h.$$

Since

$$\sum_S w(S) = \sum_{s=1}^{n} \sum_{S : |S| = s} w(S) = \sum_{s=1}^{n} \frac{\binom{n}{s}}{s\binom{n}{s}} = \sum_{s=1}^{n} \frac{1}{s} = H_n,$$

we can further rearrange this equation to obtain

$$|\mathcal{H}| - kH_n = \sum_S \left(\sum_{h : h \ni S} x_h - k \right) w(S) + \sum_h \left(1 - \sum_{S : S \in h} w(S) \right) x_h. \tag{5}$$

Consider the terms on the right-hand side of (5). Since $\mathcal{H}$ is an almost k-cover, every set is covered at least k times, so $\sum_{h:h\ni S} x_h - k$ is always a non-negative integer. Furthermore, by Theorem 3, each hyperplane has weight either exactly 1 or at most $1-\frac{1}{n}$, which means $1-\sum_{S:S\in h} w(S)$ is non-negative, and is at least $\frac{1}{n}$ when it is positive. Thus, each individual summand on the right-hand side is non-negative, and the positive summands are at least $\min\left(\min_S w(S), \frac{1}{n}\right) = \frac{1}{n\binom{n-1}{\lfloor n/2\rfloor}}$.

Now we look at the left-hand side. Since it must be non-negative, we have $|\mathcal{H}| \geqslant \lceil kH_n\rceil$. Suppose we had equality. By our assumption on k and n, $0 < \lceil kH_n\rceil - kH_n < \frac{1}{n\binom{n-1}{\lfloor n/2\rfloor}}$, but we previously established that the right-hand side, if positive, must be at least $\frac{1}{n\binom{n-1}{\lfloor n/2\rfloor}}$. Hence, we must have $f(n,k) = |\mathcal{H}| \geqslant \lceil kH_n\rceil + 1$.

For the second assertion, let $H_n = \frac{c_n}{d_n}$, where c_n and d_n are coprime. We can then find some $1 \leqslant k_0 \leqslant d_n$ such that $k_0 c_n \equiv -1 \pmod{d_n}$. It then follows that for any $n \geqslant 2$ and $k \equiv k_0 \pmod{d_n}$, we have $\lceil kH_n\rceil - kH_n = \frac{1}{d_n}$.

As shown by Boyd [5], $d_n = \mathrm{e}^{(1+o(1))n}$. On the other hand, $n\binom{n-1}{\lfloor n/2\rfloor} \leqslant n2^{n-1}$. Hence, for n sufficiently large, we have $\lceil kH_n\rceil - kH_n < \frac{1}{n\binom{n-1}{\lfloor n/2\rfloor}}$. Thus, for this infinite sequence of values of k, $f(n,k) \geqslant \lceil kH_n\rceil + 1$. □

Remark 11. Note that in the proof of the second statement, it suffices to have $k \equiv k_0 \pmod{d_n}$ where k_0 is such that $c_n k_0 \equiv -r \pmod{d_n}$ for some $1 \leqslant r < \frac{d_n}{n\binom{n-1}{\lfloor n/2\rfloor}}$. This shows that the linear programming lower bound is not tight for at least $(\mathrm{e}/2)^{(1+o(1))n}$ of the $\mathrm{e}^{(1+o(1))n}$ residue classes.

5 Concluding remarks

In this paper, we have studied the Clifton–Huang weighting of hyperplanes, characterising the hyperplanes of maximum weight, and proving some stability, showing that all other hyperplanes have weight bounded away from 1. We have then shown how this result can be used to determine further values of the extremal function $f(n,k)$, which is the minimum number of hyperplanes needed to cover all nonzero points of Q^n at least k times while avoiding the origin completely. Several open problems remain, and we shall highlight some possible directions for further research below.

Maximal hyperplanes. We have shown that the weight of any hyperplane is either 1 or at most $1-\frac{1}{n}$. Recall that, provided $\alpha \notin \{-(n-2), -(n-3), \ldots, -1, 0, 1\}$, the hyperplane $h_\alpha = \{x : \alpha x_1 + \sum_{i=2}^n x_i = 1\}$ shows the latter bound to be best possible. However, as explained in Section 4, when we are setting up the integer linear program for the covering problem, we need only consider hyperplanes that intersect $\{0,1\}^n$ maximally. Since h_1 covers all the points that h_α does, and more, the above hyperplane is not maximal.

We could hope for stronger applications, then, if we could show that the weight of a *maximal* hyperplane is either 1 or much smaller. Unfortunately, there is not much we can gain here — we can find maximal hyperplanes of weight $1-\frac{1}{n-1}$. For example, it

is known that the following hyperplanes, together with permutations of their coefficients, are maximal hyperplanes of weight $1 - \frac{1}{n-1}$:

- $x_1 + \cdots + x_{n-2} - kx_{n-1} - (n-2-k)x_n = 1$, $k = 1, \ldots, \lfloor \frac{n-2}{2} \rfloor$.
- $x_1 + \cdots + x_{n-2} + 2x_{n-1} - kx_n = 1$, $k = 1, \ldots, n-3$.
- $x_1 + \cdots + x_{n-3} + 2x_{n-2} - kx_{n-1} - (n-2-k)x_n = 1$, $k = 1, \ldots, \lfloor \frac{n-2}{2} \rfloor$.

Still, it would be interesting to prove this larger separation for maximal hyperplanes, which could be significant for small values of n.

Question 12. Let h be a hyperplane that intersects Q^n maximally. Is it true that either $w(h) = 1$ or $w(h) \leqslant 1 - \frac{1}{n-1}$?

One could seek to use our proof methods to establish this stronger bound for maximal hyperplanes. It would need to be shown that their coefficients admit $n(n-2)!$ bad or redundant permutations, instead of just $(n-1)!$. With somewhat more involved arguments, whose details we omit, we could show this in some special cases, namely when we have a coefficient larger than 1, or when we have a positive non-integral coefficient. However, it remains to resolve the problem in all other cases.

Another direction that could help with reducing the search space for a wider range of parameters would be to characterise those hyperplanes whose weight is close to 1. Initial steps in that direction were taken in Theorem 8, and it would be great to complete the picture.

Question 13. Can we characterise all hyperplanes h with $w(h) = 1 - O\left(\frac{1}{n}\right)$?

Improved lower bounds. In this setting, our lower bounds come from the fractional relaxation of the integer linear program, whose value was proven to be $H_n k$, where H_n is the nth Harmonic number, by Clifton and Huang [6]. This immediately gives $f(n,k) \geqslant \lceil H_n k \rceil$, and Clifton and Huang further proved that this lower bound is asymptotically tight.

However, it is not always sharp. As we have shown in Theorem 10, for every sufficiently large n there are infinitely many choices of k for which we have $f(n,k) \geqslant \lceil H_n k \rceil + 1$. Unfortunately, our methods do not allow us to prove any larger separation between the fractional and integer solutions. This would entail ruling out the existence of almost k-covers of size $\lceil H_n k \rceil + 1$, but these could contain a hyperplane of very small weight, and we know nothing about these hyperplanes. That said, in all the computational results we have obtained, we always have $f(n,k) \leqslant \lceil H_n k \rceil + 1$. It would be interesting to see how large $f(n,k) - \lceil H_n k \rceil$ can be, and to develop methods for proving stronger lower bounds.

Question 14. Given n and sufficiently large k, can we have $f(n,k) \geqslant \lceil H_n k \rceil + 2$?

It is worth reiterating a question of Clifton and Huang, who asked if the difference between the integer and fractional problems was bounded by an absolute constant.

Question 15 (Clifton–Huang [6])**.** Is there an absolute constant $C > 0$, such that for every n there are only finitely many k with $f(n,k) \geqslant \lceil H_n k \rceil + C$?

Large dimensions. Finally, we note that while this linear programming approach is very fruitful in the cases we have considered, where n is fixed and k is large, Clifton and Huang showed the problem behaves very differently when k is fixed and n is large. The polynomial method has been fruitful in establishing lower bounds in this range, but Sauermann and Wigderson [11] showed that the solution to the algebraic problem is smaller than Clifton and Huang's conjectured value of $f(n,k)$.

Conjecture 16 (Clifton–Huang [6])**.** For $k \geqslant 2$ and n sufficiently large, we have $f(n,k) = n + \binom{k}{2}$.

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Appendix

A An almost 20-cover of Q^6

We used an integer linear program solver to find an almost 20-cover of Q^6 that consists of 49 hyperplanes. Each row of Table 2 lists the coefficients c_i that give the hyperplane of the form $\sum_{i=1}^{6} c_i x_i = 1$.

Table 2: The coefficients for hyperplanes $\vec{c} \cdot \vec{x} = 1$ in an almost 20-cover in Q^6, where $\vec{c} = (c_1, c_2, c_3, c_4, c_5, c_6)$

c_1	c_2	c_3	c_4	c_5	c_6	c_1	c_2	c_3	c_4	c_5	c_6
1	1	1	0	−1	−1	0	1	0	0	1	0
1	1	1	0	−2	0	0	1	0	0	1	0
1	1	1	−1	−1	0	0	1	0	0	0	1
1	1	0	1	−1	0	0	1	0	0	0	0
1	1	0	−2	1	1	0	1	−1	1	1	1
1	1	−1	1	1	−1	0	0	1	1	1	−1
1	1	−1	1	0	−1	0	0	1	1	0	0
1	1	−1	0	0	1	0	0	1	1	0	0
1	1	−2	1	0	0	0	0	1	0	1	0
1	0	1	0	1	−1	0	0	1	0	0	1
1	0	1	0	0	0	0	0	1	0	0	1
1	0	1	0	0	−1	0	0	1	0	0	0
1	0	1	−1	−1	1	0	0	1	−1	1	0
1	0	0	1	1	1	0	0	0	1	1	0
1	0	0	1	−1	0	0	0	0	1	0	1
1	0	0	0	1	0	0	0	0	1	0	0
1	0	0	0	0	1	0	0	0	0	1	1
1	0	−1	1	1	0	0	0	0	0	1	0
1	0	−1	1	0	1	0	0	0	0	1	0
1	−2	0	0	1	1	0	0	0	0	0	1
0	1	1	1	1	0	0	0	−1	1	0	1
0	1	1	1	0	−1	0	0	−1	0	1	1
0	1	1	0	0	1	0	−1	1	0	1	1
0	1	1	0	0	0	−1	1	0	1	−1	1
0	1	0	1	−2	1						