Nonadjacent thin edges in simple braces

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Abstract

A bipartite graph $G[A, B]$ is a *brace* if either the underlying simple graph of G is one of the complete bipartite graphs K_2 and $K_{2,2}$, or G has at least six vertices and $G - a - a' - b - b'$ has a perfect matching, for every four distinct vertices $a, a' \in A$ and $b, b' \in B$. An edge e of G is *thin* if the graph $\widehat{G - e}$, obtained from $G - e$ by repeatedly bicontracting vertices of degree two, is a brace. In this paper, we prove that each simple brace of order six or more contains at least two nonadjacent thin edges, thereby solving a problem posed by Carvalho, Lucchesi and Murty.

Mathematics Subject Classifications: 05C70, 05C75

1 Introduction

The graphs considered in this paper have a finite number of vertices and edges, and no loops. For general graph-theoretic terminology, we follow Bondy and Murty [1].

Let G be a graph with vertex set V and edge set E. For any $S \subseteq V$, let $N_G(S)$ be the set of vertices of $V - S$ which are adjacent to vertices of S and let $\overline{S} = V - S$ be the complement of S in V. The edge subset $\partial_G(S)$ of E is a *cut* which consists of those edges with exactly one end in S and one end in $\overline{S}$: the sets S and $\overline{S}$ are its *shores*. A cut is *trivial* if one of its shores is a singleton. The degree of a vertex v in G is denoted $d_G(v)$. In each of these notations, if G is clear from the context we omit the subscript G.

Given a set X of vertices of G, we denote by $G/(X \to x)$ the graph obtained from G by the contraction of X to a single new vertex x and subsequent removal of resulting loops. If the name of the contraction vertex x is irrelevant, we simply write G/X. Given a cut $C := \partial(X)$ of G, we define the graphs G/X and $G/\overline{X}$ to be the two *C-contractions* of G.

A graph is called *matching covered* if it is connected, has at least one edge and each of its edges is contained in some perfect matching. Suppose that our graph G is matching

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covered. A cut C of G is *tight* if $|C \cap M| = 1$ for every perfect matching M of G. Clearly, every trivial cut is a tight cut. We call a matching covered graph which is free of nontrivial tight cuts a *brace* if it is bipartite, and a *brick* otherwise. Incidentally, the definition of brace presented in the abstract may be regarded as a characterization of braces [15, Theorem 5.17].

For the terminologies related to matching covered graphs, we follow the classic book by Lovász and Plummer [14]. It was shown by Lovász [13] that every matching covered graph may be decomposed into a unique list of braces and bricks. Hence, braces are fundamental graphs of matching covered graphs. The importance of braces stems from the fact that there are several important problems from matching theory that can be reduced to braces and bricks (for example, characterizing Pfaffian graphs [12, 22]).

Let v_0 be a vertex of G with degree two and let v_1 and v_2 be its two distinct neighbours. The *bicontraction* of vertex v_0 consists in contracting the set $\{v_0, v_1, v_2\}$ to a single vertex; the result of the bicontraction of v_0 is the graph $G/\{v_0, v_1, v_2\}$. The *retract of* G, denoted $\widehat{G}$, is the graph obtained from G by repeatedly applying bicontractions to vertices of degree two. It will be shown, in Section 2, that $\widehat{G}$ is unique, up to isomorphism. An edge e of a brace G is *thin* if $\widehat{G - e}$ is a brace. For instance, each edge of $K_{3,3}$ is thin.

The existence of thin edges attracted the attention of many mathematicians, as seen, for example, in [3, 4, 5, 6, 8, 9, 10, 16, 17, 18, 19, 20, 21]. In particular, Carvalho, Lucchesi and Murty [2, Theorem 32] showed that

Theorem 1. *Each brace of order six or more has at least two (possibly adjacent) thin edges.*

Since most known families of braces have many thin edges, this observation led them to made the following conjecture [2].

Conjecture 2. There exists a positive constant α such that every brace on n vertices has αn thin edges.

In that same paper, they stated that "New ideas seem to be necessary to approach even weaker conjectures. For example we do not know if every brace of order six or more has two nonadjacent thin edges." That is precisely the main result of this paper.

Theorem 3 (The Main Theorem)**.** *Every simple brace of order six or more has at least two nonadjacent thin edges.*

The proof of the Main Theorem will be given in Section 4. In Section 2 we present some basic properties of matching covered graphs. In Section 3 we present important properties of braces.

2 Basic Properties of Matching Covered Graphs

In this section we present some basic properties of matching covered graphs and, in particular, of bipartite matching covered graphs. The next three assertions are fundamental properties of matching covered graphs.

Proposition 4. *Every matching covered graph is 2-connected.* □

Proposition 5 ([15, Corollary 2.10])**.** *A matchable bipartite graph $G[A, B]$ is matching covered if and only if $|N(S)| \geqslant |S| + 1$ for every nonempty proper subset S of B.* □

Proposition 6. *Let $C := \partial(X)$ be a cut of a graph G. If both C-contractions of G are matching covered then G is also matching covered.* □

We then deduce the following simple property.

Proposition 7. *Let v be a vertex of degree two of a bipartite graph G. If the bicontraction of v yields a matching covered graph then G is also matching covered.*

Proof. Let X be the set consisting of v and its two neighbours. The underlying simple graph of $G' := G/\overline{X}$ is a quadrilateral, hence G' is matching covered. If the bicontraction of v produces a graph which is matching covered then G is matching covered, by Proposition 6. □

If C is a tight cut of a matching covered graph G, then both C-contractions of G are matching covered. From Proposition 4, we deduce the following basic property.

Proposition 8 ([15, Proposition 4.1])**.** *Let $C := \partial(X)$ be a tight cut of a matching covered graph G. Then, the subgraph $G[X]$ of G induced by X is connected.* □

Let X be an odd set of vertices of a bipartite graph $G[A, B]$. As $|X|$ is odd, then $|X \cap A| \neq |X \cap B|$. Whichever of the parts $X \cap A$ and $X \cap B$ of X is larger than the other is called the *majority part of* X, and is denoted X_+; the other part is called the *minority part of* X and is denoted X_-.

Proposition 9. *Let G be a bipartite matching covered graph and let X be a set of vertices of G. The cut $C := \partial(X)$ is tight if and only if (i) $|X_+| = |X_-| + 1$ and (ii) every edge of C joins a vertex in X_+ to a vertex in $\overline{X}_+$.* □

Corollary 10. *Let e be an edge of a brace G. Let C be a nontrivial cut of G such that $C - e$ is tight in $G - e$. The edge e is the only edge of G that joins a vertex in X_- to a vertex in $\overline{X}_-$.*

Proof. By Proposition 9, $|X_+| = |X_-| + 1$ and every edge of $C - e$ joins a vertex in X_+ to a vertex in $\overline{X}_+$. The graph G is a brace, hence C is not tight in G. By Proposition 9, the edge e joins a vertex in X_- to a vertex in $\overline{X}_-$. □

2.1 Retracts

A vertex of a graph G is *bicontractible* if it has degree two and is adjacent to two distinct vertices. Thus, a vertex of the 2-cycle is not bicontractible. Recall that the retract of a matching covered graph G is denoted by $\widehat{G}$ and is the graph obtained from G by repeatedly bicontracting bicontractible vertices. Evidently, the retract obtained may depend on the order in which the bicontractions are performed. However, that order is not really relevant, as indicated by the next result.

Theorem 11. *Any two applications of repeated bicontractions of bicontractible vertices of a matching covered graph G produce isomorphic graphs.*

Proof. Let S_1 and S_2 be two sequences of bicontractions applied to G, thereby yielding the graphs G_1 and G_2 respectively, both free of bicontractible vertices. Let us prove the assertion by induction on $|V|$. If G has no bicontractible vertices then the assertion holds trivially. We may thus assume that G has bicontractible vertices. For $i = 1, 2$, let v_i be the first vertex in S_i and let H_i be the result of the bicontraction of v_i.

If $v_1 = v_2$ then $H_1 \cong H_2$ hence $G_1 \cong G_2$, by induction. Consider next the case in which v_1 and v_2 are adjacent. Then, G has a path $P := (v_0, v_1, v_2, v_3)$, which contains the edge v_1v_2. The bicontraction of v_1, or of v_2, replaces the path P by an edge joining v_0 to v_3. Thus, $H_1 \cong H_2$, hence $G_1 \cong G_2$, by induction.

We may thus assume that v_1 and v_2 are distinct and nonadjacent. Let S_{12} be a sequence of bicontractions of vertices of G which starts with the bicontractions of v_1 and v_2, in that order, thereby producing the graph G_{12}. Likewise, let S_{21} be a sequence of bicontractions of vertices of G which starts with the bicontractions of v_2 and v_1, in that order, thereby producing the graph G_{21}. By induction, $G_1 \cong G_{12}$ and $G_2 \cong G_{21}$. As v_1 and v_2 are distinct and not adjacent, $G_{12} \cong G_{21}$. In sum, $G_1 \cong G_{12} \cong G_{21} \cong G_2$, hence $G_1 \cong G_2$. □

2.1.1 Reduction of shores due to bicontractions

Let $C := \partial(X)$ be a tight cut of a matching covered graph G and let x be a vertex of G. Due to the possible bicontraction of x, the resulting graph H may have the shore X of C be contracted to a shore X_H of a tight cut of H. (It is also possible that $X_H = X$.)

Proposition 12.

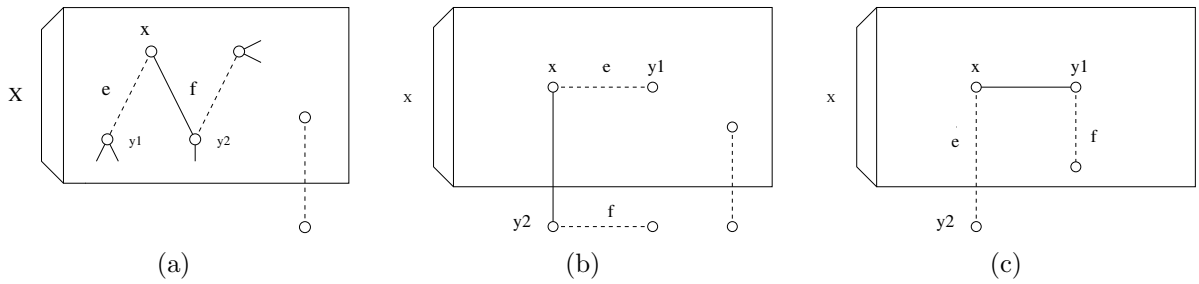
$$|X_H| = \begin{cases} |X| - 2, & \text{if } x \in X \text{ and } d(x) = 2 \\ |X|, & \text{otherwise.} \end{cases}$$

Proof. Suppose that $x \in X$. If $d(x) \geqslant 3$ then no bicontraction is applied to x in order to derive graph H. Hence, $X_H = X$.

We may thus assume that $d(x) = 2$. Let y_1 and y_2 be the two neighbours of x and let $Y := \{x, y_1, y_2\}$. Let $H := G/(Y \to y)$ and let N be a perfect matching of H. The edge f of N incident with y in H is, in G, incident with one of y_1 and y_2. Let $M := N + e$, where e is the edge of $\partial(x)$ not adjacent to f. Then, M is a perfect matching of G. By hypothesis, C is tight in G hence $|M \cap C| = 1$.

Suppose that $Y \subseteq X$. See Figure 1(a). Then, $X_H = X/(Y \to y)$ and $|X_H| = |X| - 2$. Moreover, $N \cap \partial(X_H) = N \cap C = M \cap C$, hence $|N \cap \partial(X_H)| = 1$. This conclusion holds for every perfect matching N of H, hence $\partial(X_H)$ is tight in H.

Alternatively, suppose that one of the neighbours of x, say, y_2, is not in X. See Figures 1(b) and 1(c). Then, $xy_2 \in C$. Moreover, by Proposition 8, the vertex y_1 is in X. Let $X_H := X - Y$. Then, $|X_H| = |X| - 2$. If $e = xy_1$ then N is a perfect matching of H and $N \cap \partial(X_H) = M \cap C$. If $e = xy_2$ then the edge f of M incident with y_1 is the only



(a) (b) (c)

Figure 1: Illustration of the proof of Proposition 12.

edge of $N \cap \partial(X_H)$. In both alternatives, we deduce that N has just one edge in $\partial(X_H)$. This conclusion holds for each perfect matching N of H, hence $\partial(X_H)$ is tight in H.

Note that if $x \in \overline{X}$ the previous reasoning implies that $|X|$ is not altered by a possible bicontraction of x. □

Let $C := \partial(X)$ be a tight cut of a matching covered graph G and let x_1 and x_2 be two distinct vertices of G. Due to the possible bicontractions of x_1 or x_2, the resulting graph H may have the shore X of C be contracted to a shore X_H of a tight cut of H. It is also possible that $X_H = X$.

Corollary 13. *Let* $\eta := \{v : v \in X \cap \{x_1, x_2\},\ d(v) = 2\}$. *Then,*

$$|X_H| = \begin{cases} |X| - |\eta|, & \text{if } \eta = \{x_1, x_2\} \text{ and } x_1 \text{ and } x_2 \text{ are adjacent;} \\ |X| - 2|\eta|, & \text{otherwise.} \end{cases}$$

Proof. By Proposition 12, the contraction of a vertex in η reduces $|X|$ by two. We conclude that if at most one of x_1 and x_2 is in η then the assertion holds. So, suppopse that both x_1 and x_2 are in η. The bicontraction of x_1 produces a tight cut C' of the resulting graph H' and the corresponding shore X' of C' satisfies the equality $|X'| = |X| - 2$. If x_1 and x_2 are adjacent in G then x_2 is not a vertex of H', hence $H = H'$ and $C_H = C'$. Alternatively, if x_1 and x_2 are not adjacent then x_2 may be bicontracted, thereby resulting in the graph H and $|X_H| = |X| - 4$. In all alternatives, the assertion holds. □

3 Braces

In this section we prove important properties related to braces.

Proposition 14 ([7, Proposition 2.4])**.** *If G is a simple brace of order six or more then an edge e of G is thin if and only if every nontrivial tight cut of $G - e$ has a shore consisting of precisely three vertices.* □

An edge e of a matching covered graph G is *removable* if $G - e$ is matching covered.

Theorem 15 ([15, Theorem 8.3])**.** *If G is a brace of order six or more then every edge of G is removable.* □

From Proposition 14 and Theorem 15, we see that for each nonthin edge e in a brace G of order six or more, there exists a cut C in G whose shores both have five or more vertices and such that $C - e$ is tight in $G - e$. We refer to the cut C as a *certificate that edge e is not thin in G*.

Proposition 16. *In a brace G of order six or eight, every edge is thin.*

Proof. Suppose that $|V| \in \{6, 8\}$. Let e be an edge of G. By Theorem 15, e is removable in G. Let C be a cut of G such that $C - e$ is nontrivial and tight in $G - e$. As $|V| < 10$, at least one of the shores of C has only three vertices. This conclusion holds for every nontrivial cut C of G such that $C - e$ is tight in $G - e$. By Proposition 14, e is thin in G. This conclusion holds for each edge e of G. □

3.1 Prisms, Möbius ladders, biwheels and strictly thin edges

A thin edge e of a simple brace G is *strictly thin* if $\widehat{G - e}$ is simple. We now describe three families of braces whose relevance was first established by McCuaig [17] and are precisely the simple braces of order six or more which do not have strictly thin edges.

A *prism* P_{4n}, $n \geqslant 2$, is the graph obtained from two disjoint cycles, both of length $2n$, $(u_1, u_2, \ldots, u_{2n}, u_1)$ and $(v_1, v_2, \ldots, v_{2n}, v_1)$, by the addition of the $2n$ edges $u_i v_i$, $i = 1, 2, \ldots, 2n$. Figure 2(a) shows the prism P_{12}. The $2n$ edges $u_i v_i$, $i = 1, 2, \ldots, 2n$ are nondjacent thin edges of the prism.

A *Möbius ladder* M_{4n+2}, $n \geqslant 1$, is the graph obtained from a cycle of length $4n + 2$, $(v_1, v_2, \ldots, v_{4n+2}, v_1)$, by the addition of the $2n + 1$ *chords* $v_i v_{i+2n+1}$, $1 \leqslant i \leqslant 2n + 1$ of the cycle, where the addition in the suffixes is understood to be taken modulo $4n + 2$. Figure 2(b) shows the Möbius ladder M_{10}. Each one of the $2n+1$ chords are nonadjacent thin edges of the Möbius ladder.

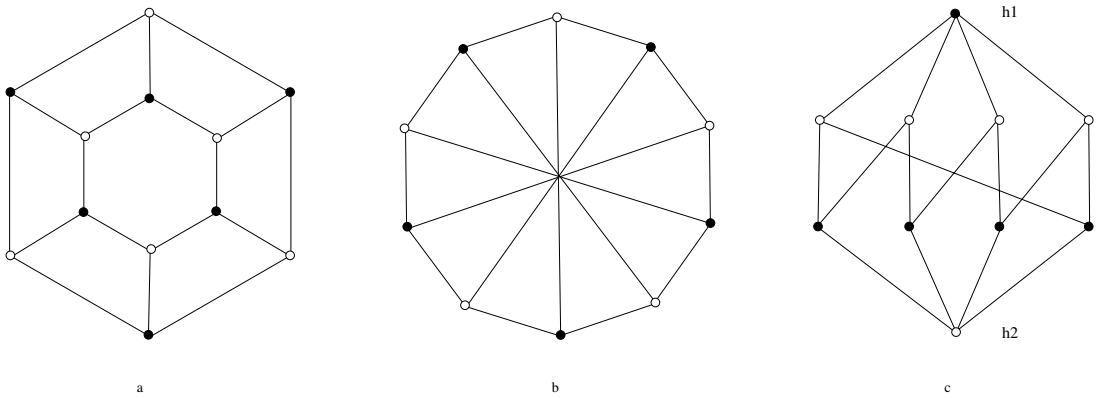


Figure 2: (a) prism P_{12}, (b) Möbius ladder M_{10}, (c) biwheel B_{10}.

A *biwheel* B_{2n}, $n \geqslant 4$, is the graph obtained from a cycle $(v_1, v_2, \ldots, v_{2n-2}, v_1)$ of length $2n-2$, called the *rim* of B_{2n}, by the addition of two vertices, h_1 and h_2, called the *hubs* of B_{2n}, and by the addition of edges $h_1 v_1, h_1 v_3, \ldots, h_1 v_{2n-3}$ and edges $h_2 v_2, h_2 v_4, \ldots, h_2 v_{2n-2}$.

Figure 2(c) shows the biwheel B_{10}. The biwheels have two nonadjacent thin edges, each one incident with one the two hubs.

We denote by $\mathcal{G}$ the collection of graphs in those three families. The next result is a fundamental contribution of McCuaig [17].

Theorem 17. *A brace of order six or more has a strictly thin edge if and only if it is not a member of the family* $\mathcal{G}$. □

3.2 Quadrilaterals

Let $C := \partial(X)$ and $D := \partial(Y)$ be two cuts of a graph G. The four sets $X \cap Y$, $X \cap \overline{Y}$, $\overline{X} \cap Y$ and $\overline{X} \cap \overline{Y}$ are the *quadrants relative to cuts* C *and* D. If the four quadrants are nonempty then we say that the cuts C and D *cross*.

Let $Q := (x_1, x_2, x_3, x_4, x_1)$ be a 4-cycle of a brace G of order six or more. Suppose that neither x_1x_2 nor x_1x_4 is thin in G. Let $C := \partial(X)$ and $D := \partial(Y)$ be some certificates that the edges x_1x_2 and x_1x_4 are not thin in G, respectively. Adjust notation so that $x_1 \in \overline{X} \cap \overline{Y}$. Thus, $x_1 \in \overline{X}_-$, $x_2 \in X_-$, $x_4 \in \overline{X}_+$ and $x_3 \in X_+$. Likewise, $x_1 \in \overline{Y}_-$, $x_4 \in Y_-$, $x_2 \in \overline{Y}_+$ and $x_3 \in Y_+$. Figure 3(a) illustrates the four quadrants of the crossing cuts C and D.

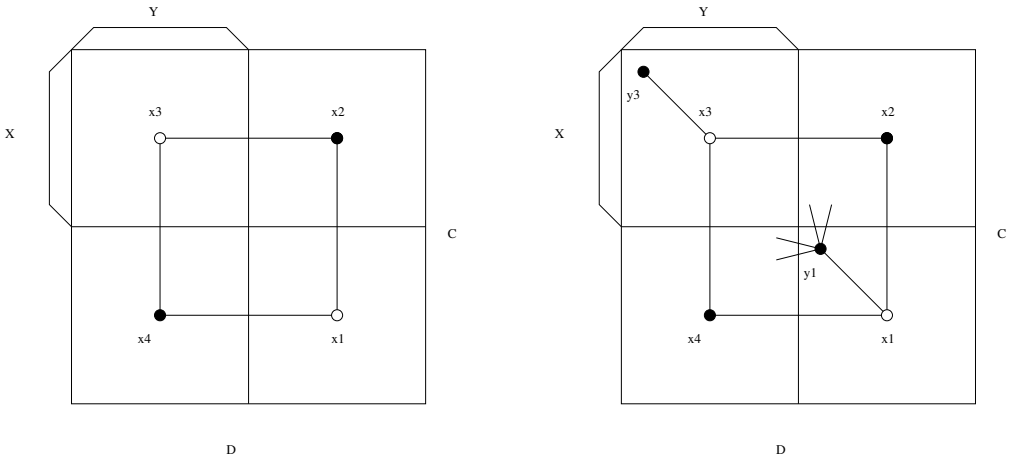


Figure 3: The cuts C and D.

Lemma 18. *Let* $G[A, B]$ *be a brace of order six or more, let* $(x_1, x_2, x_3, x_4, x_1)$ *be a 4-cycle of* G *such that neither* x_1x_2 *nor* x_1x_4 *is thin. Let* $C := \partial(X)$ *and* $D := \partial(Y)$ *be some certificates that the edges* x_1x_2 *and* x_1x_4 *are not thin in* G*, respectively. Suppose that* $d(x_3) = 3$ *and let* y_3 *be the vertex of* $N(x_3) \setminus \{x_2, x_4\}$*. Then, the following properties hold:*

(i) *the vertex* y_3 *is in the same quadrant that contains* x_3*;*

(ii) *the quadrant that contains* x_1 *consists of just two vertices,* x_1 *and say,* y_1*; moreover,* $N(x_1) = \{x_2, x_4, y_1\}$.

See Figure 3(b).

Proof. By hypothesis, the cut C is a certificate of the edge x_1x_2. Thus, $|X|, |\overline{X}| \geqslant 5$ and $C - x_1x_2$ is tight in $G - x_1x_2$. Likewise, the cut D is a certificate of the edge x_1x_4, hence $|Y|, |\overline{Y}| \geqslant 5$ and $D - x_1x_4$ is tight in $G - x_1x_4$.

Adjust notation, by interchanging X and $\overline{X}$ if necessary, so that $x_1 \in \overline{X}$. Likewise, adjust notation so that $x_1 \in \overline{Y}$. Thus, $x_1 \in \overline{X} \cap \overline{Y}$, $x_2 \in X \cap \overline{Y}$, $x_4 \in \overline{X} \cap Y$ and $x_3 \in X \cap Y$.

Adjust notation, by interchanging A and B if necessary, so that $x_3 \in A$. Then, $X_+, Y_+ \subset A$ and $x_1 \in A$. The cut $C - x_1x_2$ is tight in $G - x_1x_2$ and $D - x_1x_4$ is tight in $G - x_1x_4$. By Proposition 5,

$$\begin{array}{rcl} |X \cap A| & = & |X \cap B| + 1 \\ |Y \cap A| & = & |Y \cap B| + 1. \end{array} \tag{1}$$

Claim 19. *The vertex y_3 is in $X \cap Y$.*

Proof. Assume, to the contrary, that $y_3 \in \overline{X}$. The edge x_3x_2 is then the only edge of $\partial(x_3)$ having both ends in X. See Figure 4.

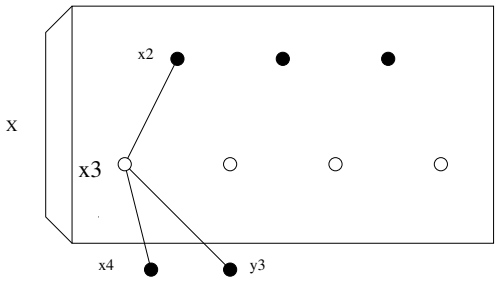


Figure 4: The vertex y_3 is not in X.

Let $W := X - x_3 - x_2$. Then, $N(W \cap B) \subseteq W \cap A$. From (1) we infer that $|W \cap A| = |W \cap B| + 1$, hence $\partial(W)$ is tight in G. Thus, W is a singleton, hence $|X| = 3$, in contradiction to the hypothesis that $|X| \geqslant 5$. We conclude that $y_3 \in X$. By symmetry, $y_3 \in Y$. Indeed, $y_3 \in X \cap Y$. □

Claim 20. $|X \cap Y \cap A| \geqslant |X \cap Y \cap B| + 2$.

Proof. Let vw be an edge of G such that $v \in X \cap Y \cap B$. The edge x_1x_2 is the only edge of C incident with $X \cap B$ and both its ends are in $\overline{Y}$, hence $w \in X \cap A$. Likewise, x_1x_4 is the only edge of D incident with a vertex in $Y \cap B$ and both its ends are in $\overline{X}$, hence $w \in Y \cap A$. In sum, $w \in X \cap Y \cap A$. This conclusion holds for each edge vw such that $v \in X \cap Y \cap B$, hence $N(X \cap Y \cap B) \subseteq X \cap Y \cap A$.

By Proposition 5, $|X \cap Y \cap A| \geqslant |X \cap Y \cap B| + 1$. Assume, to the contrary, that equality holds. By Proposition 9, the cut $\partial(X \cap Y)$ is tight, hence it is trivial. Consequently, $X \cap Y = \{x_3\}$. This is a contradiction, as y_3 also lies in $X \cap Y$. We conclude that $|X \cap Y \cap A| \geqslant |X \cap Y \cap B| + 2$. □

Let $Z := X \cup Y + x_1$. Then,

$$\begin{array}{rcl} |Z \cap A| & = & |(X \cup Y) \cap A| + 1 \\ (X \cup Y) \cap A| & = & |X \cap A| + |Y \cap A| - |X \cap Y \cap A| \\ |X \cap Y \cap B| + 2 & \leqslant & |X \cap Y \cap A| \\ |X \cap B| + |Y \cap B| & = & |Z \cap B| + |X \cap Y \cap B|. \end{array} \tag{2}$$

Let us now add equations (1) and (2); after simplifications, we deduce that $|Z \cap A| \leqslant |Z \cap B| + 1$. As $N(Z \cap B) \subseteq Z \cap A$, we conclude, by Proposition 5, that equality holds. By Proposition 9, $\partial(Z)$ is tight, hence trivial. Thus, $\overline{Z}$ is a singleton, say, $\{y_1\}$. Moreover, as x_1x_2 and x_1x_4 are the only edges of $\partial(\overline{X} \cap \overline{Y})$ incident with a vertex in $\overline{X} \cap \overline{Y} \cap A$, we conclude that $\overline{X} \cap \overline{Y} = \{x_1, y_1\}$ and $N(x_1) = \{x_2, x_4, y_1\}$. □

Lin and Lu proved originally the following assertion [11].

Theorem 21. *Let $G[A, B]$ be a simple brace and let Q be a 4-cycle of G. If two adjacent vertices of Q have degree three then one of the edges of Q is thin in G.*

Proof. Let $Q := (x_1, x_2, x_3, x_4, x_1)$ be a 4-cycle of G. Adjust notation so that $x_1, x_3 \in B$ and $x_2, x_4 \in A$. Suppose that $d(x_3) = 3 = d(x_4)$. As G is a simple brace, it follows that G has six or more vertices. Assume, to the contrary, that no edge of Q is thin in G. For $i = 1, 2, 3, 4$, let $C_i := \partial(X_i)$ be a cut of G that certifies that the edge x_ix_{i+1} is not thin and adjust notation so that $X_{i+} \subset A$ (indices are taken modulo four). As will be seen, the hypothesis that no edge of Q is thin implies that the four vertices of Q have degree three. For each vertex x_i of Q that has degree three, let y_i denote the vertex adjacent to x_i which is not in Q.

The cuts C_1 and C_2

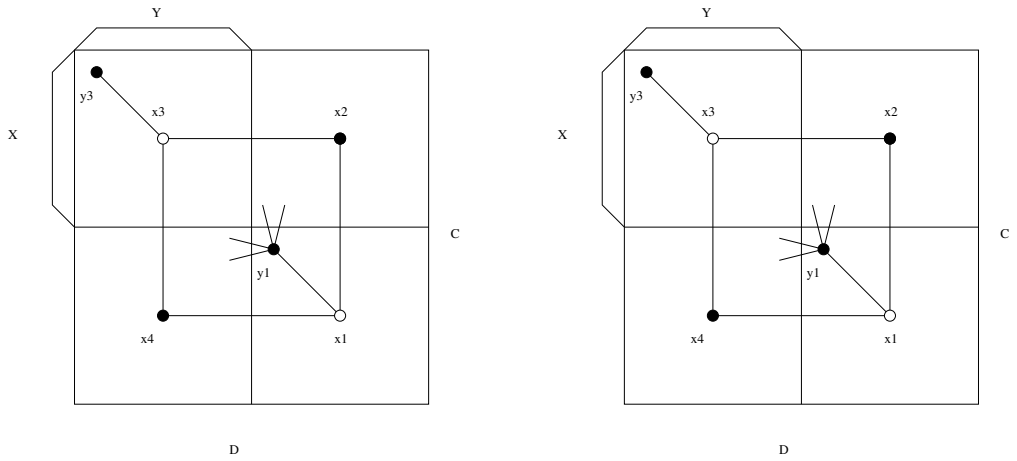
Let us consider the edges x_1x_2 and x_2x_3, certified respectively by the cuts C_1 and C_2. Then, $x_2 \in \overline{X_1} \cap \overline{X_2}$, $x_1 \in X_1 \cap \overline{X_2}$, $x_3 \in \overline{X_1} \cap X_2$ and $x_4 \in X_1 \cap X_2$. See Figure 5(a). By hypothesis, $d(x_4) = 3$. By Lemma 18,

$$\{x_4, y_4\} \subseteq X_1 \cap X_2 \tag{3}$$
$$d(x_2) = 3 \tag{4}$$
$$\{x_2, y_2\} = \overline{X_1} \cap \overline{X_2}. \tag{5}$$

The cuts C_3 and C_4

Let us consider the edges x_3x_4 and x_4x_1, certified respectively by the cuts C_3 and C_4. Then, $x_4 \in \overline{X_3} \cap \overline{X_4}$, $x_3 \in X_3 \cap \overline{X_4}$, $x_1 \in \overline{X_3} \cap X_4$ and $x_2 \in X_3 \cap X_4$. See Figure 5(b). By (4), $d(x_2) = 3$. By Lemma 18,

$$\{x_2, y_2\} \subseteq X_3 \cap X_4. \tag{6}$$
$$\{x_4, y_4\} = \overline{X_3} \cap \overline{X_4}. \tag{7}$$



(a) (b)

Figure 5: The pairs C_1, C_2 and C_3, C_4.

The cuts C_4 and C_1

Let us consider the edges x_4x_1 and x_1x_2, certified respectively by the cuts C_4 and C_1. Then, $x_1 \in X_4 \cap X_1$, $x_4 \in \overline{X_4} \cap X_1$, $x_2 \in X_4 \cap \overline{X_1}$ and $x_3 \in \overline{X_4} \cap \overline{X_1}$. See Figure 6(a). By hypothesis, $d(x_3) = 3$. By Lemma 18,

$$\begin{aligned} \{x_3, y_3\} &\subseteq \overline{X_4} \cap \overline{X_1} && (8)\\ d(x_1) &= 3 && (9)\\ \{x_1, y_1\} &= X_4 \cap X_1. && (10) \end{aligned}$$

The cuts C_2 and C_3

Let us consider the edges x_2x_3 and x_3x_4, respectively of the the cuts C_2 and C_3. Then, $x_3 \in X_2 \cap X_3$, $x_2 \in \overline{X_2} \cap X_3$, $x_4 \in X_2 \cap \overline{X_3}$ and $x_1 \in \overline{X_2} \cap \overline{X_3}$. See Figure 6(b). By (9), $d(x_1) = 3$. By Lemma 18,

$$\begin{aligned} \{x_1, y_1\} &\subseteq \overline{X_2} \cap \overline{X_3} && (11)\\ \{x_3, y_3\} &= X_2 \cap X_3. && (12) \end{aligned}$$

From (7) and (3) we infer that

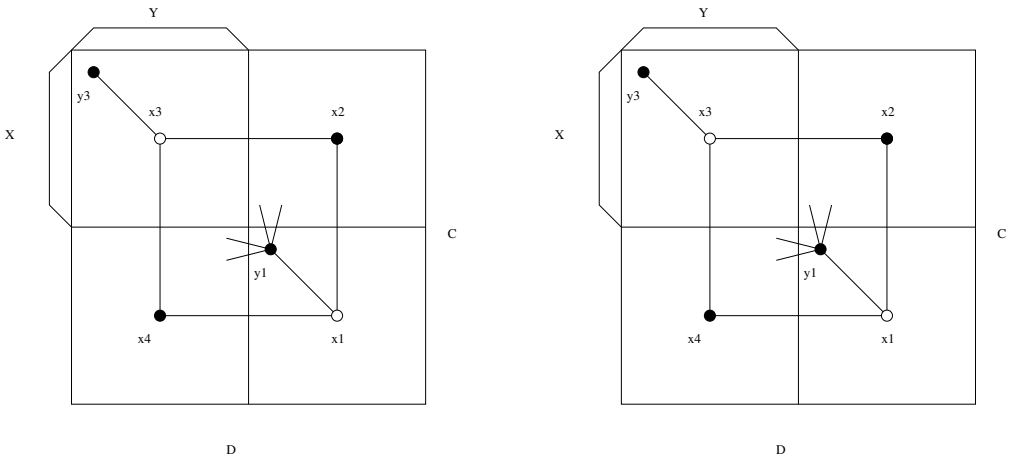
$$\overline{X_3} \cap \overline{X_4} = \{x_4, y_4\} = X_1 \cap X_2 \cap \overline{X_3} \cap \overline{X_4}.$$

From (5) and (6), we deduce that

$$\overline{X_1} \cap \overline{X_2} = \{x_2, y_2\} = \overline{X_1} \cap \overline{X_2} \cap X_3 \cap X_4.$$

From (10) and (11) we infer that

$$X_1 \cap X_4 = \{x_1, y_1\} = X_1 \cap \overline{X_2} \cap \overline{X_3} \cap X_4.$$



(a) (b)

Figure 6: The pairs C_4, C_1 and C_2, C_3.

From (12) and (8), we deduce that

$$X_2 \cap X_3 = \{x_3, y_3\} = \overline{X_1} \cap X_2 \cap X_3 \cap \overline{X_4}.$$

Figure 7 depicts the empty sets as dashed lines. It turns out that every vertex y_i, $1 \leqslant i \leqslant 4$, has degree one. For example, let z be a vertex adjacent to y_1. The vertex y_1 is in $\overline{X_2} \cap A$, in turn a subset of $\overline{X_2}_-$. The edge x_2x_3 is the only edge of C_2 incident with a vertex in $\overline{X_2}_-$. Thus, $z \in \overline{X_2}$. Likewise, y_1 is in $\overline{X_3} \cap A$, in turn a subset of $\overline{X_3}_-$ and x_3x_4 is the only edge of C_3 incident with a vertex in $\overline{X_3}_-$, hence $z \in \overline{X_3}$. In sum, $z \in \overline{X_2} \cap \overline{X_3}$. The only nonempty subset of $\overline{X_2} \cap \overline{X_3}$ is $X_1 \cap \overline{X_2} \cap \overline{X_3} \cap X_4$, which consists solely of x_1 and y_1. Thus, $z = x_1$, hence y_1 has degree one in G. This is a contradiction, because G is not even matching covered! □

Lemma 22. *Let $Q := (y_0, y_1, y_2, y_3, y_0)$ be a 4-cycle of a brace G, where $d(y_0) = 3$. Let $e_0 := x_0y_0$, $e_1 := y_1y_2$ and $e_2 := y_2y_3$, where $e_0 \notin E(Q)$. See Figure 8(a). Suppose that the graph $\widehat{G - e_0 - e_1}$ is a brace. If e_2 is not thin in G then G has two vertices x_1 and x_2, both distinct from y_2, such that $N(x_0) = \{x_1, x_2, y_0\}$ and $N(y_3) \subseteq \{x_1, x_2, y_0, y_2\}$. See Figure 8(b).*

Proof. Let $H := \widehat{G - e_0}$, and $L := \widehat{G - e_0 - e_2}$. Note that, in H, the edges e_1 and e_2 are parallel, hence $H - e_1 \cong H - e_2$. By Theorem 11,

$$L = \widehat{G - e_0 - e_2} \cong \widehat{H - e_2} \cong \widehat{H - e_1} \cong \widehat{G - e_0 - e_1}.$$

By hypothesis, $\widehat{G - e_0 - e_1}$ is a brace, hence so too is L. Suppose that e_2 is not thin in G. By Proposition 14, G has a nontrivial cut $C := \partial(X)$ such that $C - e_2$ is tight in $G - e_2$ and $|X|, |\overline{X}| \geqslant 5$. See Figure 8(a). By Proposition 9,

$$|X_+| = |X_-| + 1 \quad \text{and} \quad |\overline{X}_+| = |\overline{X}_-| + 1. \tag{13}$$

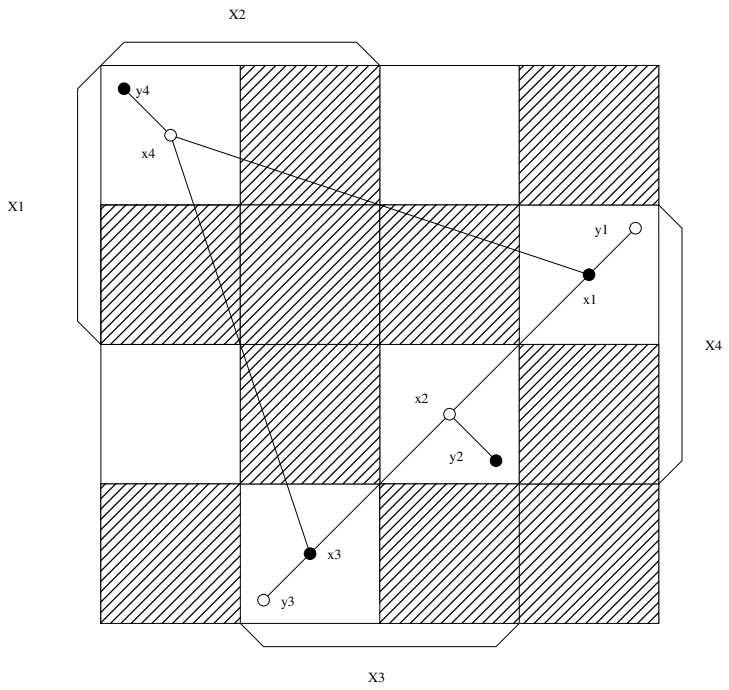


Figure 7: The empty sets represented by dashed lines.

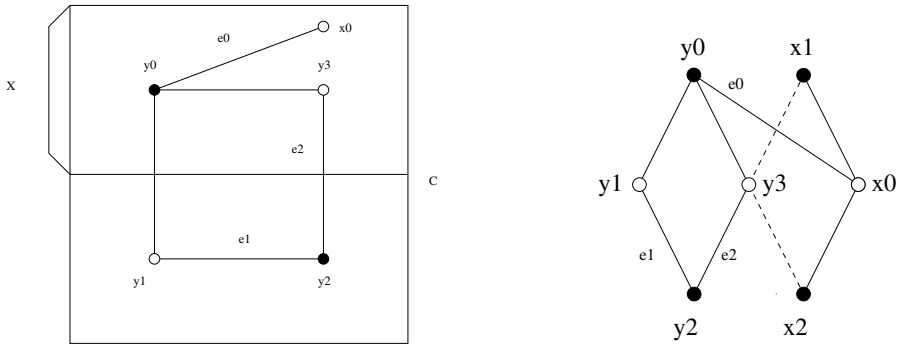


Figure 8: Illustration for Lemma 22.

By Corollary 10, e_2 is the only edge of C that is incident with a vertex in X_-. Adjust notation so that $y_3 \in X_-$. The vertex y_0, which is adjacent to y_3, must be in X_+. Likewise, the vertex y_2 is in $\overline{X}_-$, hence $y_1 \in \overline{X}_+$. See Figure 8(a).

Claim 23. $N(X_- - x_0 - y_3) \subseteq X_+ - y_0$.

Proof. Let xy be an edge of G where $x \in X_- - x_0 - y_3$. The edge y_3y_2 is the only edge of C incident with a vertex in X_-. As $x \neq y_3$, we conclude that $y \in X_+$. Moreover, by hypothesis, $N(y_0) = \{x_0, y_1, y_3\}$ and $y_1 \in \overline{X}$, hence $y \neq y_0$. □

Claim 24. *The vertex x_0 is in X_-.*

Proof. Assume, to the contrary, that $x_0 \notin X_-$. The vertex x_0 is adjacent to y_0, in turn a vertex of X_+. Thus, $x_0 \in \overline{X}$. Let $Y := X - y_0 - y_3$. Then, $Y_+ = X_+ - y_0$ and $Y_- = X_- - y_3$. Moreover, by Claim 23, $N(Y_-) \subseteq Y_+$. By (13), $|Y_+| = |Y_-| + 1$, hence $\partial(Y)$ is tight in G, by Proposition 9. As G is a brace and $y_0, y_3 \in \overline{Y}$, we infer that Y is a singleton, hence $|X| = 3$, a contradiction to the hypothesis that $|X| \geqslant 5$. □

Let X_L and $\overline{X}_L$ denote the result in L of the contractions of X and $\overline{X}$, respectively, due to bicontractions of ends of e_0 and e_2, such that $\partial_L(X_L)$ is tight in L. Then, one of X_L and $\overline{X}_L$ must be a singleton, because L is a brace. The vertex y_2 is the only end in $\overline{X}$ that may possibly have degree two in $G - e_0 - e_2$. Thus, $|\overline{X}_L| \geqslant |\overline{X}| - 2 \geqslant 3$. Hence, X_L is a singleton. The vertices in X that are ends the edges e_0 and e_2 are x_0, y_0 and y_3. The vertices y_0 and y_3 are adjacent and y_0 has degree three. By Corollary 13,

$$|X_L| = \begin{cases} |X| - 2, & \text{if } d(x_0) \geqslant 4 \\ |X| - 4, & \text{if } d(x_0) = 3. \end{cases}$$

The set X_L must be a singleton and $|X| \geqslant 5$, hence $|X| = 5$ and $d(x_0) = 3$. Let x_1 and x_2 be the two neighbours of x_0 distinct from y_0.

The vertices x_1 and x_2 must be in X_+, together with y_0. By (13), $X_+ = \{x_1, x_2, y_0\}$ and $X_- = \{x_0, y_3\}$. See Figure 9.

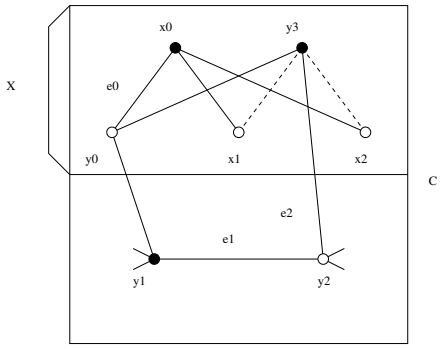


Figure 9: At least one of the dashed lines represents an edge.

The vertex y_3 must be adjacent to at least one of the vertices x_1 and x_2. Moreover, $N(y_3) \subseteq \{x_1, x_2, y_0, y_2\}$.

The vertex y_2 is in $\overline{X}$, whereas the vertices x_1 and x_2 are in X, hence $y_2 \notin \{x_1, x_2\}$. □

Corollary 25. *Let $Q := (y_0, y_1, y_2, y_3, y_0)$ be a 4-cycle and e_0, e_1 and e_2 be edges of a brace G as in statement of Lemma 22. Suppose that e_1 is thin in G and e_0 is thin in $H := \widehat{G - e_1}$. Then, e_2 is thin in G or G has a thin edge not adjacent to e_1.*

Proof. By Theorem 11, $\widehat{G - e_0 - e_1} \cong \widehat{H - e_0}$. By hypothesis, e_0 is thin in H, hence $\widehat{H - e_0}$ is a brace. Thus, $\widehat{G - e_0 - e_1}$ is also a brace.

Suppose that e_2 is not thin in G. By Lemma 22, the brace G has two vertices x_1 and x_2, both distinct from y_2, and such that $N(x_0) = \{x_1, x_2, y_0\}$ and $N(y_3) \subseteq \{x_1, x_2, y_0, y_2\}$.

See Figure 8(b). The vertex y_3 must be adjacent to at least one of the vertices x_1 and x_2. Let $j \in \{1, 2\}$ such that y_3 and x_j are adjacent. Consider now the 4-cycle $Q := (x_0, y_0, y_3, x_j, x_0)$. The vertices x_0 and y_0 are adjacent and both have degree three. By Theorem 21, Q contains an edge, say, f, which is thin in G. The vertices x_1 and x_2 are both distinct from y_2, hence the edge e_1 is not incident with any vertex of Q. We conclude that e_1 and f are not adjacent thin edges of G. □

Corollary 26. *Let Q be a 4-cycle and e_0, e_1 and e_2 be the edges of a brace G as in the statement of Lemma 22. Suppose that e_0 is thin in G but e_2 is not thin in G. Then, G has two vertices x_1 and x_2, both distinct from y_2, such that $N(x_0) = \{x_1, x_2, y_0\}$ and $N(y_3) \subseteq \{x_1, x_2, y_0, y_2\}$. See Figure 8(b).*

Proof. By hypothesis, e_0 is thin in G, hence $H := \widehat{G - e_0}$ is a brace. In the brace H, the edges e_1 and e_2 are parallel, hence $H - e_1$ is also a brace. By Theorem 11, $\widehat{G - e_0 - e_1} \cong H - e_1$, hence $\widehat{G - e_0 - e_1}$ is a brace. Suppose that e_2 is not thin in G. By Lemma 22, G has two vertices x_1 and x_2, both distinct from y_2, such that $N(x_0) = \{x_1, x_2, y_0\}$ and $N(y_3) \subseteq \{x_1, x_2, y_0, y_2\}$. □

Corollary 27. *Let Q be a 4-cycle and e_0, e_1 and e_2 be edges of a brace G as in statement of Lemma 22. Suppose that e_0 is thin in G. Then, at least one of the edges e_1 and e_2 is thin in G.*

Proof. By hypothesis, the edge e_0 is thin in G, hence H is a brace. The edges e_1 and e_2 are parallel in H, thus $H - e_1$ and $H - e_2$ are braces.

Suppose that neither e_1 nor e_2 is thin in G. By Corollary 26, the brace G has two vertices x_1 and x_2, both distinct from y_2 and such that $N(x_0) = \{x_1, x_2, y_0\}$ and $N(y_1), N(y_3) \subseteq \{x_1, x_2, y_0, y_2\}$. See Figure 10.

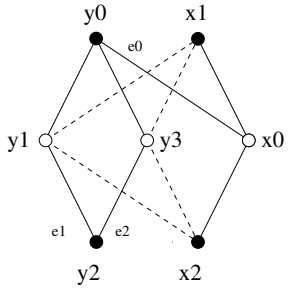


Figure 10: Illustration for Corollary 27.

Let $X := \{x_1, x_2, y_0, y_2\}$ and let $Y := \{x_0, y_1, y_3\}$. Notice that $N(Y) \subseteq X$. Moreover, $|X| = 4$ and $|Y| = 3$, hence $|X| = |Y| + 1$. Thus, $D := \partial(X \cup Y)$ is a tight cut of the brace G, hence D is trivial. We conclude that G has order eight. By Proposition 14, every edge of G is thin. In particular, e_1 and e_2 are thin in G. □

4 Proof of the Main Theorem

Proof. Let $G[A, B]$ be a simple brace of order six or more. We prove that G has two nonadjacent thin edges, by induction on $|E|$. We begin the proof by considering a simple case in which the inductive hypothesis is not used.

Suppose that $|V| < 10$. This case is actually the basis of the inductive argument. By Proposition 16, every edge of G is thin. The assertion of the Theorem holds immediately.

We may thus assume that G has 10 or more vertices. Recall that in Section 3.1 we define $\mathcal{G}$ to be the collection of prisms, Möbius ladders and biwheels. If $G \in \mathcal{G}$ then as noted, G has two or more nonadjacent thin edges. We may thus assume that G is not in the family $\mathcal{G}$. By Theorem 17, G has a strictly thin edge, say, e. Then, $H := \widehat{G - e}$ is a simple brace of order six or more.

By induction, H has two nonadjacent thin edges, say, f and f'. Clearly, the edges f and f' are not adjacent in G. If both f and f' are thin in G then the assertion of the Theorem holds. We may thus assume that one of them, say f, is not thin in G. By Proposition 14, the brace G has a cut C whose shores both have five or more vertices and such that $C - f$ is tight in $G - f$. Recall that we refer to the cut C as a certificate that edge f is not thin in G.

Claim 28. *The graph $G - e - f$ is matching covered and the cut $C - e - f$ is tight in $G - e - f$.*

Proof. The graph H is a brace of order six or more. By Theorem 15, the graph $H - f$ is matching covered. That graph is obtained from the graph $G - e - f$ by the bicontraction of two or fewer vertices. By the application of Proposition 7 at most twice, we deduce that $G - e - f$ is matching covered.

The cut $C - f$ is tight in $G - f$. Every perfect matching of $G - e - f$ is a perfect matching of $G - f$, hence it has just one edge in $C - e - f$. We conclude that $C - e - f$ is tight in $G - e - f$. □

The possible bicontractions of the ends of e reduce at least one shore of C, say, X, to a set X_H of at most three vertices, where C_H denotes the cut $\partial_H(X_H)$ such that $C - f$ is tight in $G - f$, $C_H - f$ is tight in $H - f$ and $|X_H| \leqslant 3$. The brace G is simple, hence the ends of e are not adjacent in $G - e$. By Corollary 13, we immediately deduce the following assertion.

Claim 29. *Let $\eta := \{v : v \in X,\ v$ is an end of e and $d(v) = 3\}$. Then, $|X_H| = |X| - 2|\eta|$.* □

Thus, at least one end of e is in X and has degree three. Let us assume that $X_+ \subset A$. We denote by r_0 and s_0 the ends of e in A and in B, respectively. The edge e has at least one end in X. If it has an end in X_- then that end is s_0 and naturally $r_0 \in X_+$. Alternatively, if e has no end in X_- then it must have an end in X_+, which is r_0. We conclude that $r_0 \in X_+$ in all alternatives.

We denote by t_0 and u_0 the ends of f in A and in B, respectively. We denote the neighbours of r_0 by r_i, $1 \leqslant i \leqslant d(r_0) - 1$. We adopt similar notations for $N(s_0) - r_0$, for $N(t_0) - u_0$ and for $N(u_0) - t_0$.

The nontrivial cut $C - f$ is tight in $G - f$ and G is a brace, thus, by Corollary 10, f is the only edge of G that joins a vertex in X_- to a vertex in $\overline{X}_-$; hence, $u_0 \in X_-$ and $t_0 \in \overline{X}_-$. Moreover, $C - e - f$ is tight in $G - e - f$, hence $(G - e - f)[X]$ is connected, by Proposition 8. Thus, at least one neighbour of r_0 in $G - e$ is in X_-. We adopt the convention that $r_1 \in X_-$. The hypothesis that e is strictly thin immediately implies the following statement.

Claim 30. *If $d(r_0) = 3$ then no vertex of $G - r_0$ is adjacent to both r_1 and r_2. Likewise, if $d(s_0) = 3$ then no vertex in $G - s_0$ is adjacent to both s_1 and s_2. Moreover, if $d(r_0) = 3 = d(s_0)$ then the set $F := \{r_i s_j : i, j \in \{1, 2\}\}$ contains at most one edge.*

We now indicate the five possible configurations of cut C.

The five configurations

In the first configuration, the edge e is in C, a certificate cut of edge f.

Configuration: the edge e is in C

The vertex r_0 is in X_+ and $r_1 \in X_-$. The vertex s_0 is in $\overline{X}$. By Claim 29, $|X| = 5$ and $d(r_0) = 3$. Let u_1 and u_2 denote the two vertices of $X_+ - r_0$. The vertex of X_- distinct from the end u_0 of f must be adjacent to the three vertices of X_+. In particular, it is adjacent to r_0, thus we may assume, without loss of generality, that it is r_1.

The vertex u_0 must be adjacent to at least two vertices of X_+, hence it must be adjacent to at least one of the vertices u_1 and u_2. Without loss of generality, we may assume that u_0 is adjacent to u_1. Then, u_0 is not adjacent to r_0, otherwise the bicontraction of r_0 in $G - e$ would render the edges r_1u_1 and u_0u_1 parallel in H. This is not possible, as e is strictly thin. Thus, u_0 is adjacent to u_2. We conclude that $N(u_0) = \{t_0, u_1, u_2\}$ and $r_2 \in \overline{X}$. See Figure 11(a).

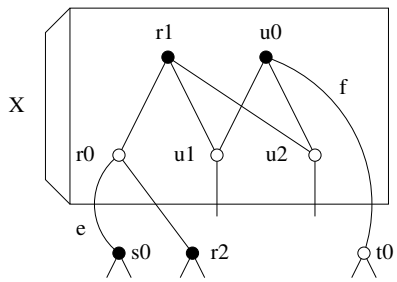
In the last four configurations, $e \in E(G[X])$, with $r_0 \in X_+$ and $s_0 \in X_-$. The edge f is in H. Thus, if f is incident with vertex s_0, then $d(s_0) \geqslant 4$. Hence, regardless of its degree, s_0 is adjacent in $G - e$ to at least two vertices in X_+. We adjust notation so that s_1 and s_2 are in X_+. Recall that we have assumed, in all cases, that r_1 is in X_-. In sum,

Claim 31. *If $e \in E(G[X])$ then $\{r_0, s_1, s_2\} \subseteq X_+$ and $\{r_1, s_0\} \subseteq X_-$.*

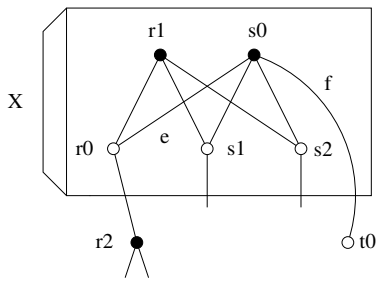
As $|X_H| \leqslant 3$ and $|X| \geqslant 5$, it follows, by Claim 29, that at least one of $d(r_0)$ and $d(s_0)$ is equal to three.

Configuration: $e \in E(G[X])$, $d(r_0) = 3$ and $d(s_0) \geqslant 4$

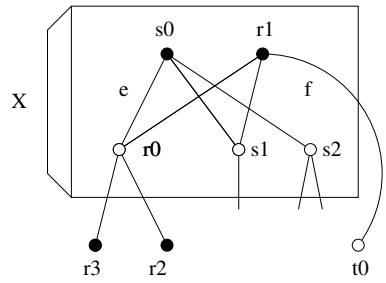
By Claim 29, $|X| = 5$, hence $|X_+| = 3$ and $|X_-| = 2$. By Claim 31, $X_+ = \{r_0, s_1, s_2\}$ and $X_- = \{r_1, s_0\}$. The hypothesis of the case assumes that $d(s_0) \geqslant 4$. It follows that $d(s_0) = 4$ and s_0 is the end u_0 of f. See Figure 11(b).



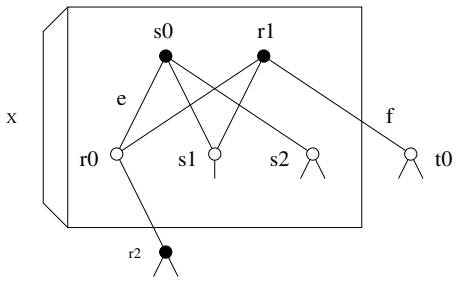
(a) $e \in C$



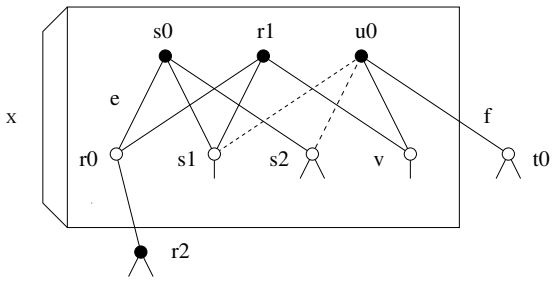
(b) $d(r_0) = 3, d(s_0) \geqslant 4$



(c) $d(r_0) \geqslant 4, d(s_0) = 3$



(d) $d(r_0) = 3 = d(s_0), |X| = 5$



(e) $d(r_0) = 3 = d(s_0), |X| = 7$

Figure 11: The five configurations.

Configuration: $e \in E(G[X])$, $d(r_0) \geqslant 4$ and $d(s_0) = 3$

As $d(r_0) \geqslant 4$, we deduce, by Claim 29, that $|X| = 5$, hence $|X_+| = 3$ and $|X_-| = 2$. By Claim 31, $X_+ = \{r_0, s_1, s_2\}$ and $X_- = \{r_1, s_0\}$. As $d(s_0) = 3$, then $N(s_0) = X_+$ hence s_0 is not an end of f. Thus, r_1 is u_0, the end of f in X_-. The vertex r_1 must be adjacent to at least one of s_1 and s_2. By Claim 30, the set F is a singleton. We assume that r_1 is adjacent to s_1, hence $F = \{r_1 s_1\}$. See Figure 11(c).

Configuration: $e \in E(G[X])$, $d(r_0) = 3 = d(s_0)$ and $|X| = 5$

By Claim 31, $X_+ = \{r_0, s_1, s_2\}$ and $X_- = \{s_0, r_1\}$. As $f \in H$ and $d(s_0) = 3$, the edges e and f are nonadjacent. Thus, r_1 is the end u_0 of edge f in X_-. Moreover, r_1 must be adjacent to a vertex in $\{s_1, s_2\}$ We adjust notation so that r_1 and s_1 are adjacent. By Claim 30, $F = \{r_1 s_1\}$. See Figure 11(d).

Configuration: $e \in E(G[X])$, $d(r_0) = 3 = d(s_0)$ and $|X| = 7$

By Claim 31, $\{r_0, s_1, s_2\} \subset X_+$ and $\{r_1, s_0\} \subset X_-$. As $d(s_0) = 3$, it follows that $N(s_0) = \{r_0, s_1, s_2\}$. Let v be a vertex of G not adjacent to s_0 such that $X_+ = \{r_0, s_1, s_2, v\}$.

The vertex of $X_- - s_0 - u_0$ must be adjacent to three vertices of X_+. No vertex of $X_- - s_0$ is adjacent to both s_1 and s_2. Thus, a vertex of X_- is adjacent to r_0, to v and to one of s_1 and s_2. We have assumed that $r_1 \in X_-$. Thus, we may assume that r_1 is adjacent to one of s_1 and s_2 and also to v. We adjust notation so that r_1 and s_1 are adjacent. By Claim 30, $F = \{r_1 s_1\}$.

The third vertex of X_- is u_0, it must be adjacent to at least two vertices of X_+. If the vertex u_0 is not adjacent to any vertex in $\{s_1, s_2\}$ then u_0 is adjacent to both r_0 and v, hence, in H, the edges $r_1 v$ and $u_0 v$ are parallel, in contradiction to the hypothesis that e is strictly thin. Thus, u_0 is adjacent to (precisely) one of s_1 and s_2, say, s_j. It follows that u_0 is not adjacent to r_0, otherwise $u_0 = r_2$ and $r_2 s_j \in F$, a contradiction. We conclude that $N(u_0) = \{s_j, v, t_0\}$. See Figure 11(e).

A simple inspection of the five configurations in Figure 11 illustrates the following assertion.

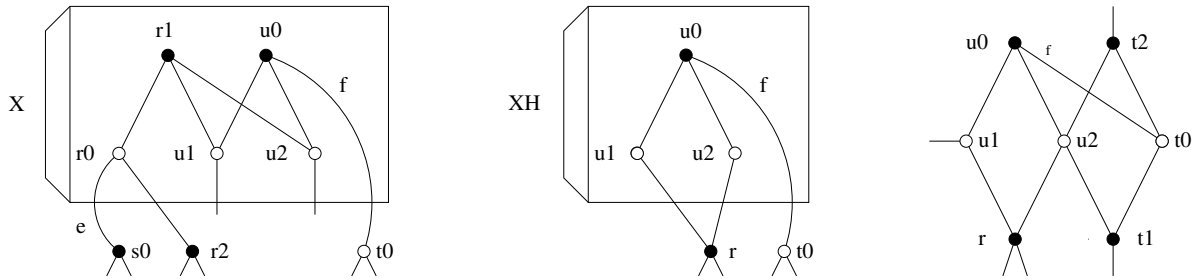
Claim 32. *Let f be an edge that is thin in H. If f joins a vertex in $\{r_i : i = 0, 1, 2\}$ to a vertex in $\{s_i : i = 0, 1, 2\}$ then f is thin in G.* □

If f' is not thin in G then similar configurations hold for f' and its cut certificate $C' := \partial(X')$.

We assume that f is not thin in G and consider its certificate cut $C := \partial(X)$. There are five cases, corresponding to the five configurations. Sometimes we may have to consider a change of notation, in which $X_+ \subset B$, implying the reversal of the roles of r_i and s_i, and of t_i and u_i.

Case 1. The edge e is in C.

This configuration is shown in Figure 12(a). The graph H is depicted in Figure 12(b).



(a) G (b) H (c) Q_1, Q_2

Figure 12: Illustration for Case 1.

Note that $Q := (u_0, u_1, r_1, u_2, u_0)$ is a quadrilateral of G. Moreover, both r_1 and u_0 have degree three in G. If, in addition, one of u_1 and u_2 also has degree three in G, then, by Theorem 21, Q contains an edge which is thin in G. In that case, the assertion of the Theorem holds, as e is not incident with any vertex of Q. We may thus assume that $d(u_1), d(u_2) \geqslant 4$.

Let r be the vertex of H obtained by the bicontraction of vertex r_0 in $G-e$. Note that $Q_H := (u_0, u_1, r, u_2, u_0)$ is a quadrilateral of H, the edge f is not in Q_H, but is incident with vertex u_0, which has degree three.

Suppose that one of ru_1 and ru_2 is not thin in H. Without loss of generality, assume that ru_2 is not thin in H. By Lemma 22, $N_H(t_0) = \{u_0, t_1, t_2\}$ and $N_H(u_2) \subseteq \{r, u_0, t_1, t_2\}$. As $d(u_2) \geqslant 4$, we deduce that $N_H(u_2) = \{r, u_0, t_1, t_2\}$. See Figure 12(c). Thus, H has two quadrilaterals, namely, $Q_1 = (u_0, u_2, t_1, t_0, u_0)$ and $Q_2 = (u_0, u_2, t_2, t_0, u_0)$. Since $N_H(t_0) = \{u_0, t_1, t_2\}$, at least one of t_1 and t_2 is distinct from s_0. Applying Theorem 21 to the quadrilateral Q_1, we deduce that one of the edges of Q_1 is thin in G. No edge of Q_1 is adjacent to e, hence the assertion of the Theorem holds. We may thus assume that both ru_1 and ru_2 are thin in H. That is, both edges r_1u_1 and r_1u_2 are thin in H.

We shall now prove that one of these two edges is thin in G. For this, assume the contrary. For $i = 1, 2$, let $C_i := \partial(X_i)$ be a certificate cut associated with r_1u_i. That is, $|X_i|, |\overline{X_i}| \geqslant 5$ and $C_i - r_1u_i$ is tight in $G - r_1u_i$. Adjust notation so that $r_1 \in \overline{X_1} \cap \overline{X_2}$. The vertex u_1 is in X_1 and the vertex u_2 is in X_2. By Lemma 18, $\overline{X_1} \cap \overline{X_2} = \{r_1, r_0\}$. Thus, $s_0 \notin \overline{X_1} \cap \overline{X_2}$. That is, $s_0 \in X_1 \cup X_2$. Without loss of generality, assume that $s_0 \in X_1$. In Figure 13 we consider two cases, depending on whether $s_0 \in X_2$.

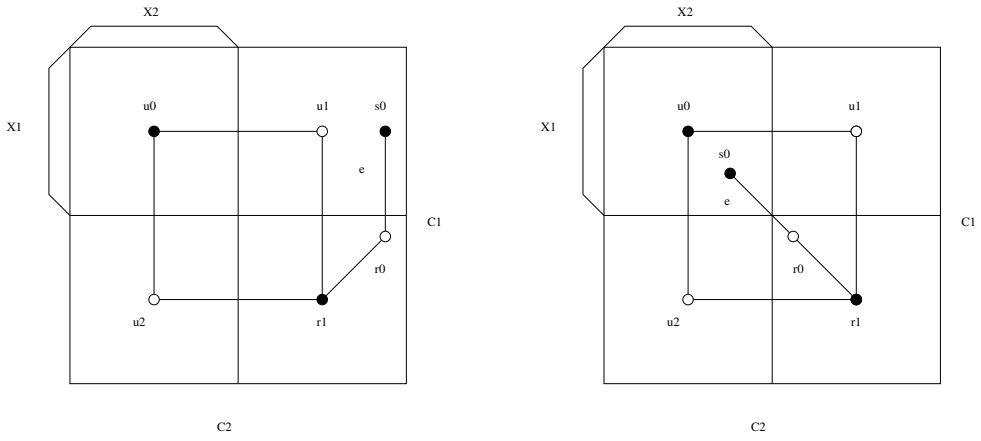


Figure 13: The cuts C_1 and C_2.

The edge r_1u_1 is thin in H. Thus, in the graph $L := \widehat{G - e - r_1u_1}$, one of X_1 and $\overline{X_1}$ is contracted to a single vertex, say, Y. Let us consider first the shore $\overline{X_1}$. The only ends of edges e and r_1u_1 in $\overline{X_1}$ are r_0 and r_1. Both have degree three but they are adjacent in $G - e - r_1u_1$. By Corollary 13, $|Y| = |\overline{X_1}| - 2$, hence $|Y| \geqslant 3$, a contradiction.

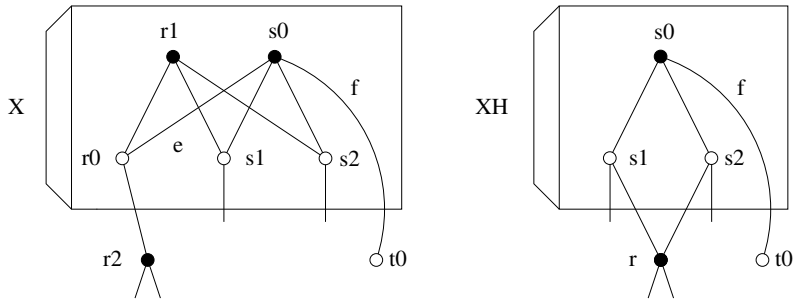
We may thus assume that X_1 is contracted to the singleton Y. The brace L is the retract of $G - e - r_1u_1$. In X_1, the only ends of the edges e and r_1u_1 are s_0 and u_1. The

vertex u_1 has degree four or more, hence, by Corollary 13, $|Y| = |X_1| - 2 \geqslant 3$. In both alternatives we derived a contradiction. Indeed, one of the edges r_1u_1 and r_1u_2 is thin in G. Neither of these two edges is adjacent to e. The assertion of the Theorem holds in Case 1. △

We may thus assume that $e \notin C$. That is, $e \in E(G[X])$. If f' is not thin in G then we may also assume that $e \in E(G[X'])$.

Case 2. $e \in E(G[X])$, $d(r_0) = 3$ and $d(s_0) \geqslant 4$.

By Claim 29, $|X| = 5$, hence $|X_+| = 3$ and $|X_-| = 2$. By Claim 31, $X_+ = \{r_0, s_1, s_2\}$ and $X_- = \{r_1, s_0\}$. The hypothesis of the case assumes that $d(s_0) \geqslant 4$. It follows that $d(s_0) = 4$ and s_0 is the end u_0 of f. See Figure 14(a).



(a) G (b) H

Figure 14: Illustration for Case 2.

Let r be the contraction vertex of H. See Figure 14(b). Let Q be the quadrilateral (r, s_1, s_0, s_2, r) in H. The edge e is strictly thin, hence H is simple. Thus, the vertex r, a contraction vertex of H, has degree four or more in H. The vertex s_0 has degree three in H. By Lemma 18, one of the edges rs_1 and rs_2 is thin in H. Without loss of generality, suppose that rs_1 is thin in H. That is, the edge r_1s_1 of G is thin in H. By Claim 32, the edge r_1s_1 is thin in G. The edges r_1s_1 and e are not adjacent. The assertion of the Theorem holds in Case 2. △

Case 3. $e \in E(G[X])$, $d(r_0) \geqslant 4$ and $d(s_0) = 3$.

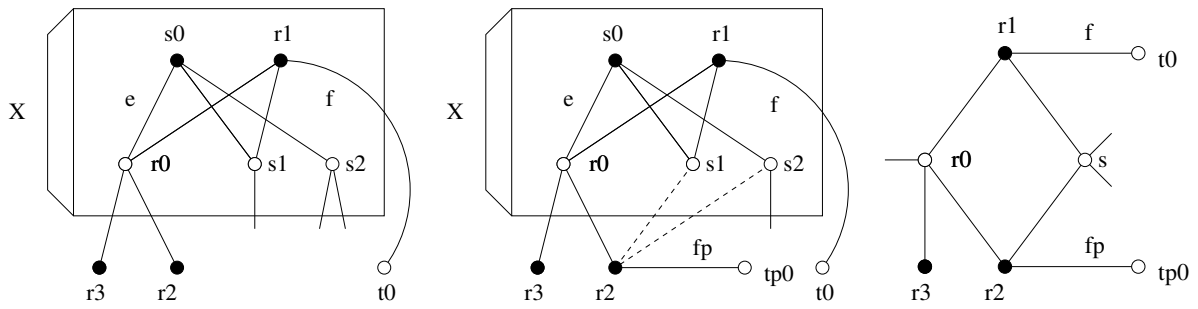
Figure 15(a) depicts the configuration in analysis.

The graph G has the quadrilateral $(r_1, r_0, s_0, s_1, r_1)$. The edge e is thin in G and f is thin in H. By Corollary 25, either s_0s_1 is thin in G or G has a thin edge not adjacent to e. We may thus assume that s_0s_1 is thin in G.

Recall that H has two nonadjacent thin edges, f and f'.

Case 3.1. The edges e and f' are adjacent.

If f' is not thin in G then Case 2 applies, with f' playing the role of f. If f' is thin in G then f' and s_0s_1 are nonadjacent thin edges of G. In both alternatives the assertion holds.



(a) G (b) f' (c) H

Figure 15: Illustration for Case 3.

We may thus assume that e and f' are not adjacent. If f' is thin in G then the assertion of the Theorem holds. We may then suppose that f' is not thin in G. The edges f and f' are not adjacent, hence f' is not incident with r_1. The vertex r_0 has degree four or more in G and the vertex s_0 has degree three in G. This difference in the degrees implies that likewise edge f, the edge f' must be incident with r_0 or with a vertex in $N(r_0) - s_0 - r_1$. The edges e and f' are not adjacent, hence f' is incident with a vertex in $N(r_0) - r_1 - s_0$. Without loss of generality, we may suppose that f' is incident with r_2.

The end r_2 of f' has similar properties of the end r_1 of f. Thus, r_2 has degree three and r_2 is adjacent to r_0 and to precisely one vertex in $\{s_1, s_2\}$. See Figure 15(b).

Let $j \in \{1, 2\}$ such that r_2 is adjacent to s_j. The graph H then has the quadrilateral $Q := (r_1, s, r_2, r_0, r_1)$. See Figure 15(c). As $d_H(r_1) = 3$, then, by Corollary 27, one of the edges r_0r_2 and r_2s, denoted f'', is thin in H. If e and f'' are adjacent then Case 2 applies, with f'' playing the role of f'. Thus, $f'' \neq r_0r_2$, hence r_2s is thin in H. That is, the edge r_2s_j is thin in H. By Claim 32, r_2s_j is thin in G. The assertion of the Theorem holds, with e and r_2s_j two nonadjacent thin edges of G. In all alternatives under consideration, the assertion of the Theorem holds in Case 3. △

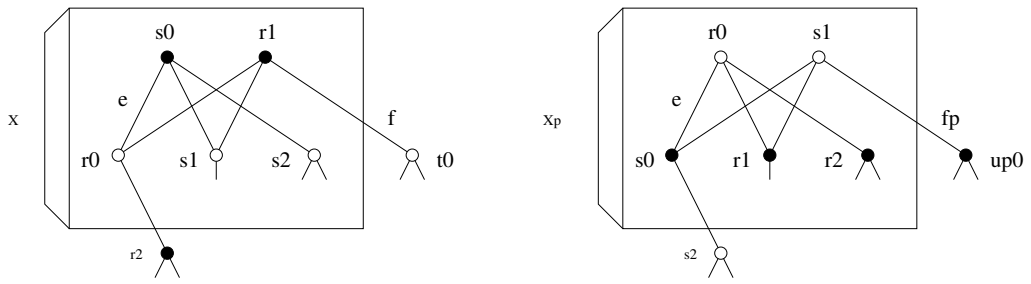
It remains to consider the alternative in which $d(r_0) = 3 = d(s_0)$. In this case, neither r_0 nor s_0 are vertices of H. Thus, neither f nor f' is adjacent to e in G. If one of f and f' is thin in G then the assertion of the Theorem holds. We may thus assume that neither f nor f' is thin in G. Recall that $C := \partial(X)$ and $C' := \partial(X')$ are certificate cuts of f and f', respectively.

If $e \in C \cup C'$ then Case 1 applies. We may thus assume that e is in both $G[X]$ and $G[X']$.

Case 4. $e \in E(G[X]), E(G[X'])$, $d(r_0) = 3 = d(s_0)$ and $|X| = 5 = |X'|$.

The vertex r_1 is the end of f in X_-. The end of f' in X'_- cannot be r_1, because f and f' are not adjacent. We have assumed that the set F, as defined in the statement of Claim 30, is the singleton $\{r_1s_1\}$. Thus, f' is not incident with r_2, otherwise r_2 would have to be adjacent to a vertex in $\{s_1, s_2\}$. See Figure 16(a).

We deduce that f' is incident with a vertex in $\{s_1, s_2\}$, and the roles of r_i and s_i are interchanged, as are interchanged the roles of t_i and u_i. The end of f' in $\{s_1, s_2\}$ must be



(a) $|X| = 5$ (b) $|X'| = 5$

Figure 16: The cuts C and C'.

adjacent to a vertex in $\{r_1, r_2\}$. As $F = \{r_1 s_1\}$, we infer that s_1 is the end of f' in X'_-. See Figure 16(b).

Consider the quadrilateral $(r_1, r_0, s_0, s_1, r_1)$ of G and the edge f, an edge not in Q but incident with vertex r_1, which has degree three. The edge e, which is $r_0 s_0$, is thin in G and f is thin in H. By Corollary 25, either $s_0 s_1$ is thin in G or G has a thin edge not adjacent to e. We may thus assume that $s_0 s_1$ is thin in G.

Applying to f' the same reasoning we applied to f, we deduce that either G has a thin edge not adjacent to e or $r_0 r_1$ is thin in G. In the latter alternative, the assertion of the Theorem also holds, with $r_0 r_1$ and $s_0 s_1$ two nonadjacent thin edges of G. In all alternatives, the assertion of the Theorem holds in Case 4. △

To finish the analysis of the case $d(r_0) = 3 = d(s_0)$ and $e \in E(G[X]), E(G[X'])$, we must consider the possibility that one of $|X|$ or $|X'|$ is equal to seven.

Case 5. $e \in E(G[X]), E(G[X'])$, $d(r_0) = 3 = d(s_0)$ and either $|X| = 7$ or $|X'| = 7$.

Adjust notation so that $|X| = 7$. Figure 17(a) depicts the cut C. Recall that H is the brace $\widehat{G - e}$. Let r and s denote, respectively, the result of the bicontractions of vertices r_0 and s_0 in $G - e$. See Figure 17(b). By Corollary 27, at least one of rs and rv is thin in H.

Case 5.1. The edge rs is thin in H.

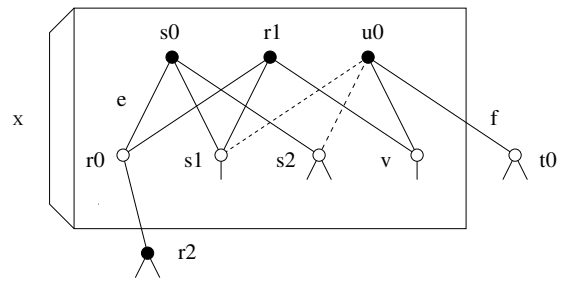
The edge rs is in fact the edge $r_1 s_1$ in G. Thus, $r_1 s_1$ is thin in H. By Claim 32, $r_1 s_1$ is thin in G. Moreover, $r_1 s_1$ and e are nonadjacent, hence the assertion of the Theorem holds. We may thus assume that the edge rs is not thin in H.

The edge e is strictly thin, thus H is a simple brace. Moreover, s is a contraction vertex of H, hence $|N_H(s)| \geqslant 4$. By Corollary 26, G has two vertices t_1 and t_2, both distinct from r, such that $N_H(t_0) = \{t_1, t_2, u_0\}$ and $N_H(s) = \{r, t_1, t_2, u_0\}$. See Figure 17(c).

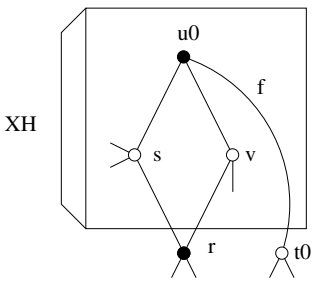
Vertex r_2 is not adjacent to any of s_1 and s_2. Thus, $N(s_1), N(s_2) \subseteq \{s_0, r_1, t_1, t_2, u_0\}$. Recall that for some $j \in \{1, 2\}$, the vertex u_0 is adjacent to s_j.

Case 5.2. s_j is adjacent to a vertex in $\{t_1, t_2\}$, say, t_1.

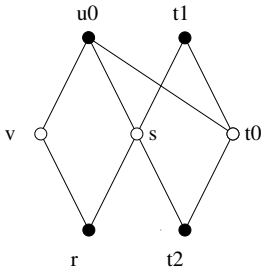
The graph G then has the quadrilateral $Q_1 := (u_0, t_0, t_1, s_j, u_0)$. Moreover, both u_0 and t_0 have degree three. See Figure 17(d). By Theorem 21, Q_1 contains an edge which



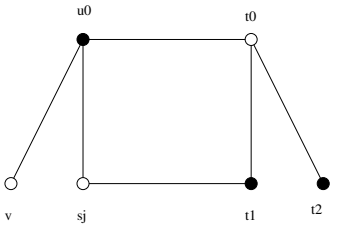
(a) G



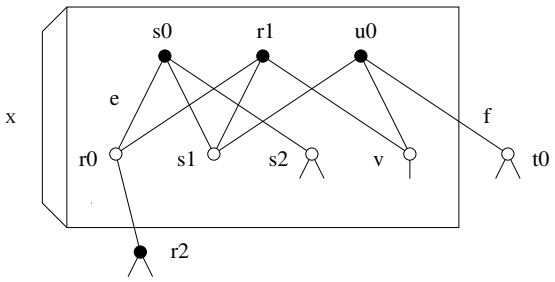
(b) H



(c) $N(s) = \{r, t_1, t_2, u_0\}$



(d) $s_j t_k \in E$



(e) $s_1 u_0 \in E$ and $d(s_1) = 3$

Figure 17: Illustration of Case 5.

is thin in G. Clearly, that edge is not adjacent to edge e. Thus, the assertion of the Theorem holds.

Case 5.3. s_j is not adjacent to any vertex in $\{t_1, t_2\}$.

If $j = 2$ then, as $d(s_2) \geqslant 3$, s_2 is adjacent to a vertex in $\{t_1, t_2\}$, hence Case 5.2 applies. We may thus assume that $j = 1$. If $d(s_1) \geqslant 4$ then s_1 is adjacent to a vertex in $\{t_1, t_2\}$, hence Case 5.2 applies. We may assume that $d(s_1) = 3$, hence $N(s_1) = \{s_0, r_1, u_0\}$.

The brace G has the quadrilateral $Q_2 := (r_1, s_1, u_0, v, r_1)$ and $d(r_1) = 3 = d(s_1)$. See Figure 17(e). By Theorem 21, Q_2 contains an edge which is thin in G. Clearly, that edge and e are nonadjacent thin edges of G. The assertion of the Theorem holds in all alternatives considered in Case 5. △

In each one of the five cases considered, we concluded that the assertion holds. The proof of the Main Theorem is complete. □

Acknowledgements

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