On the thresholds of degenerate hypergraphs

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Abstract

Fix $d > 0, \alpha, \epsilon \in (0,1]$. An n-vertex k-uniform hypergraph G is (d,α,ϵ)-*degenerate* if $m_1(G) \leqslant d + \frac{d}{\log_2 n}$ and $e(G[U]) \leqslant d(|U|-1) - \alpha$ for every subset $U \subseteq V(G)$ of size $2 \leqslant |U| \leqslant \epsilon n$. These hypergraphs encompass several natural (hyper)graph classes, including d-degenerate hypergraphs for integers $d \geqslant 2$, planar graphs, and powers of cycles. In this paper, we consider the appearance threshold of a (d,α,ϵ)-degenerate hypergraph with bounded maximum degree in the Erdős–Rényi model. We show that $G^{(k)}(n,p_n)$ contains G_n with high probability whenever $p_n = \omega(n^{-1/d})$, improving previous results of Riordan and Kelly–Müyesser–Pokrovskiy.

Mathematics Subject Classifications: 05C88, 05C89

1 Introduction

Since its introduction by Erdős and Rényi in 1960 [3], the random graph model has been one of the main objects of study in probabilistic combinatorics. Given positive integers n, k and a real number $p \in [0,1]$, $G^{(k)}(n,p)$ is the random k-uniform hypergraph on n vertices, where each k-set is included as an edge independently with probability p. A central topic in random graph theory is to determine the threshold for the emergence of a given graph property. More precisely, given a sequence (G_n) of n-vertex k-uniform hypergraphs, $\mathrm{th}(G_n)$ is a *threshold function*[1] for the containment of G_n, if for every sequence (p_n), $p_n/\mathrm{th}(G_n) \to \infty$, then $G^{(k)}(n,p_n)$ contains a copy of G_n with probability $1-o_n(1)$; $p_n/\mathrm{th}(G_n) \to 0$, then $G^{(k)}(n,p_n)$ contains no copies of G_n with probability $1-o_n(1)$. A stronger, two-sided constant-factor conclusion is that there are constants $C_2 > C_1 > 0$ such that if $p_n \geqslant C_2\mathrm{th}(G_n)$, then $\mathbb{P}[G_n \subseteq G^{(k)}(n,p_n)] = 1-o_n(1)$; if $p_n \leqslant C_1\mathrm{th}(G_n)$, then

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[1] With this definition, constant factors in the expression of $\mathrm{th}(G_n)$ do not matter and we usually omit them.

$\mathbb{P}[G_n \nsubseteq G^{(k)}(n, p_n)] = 1 - o_n(1)$. We say $\mathrm{th}(G_n)$ is a sharp threshold if for any constant $\gamma \in (0,1)$, the two probability limits hold with $C_1 = (1-\gamma)$ and $C_2 = (1+\gamma)$.

In [10], Kahn and Kalai proposed the following conjecture. For an n-vertex graph G, let $p_E(G)$ be the smallest p such that for every nonempty subgraph $F \subseteq G$,

$$n! p^{e(F)} / (|\mathrm{Aut}(F)|(n - v(F))!) \geqslant 1.$$

Conjecture 1. There is a universal constant $K > 0$ such that for any sequence (G_n), a threshold function for the appearance of (G_n) in $G(n,p)$ is at most $K p_E(G_n) \log v(G_n)$.

Johansson, Kahn, and Vu [8] proved Conjecture 1 for F-factors when F is a (fixed) strictly 1-balanced graph and their result also extends to hypergraphs. It is natural to ask more general questions: let $\mathcal{F}(n, \Delta)$ be the collection of all n-vertex graphs with maximum degree at most Δ. What is the threshold behavior for the appearance of an arbitrary graph $G \in \mathcal{F}(n, \Delta)$ in $G(n,p)$? Ferber, Luh, and Nguyen [5] considered a similar problem for the family $\mathcal{F}((1-\epsilon)n, \Delta)$ of all almost spanning graphs with maximum degree at most Δ. In [6], Frankston, Kahn, Narayanan, and Park improved Ferber, Luh, and Nguyen's result and showed that for a graph $G \in \mathcal{F}(n, \Delta)$, with high probability $G(n,p)$ contains a copy of G, provided that $p = \omega((n^{-1} \log^{1/\Delta} n)^{2/(\Delta+1)})$ for each fixed integer $\Delta > 0$.

Let $k \in \mathbb{N}$ with $k \geqslant 2$. For a k-uniform hypergraph G and an integer i with $1 \leqslant i \leqslant 2$, we let[2]

$$m_i(G) = \max \{ e(F)/(v(F) - i) : F \subseteq G \text{ and } v(F) > i \}.$$

A natural class of (hyper)graphs to consider is the class of degenerate (hyper)graphs. A (hyper)graph G is called d-degenerate if every nonempty sub(hyper)graph of G contains a vertex of degree at most d (therefore, it is enough to consider $d \in \mathbb{N}$). Using the second-moment method, Riordan [19] showed a general result which implies that $\mathrm{th}(G_n) \leqslant n^{-1/m_2(G_n)}$ for bounded-degree graphs with at least $\lfloor n/2 \rfloor + 1$ edges. Under these hypotheses, letting (G_n) be a sequence of d-degenerate graphs, Riordan's result implies that if $d \geqslant 3$, then $\mathrm{th}(G_n) \leqslant n^{-1/d}$. However, for $d = 2$, it only gives a suboptimal result $\mathrm{th}(G_n) \leqslant n^{-1/3}$, as $m_2(K_3) = 3$ (and K_3 is 2-degenerate). Parczyk and Person [15] extend Riordan's result to the hypergraph setting. Lastly, 1-degenerate graphs of bounded maximum degree are forests, so their threshold is at most $\log_2 n / n$.

In this paper, we study the threshold behavior of a wider class of hypergraphs, namely (d, α, ϵ)-degenerate hypergraphs with bounded maximum degree.

Definition 2. Fix $\epsilon, \alpha \in (0,1], d > 0$ and an integer $k \geqslant 2$. Let $n \in \mathbb{N}$ with $n \geqslant 2$. An n-vertex k-uniform hypergraph G is a (d, α, ϵ)-*degenerate* hypergraph if $m_1(G) \leqslant d + \frac{d}{\log_2 n}$ and $e(G[U]) \leqslant d(|U| - 1) - \alpha$ for every subset $U \subseteq V(G)$ with $2 \leqslant |U| \leqslant \epsilon n$.

[2]Note that this definition of $m_2(G)$ is different from the one used for the so-called random Turán theorem and random Ramsey theorem (sometimes also denoted by $m_2(G)$), which is $\max \left\{ \frac{e(F)-1}{v(F)-2} : F \subseteq G \text{ and } v(F) > 2 \right\}$, defined with respect to the expected number of copies of G containing a given edge.

The (d, α, ϵ)-degenerate hypergraphs include several natural (hyper)graph classes. Appendices A.1 and A.2 verify Example 3(4).

Example 3.
1. For any integers $k, d \geqslant 2$, every d-degenerate k-uniform hypergraph G is $(d, 1/2, 1)$-degenerate, as for every subhypergraph $F \subseteq G$, delete vertices of degree at most d one at a time until k remain; at most $d(v(F)-k)$ edges are lost and at most one edge survives, then $e(F) \leqslant d(v(F)-k)+1 \leqslant d(v(F)-1)-1$ if $k \leqslant v(F) \leqslant v(G)$ and $e(F) = 0$ if $1 \leqslant v(F) < k$.
2. Every planar graph is $(3, 1/2, 1)$-degenerate as $e(F) \leqslant 3v(F) - 6$ for every planar F with $v(F) \geqslant 3$. For $v(F) = 2$, one has $e(F) \leqslant 1 \leqslant 3(2-1) - 1/2$.
3. By [2, Corollary 7.3.2], every K_4-minor-free graph is $(2, 1/2, 1)$-degenerate.
4. For sufficiently large n, the r-th power of a cycle is $(r, 1/2, 1/10)$-degenerate for fixed $r \geqslant 2$. More generally, the r-th power of a k-uniform tight cycle is $\left(\binom{r+k-2}{k-1}, 1/2, \frac{1}{2(k+r-2)}\right)$-degenerate for fixed $r, k \geqslant 2$.

For bounded-degree (d, α, ϵ)-degenerate hypergraphs, [13, Proposition 1.17] (see Proposition 9 below) implies that $\mathrm{th}(G_n) \leqslant n^{-1/m_1(G_n)} \log_2 n$, and since we have $m_1(G_n) \leqslant d + \frac{d}{\log_2 n}$, it follows that $\mathrm{th}(G_n) \leqslant n^{-1/d} \log_2 n$. The following theorem removes this logarithmic factor under the local deficit assumption.

Theorem 4. *Fix $d > 0, \epsilon, \alpha \in (0, 1]$ and integers $k \geqslant 2, \Delta \geqslant 1$. Assume (G_n) is a sequence of nonempty n-vertex (d, α, ϵ)-degenerate k-uniform hypergraphs with $\Delta(G_n) \leqslant \Delta$. Then $\mathbb{P}[G_n \subseteq G^{(k)}(n, p_n)] = 1 - o_n(1)$ whenever $p_n n^{1/d} \to \infty$. In particular, $\mathrm{th}(G_n) = O(n^{-1/d})$.*

We next give an application to locally sparse regular graphs. Fix an integer $D \geqslant 3$ and $0 < \epsilon < 1/2$. Let G be an n-vertex graph. For every subset $U \subseteq V(G)$, we define its edge boundary ∂U as $\partial U := \{e \in E(G) : |e \cap U| = 1\}$. We say that G is (D, ϵ)-*locally sparse* if for every subset $U \subseteq V(G)$ of size $D \leqslant |U| \leqslant \epsilon n$, we have $|\partial U| \geqslant D + 1$. Let G be a (D, ϵ)-locally sparse D-regular graph. For $2 \leqslant |U| \leqslant D-1$, $e(G[U]) \leqslant \binom{|U|}{2} \leqslant (|U|-1)(D-1)/2$. For $D \leqslant |U| \leqslant \epsilon n$, $2e(G[U]) \leqslant D(|U| - 1) - 1$. For $|U| \geqslant \epsilon n$, $e(G[U]) \leqslant D|U|/2$. Thus, one has $e(G[U])/(|U| - 1) \leqslant D/2 + D/[2(\epsilon n - 1)] \leqslant D/2 + D/(2 \log_2 n)$. Hence, G is $(D/2, 1/2, \epsilon)$-degenerate. Theorem 4 and the first moment method yield Corollary 5.

Corollary 5. *Fix an integer $D \geqslant 3$ and $\epsilon \in (0, 1/2)$. Let (G_n) be a sequence of (D, ϵ)-locally sparse D-regular graphs on $[n]$. Then $\mathrm{th}(G_n) = \Theta(n^{-\frac{2}{D}})$.*

Now, we show that the edge-boundary requirement in the definition cannot be weakened from $D + 1$ to D. Let $D, n \in \mathbb{N}$ and $0 < \epsilon < 1/2$, where D divides n. We construct such a graph G_n that contains a K_D-factor, so [8] gives $\mathrm{th}(G_n) = \Omega\left(n^{-2/D} \log^{2/[D(D-1)]} n\right)$; Appendix A.3 verifies its cut property. This implies the lower bound of edge boundary in the definition of (D, ϵ)-locally sparse graphs is the best possible for D-regular graphs. Take $D \geqslant 3$ fixed, $M = n/D$ sufficiently large, and index the D-cliques cyclically modulo

M. When D is even, join the last $D/2$ vertices of block i to the first $D/2$ of block $i+1$ by a matching. When D is odd, require M to be even, match the corresponding $(D-1)/2$ vertices between consecutive blocks, and match the remaining distinguished vertices of blocks i and $i + M/2$ for $1 \leqslant i \leqslant M/2$.

Remarks.

- The upper bound $n^{-1/d}$ of $\mathrm{th}(G_n)$ is best possible if G_n is asymptotically maximal subject to the (d, α, ϵ)-degeneracy condition. Indeed, if G_n satisfies $e(G_n) \geqslant dn - O(1)$, then we have $p_E(G_n) \geqslant n^{-1/d}$ by computing the expected number of copies of G_n.

- Our result on thresholds improves on Proposition 9 for (d, α, ϵ)-degenerate graphs by eliminating the logarithmic factor. This contributes to a better understanding of the ratio $\mathrm{th}(G_n)/p_E(G_n)$ (see Conjecture 1) – all known results on this ratio are at most $\log_2 n$ and Theorem 4 shows that the logarithmic loss can be removed for (d, α, ϵ)-degenerate k-graphs G_n with $e(G_n) \geqslant dv(G_n) - O(1)$. This question is also studied in a recent paper by Balogh, Bernshteyn, Delcourt, Ferber, and Pham in [1] for hypergraphs of bounded VC-dimension.

- Riordan's result [19] showed that $\mathrm{th}(G_n) \leqslant n^{-1/d}$ for graphs G_n of bounded maximum degree with $m_2(G_n) \leqslant d$ for $d \geqslant 3$. These graphs G_n are $(d, 1/2, 1)$-degenerate. Thus, our result improves on Riordan's result by allowing arbitrarily small $\alpha > 0$.

- The assumption $\alpha > 0$ in the definition of (d, α, ϵ)-degenerate hypergraphs is necessary for removing the logarithmic loss in general: if the defining inequalities are extended to allow $\alpha = 0$, let G_n be a Hamilton cycle [18] or a spanning tree with bounded maximum degree [14]; then G_n is $(1, 0, 1/2)$-degenerate and has a threshold $\log_2 n/n$ (see also the examples above).

2 Spreadness

Recently, random graph theory was revolutionized by Frankston, Kahn, Narayanan, and Park's [6] proof of the *fractional expectation threshold vs. threshold* conjecture of Talagrand [21] and by Park and Pham's [16] proof of the even stronger (abstract version of) Kahn–Kalai conjecture [10]. A crucial corollary of Talagrand's conjecture relates the so-called *spread measure* to thresholds. Roughly speaking, for the study of random graphs, this connection implies that if one can find a probability measure on the copies of G_n in $K_n^{(k)}$ with good *spread*, then with high probability G_n appears in $G^{(k)}(n, p_n)$.

Let $n \in \mathbb{N}$ and let H be an n-vertex hypergraph with $N = e(H)$ edges. Let H_p denote the random subhypergraph obtained by retaining each edge of H independently with probability p.

Definition 6. Let $q \in [0,1]$ and $\mathcal{H}$ be a hypergraph on the vertex set V. Let μ be a probability measure on the edge set of $\mathcal{H}$. We say μ is *q-spread* if for every $S \subseteq V$, we have

$$\mu\left(\{A \in E(\mathcal{H}) : S \subseteq A\}\right) \leqslant q^{|S|}.$$

Pham, Sah, Sawhney, and Simkin introduced a notion of *vertex spreadness* in [17], where they consider a random embedding of $T \to H$ and analyze the probability that prescribed images of a collection of distinct vertices. Kelly, Müyesser, and Pokrovskiy [13] extended their result to general settings.

Definition 7 (Vertex-spread)**.** Let X and Y be finite sets, and let μ be a probability distribution over injections $\psi : X \to Y$. For $q \in [0,1]$, we say that μ is *q-vertex spread* if for every integer $0 \leqslant s \leqslant |X|$ and every choice of pairwise distinct vertices $x_1, \dots, x_s \in X$ and pairwise distinct $y_1, \dots, y_s \in Y$,

$$\mu\left(\{\psi(x_i) = y_i, \forall i \in [s]\}\right) \leqslant q^s.$$

A *hypergraph embedding* $\psi : G \to H$ of a hypergraph G into a hypergraph H is an injective map $\psi : V(G) \to V(H)$ which maps edges of G to edges of H, so there is an embedding of G into H if and only if H contains a subgraph isomorphic to G.

Fact 8. *Let G be an n-vertex k-uniform hypergraph. Then the uniform distribution μ on embeddings ψ of G in $K_n^{(k)}$ is (e/n)-vertex spread.*

Proof. Fix $0 \leqslant s \leqslant n$ and distinct vertices $x_1, \dots, x_s \in V(G)$ and distinct images $y_1, \dots, y_s \in V(K_n^{(k)})$. Then

$$\mu\left(\{\psi : \psi(x_i) = y_i,\ \forall i \in [s]\}\right) = \frac{(n-s)!}{n!} \leqslant (e/n)^s. \qquad \square$$

The following result, proved by Kelly, Müyesser, and Pokrovskiy in [13], converts vertex-spread embeddings into an edge-spread measure on copies. Moreover, combined with the main theorem in [6], it gives an upper bound for $\mathrm{th}(G_n)$.

Proposition 9 (Proposition 1.17 and its proof in [13])**.** *Fix integers $k \geqslant 2, \Delta \geqslant 1$ and a constant $C' \geqslant 1$. The following holds for sufficiently large n. Let H and G be two nonempty n-vertex k-uniform hypergraphs. Let $\mathcal{H}$ be the set of subgraphs of H which are isomorphic to G and suppose there is a (C'/n)-vertex spread distribution μ on embeddings $\psi : G \to H$ and $\Delta(G) \leqslant \Delta$. Then there exists a $\left(B^{k-1}/n^{1/m_1(G)}\right)$-spread distribution μ' on $\mathcal{H}$ such that for any subset $S \subseteq E(H)$ with v vertices and c components, we have*

$$\mu'\left(\{A \in \mathcal{H} : S \subseteq A\}\right) \leqslant (B/n)^{v-c}, \tag{1}$$

where $B := k\Delta C'^2$. In particular, for every sequence (G_n) of such hypergraphs, we have $\mathrm{th}(G_n) \leqslant n^{-1/m_1(G_n)} \log_2 n$.

We first state the following stronger notion of spreadness introduced by Spiro [20]. We remark that a similar notion of "superspread" was also introduced by Espuny Díaz and Person [4].

Definition 10. Let $q \in [0,1]$, and let $r_0 > r_1 > \ldots > r_\ell = 1$ be a decreasing sequence of positive integers. Let $\mathcal{H}$ be an r_0-uniform hypergraph on V, and let μ be a probability measure on $E(\mathcal{H})$. We say μ is $(q; r_0, \ldots, r_\ell)$-*spread* if the following holds for all $i \in [\ell]$:

$$\mu\big(\{A \in E(\mathcal{H}) \ : \ |A \cap S| \geqslant t\}\big) \leqslant q^t \text{ for all } t \in [r_i, r_{i-1}] \text{ and } S \in \bigcup_{A \in E(\mathcal{H})} \bigcup_{j=r_i}^{r_{i-1}} \binom{A}{j}.$$

The following result is proved by Spiro [20] and is also a stronger version of the main theorem in [6, Theorem 1.6].

Theorem 11 (Theorem 1.4 in [20])**.** *There exists an absolute constant $K_{11} > 0$ such that the following holds for all $C \geqslant K_{11}$. Let $r_0 > r_1 > \ldots > r_\ell = 1$ be a decreasing sequence of positive integers. Let $\mathcal{H}$ be an r_0-uniform hypergraph on V. If there is a $(q; r_0, \ldots, r_\ell)$-spread distribution on $E(\mathcal{H})$, then a set W of size $\lfloor C\ell q|V| \rfloor$ chosen uniformly at random from V contains an edge of $\mathcal{H}$ with probability at least $1 - K_{11}/(C\ell)$.*

Spiro already noted that the proof of Kahn, Narayanan, and Park in [11] yields that the uniform distribution on squares of Hamilton cycles is $(Cn^{-1/2}; 2n, C_0 n^{1/2}, 1)$-spread. The following standard result allows us to transfer asymptotic statements from the uniform model $H_n(M)$ to the binomial model $(H_n)_{p_n}$.

Proposition 12 (Proposition 1.12 in [7])**.** *Let (H_n) be a sequence of nonempty n-vertex k-uniform hypergraphs with $N = N(n)$ edges. For an integer $0 \leqslant M \leqslant N$, let $H_n(M)$ denote the uniformly random M-edge subhypergraph of H_n. Let $\mathcal{P}_n$ be an arbitrary property of subhypergraphs of H_n, and $p_n \in [0,1]$ and $0 \leqslant a \leqslant 1$. If for every sequence of integers $M = M(n)$ such that $M = Np_n + O(\sqrt{Np_n(1-p_n)})$ it holds that $\mathbb{P}[H_n(M) \in \mathcal{P}_n] = a + o_n(1)$, then also $\mathbb{P}[(H_n)_{p_n} \in \mathcal{P}_n] = a + o_n(1)$.*

Next, we state our main result, partially improving Proposition 9. Indeed, we show that for a (d, α, ϵ)-degenerate k-uniform hypergraph G, the distribution μ' given by Proposition 9 satisfies Spiro spreadness with constant ℓ, and thus obtain a saving of the logarithmic factor.

Theorem 13. *Fix integers $k \geqslant 2, \Delta \geqslant 1$, and constants $\epsilon, \alpha \in (0,1], d > 0, C' \geqslant 1$. There exists a constant $\tilde{C} = \tilde{C}(\epsilon, d, k, \Delta, C') > 0$ such that the following holds for all sufficiently large n. Let G be an n-vertex (d, α, ϵ)-degenerate k-uniform hypergraph with at least two edges and $\Delta(G) \leqslant \Delta$. Let H be an n-vertex k-uniform hypergraph. If there is a (C'/n)-vertex spread distribution μ on embeddings $\psi : G \to H$, then there exists a $\left(\tilde{C}/n^{1/d}; e(G), r_{i_0}, \ldots, r_\ell\right)$-spread distribution μ' on copies of G in H where $\beta = \min\{\alpha/(2d), 1/4\}$, $\ell := \lceil 1/\beta \rceil$ and $r_i := \lfloor \frac{\epsilon}{2k} n^{1-i\beta} \rfloor$ for $0 \leqslant i \leqslant \ell - 1$, $r_\ell = 1$, and let $0 \leqslant i_0 \leqslant \ell$ be the smallest index satisfying $e(G) > r_{i_0}$.*

We will include a slight refinement of Theorem 13 in Section 4, using a recent work of Kelly [12]. With additional (and separate) work on the vertex spreadness, Theorem 13 can be used to obtain *robust thresholds* for (d, α, ϵ)-degenerate hypergraphs, similar to some results in [13] which were obtained from establishing vertex spreadness in host graphs and applying Proposition 9. One prominent such example is the robust threshold for the rth power of a Hamilton cycle in graphs, which was recently determined by Joos–Lang–Sanhueza-Matamala [9].

3 Spiro spread for (d, α, ϵ)-degenerate hypergraphs

We first introduce a simple counting result, and a specialized version was proved in [13].

Lemma 14. *Fix $\Delta, k \in \mathbb{N}$ with $k \geqslant 2$. Let G be an n-vertex k-uniform hypergraph with $\Delta(G) \leqslant \Delta$. If S is a nonempty subset of $E(G)$, then for integers $1 \leqslant c \leqslant t \leqslant |S|$, the number of subgraphs of G with t edges, all in S, no isolated vertices and c components is at most*

$$(8k\Delta)^t \binom{|S|}{c}. \tag{2}$$

Proof. Let F be the edge-supported subhypergraph corresponding to a t-element subset of S. Let $F_1, F_2, \ldots, F_c$ be the c components of F with $e(F_i) = t_i$ for $i \in [c]$. If $\Delta = 1$, G is a disjoint union of a matching and isolated vertices. Therefore, F has $t = c$ edges. Thus, there are at most $\binom{|S|}{c}$ such subhypergraphs. Now suppose $\Delta \geqslant 2$. Work in the line graph on S; its maximum degree is at most $k(\Delta - 1)$. Choose the least edge of each component in a fixed order, giving $\binom{|S|}{c}$ possible root sets. Order the roots and choose a positive composition $\sum_{i=1}^{c} t_i = t$, at most 2^t choices. A connected set of t_i line graph vertices containing a fixed root has a canonical rooted plane spanning tree. There are at most $4^{t_i - 1}$ unlabelled rooted plane tree shapes and at most $(k(\Delta - 1))^{t_i - 1}$ choices of neighbor images. The image set recovers the component; allowing invalid encodings only increases the bound. Thus, multiplication proves (2). □

Proof of Theorem 13. Let $1/n \ll 1/C' \leqslant 1$, $\Delta \geqslant 1$, $d > 0$, and $\alpha \in (0, 1]$. Set $m := e(G) \geqslant 2$. Applying the local inequality to one edge gives $d \geqslant \frac{1+\alpha}{k-1}$. Let μ be a (C'/n)-vertex spread distribution on embeddings ψ of G in H. Let $\mathcal{H}$ be the m-uniform hypergraph with vertex set $E(H)$ and edge set consisting of subgraphs of H isomorphic to G. As $m_1(G) \leqslant d + d/\log_2 n$, one has $n^{-1/m_1(G)} \leqslant n^{-\frac{1}{d(1+1/\log_2 n)}} \leqslant 2^{1/d} n^{-1/d}$. By Proposition 9, the pushforward distribution μ' on $E(\mathcal{H})$ is q-spread, where $q := 2^{1/d}(B)^{k-1} n^{-1/d}$ and $B = k\Delta C'^2$.

Let $\beta = \min\{\alpha/(2d), 1/4\}$, $\ell := \lceil 1/\beta \rceil$ and $r_i := \lfloor \frac{\epsilon}{2k} n^{1-i\beta} \rfloor$ for $0 \leqslant i \leqslant \ell - 1$, $r_\ell = 1$, and let i_0 be the smallest index satisfying $0 \leqslant i_0 \leqslant \ell$ and $m > r_{i_0}$. Then $\theta = 1 - (\ell - 1)\beta$ satisfies $0 < \theta \leqslant \beta$, so $r_{\ell-1} \to \infty$ and $r_{\ell-1} \leqslant n^\beta$. Moreover, $r_{i-1}/r_i \leqslant 2n^\beta$ for $1 \leqslant i < \ell$. Now we show that μ' is $\left(\frac{\tilde{C}}{n^{1/d}}; m, r_{i_0}, \ldots, r_\ell\right)$-spread.

Fix a copy A_0 of G in H and a nonempty $S \subseteq A_0$. Write $s = |S|$, and let $1 \leqslant t \leqslant s$.

Coarse range. If $m > r_0$, for $s, t \in [r_0, m]$, then $s/t \leqslant 4\Delta/\epsilon$ because of $m \leqslant \Delta n/k$. Since μ' is q-spread,

$$\mu'\big(\{A \in E(\mathcal{H}) : |A \cap S| \geqslant t\}\big) \leqslant \binom{s}{t} q^t \leqslant \big((8e\Delta/\epsilon)2^{1/d} B^{k-1} n^{-1/d}\big)^t, \tag{3}$$

as $q = 2^{1/d} B^{k-1} n^{-1/d}$.

Next, for $t \leqslant s \leqslant \min\{m, r_0\} \leqslant \epsilon n/(2k)$, let F be an edge-supported subhypergraph corresponding to a t-element subset of S with v vertices and c components $F_1, \ldots, F_c$. As G is (d, α, ϵ)-degenerate, one has $e(F_i) \leqslant d(v(F_i) - 1) - \alpha$ for $i \in [c]$. Therefore, $t \leqslant d(v - c) - \alpha c$. Then

$$\begin{aligned} \mu'\big(\{A \in E(\mathcal{H}) : |A \cap S| \geqslant t\}\big) &\leqslant \sum_{F \in \binom{S}{t}} \mu'\big(\{A \in E(\mathcal{H}) : F \subseteq A\}\big) \\ &\overset{(1),(2)}{\leqslant} \sum_{c=1}^{t} (8k\Delta)^t \binom{|S|}{c} \left(\frac{B}{n}\right)^{\frac{t}{d} + \frac{\alpha c}{d}} \\ &\leqslant \left(\frac{8k\Delta B^{1/d}}{n^{1/d}}\right)^t \sum_{c=1}^{t} \binom{|S|}{c} x^c. \end{aligned} \tag{4}$$

where $x = \left(\frac{B}{n}\right)^{2\beta}$ and $B/n < 1$. The second inequality follows from (1), (2) and $v \geqslant \frac{t}{d} + \frac{\alpha c}{d} + c$. The third inequality follows from $2\beta \leqslant \alpha/d$.

Intermediate range. If $r_i \leqslant t \leqslant s \leqslant \min\{m, r_{i-1}\}$, $1 \leqslant i < \ell$, one has $s/t \leqslant 2n^\beta$. Hence,

$$\sum_{c=1}^{t} \binom{|S|}{c} x^c \leqslant (1 + x)^s - 1 \leqslant e^{sx} \leqslant \exp\left(2B^{2\beta} t n^{-\beta}\right) \leqslant e^t$$

for all sufficiently large n. Therefore,

$$\mu'\big(\{A \in E(\mathcal{H}) : |A \cap S| \geqslant t\}\big) \leqslant \left(\frac{8ek\Delta B^{1/d}}{n^{1/d}}\right)^t. \tag{5}$$

Terminal range. If $t \leqslant s \leqslant \min\{m, r_{\ell-1}\} \leqslant n^\beta$, then $sx \leqslant B^{2\beta} n^{-\beta} \leqslant 1/2$ for large n, so

$$\sum_{c=1}^{t} \binom{|S|}{c} x^c \leqslant e^{sx} - 1 \leqslant 2sx \leqslant 2B^{2\beta} n^{-\beta}, \tag{6}$$

using $e^y - 1 \leqslant 2y$ for $0 \leqslant y \leqslant 1/2$. Substituting this bound into (4) gives

$$\mu'\big(\{A \in E(\mathcal{H}) : |A \cap S| \geqslant t\}\big) \leqslant n^{-\beta/2} \left(\frac{8k\Delta B^{1/d}}{n^{1/d}}\right)^t. \tag{7}$$

Note that (7) gives a stronger bound for the range $[1, \min\{m, r_{\ell-1}\}]$, which will be used in Section 4. Choose $\tilde{C} = 8ek(2B)^{k-1+\frac{1}{d}}\Delta/\epsilon$, then μ' is $\left(\frac{\tilde{C}}{n^{1/d}}; m, r_{i_0}, \ldots, r_\ell\right)$-spread. □

Proof of Theorem 4. If $e(G_n) = 1$, the local inequality implies $d \geqslant \frac{1+\alpha}{k-1}$. Let X be the number of edges in $G^{(k)}(n, p_n)$. When $p_n = \omega(n^{-1/d})$, one has $\mathbb{E}[X] = \binom{n}{k} p_n \to \infty$ and $\mathrm{Var}(X) = (1 - p_n)\mathbb{E}[X]$. By the second moment method, we have

$$\mathbb{P}[G_n \subseteq G^{(k)}(n, p_n)] = 1 - o_n(1).$$

Set $H_n = K_n^{(k)}$ and $N = \binom{n}{k}$. For the case $e(G_n) \geqslant 2$, by Fact 8 and Theorem 13, there is a $(q; e(G_n), r_{i_0}, \ldots, r_\ell)$-spread distribution μ' on the copies of G_n in $K_n^{(k)}$, where $q = \tilde{C} n^{-1/d}$. When $p_n = \omega(1/n^{1/d})$, set $M_n = \lfloor p_n N \rfloor$. As $p_n/((\ell+1)q) \to \infty$, then $M_n \geqslant \lfloor K_{11}(\ell+1)qN \rfloor$. Since $d \geqslant \frac{1+\alpha}{k-1}$, we have $\lfloor K_{11}(\ell+1)qN \rfloor \to \infty$. Hence, by Theorem 11 and Proposition 12, $\mathbb{P}[G_n \subseteq G^{(k)}(n, p_n)] \geqslant \mathbb{P}[G_n \subseteq K_n^{(k)}(M_n)] - o_n(1) \geqslant 1 - \frac{K_{11}(\ell+1)q}{p_n} - o_n(1) \geqslant 1 - o_n(1)$. Thus, we have $\mathrm{th}(G_n) = O(n^{-1/d})$. □

4 A fixed-constant high-probability bound

While we were finalizing this project, we learned from Tom Kelly that he introduced yet another strengthening of spreadness [12], which can boost our result to a two-sided constant-factor conclusion.

Definition 15. Let $q, \gamma \in [0, 1]$, and let $r, r' \in \mathbb{N}$ be positive integers such that $r > r'$, and let $\mathcal{H}$ be a hypergraph on vertex set V. We say the probability measure μ on $E(\mathcal{H})$ is $(q, \gamma; r, r')$-*spread* if the following holds:

$$\mu\big(\{A \in E(\mathcal{H}) \ : \ |A \cap S| \geqslant t\}\big) \leqslant \gamma q^t \text{ for all } t \in [r', r] \text{ and } S \in \bigcup_{A \in E(\mathcal{H})} \bigcup_{j=r'}^{r} \binom{A}{j}.$$

The following is a slightly stronger version of the result of Spiro [20, Theorem 2.7] that incorporates Definition 15. We include Kelly's proof of Theorem 16 in Appendix B.

Theorem 16 (Theorem 8.3 in [12])**.** *Let $q, \gamma \in [0, 1]$, and $r_0 > r_1 > \cdots > r_\ell = 1$ be a decreasing sequence of positive integers, and $\mathcal{H}$ be an r_0-uniform hypergraph on V, and let $C_{16} \geqslant 8$. If there is a $(q; r_0, \ldots, r_\ell)$-spread distribution on $E(\mathcal{H})$ which is $(q, \gamma; r_{\ell-1}, 1)$-spread and W is a set of size $\lfloor 2\ell C_{16} q|V| \rfloor$ chosen uniformly at random from V, then W contains an edge of $\mathcal{H}$ with probability at least*

$$1 - 6\ell^2 \left(\frac{C_{16}}{4}\right)^{-r_{\ell-1}/2} - 2\exp\left(-\frac{\ell C_{16} q|V|}{6}\right) - \frac{8\gamma}{C_{16}\ell}.$$

For the negative terms in the probability above, the first term is exponential in $r_{\ell-1}$, the second is exponential in $\ell q|V|$, and the last is linear in γ/ℓ. Therefore, if we have $r_{\ell-1} \to \infty$, $\ell q|V| \to \infty$, and $\gamma/\ell \to 0$, then the claimed probability above is $1 - o(1)$.

Using Theorem 16, we can improve the error probability in Theorem 11 and obtain a two-sided constant-factor conclusion for (d, α, ϵ)-degenerate hypergraphs with $e(G_n) \geqslant dn - O(1)$. The terminal estimate strengthens the upper bound to containment with high

probability at a sufficiently large fixed constant multiple of $n^{-1/d}$. Let X be the number of copies of G_n in $G^{(k)}(n,p_n)$. When $e(G_n) \geqslant dn-b$ for some fixed integer $b>0$, choose $p_n = cn^{-1/d}$ for a small constant $c \in (0,1)$, the first-moment lower bound shows that $\mathbb{E}[X] \leqslant n! p_n^{e(G_n)} \leqslant n^n p_n^{dn-b} \leqslant n^{b/d} c^{dn-b} = o_n(1)$. This is the correct threshold order.

Theorem 17. *Fix $k, \Delta \in \mathbb{N}$ with $k \geqslant 2$, $d > 0, \alpha, \epsilon \in (0,1], C' \geqslant 1$. There exist constants $\tilde{C}, C_* > 0$ such that the following holds for sufficiently large n. Let (G_n) be a sequence of n-vertex (d,α,ϵ)-degenerate k-uniform hypergraphs with at least two edges and $\Delta(G_n) \leqslant \Delta$. Let (H_n) be a sequence of n-vertex k-uniform hypergraphs. If there is a (C'/n)-vertex spread distribution μ on embeddings $\psi_n : G_n \to H_n$, then there exists a $\left(\tilde{C}/n^{1/d}; e(G_n), r_{i_0}, \ldots, r_\ell\right)$-spread distribution μ' on copies of G_n in H_n which is $(\tilde{C}/n^{1/d}, n^{-\beta/2}; R_n, 1)$-spread, where $R_n = \min\{r_{\ell-1}, e(G_n)\}$, $\beta = \min\{\alpha/(2d), 1/4\}$, $\ell := \lceil 1/\beta \rceil$ and $r_i := \lfloor \frac{\epsilon}{2k} n^{1-i\beta} \rfloor$ for $0 \leqslant i < \ell$, $r_\ell = 1$ and i_0 is the smallest index satisfying $0 \leqslant i_0 \leqslant \ell$ and $e(G_n) > r_{i_0}$. Moreover, if $p_n \geqslant C_* n^{-1/d}$, then $\mathbb{P}[G_n \subseteq (H_n)_{p_n}] = 1 - o_n(1)$.*

Proof. Choose $C = \max\{K_{11}, C_{16}\} \geqslant 32$. Set $q_n = \tilde{C}/n^{1/d}$, $C_* = 2(\ell+1)C\tilde{C}$ and $p_n = C_* n^{-1/d}$. Let $M_n = \lfloor p_n |E(H_n)| \rfloor$. For a sequence of G_n, a subsequence of $R_n = \min\{r_{\ell-1}, e(G_n)\}$ either be bounded by a constant or tend to infinity. It suffices to prove $P[G_n \subseteq (H_n)_{p_n}] \to 1$ along every subsequence on which R_n is bounded or tends to infinity, because every subsequence of (R_n) has a further subsequence of one of these two kinds. As μ' is $(\tilde{C}/n^{\frac{1}{d}}, n^{-\beta/2}; R_n, 1)$-spread, then we have

$$1 \leqslant \sum_{e \in E(H_n)} \mu'(e \in G_n) \leqslant n^{-\beta/2} |E(H_n)| \frac{\tilde{C}}{n^{\frac{1}{d}}},$$

which implies $|E(H_n)| \geqslant \Omega(n^{\frac{1}{d}+\beta/2})$. For any subsequence $R_n \geqslant 2$ is bounded by a constant $L > 0$ independent of n, by (7), then μ' is $(\tilde{C}/n^{(\frac{1}{d}+\beta/2L)}; R_n, 1)$-spread. Let $p_0 = C_* / n^{\frac{1}{d}+\frac{\beta}{5L}}$, then we have $p_0 |E(H_n)| \geqslant \Omega(n^{\beta/4})$ as $L \geqslant 2$. Since $R_n = e(G_n)$ for large n, and $p_0/q_0 = C_* n^{3\beta/(10L)}/\tilde{C} \to \infty$ with $q_0 := \tilde{C} n^{-1/d-\beta/(2L)}$. Thus, by Theorem 11 and Proposition 12, we have $\mathbb{P}[G_n \subseteq (H_n)_{p_0}] \geqslant 1/2$. Thus, we can expose $\lfloor n^{\beta/(5L)} \rfloor$ independent $(H_n)_{p_i}$ with $p_i = p_0$. As $\lfloor n^{\beta/(5L)} \rfloor \cdot p_0 \leqslant p_n$, we couple $\cup_i (H_n)_{p_i}$ as a subgraph of $(H_n)_{p_n}$ surely. Then, $\mathbb{P}[G_n \subseteq (H_n)_{p_n}] \geqslant 1 - 1/2^{\lfloor n^{\beta/(5L)} \rfloor} = 1 - o_n(1)$.

Otherwise, the subsequence $R_n = f(n)$ is a function depending on n such that $f(n) \to \infty$ as $n \to \infty$. Note that our proof of Theorem 13 gives that μ' is $(\tilde{C}/n^{1/d}, n^{-\beta/2}; R_n, 1)$-spread by (7). As $M_n = p_n |E(H_n)| \geqslant \Omega(n^{\beta/2})$, we have $M_n \to \infty$. By Theorem 16 and Proposition 12, $\mathbb{P}[G_n \subseteq (H_n)_{p_n}] \geqslant \mathbb{P}[G_n \subseteq H_n(M_n)] - o_n(1) \geqslant 1 - o_n(1)$.

Combining both cases, we have $\mathbb{P}[G_n \subseteq (H_n)_{p_n}] = 1 - o_n(1)$. □

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Declaration on the use of generative AI

The original version of this paper was prepared and submitted by the authors without the use of any AI model. After receiving the referees' reports, the authors did the revision of the manuscript and used ChatGPT to check again the manuscript and responses to the referees. The authors take full responsibility for the final manuscript.

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A Density and cut verifications

A.1 Powers of cycles

Fix an integer $r \geqslant 2$, and let $G = C_n^r$, where n is sufficiently large in terms of r. We show that G is $(r, 1/2, 1/10)$-degenerate. First, let $U \subseteq V(G)$ with $2 \leqslant u := |U| \leqslant n/10$. Since $n > 2r$, we may orient each edge of G clockwise over a distance of at most r. Every vertex then has outdegree r. Let b be the number of maximal intervals of consecutive vertices of the underlying cycle that are contained in U.

Suppose first that $b \geqslant r+1$. The last vertex of each interval has its clockwise successor outside U. Thus, at least b of the ru directed edges with initial vertex in U have their terminal vertex outside U. Consequently, $ru - e(G[U]) \geqslant b \geqslant r+1$, and hence $e(G[U]) \leqslant r(u-1) - 1$.

Now suppose that $b \leqslant r$. The complement of U consists of b intervals, so one of them contains at least $\frac{n-u}{b} \geqslant \frac{n-u}{r} \geqslant \frac{9n}{10r}$ consecutive vertices. For sufficiently large n, this interval contains at least r vertices. Cutting the cyclic ordering within this interval gives a linear ordering in which every directed edge of $G[U]$ points forward. In this ordering, the last vertex of U has no outgoing edge in $G[U]$, and the penultimate vertex has at most one. Each of the remaining vertices has at most r. Therefore, $e(G[U]) \leqslant r(u-2) + 1 \leqslant r(u-1) - 1$, where the last inequality uses $r \geqslant 2$. This proves the required local bound in both cases.

It remains to verify the global density condition. Since G is $2r$-regular, every set $U \subseteq V(G)$ of size $u > n/10$ satisfies $e(G[U]) \leqslant ru$. Thus, for sufficiently large n, $\frac{e(G[U])}{u-1} \leqslant r + \frac{r}{u-1} \leqslant r + \frac{r}{\log_2 n}$. For $2 \leqslant u \leqslant n/10$, the local bound gives $e(G[U])/(u-1) \leqslant r$. Since every subgraph of G is contained in the subgraph induced by its vertex set, it follows that $m_1(G) \leqslant r + \frac{r}{\log_2 n}$. Hence C_n^r is $(r, 1/2, 1/10)$-degenerate.

A.2 Powers of tight hypergraph cycles

Fix integers $k, r \geqslant 2$, and let G be the r-th power of a k-uniform tight cycle on n vertices. Put $a = k + r - 2$ and $d = \binom{a}{k-1}$. We show that G is $(d, 1/2, 1/(2a))$-degenerate whenever n is sufficiently large in terms of k and r. Assume that $n > 2a$. Every edge of G has a unique first vertex in the cyclic ordering, with its remaining $k-1$ vertices among the next a positions. Each vertex is therefore the first vertex of exactly d edges, giving $e(G) = dn$. By cyclic symmetry, G is regular, and its degree is kd.

Let $U \subseteq V(G)$ with $2 \leqslant u := |U| \leqslant n/(2a)$. For $2 \leqslant u < k$, the induced hypergraph has no edges and $0 \leqslant d(u-1) - 1/2$, since $d \geqslant k \geqslant 2$. For $|U| \geqslant k$, the average cyclic distance between consecutive vertices of U is $n/u \geqslant 2a$. Hence, some gap between consecutive vertices of U contains at least a vertices outside U. Cut the cyclic ordering within this gap. No edge of $G[U]$ crosses the cut, so we may assign each edge of $G[U]$ to its first vertex in the resulting linear ordering.

Each vertex is assigned at most d edges. The last vertex of U is assigned none, while the penultimate vertex is assigned at most one; indeed, it is assigned none when $k \geqslant 3$. Since $d \geqslant k \geqslant 2$, we obtain $e(G[U]) \leqslant d(u-2) + 1 \leqslant d(u-1) - 1$. In particular, the local condition holds with $\alpha = 1/2$ and $\epsilon = 1/(2a)$.

Finally, if $U \subseteq V(G)$ has size $u > n/(2a)$, then kd-regularity gives $k\, e(G[U]) \leqslant \sum_{v \in U} \deg_G(v) = kdu$. Consequently, for sufficiently large n, $\frac{e(G[U])}{u-1} \leqslant d + \frac{d}{u-1} \leqslant d + \frac{d}{\log_2 n}$. For $k \leqslant u \leqslant n/(2a)$, the local estimate gives $e(G[U])/(u-1) \leqslant d$, while sets of size less than k contain no edges. Taking the maximum over all subgraphs yields $m_1(G) \leqslant d + \frac{d}{\log_2 n}$. Together with the local estimate, this shows that G is $(d, 1/2, 1/(2a))$-degenerate.

A.3 Cuts in cyclic clique construction

We show that the construction G following Corollary 5 has $\min_{D \leqslant |U| \leqslant \epsilon n} |\partial U| = D$. Let U be a vertex set of size $D \leqslant |U| \leqslant \epsilon n$. Write $u_i = |U \cap V(K_i)|$, where K_i is the ith D-clique.

If U is a union of whole blocks, an even-D construction has at least two transitions, each contributing $D/2$ edges. For odd D, four or more transitions contribute at least $2(D-1) \geqslant D$. If there are exactly two, the selected blocks form an interval. Under $|U| \leqslant \epsilon n < n/2$, its antipodal matching has at least one crossing edge, in addition to the $D-1$ edges from the cyclic matchings.

If U contains partial blocks, then internal clique edges contribute $\sum_i u_i(D - u_i)$ to ∂U. If at least two cliques are partially occupied, this contribution is at least $2(D-1) \geqslant D$. Suppose that exactly one block, say K_i, is partially occupied, and set $W := U \setminus V(K_i)$. Since $|U| \geqslant D$ and $|U \cap V(K_i)| \leqslant D-1$, we have $W \neq \emptyset$. Moreover, W is a union of whole

blocks and $|W| \leqslant |U| \leqslant \epsilon n$. Hence, by the preceding case, $|\partial W| \geqslant D$. All edges in ∂W are edges of the added matching, since no clique edge joins two distinct blocks. Among these boundary edges, at most $|U \cap V(K_i)| \leqslant D-1$ can have their other endpoint in $U \cap V(K_i)$, because every vertex is incident to exactly one added matching edge. Therefore, at least one edge of ∂W has its other endpoint outside U, and hence belongs to ∂U.

On the other hand, since K_i is partially occupied, the clique K_i contributes $|U \cap V(K_i)|\big(D - |U \cap V(K_i)|\big) \geqslant D-1$ edges to ∂U. Together with the external matching edge found above, this gives $|\partial U| \geqslant (D-1)+1 = D$. Hence $|\partial U| \geqslant D$ whenever $D \leqslant |U| \leqslant \epsilon n$. A single whole block has boundary exactly D, proving the claimed minimum.

For $2 \leqslant u < D$, the complete-graph bound gives $e(G[U]) \leqslant D(u-1)/2$. For $D \leqslant u \leqslant \epsilon n$, the cut bound gives the same inequality. For larger sets, the degree bound gives $m_1 \leqslant D/2 + O(1/n)$.

B A proof of Theorem 16

Lemma 18. *Let $q, \gamma, \alpha \in (0,1]$ with $\alpha \geqslant 4q$, let $r \in \mathbb{N}$, and let $\mathcal{H}$ be an r-uniform hypergraph on V. If there exists a $(q, \gamma; r, 1)$-spread distribution on $E(\mathcal{H})$ and W is a set of size $\alpha|V|$ chosen uniformly at random from V, then W contains an edge of $\mathcal{H}$ with probability at least*

$$1 - 2\exp\Big(-\frac{\alpha|V|}{6}\Big) - \frac{4\gamma q}{\alpha}.$$

Lemma 18 allows us to replace the term $40(C\ell)^{-1}$ in [20, Theorem 2.7] with $8\gamma(C\ell)^{-1}$, which yields Theorem 16. The proof of Lemma 18 is essentially the same as the proof of [20, Lemma 2.5], which uses a fairly standard second moment argument, so we omit it.

Proof of Theorem 16. Write $C := C_{4.2}$. The proof is nearly identical to Spiro's proof of [20, Theorem 2.7], so we only sketch the proof here.

We choose $W_1, \dots, W_\ell \subseteq V$ uniformly at random subject to $|W_i| = Cq|V|$ for each $i \in [\ell-1]$ and $|W_\ell| = \ell Cq|V|$, and we assume that $W \supseteq \bigcup_{i=1}^{\ell} W_i$. We define $\mathcal{H}^1, \dots, \mathcal{H}^\ell$ in the same way as Spiro does, and, as Spiro does, for $i \in [\ell-1]$ we say that W_i is *successful* if $|\mathcal{H}^{i+1}| \geqslant \big(1 - \frac{1}{2\ell}\big)|\mathcal{H}^i|$. The set W_ℓ plays the role of W' in Spiro's proof, and we say that W_ℓ is *successful* if it contains an edge of $\mathcal{H}^\ell$. By the choice of $\mathcal{H}^1, \dots, \mathcal{H}^\ell$, if W_ℓ is successful, then W contains an edge of $\mathcal{H}$. In particular, if W_i is successful for every $i \in [\ell]$, then W contains an edge of $\mathcal{H}$.

For $i \in [\ell]$, let $\mathcal{S}_i$ be the event that W_i is successful. The probability that W contains

an edge of $\mathcal{H}$ is at least the probability that $W_1, \dots, W_\ell$ are all successful. Moreover,

$$\begin{aligned}\mathbb{P}\Big[\bigcap_{i=1}^{\ell} \mathcal{S}_i\Big] &= \prod_{i=1}^{\ell} \mathbb{P}\Big[\mathcal{S}_i \,\Big|\, \bigcap_{j=1}^{i-1} \mathcal{S}_j\Big] = \mathbb{P}\Big[\mathcal{S}_\ell \,\Big|\, \bigcap_{i=1}^{\ell-1} \mathcal{S}_i\Big] \prod_{i=1}^{\ell-1} \mathbb{P}\Big[\mathcal{S}_i \,\Big|\, \bigcap_{j=1}^{i-1} \mathcal{S}_j\Big] \\ &\geqslant \mathbb{P}\Big[\mathcal{S}_\ell \,\Big|\, \bigcap_{i=1}^{\ell-1} \mathcal{S}_i\Big] \left(1 - \sum_{i=1}^{\ell-1} \mathbb{P}\Big[\overline{\mathcal{S}_i} \,\Big|\, \bigcap_{j=1}^{i-1} \mathcal{S}_j\Big]\right) \\ &\geqslant \mathbb{P}\Big[\mathcal{S}_\ell \,\Big|\, \bigcap_{i=1}^{\ell-1} \mathcal{S}_i\Big] - \sum_{i=1}^{\ell-1} \mathbb{P}\Big[\overline{\mathcal{S}_i} \,\Big|\, \bigcap_{j=1}^{i-1} \mathcal{S}_j\Big].\end{aligned}$$

Hence, it suffices to bound $\mathbb{P}\big[\mathcal{S}_\ell \mid \bigcap_{i=1}^{\ell-1} \mathcal{S}_i\big]$ from below and $\sum_{i=1}^{\ell-1} \mathbb{P}\big[\overline{\mathcal{S}_i} \mid \bigcap_{j=1}^{i-1} \mathcal{S}_j\big]$ from above.

We bound the sum from above as in Spiro's proof. By [20, Claim 2.10],

$$\mathbb{P}\Big[\overline{\mathcal{S}_i} \,\Big|\, \bigcap_{j=1}^{i-1} \mathcal{S}_j\Big] \leqslant 6\ell \Big(\frac{C}{4}\Big)^{-r_i/2} \qquad \text{for every } i \in [\ell - 1].$$

The proof of this fact uses [20, Claim 2.9], which states that for $i \in [\ell]$, if $W_1, \dots, W_{i-1}$ are successful (that is, on the event $\bigcap_{j=1}^{i-1} \mathcal{S}_j$), then $\mathcal{H}^i$ is $(2q; r_{i-1}, \dots, r_\ell)$-spread. Moreover, the same proof yields that if $W_1, \dots, W_{\ell-1}$ are successful, then $\mathcal{H}^\ell$ is $(q, 2\gamma; r_{\ell-1}, 1)$-spread. We need this to bound $\mathbb{P}\big[\mathcal{S}_\ell \mid \bigcap_{i=1}^{\ell-1} \mathcal{S}_i\big]$ from below: applying Lemma 18 to $\mathcal{H}^\ell$ with $\ell C q$ in the role of α and 2γ in the role of γ, we obtain

$$\mathbb{P}\Big[\mathcal{S}_\ell \,\Big|\, \bigcap_{i=1}^{\ell-1} \mathcal{S}_i\Big] \geqslant 1 - 2\exp\Big(-\frac{\ell C q |V|}{6}\Big) - \frac{8\gamma}{C\ell}.$$

Since $r_i \geqslant r_{\ell-1}$ for every $i \in [\ell - 1]$ and $C/4 \geqslant 2$, the sum is at most $6\ell(\ell - 1)(C/4)^{-r_{\ell-1}/2} \leqslant 6\ell^2 (C/4)^{-r_{\ell-1}/2}$. Combining the three inequalities above, we conclude that W contains an edge of $\mathcal{H}$ with probability at least

$$1 - 6\ell^2 \Big(\frac{C}{4}\Big)^{-r_{\ell-1}/2} - 2\exp\Big(-\frac{\ell C q |V|}{6}\Big) - \frac{8\gamma}{C\ell}. \qquad \square$$