The rank-nullity ring of a matroid

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Abstract

We introduce the rank-nullity ring of a matroid M, which is a subring of the Chow ring of the permutahedral toric variety. This subring contains the tautological Chern classes of M, a fact we deduce from a highly symmetric formula for these classes. When the matroid M is a uniform matroid, the rank-nullity ring coincides with the subring of S_n-invariants of the Chow ring of the permutahedral toric variety. In this case, we compute its Hilbert function explicitly and provide a Gröbner basis for the ideal of relations among its generators.

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1 Introduction

A matroid is a combinatorial abstraction of the concept of independence that arises in various areas of mathematics, including linear algebra and graph theory. Several geometric models of matroids have been explored in the literature, among them the matroid polytope, the Bergman fan, and the conormal fan; see [4] for a survey.

The framework of *tautological classes of matroids* introduced in [6] provides a unified perspective on these seemingly distinct models, allowing the authors to recover and generalize numerous known results, prove conjectures about their connections, and establish several new findings.

More specifically, given a matroid M, the authors of [6] define the *tautological sub K-class* $[S_M]$ and the *tautological quotient K-class* $[Q_M]$ as certain torus-equivariant classes on the permutohedral toric variety, generalizing the classes of the universal subbundle and quotient bundle associated with a linear subspace (see [6, Definition 3.9]). Their Chern roots $c_i(S_M), c_i(Q_M)$, called the *tautological Chern classes* of M, lie in the Chow ring $A^*(U_{n,n})$ of the permutohedral toric variety, which has a standard presentation by

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generators and relations:

$$A^*(U_{n,n}) = \frac{\mathbb{Z}[x_S : \emptyset \subsetneq S \subseteq [n]]}{\langle x_S x_T : S \not\subseteq T \text{ and } S \not\supseteq T\rangle + \langle \sum_{S \ni e} x_S : e \in [n]\rangle}.$$

In [6, Remark III.1], the authors provide a combinatorial formula for the generating functions of $c_i(S_M)$ and $c_i(Q_M)$, which we adapt in this work to establish the following explicit description.

Theorem 3. *Let M be a matroid of rank r on the set $[n] := \{1, \ldots, n\}$, with rank function $\mathrm{r_M} : 2^{[n]} \to \mathbb{Z}_{\geqslant 0}$. For any $0 \leqslant k \leqslant r$, the Chern class $c_k(S_M)$ of the tautological subbundle S_M is given by*

$$c_k(S_M) = \sum \alpha_S(S_1, \ldots, S_t, p_1, \ldots, p_t)\, x_{S_1}^{p_1} x_{S_2}^{p_2} \cdots x_{S_t}^{p_t} \quad \in A^k(U_{n,n}),$$

where the sum is taken over all chains of subsets $\emptyset \subsetneq S_1 \subsetneq \cdots \subsetneq S_t \subseteq E$ and all positive integers $p_1, \ldots, p_t$ satisfying $\sum_{i=1}^t p_i = k$, with coefficients given by

$$\alpha_S(S_1, \ldots, S_t, p_1, \ldots, p_t) := (-1)^k \binom{\mathrm{r_M}(S_1)}{p_1} \binom{\mathrm{r_M}(S_2) - \tilde{p}_1}{p_2} \cdots \binom{\mathrm{r_M}(S_t) - \tilde{p}_{t-1}}{p_t}$$

where $\tilde{p}_i = \sum_{j=1}^i p_j$.

A similar formula holds for the Chern classes $c_k(Q_M)$ of the tautological quotient bundle Q_M; see Theorem 3 in Section 3.1.

In Theorem 5, we specialize the formula above to obtain an explicit description of the Poincaré duals $\mathrm{ch}_k(M)$ of the Chern-Schwartz-MacPherson cycles of the matroid M (see [12]) in the Chow ring $A^*(M)$ of M. The second author has also used an earlier version of Theorem 5 to prove results regarding the distribution and emergence of *Chern numbers of matroids* of rank 3, which are the numbers obtained by intersecting various CSM-cycles of a matroid (see [13]).

Motivated by a preliminary version of this paper in which our formula for the CSM classes $\mathrm{ch}_k(M)$ was still conjectural, an independent proof of it was also given in [7, Theorem 3.1 and Theorem 4.2]. In that work, the authors also provide a geometric argument in the case the matroid M is realizable over $\mathbb{C}$.

The symmetry of the formula in Theorem 3, in which the coefficients α_S depend only on the sizes of the subsets S_j, implies that the tautological Chern classes $c_i(S_M), c_i(Q_M)$ of a matroid live in a particular subring $R^*(M)$ of $A^*(U_{n,n})$, which we call the *rank-nullity ring* of the matroid M.

Definition 7. *Let M be a matroid on the set $[n]$ with rank function $\mathrm{r_M}$ and nullity function $\mathrm{n_M}$ (given by $\mathrm{n_M}(S) := |S| - \mathrm{r_M}(S)$). For $0 \leqslant i, j \leqslant n$, let*

$$y_{i,j} := \sum_{\substack{\emptyset \neq S \subseteq [n] \\ \mathrm{r_M}(S) = i,\ \mathrm{n_M}(S) = j}} x_S \quad \in A^*(U_{n,n}).$$

The rank-nullity ring *of M is the subring*

$$R^*(M) := \mathbb{Z}[y_{i,j} \,:\, 0 \leqslant i, j \leqslant n] \quad \subseteq A^*(U_{n,n}).$$

Our main motivation for studying the rank-nullity ring is that, as we show in Theorem 9, it contains all the tautological Chern classes: for any k we have

$$c_k(S_M), c_k(Q_M) \in R^*(M).$$

The rank-nullity ring $R^*(M)$ is in general much smaller than the Chow ring $A^*(U_{n,n})$; we quantify in Theorem 10 exactly how many linearly independent generators this subring has.

In Theorem 11 we note that the rank-nullity ring $R^*(M)$ is invariant under any automorphism of the matroid M; in other words, we have

$$R^*(M) \subseteq A^*(U_{n,n})^{\mathrm{Aut}(M)},$$

where $\mathrm{Aut}(M)$ denotes the automorphism group of M. The action of $\mathrm{Aut}(M)$ on the Chow ring $A^*(M)$ of M was recently studied in [2], where the authors lift the Poincaré duality and Hard Lefschetz properties to the equivariant setting.

The rank-nullity ring of the uniform matroid $U_{r,n}$ of rank r on n elements is of particular interest. In this case, we show that it agrees with the subring of S_n-invariants of the permutohedral Chow ring $A^*(U_{n,n})$. We also compute a $\mathbb{Z}$-basis and the Hilbert function of this subring.

Theorem (Theorem 13, Theorem 14, and Theorem 19)**.** *For any positive integers $r \leqslant n$, the rank-nullity ring of the uniform matroids satisfies*

$$R^*(U_{r,n}) = A^*(U_{n,n})^{S_n}.$$

It is a free $\mathbb{Z}$-module with basis

$$\mathcal{B}_n = \{z_{s_1}^{p_1} z_{s_2}^{p_2} \cdots z_{s_l}^{p_l} : 1 \leqslant s_1 < s_2 < \cdots < s_l \leqslant n \text{ and } 1 \leqslant p_i < s_i - s_{i-1}\},$$

where $z_j := \sum_{|S|=j} x_S$ and by convention $s_0 = 0$. Its Hilbert function is given by

$$\mathrm{rank}_{\mathbb{Z}}(R^d(U_{r,n})) = \binom{n-1}{d}.$$

We also investigate the ideal of relations among the generators z_j of the rank-nullity ring $R^*(U_{r,n}) = A^*(U_{n,n})^{S_n}$, and provide a Gröbner basis for it.

Theorem 16. *The rank-nullity ring $R^*(U_{r,n})$ of the uniform matroids is the quotient of the free polynomial ring $\mathbb{Z}[Z_1, \ldots, Z_n]$ by the ideal $\mathcal{I}$ generated by the polynomials*

$$Z_a \cdot \left(\sum_{i=b}^{n} \binom{i-a-1}{b-a-1} (Z_i + Z_{i+1} + \cdots + Z_n)^{b-a} \right) \qquad \text{for } 0 \leqslant a < b \leqslant n$$

(with the convention that $Z_0 = 1$), where the class $[Z_i]$ in this quotient is equal to the element $z_i \in R^*(U_{r,n})$. In fact, the polynomials above are a Gröbner basis for $\mathcal{I}$.

Finally, in the last section of the paper we further explore the behavior of the Hilbert functions of general rank-nullity rings (see Table 1 for examples). In Theorem 23 we prove that for any matroid M on the ground set $[n]$, the Hilbert function of $R^*(M)$ satisfies

$$\operatorname{rank}_{\mathbb{Z}}(R^0(M)) \leqslant \operatorname{rank}_{\mathbb{Z}}(R^1(M)) \leqslant \ldots \leqslant \operatorname{rank}_{\mathbb{Z}}(R^{\lfloor \frac{n}{2} \rfloor}(M)).$$

We do not know whether these Hilbert functions always satisfy stronger properties.

Question 24. *Is the Hilbert function of the rank-nullity ring of a matroid M always unimodal? Is it log-concave?*

2 Background and notation

In this section we briefly recall all the background necessary for the paper. Readers already familiar with these topics might want to just skim through it or jump directly to Section 3.

2.1 The Chow ring of a matroid

Feichtner and Yuzvinsky [9] defined a graded commutative ring associated to an arbitrary matroid, as we recall below. The definition is motivated by the fact that, when a matroid arises from a complex hyperplane arrangement, the ring is isomorphic to the Chow ring of de Concini and Procesi *wonderful compactification* of the complement of the arrangement [8].

Definition 1. Let M be a matroid on the ground set $E = [n] = \{1, \ldots, n\}$. The *Chow ring* of M is the quotient

$$A^*(M) = \frac{\mathbb{Z}[x_F : F \text{ is a nonempty flat of } M]}{\langle x_F x_{F'} : F \not\subseteq F' \text{ and } F \not\supseteq F' \rangle + \langle \sum_{F \ni e} x_F : e \in E \rangle}. \tag{2.1}$$

The Chow ring of M is a graded Artinian ring of socle degree $\operatorname{rank}(M) - 1$, and its Hilbert series is a polynomial of that degree. Adiprasito, Huh, and Katz [1] proved that the Chow ring of any matroid satisfies several nice properties that we now recall.

Poincaré duality states that, if the matroid M has rank r, there is a group isomorphism

$$A^{r-1}(M) \cong \mathbb{Z}$$

and moreover, for any $0 \leqslant k \leqslant r-1$, the multiplication map

$$A^k(M) \times A^{r-1-k}(M) \longrightarrow A^{r-1}(M) \cong \mathbb{Z}$$

induces an isomorphism

$$A^{r-1-k}(M) \cong \operatorname{Hom}_{\mathbb{Z}}(A^k(M), \mathbb{Z}).$$

As a consequence, the *Hilbert function* of the ring $A^*(M)$, which returns the dimension of its various graded components, is symmetric.

The *Hard Lefschetz* property states that there is an element $l \in A^1(M)$ such that, for every nonnegative integer $q \leqslant (r-1)/2$, multiplication by l defines an isomorphism of vector spaces

$$\begin{aligned} L_l^q : A^q(M)_{\mathbb{R}} &\xrightarrow{\simeq} A^{r-1-q}(M)_{\mathbb{R}} \\ a &\longmapsto l^{r-1-2q} \cdot a. \end{aligned}$$

This implies the weaker Lefschetz property: multiplication by the element $l \in A^1(M)$ from $A^q(M)_{\mathbb{R}}$ to $A^{q+1}(M)_{\mathbb{R}}$ is injective when $q \leqslant (r-1)/2$ and surjective when $q \geqslant (r-1)/2$. It follows that the Hilbert function of the Chow ring $A^*(M)$ is unimodal, with peak around $q = (r-1)/2$. In fact, it is conjectured that the Hilbert function of $A^*(M)$ is log-concave, and further, that it is the sequence of coefficients of a real-rooted polynomial [16, 10].

2.2 The Bergman fan and Minkowski weights

Ardila and Klivans [3] assigned to any matroid M a rational balanced polyhedral fan Σ_M, called the Bergman fan of M, having a cone for each chain of flats.

We denote by e_i, with $i \in E$, the standard basis vectors of $\mathbb{Z}^E$, and by N the quotient lattice $\mathbb{Z}^E/\mathbb{Z}\mathbf{1}$, where the vector $\mathbf{1} = \sum_{i \in E} e_i \in \mathbb{Z}^E$. We also write u_i for the image of e_i in N, and $u_S = \sum_{i \in S} u_i$ for any subset $S \subseteq E$.

Definition 2. Let M be a loopless matroid of rank r on the set E. The *Bergman fan* Σ_M of M is the pure $(r-1)$-dimensional polyhedral fan in $N_{\mathbb{R}} := N \otimes \mathbb{R}$ consisting of the cones

$$\sigma_{\mathcal{F}} := \operatorname{cone}(u_{F_1}, u_{F_2}, \ldots, u_{F_k}) \quad \subseteq N_{\mathbb{R}}$$

for each chain of flats $\mathcal{F} : \emptyset \subsetneq F_1 \subsetneq \cdots \subsetneq F_k \subsetneq E$ in M.

The Bergman fan $\Sigma_{U_{n,n}}$ of the free matroid $U_{n,n}$ on n elements is a complete fan in $N_{\mathbb{R}}$, called the *permutahedral fan*.

The group $\mathrm{MW}_*(\Sigma_M)$ of *Minkowski weights* supported on the Bergman fan Σ_M consists of weight functions ω from the set of cones of Σ_M to $\mathbb{Z}$ that satisfy the balancing condition — see, for instance, [1, Section 5.1] for a precise definition. There is a duality between Minkowski weights and elements of the Chow ring, as we now explain following the exposition in [1, Section 5.2]. For a chain of flats $\emptyset \subsetneq F_1 \subsetneq \cdots \subsetneq F_k \subsetneq E$ in M corresponding to the cone $\sigma \in \Sigma_M$, let $x_\sigma := x_{F_1} x_{F_2} \cdots x_{F_k} \in A^k(M)$. The elements x_σ, with σ a k-dimensional cone of Σ_M, generate the group $A^k(M)$. The map

$$\begin{aligned} t_{\Sigma_M} : \mathrm{MW}_k(\Sigma_M) &\xrightarrow{\simeq} \mathrm{Hom}(A^k(M), \mathbb{Z}) \\ \omega &\longmapsto t_{\Sigma_M}\omega, \qquad \text{where } t_{\Sigma_M}\omega\,(x_\sigma) := \omega(\sigma) \end{aligned} \tag{2.2}$$

is an isomorphism of groups, which is an analogue of the Kronecker duality map in algebraic topology. This isomorphism gives rise to the *cap product*:

$$\begin{aligned} A^l(M) \times \mathrm{MW}_k(\Sigma_M) &\longrightarrow \mathrm{MW}_{k-l}(\Sigma_M) \\ (\xi, \omega) &\longmapsto \xi \cap \omega, \qquad \text{where } \xi \cap \omega\,(\sigma) := t_{\Sigma_M}\omega\,(\xi \cdot x_\sigma). \end{aligned} \tag{2.3}$$

Let ω_M be the *Bergman class* of M, i.e., the top-dimensional Minkowski weight in $\mathrm{MW}_{r-1}(\Sigma_M)$ where every maximal cone has weight 1. As noted in [5, Corollary 4.2.4], we have an isomorphism

$$\begin{aligned} \delta_{\Sigma_M} : A^*(M) &\xrightarrow{\simeq} \mathrm{MW}_{r-1-*}(\Sigma_M) \\ \xi &\longmapsto \xi \cap \omega_M, \end{aligned} \tag{2.4}$$

which is equal to the composition of the isomorphism in Equation (2.2) with Poincaré duality.

Finally, let $\Sigma_{U_{n,n}}$ be the permutahedral fan, i.e., the Bergman fan of the free matroid $U_{n,n}$. As noted in the discussion right after [5, Proposition 2.1.7], the inclusion of fans $\iota : \Sigma_M \hookrightarrow \Sigma_{U_{n,n}}$ induces the surjective pullback map

$$\begin{aligned} \iota^* : A^*(U_{n,n}) &\longrightarrow A^*(M) \\ x_S &\longmapsto \begin{cases} x_S & \text{if } S \subseteq E \text{ is a flat of } M, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{2.5}$$

Any Minkowski weight $\omega \in \mathrm{MW}_k(\Sigma_M)$ can also be seen as a Minkowski weight $\iota_*(\omega) \in \mathrm{MW}_k(\Sigma_{U_{n,n}})$. If $\xi \in A^l(U_{n,n})$ and $\omega \in \mathrm{MW}_k(\Sigma_M)$, we then have

$$\xi \cap \iota_*(\omega) = \iota_*(\iota^*(\xi) \cap \omega).$$

2.3 Tautological classes of matroids

In the paper [6], Berget, Eur, Spink, and Tseng introduced the tautological classes of a matroid. This powerful framework allowed for the unification of various results from several separate geometric models for matroids.

Let $E = [n]$, and consider the $(n-1)$-dimensional permutahedral toric variety X_E, which is the projective toric variety associated to the permutahedral fan $\Sigma_{U_{n,n}}$. Denote by $K_0^T(X_E)$ be the T-equivariant K-ring of X_E. The Chow ring of X_E is the Chow ring $A^*(U_{n,n})$ of the free matroid $U_{n,n}$. Suppose M is a matroid on the set E. In the case M is realizable by a linear subspace $L \subseteq \mathbb{C}^E$, the T-equivariant K-classes $[S_L], [Q_L] \in K_0^T(X_E)$ of the tautological subbundle S_L and quotient bundle Q_L on X_E (see [6, Definition 1.2]) depend only on the matroid M. This leads to a combinatorial definition of the *tautological K-classes* of any matroid M, which are elements $[S_M], [Q_M] \in K_0^T(X_E)$ (see [6, Definition 3.9]). The *tautological Chern classes* $c_i(S_M), c_i(Q_M) \in A^i(U_{n,n})$ of a matroid M are the non-equivariant Chern classes of $[S_M], [Q_M]$.

A formula for the Chern roots of $[S_M]$ and $[Q_M]$ is provided in [6, Remark III.1], as we now explain. Fix a sequence of matroids $M_0, M_1, \ldots, M_n$ on the set $[n]$ such that M_i has rank i, the matroid $M_r = M$, and M_i is a quotient of M_j for every $i < j$. Such a sequence has the property that $M_0 = U_{0,n}$ and $M_n = U_{n,n}$. Denote by $\mathrm{r}_{\mathrm{M_i}}$ the rank function of M_i.

Then, we have the following expressions for the Chern polynomials of $[S_M]$ and $[Q_M]$:

$$\sum_{k=0}^{r} c_k(S_M)t^k = \prod_{i=0}^{r-1}\left(1 - t\sum_{\emptyset\subsetneq S\subseteq E}\big(\mathrm{r}_{\mathrm{M}_{i+1}}(S) - \mathrm{r}_{\mathrm{M}_i}(S)\big)x_S\right) \quad \in A^*(U_{n,n})[t], \tag{2.6}$$

$$\sum_{k=0}^{n-r} c_k(Q_M)t^k = \prod_{i=r}^{n-1}\left(1 - t\sum_{\emptyset\subsetneq S\subseteq E}\big(\mathrm{r}_{\mathrm{M}_{i+1}}(S) - \mathrm{r}_{\mathrm{M}_i}(S)\big)x_S\right) \quad \in A^*(U_{n,n})[t]. \tag{2.7}$$

2.4 Chern-Schwartz-MacPherson (CSM) cycles of matroids.

López de Medrano, Rincón, and Shaw introduced in [12] certain Minkowski weights $\mathrm{csm}_k(M)$ on the Bergman fan Σ_M of a rank-r matroid M, one for each dimension $0 \leqslant k \leqslant r-1$, called the *Chern-Schwartz-MacPherson (CSM) cycles* of M. The weights of the maximal cones of $\mathrm{csm}_k(M)$ are given by products of beta invariants of certain minors of M; see [12, Definition 2.8] for details. This construction is motivated by the fact that, when the matroid M arises from a complex hyperplane arrangement, these Minkowski weights encode the Chern-Schwartz-MacPherson classes of the complement of the arrangement inside its wonderful compactification.

As shown in [6], CSM cycles of matroids are one of the various invariants that are nicely obtained from their tautological Chern classes. Concretely, given a matroid M of rank r, [6, Theorem C] states that

$$\mathrm{csm}_k(M) = \iota^*(c_{r-1-k}(S_M)) \cap \omega_M \quad \in \mathrm{MW}_k(\Sigma_M), \tag{2.8}$$

where ι^* is as in Equation (2.5) and ω_M is the Bergman class of Σ_M. Keeping in mind the isomorphism in Equation (2.4), this shows that

$$\mathrm{ch}_k(M) := \iota^*(c_k(S_M)) \quad \in A^k(M) \tag{2.9}$$

is the element of $A^k(M)$ satisfying

$$\delta_{\Sigma_M}(\mathrm{ch}_k(M)) = \mathrm{csm}_{r-1-k}(M). \tag{2.10}$$

3 The rank-nullity ring of a matroid

3.1 A formula for the tautological Chern classes

In this subsection we give an explicit formula for the Chern classes $c_k(S_M)$ and $c_k(Q_M)$ of the tautological subbundle and quotient bundle of a matroid M in the presentation given in Theorem 1 of the Chow ring of the free matroid $A^*(U_{n,n})$. We denote by r_M the rank function of M, and by n_M its nullity function, namely, for $S \subseteq E$,

$$\mathrm{n}_\mathrm{M}(S) = |S| - \mathrm{r}_\mathrm{M}(S).$$

Theorem 3. *Let M be a matroid of rank r on the set $E = [n]$. For $0 \leqslant k \leqslant r$, the Chern class $c_k(S_M)$ of the tautological subbundle S_M is given by*

$$c_k(S_M) = \sum \alpha_S(S_1, \ldots, S_t, p_1, \ldots, p_t)\, x_{S_1}^{p_1} x_{S_2}^{p_2} \ldots x_{S_t}^{p_t} \quad \in A^k(U_{n,n}),$$

where the sum is taken over all chains of subsets $\emptyset \subsetneq S_1 \subsetneq \cdots \subsetneq S_t \subseteq E$ and all positive integers $p_1, \ldots, p_t$ satisfying $\sum_{i=1}^{t} p_i = k$, with coefficients given by

$$\alpha_S(S_1, \ldots, S_t, p_1, \ldots, p_t) = (-1)^k \binom{\mathrm{r_M}(S_1)}{p_1} \binom{\mathrm{r_M}(S_2) - \tilde{p}_1}{p_2} \ldots \binom{\mathrm{r_M}(S_t) - \tilde{p}_{t-1}}{p_t}$$

where $\tilde{p}_i = \sum_{j=1}^{i} p_j$. Similarly, for any $0 \leqslant k \leqslant n - r$, the Chern class $c_k(Q_M)$ of the tautological quotient bundle Q_M is given by

$$c_k(Q_M) = \sum \alpha_Q(S_1, \ldots, S_t, p_1, \ldots, p_t)\, x_{S_1}^{p_1} x_{S_2}^{p_2} \ldots x_{S_t}^{p_t} \quad \in A^k(U_{n,n}),$$

where the sum is taken over all chains of subsets $\emptyset \subsetneq S_1 \subsetneq \cdots \subsetneq S_t \subseteq E$ and all positive integers $p_1, \ldots, p_t$ satisfying $\sum_{i=1}^{t} p_i = k$, with coefficients given by

$$\alpha_Q(S_1, \ldots, S_t, p_1, \ldots, p_t) = (-1)^k \binom{\mathrm{n_M}(S_1)}{p_1} \binom{\mathrm{n_M}(S_2) - \tilde{p}_1}{p_2} \ldots \binom{\mathrm{n_M}(S_t) - \tilde{p}_{t-1}}{p_t}$$

where $\tilde{p}_i = \sum_{j=1}^{i} p_j$.

Proof. Given a sequence of matroids $M_0, M_1, \ldots, M_n$ with properties as given in Section 2.3, we use the formula for the Chern roots of S_M and Q_M in [6, Remark III.1], recalled above in Equation (2.6) and Equation (2.7):

$$\sum_{k=0}^{r} c_k(S_M) t^k = \prod_{i=0}^{r-1} \left(1 - t \sum_{\emptyset \subsetneq S \subseteq E} \big(\mathrm{r_{M_{i+1}}}(S) - \mathrm{r_{M_i}}(S)\big) x_S\right), \tag{3.1}$$

$$\sum_{k=0}^{n-r} c_k(Q_M) t^k = \prod_{i=r}^{n-1} \left(1 - t \sum_{\emptyset \subsetneq S \subseteq E} \big(\mathrm{r_{M_{i+1}}}(S) - \mathrm{r_{M_i}}(S)\big) x_S\right).$$

An explicit formula for each of the Chern classes $c_k(S_M)$ and $c_k(Q_M)$ is obtained by expanding the products above. The fact that the sequence of matroids $M_0, M_1, \ldots, M_n$ increases rank by one every step implies that the difference $\mathrm{r_{M_{i+1}}}(S) - \mathrm{r_{M_i}}(S)$ is either 0 or 1 for all i and all $S \subseteq [n]$. For each $0 \leqslant i \leqslant n - 1$, let

$$\mathscr{S}_i = \{S \subseteq E : \mathrm{r_{M_{i+1}}}(S) - \mathrm{r_{M_i}}(S) = 1\}.$$

Moreover, for each nonempty $S \subseteq E$, let

$$\zeta_S^{S_M} = \{i \in [0, r-1] : S \in \mathscr{S}_i\},$$
$$\zeta_S^{Q_M} = \{i \in [r, n-1] : S \in \mathscr{S}_i\}.$$

Whenever $S \subseteq T$ we have the inclusions $\zeta_S^{S_M} \subseteq \zeta_T^{S_M}$ and $\zeta_S^{Q_M} \subseteq \zeta_T^{Q_M}$; see [15, Proposition 7.3.6 (iii)]. Moreover, note that $|\zeta_S^{S_M}| = \mathrm{r_M}(S)$ since $\mathrm{r_{M_0}}(S) = 0$ and $\mathrm{r_{M_r}}(S) = \mathrm{r_M}(S)$. Likewise, we have $|\zeta_S^{Q_M}| = \mathrm{n_M}(S)$ since $\mathrm{r_{M_n}}(S) = |S|$ and $\mathrm{r_{M_r}}(S) = \mathrm{r_M}(S)$.

To prove the formula for $c_k(S_M)$, we write the product in Equation (3.1) as

$$\prod_{i=0}^{r-1} \big(1 - t \sum_{S \in \mathscr{S}_i} x_S\big).$$

By the relations in the Chow ring $A^*(U_{n,n})$, if S and T are incomparable then $x_S x_T = 0$. When expanding this product, we thus obtain only monomials of the form $x_{S_1}^{p_1} x_{S_2}^{p_2} \dots x_{S_t}^{p_t}$ for a chain of subsets $\emptyset \subsetneq S_1 \subsetneq \cdots \subsetneq S_t \subseteq E$. To find the corresponding coefficient $\alpha_S(S_1, \dots, S_t, p_1, \dots, p_t)$, we need to count the number of times the monomial $x_{S_1}^{p_1} x_{S_2}^{p_2} \dots x_{S_t}^{p_t}$ is obtained in this expansion. The variable x_{S_1} appears in exactly $|\zeta_{S_1}^{S_M}| = \mathrm{r_M}(S_1)$ many of the sums $\sum_{S \in \mathscr{S}_i} x_S$, hence there are $\binom{\mathrm{r_M}(S_1)}{p_1}$ number of ways of obtaining $x_{S_1}^{p_1}$. The variable x_{S_2} appears in exactly $|\zeta_{S_2}^{S_M}| = \mathrm{r_M}(S_2)$ many of the sums $\sum_{S \in \mathscr{S}_i} x_S$ (which include all the sums that contained x_{S_1}), but we cannot choose it from the sums where we chose x_{S_1}, hence there are $\binom{\mathrm{r_M}(S_2)-p_1}{p_2}$ ways of obtaining $x_{S_2}^{p_2}$. In general, the variable x_{S_j} appears in exactly $|\zeta_{S_j}^{S_M}| = \mathrm{r_M}(S_j)$ many sums $\sum_{S \in \mathscr{S}_i} x_S$ (which include all the sums that contained the previous variables), and since we cannot choose it from the sums where we have chosen previous variables x_{S_l} with $1 \leqslant l \leqslant j-1$, there are $\binom{\mathrm{r_M}(S_j)-\tilde{p}_{j-1}}{p_j}$ ways of obtaining $x_{S_j}^{p_j}$. This proves the claimed formula for $c_k(S_M)$. A similar argument proves the formula for $c_k(Q_M)$. □

Example 4. Consider the rank-3 graphic matroid M arising from the complete graph K_4 on 4 vertices. The ground set of M is the set of edges of K_4; see Figure 1. The bases of M are the spanning trees of K_4, so the subsets of size 3 that are not bases are $\{\{1,2,5\},\{1,4,6\},\{2,3,6\},\{3,4,5\}\}$. The sequence $(U_{0,6}, U_{1,6}, U_{2,6}, M, U_{4,6}, U_{5,6}, U_{6,6})$

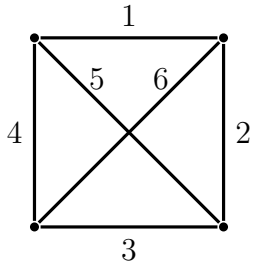


Figure 1: The complete graph K_4.

satisfies the properties given in the proof of Theorem 3. The corresponding subsets $\mathscr{S}_i$

are:

$$\begin{aligned}
\mathscr{S}_0 &= \{\emptyset \subsetneq S \subseteq E\}, & \mathscr{S}_3 &= \{\emptyset \subsetneq S \subseteq E : \mathrm{n_M}(S) \geqslant 1\},\\
\mathscr{S}_1 &= \{\emptyset \subsetneq S \subseteq E : \mathrm{r_M}(S) \geqslant 2\}, & \mathscr{S}_4 &= \{\emptyset \subsetneq S \subseteq E : \mathrm{n_M}(S) \geqslant 2\},\\
\mathscr{S}_2 &= \{\emptyset \subsetneq S \subseteq E : \mathrm{r_M}(S) = 3\}, & \mathscr{S}_5 &= \{E\}.
\end{aligned}$$

Moreover, we have that

$$\zeta_S^{S_M} = \begin{cases} \{0\} & \text{if } \mathrm{r_M}(S) = 1,\\ \{0,1\} & \text{if } \mathrm{r_M}(S) = 2,\\ \{0,1,2\} & \text{if } \mathrm{r_M}(S) = 3, \end{cases} \qquad \zeta_S^{Q_M} = \begin{cases} \{3\} & \text{if } \mathrm{n_M}(S) = 1,\\ \{3,4\} & \text{if } \mathrm{n_M}(S) = 2,\\ \{3,4,5\} & \text{if } S = E. \end{cases}$$

Note that $\zeta_S^{S_M} \subseteq \zeta_T^{S_M}$ and $\zeta_S^{Q_M} \subseteq \zeta_T^{Q_M}$ whenever $S \subseteq T$, as predicted. According to Equations (2.6) and (2.7), the Chern classes of S_M and Q_M are given by

$$\sum_{i=0}^{3} c_i(S_M)t^i = (1 - t\sum_{S\in\mathscr{S}_0} x_S)(1 - t\sum_{S\in\mathscr{S}_1} x_S)(1 - t\sum_{S\in\mathscr{S}_2} x_S),$$
$$\sum_{i=0}^{3} c_i(Q_M)t^i = (1 - t\sum_{S\in\mathscr{S}_3} x_S)(1 - t\sum_{S\in\mathscr{S}_4} x_S)(1 - t\sum_{S\in\mathscr{S}_5} x_S).$$

For $i = 1, 2, 3$, let $\mathcal{T}_i$ be the set of nonempty subsets of E of rank i. The expression for the Chern class $c_2(S_M)$ given in Theorem 3 is

$$\begin{aligned} c_2(S_M) = \sum_{S\in\mathcal{T}_2} x_S^2 + 3\sum_{S\in\mathcal{T}_3} x_S^2 + \sum_{S\in\mathcal{T}_1, T\in\mathcal{T}_2} x_S x_T + 2\sum_{S\in\mathcal{T}_1, T\in\mathcal{T}_3} x_S x_T \\ + 2\sum_{S,T\in\mathcal{T}_2,\ S\subsetneq T} x_S x_T + 4\sum_{S\in\mathcal{T}_2, T\in\mathcal{T}_3} x_S x_T + 6\sum_{S,T\in\mathcal{T}_3,\ S\subsetneq T} x_S x_T, \end{aligned}$$

where all these sums can be taken only over subsets that form a chain under inclusion.

We now give a formula for $\mathrm{ch}_k(M) \in A^k(M)$, which is the dual element to the csm cycle $\mathrm{csm}_{r-1-k}(M) \in \mathrm{MW}_{r-1-k}(M)$, as in Equation (2.10). Inspired by a previous version of this paper in which our formula was still conjectural, the authors of [7] have also provided a separate proof of an equivalent statement (see Theorem 6), including a geometric proof in the realizable case.

Corollary 5. *Let M be a matroid of rank r on $E = [n]$. Then the element $\mathrm{ch}_k(M) \in A^k(M)$ dual to $\mathrm{csm}_{r-1-k}(M) \in \mathrm{MW}_{r-1-k}(M)$ is given by*

$$\mathrm{ch}_k(M) = \sum \alpha_S(F_1, \ldots, F_t, p_1, \ldots, p_t)\, x_{F_1}^{p_1} x_{F_2}^{p_2} \cdots x_{F_t}^{p_t},$$

where the sum is taken over all chains of flats $\emptyset \subsetneq F_1 \subsetneq \cdots \subsetneq F_t \subseteq E$ and all positive integers $p_1, \ldots, p_t$ satisfying $\sum_{i=1}^{t} p_i = k$, with coefficients given by

$$\alpha_S(F_1, \ldots, F_t, p_1, \ldots, p_t) = (-1)^k \binom{\mathrm{r_M}(F_1)}{p_1} \binom{\mathrm{r_M}(F_2) - \tilde{p}_1}{p_2} \cdots \binom{\mathrm{r_M}(F_t) - \tilde{p}_{t-1}}{p_t}$$

where $\tilde{p}_i = \sum_{j=1}^{i} p_j$.

Proof. Equation (2.9) states that $\mathrm{ch}_k(M) = \iota^*(c_k(S_M))$. The result thus follows from Theorem 3 and the description of ι^* given in Equation (2.5). □

Remark 6. One possible choice for the sequence of matroids $M_0, M_1, \ldots, M_n$ in the proof of Theorem 3 is the *full Higgs lift* of the matroid M, where M_i is taken to be the matroid with bases

$$\mathcal{B}_i = \Big\{ S \in \binom{E}{i} \ : \ S \text{ contains or is contained in a basis of } M \Big\}.$$

For example, the sequence given in Theorem 4 is the full Higgs lift of the graphic matroid of K_4. With this choice, Equation (2.6) and Equation (2.7) become

$$\sum_{k=0}^{r} c_k(S_M)t^k = \prod_{i=1}^{r} \Big(1 - \sum_{\mathrm{r_M}(S)\geqslant i} x_S\Big)t, \tag{3.2}$$

$$\sum_{k=0}^{n-r} c_k(Q_M)t^k = \prod_{i=1}^{n-r} \Big(1 - \sum_{\mathrm{n_M}(S)\geqslant i} x_S\Big)t. \tag{3.3}$$

Indeed, for $0 \leqslant i \leqslant r-1$, the rank function $\mathrm{r_{M_i}}$ of the matroid M_i is given by $\mathrm{r_{M_i}}(S) = \min(i, \mathrm{r_M}(S))$, which implies that

$$\mathscr{S}_i = \{S \subseteq E : \mathrm{r_{M_{i+1}}}(S) - \mathrm{r_{M_i}}(S) = 1\} = \{S \subseteq E : \mathrm{r_M}(S) \geqslant i+1\}.$$

Similarly, for $r+1 \leqslant i \leqslant n$, the rank function $\mathrm{r_{M_i}}$ of the matroid M_i is given by $\mathrm{r_{M_i}}(S) = \min(|S|, \mathrm{r_M}(S) + i - r)$, which implies that

$$\mathscr{S}_{r+j} = \{S \subseteq E : r_{M_{r+j+1}}(S) - r_{M_{r+j}}(S) = 1\} = \{S \subseteq E \mid \ \mathrm{n_M}(S) \geqslant j+1\}.$$

After taking the cap product of Equation (3.2) with the Bergman class ω_M of M, this leads to the "staircase" formula for csm cycles of M provided in [7]:

$$\sum_{k=0}^{r} \mathrm{ch}_k(M)\, t^k = \prod_{i=1}^{r} \Big(1 - \sum_{\substack{F \text{ flat} \\ \mathrm{r_M}(F)\geqslant i}} x_F\Big)\, t \quad \in A^*(M)[t].$$

This formula is equivalent to our Theorem 5, by a similar argument to the proof of Theorem 3.

3.2 The rank-nullity ring of a matroid

In this subsection we introduce the rank-nullity ring of a matroid M as certain subring of the permutahedral Chow ring $A^*(U_{n,n})$, and show that it contains the tautological Chern classes $c_k(S_M), c_k(Q_M)$ of M.

Definition 7. Let M be a matroid on the set $[n]$ with rank function $\mathrm{r_M}$ and nullity function $\mathrm{n_M}$. For $0 \leqslant i, j \leqslant n$, consider the degree-1 element

$$y_{i,j} := \sum_{\substack{\emptyset \neq S \subseteq [n] \\ \mathrm{r_M}(S)=i,\ \mathrm{n_M}(S)=j}} x_S \quad \in A^*(U_{n,n}).$$

Note that $y_{i,j} = 0$ whenever $i + j > n$ or $(i, j) = (0, 0)$. The *rank-nullity ring* $R^*(M)$ of the matroid M is the graded subring

$$R^*(M) := \mathbb{Z}[y_{i,j} \,:\, 0 \leqslant i, j \leqslant n] \quad \subseteq A^*(U_{n,n}).$$

Note that we do not require the matroid M to be loopless.

We provide a simple example of a rank-nullity ring before stating our main result of this subsection.

Example 8. Consider the rank-3 graphic matroid M of the complete graph K_4 on 4 vertices, discussed in Theorem 4. The rank-nullity ring $R^*(M)$ is equal to

$$R^*(M) := \mathbb{Z}[y_{1,0}, y_{2,0}, y_{2,1}, y_{3,0}, y_{3,1}, y_{3,2}, y_{3,3}] \quad \subseteq A^*(U_{6,6})$$

where the elements $y_{i,j}$ are

$$y_{1,0} = \sum_{1 \leqslant i \leqslant 6} x_{\{i\}}, \qquad y_{2,0} = \sum_{1 \leqslant i < j \leqslant 6} x_{\{i,j\}},$$

$$y_{2,1} = x_{\{1,2,5\}} + x_{\{1,4,6\}} + x_{\{2,3,6\}} + x_{\{3,4,5\}}, \qquad y_{3,0} = \sum_{1 \leqslant i < j < k \leqslant 6} x_{\{i,j,k\}} - y_{2,1}$$

$$y_{3,1} = \sum_{1 \leqslant i < j < k < l \leqslant 6} x_{\{i,j,k,l\}}, \qquad y_{3,2} = \sum_{1 \leqslant i < j < k < l < m \leqslant 6} x_{\{i,j,k,l,m\}},$$

$$y_{3,3} = x_{\{1,2,3,4,5,6\}}.$$

The subring $R^*(M)$ is generated by only 7 elements, compared to the 63 variables that generate $A^*(U_{6,6})$. We will see in Theorem 20 that the degree-1 part of $R^*(M)$ is isomorphic to $\mathbb{Z}^6$, while the degree-1 part of $A^*(U_{6,6})$ is isomorphic to $\mathbb{Z}^{57}$.

Our motivation for introducing the rank-nullity ring of a matroid comes from the following fact.

Theorem 9. *Let M be a rank-r matroid on the set $[n]$ with rank function $\mathrm{r_M}$ and nullity function $\mathrm{n_M}$. The Chern classes $c_k(S_M)$ and $c_k(Q_M)$ of the tautological bundles of M belong to the rank-nullity ring $R^*(M)$. More concretely,*

$$c_k(S_M) = \sum \alpha_S(r_1, \ldots, r_t, p_1, \ldots, p_t)\, y_{r_1,n_1}^{p_1} y_{r_2,n_2}^{p_2} \cdots y_{r_t,n_t}^{p_t} \quad \textit{and}$$
$$c_k(Q_M) = \sum \alpha_Q(n_1, \ldots, n_t, p_1, \ldots, p_t)\, y_{r_1,n_1}^{p_1} y_{r_2,n_2}^{p_2} \cdots y_{r_t,n_t}^{p_t},$$

where the sums are taken over all sequences $(0,0) < (r_1,n_1) < \cdots < (r_t,n_t) \leqslant (r, n-r)$ *(under the coordinatewise partial order) and all positive integers* $p_1, \ldots, p_t$ *satisfying* $\sum_{m=1}^{t} p_m = k$*, with coefficients given by*

$$\alpha_S(r_1,\ldots,r_t,p_1,\ldots,p_t) := (-1)^k \binom{r_1}{p_1}\binom{r_2-\tilde{p}_1}{p_2}\cdots\binom{r_t-\tilde{p}_{t-1}}{p_t} \quad \textit{and}$$
$$\alpha_Q(n_1,\ldots,n_t,p_1,\ldots,p_t) := (-1)^k \binom{n_1}{p_1}\binom{n_2-\tilde{p}_1}{p_2}\cdots\binom{n_t-\tilde{p}_{t-1}}{p_t},$$

where $\tilde{p}_i = \sum_{j=1}^{i} p_j$.

Proof. For fixed $0 \leqslant i, j \leqslant n$, any two distinct subsets $S, T \subseteq [n]$ of rank i and nullity j have both size $i+j$ and are thus incomparable, which implies that $x_S x_T = 0$ in $A^*(U_{n,n})$. It follows that

$$y_{i,j}^m = \sum_{\mathrm{r_M}(S)=i,\, \mathrm{n_M}(S)=j} x_S^m \tag{3.4}$$

for all $m \geqslant 0$. Recall the formula

$$c_k(S_M) = \sum_{\substack{\emptyset \subsetneq S_1 \subsetneq \cdots \subsetneq S_t \subseteq [n] \\ p_1+\cdots+p_t=k}} \alpha_S(S_1,\ldots,S_t,p_1,\ldots,p_t)\, x_{S_1}^{p_1} x_{S_2}^{p_2} \cdots x_{S_t}^{p_t} \tag{3.5}$$

stated in Theorem 3. Note that the coefficients $\alpha_S(S_1,\ldots,S_t,p_1,\ldots,p_t)$ given there only depend on the sequence of ranks $\mathrm{r_M}(S_1),\ldots,\mathrm{r_M}(S_t)$ and the exponents $p_1,\ldots,p_t$. Abusing notation slightly, we hence denote

$$\alpha_S(S_1,\ldots,S_t,p_1,\ldots,p_t) = \alpha_S(\mathrm{r_M}(S_1),\ldots,\mathrm{r_M}(S_t),p_1,\ldots,p_t).$$

We can group together the terms in Equation (3.5) corresponding to the same sequence $(0,0) < (r_1,n_1) < \cdots < (r_t,n_t) \leqslant (r,n-r)$ of ranks and nullities to obtain

$$c_k(S_M) = \sum_{\substack{(r_1,n_1)<\cdots<(r_t,n_t) \\ p_1+\cdots+p_t=k}} \big(\alpha_S(r_1,\ldots,r_t,p_1,\ldots,p_t) \sum_{\substack{S_1 \subsetneq \cdots \subsetneq S_t \\ \mathrm{r_M}(S_j)=r_j \\ \mathrm{n_M}(S_j)=n_j}} x_{S_1}^{p_1} x_{S_2}^{p_2} \cdots x_{S_t}^{p_t}\big).$$

In the inner sum we can ignore the condition that the subsets $S_1,\ldots,S_t$ form a chain (since the extra terms are all equal to 0), and factor each summand of $c_k(S_M)$ as

$$\sum_{\substack{(r_1,n_1)<\cdots<(r_t,n_t) \\ p_1+\cdots+p_t=k}} \alpha_S(r_1,\ldots,r_t,p_1,\ldots p_t)\big(\sum_{\substack{\mathrm{r_M}(S_1)=r_1 \\ \mathrm{n_M}(S_1)=n_1}} x_{S_1}^{p_1}\big)\big(\sum_{\substack{\mathrm{r_M}(S_2)=r_2 \\ \mathrm{n_M}(S_2)=n_2}} x_{S_2}^{p_2}\big)\cdots\big(\sum_{\substack{\mathrm{r_M}(S_t)=r_t \\ \mathrm{n_M}(S_t)=n_t}} x_{S_t}^{p_t}\big) =$$
$$\sum_{\substack{(r_1,n_1)<\cdots<(r_t,n_t) \\ p_1+\cdots+p_t=k}} \alpha_S(r_1,\ldots,r_t,p_1,\ldots p_t)\, y_{r_1,n_1}^{p_1} y_{r_2,n_2}^{p_2} \cdots y_{r_t,n_t}^{p_t},$$

which is the desired formula for $c_k(S_M)$. A similar argument proves the claimed formula for $c_k(Q_M)$. □

The rank-nullity ring $R^*(M)$ of a matroid M can be much smaller than the Chow ring $A^*(U_{n,n})$. The following result quantifies exactly how many linearly independent generators (of degree 1) this subring has.

Proposition 10. *Let M be a matroid of rank r on the set $[n]$. If M is not a direct sum of a free matroid and a rank-0 matroid (both having at least one element), all linear relations among the generators $y_{i,j}$ of $R^*(M) \subseteq A^*(U_{n,n})$, where $0 \leqslant i, j \leqslant n$, are generated by the following ones:*

$$y_{0,0} = 0,$$

$$y_{i,j} = 0 \quad \textit{whenever } i + j > \max\{|F| : F \textit{ a rank-}i \textit{ flat}\},$$

$$\sum_{0 \leqslant i,j \leqslant n} (i+j)\, y_{i,j} = 0.$$

If $M \cong U_{r,r} \oplus U_{0,n-r}$ with $1 \leqslant r \leqslant n-1$ then the same statement is true but replacing the last relation by the following two:

$$\sum_{0 \leqslant i,j \leqslant n} i\, y_{i,j} = 0 \qquad \textit{and} \qquad \sum_{0 \leqslant i,j \leqslant n} j\, y_{i,j} = 0.$$

Proof. We first show that all these relations hold. By definition, $y_{i,j} = 0$ whenever there are no subsets of $[n]$ of rank i and nullity j. Note that, for fixed i, the set of numbers j for which this is the case is closed upwards, as it is always possible to remove an element from a dependent subset $S \subseteq [n]$ that decreases its nullity but not its rank. The rank-i subsets of maximum nullity are flats of rank i, which shows the first set of relations hold.

The last relation is obtained by adding over all $e \in [n]$ the linear relations $0 = \sum_{S \ni e} x_S$ that hold in the Chow ring $A^*(U_{n,n})$, namely

$$0 = \sum_{e \in [n]} \sum_{S \ni e} x_S = \sum_{\emptyset \neq S \subseteq [n]} |S|\, x_S = \sum_{0 \leqslant i,j \leqslant n} (i+j)\, y_{i,j}. \tag{3.6}$$

If M consists only of coloops and loops, this last relation is actually implied by the following two:

$$0 = \sum_{e \text{ coloop}} \sum_{S \ni e} x_S = \sum_{0 \leqslant i,j \leqslant n} i\, y_{i,j} \qquad \text{and} \qquad \sum_{e \text{ loop}} \sum_{S \ni e} x_S = \sum_{0 \leqslant i,j \leqslant n} j\, y_{i,j}. \tag{3.7}$$

To see that these relations generate all linear relations, suppose $\sum_{0 \leqslant i,j \leqslant n} \alpha_{i,j}\, y_{i,j} = 0$ for integers $\alpha_{i,j}$. After removing all the $y_{i,j}$ for which there are no subsets of $[n]$ of rank i and nullity j, we can assume that all the $y_{i,j}$ involved in this linear relation are non-zero linear combinations of the variables x_S of $A^*(U_{n,n})$. This relation must be generated by the linear relations that hold in $A^*(U_{n,n})$, so there are integers β_e for $e \in [n]$ satisfying

$$\sum_{0 \leqslant i,j \leqslant n} \alpha_{i,j}\, y_{i,j} = \sum_{e \in [n]} \left(\beta_e \sum_{S \ni e} x_S \right). \tag{3.8}$$

For any $e \in [n]$ that is not a loop, the coefficient of $x_{\{e\}}$ on the left-hand side of Equation (3.8) is $\alpha_{1,0}$, while on the right-hand side it is β_e; this shows that all coefficients β_e with e not a loop are equal to $\alpha_{1,0}$. Similarly, for any loop $e \in [n]$, the coefficient of $x_{\{e\}}$ on the left-hand side of Equation (3.8) is $\alpha_{0,1}$, while on the right-hand side it is β_e, and so all β_e with e a loop are equal to $\alpha_{0,1}$. If M consists only of coloops and loops, this shows that the linear relation is a combination of the two relations in Equation (3.7). If M has no loops, all the coefficients β_e are equal, so the relation is a multiple of the relation in Equation (3.6). Finally, if M has a loop but does not decompose as $M \cong U_{r,r} \oplus U_{0,n-r}$ with $1 \leqslant r \leqslant n-1$, there must exist a circuit C of M of size $k \geqslant 2$. Take $c \in C$ and let ℓ be a loop of M. The coefficient of x_C on the left-hand side of Equation (3.8) is $\alpha_{k-1,1}$, while on the right-hand side it is $k\alpha_{1,0}$. Furthermore, the coefficient of $x_{(C-c)\cup\ell}$ on the left-hand side of Equation (3.8) is $\alpha_{k-1,1}$, while on the right-hand side it is $(k-1)\alpha_{1,0} + \alpha_{0,1}$. This implies that $\alpha_{1,0} = \alpha_{0,1}$, and so the linear relation is again a multiple of the relation in Equation (3.6). □

We conclude this subsection with a remark on the symmetry of the rank-nullity ring of a matroid.

The *automorphism group* $\mathrm{Aut}(M)$ of a matroid M on a ground set E is the group of permutations of E that preserve the rank of every subset. For example, the automorphism group $\mathrm{Aut}(U_{r,n})$ of the uniform matroid $U_{r,n}$ is the symmetric group S_n. The group $\mathrm{Aut}(M)$ acts naturally on the Chow ring $A^*(M)$ of M; namely, a permutation $\sigma \in \mathrm{Aut}(M)$ acts as $\sigma(x_S) = x_{\sigma(S)}$.

Proposition 11. *Let M be a matroid on the set $[n]$. The rank-nullity subring $R^*(M) \subseteq A^*(U_{n,n})$ is $\mathrm{Aut}(M)$-invariant, i.e.,*

$$R^*(M) \subseteq A^*(U_{n,n})^{\mathrm{Aut}(M)}.$$

Proof. Any $\sigma \in \mathrm{Aut}(M)$ permutes the set of subsets of $[n]$ of any given rank and nullity, and thus it fixes each of the generators $y_{i,j} \in R^*(M)$. □

The following example shows that the inclusion $R^*(M) \subseteq A^*(U_{n,n})^{\mathrm{Aut}(M)}$ can be strict.

Example 12. Again, consider the rank-3 graphic matroid M of the complete graph K_4 on 4 vertices, discussed in Theorem 4. In this case, the automorphism group of M is induced from the automorphism group of the underlying graph K_4, which is isomorphic to the symmetric group S_4. Explicitly, the automorphism group of M consists of the permutations (listed in cycle notation)

$$\begin{aligned}\mathrm{Aut}(M) = \{&(),\ (16)(35),\ (14)(23),\ (45)(26),\ (46)(52),\ (12)(34),\ (63)(15),\\ &(164)(235),\ (146)(253),\ (126)(345),\ (162)(354),\ (154)(362),\ (145)(326),\\ &(125)(346),\ (152)(364),\ (65)(13),\ (13)(24),\ (24)(56),\ (1234)(56),\\ &(13)(2645),\ (1536)(24),\ (13)(2546),\ (1635)(24), (1432)(56)\} \simeq S_4.\end{aligned}$$

Since any automorphism permutes the three perfect matchings of K_4, we have

$$x_{\{1,3\}} + x_{\{2,4\}} + x_{\{5,6\}} \in A^*(U_{n,n})^{\mathrm{Aut}(M)}.$$

We can see from the description of the rank-nullity ring of M given in Theorem 8 that $x_{\{1,3\}} + x_{\{2,4\}} + x_{\{5,6\}} \notin R^*(M)$, so the inclusion $R^*(M) \subseteq A^*(U_{n,n})^{\mathrm{Aut}(M)}$ is strict.

3.3 The rank-nullity ring of the uniform matroid

If the matroid M is the uniform matroid $U_{r,n}$, subsets of $[n]$ of size at most r have all nullity zero, while subsets of size more than r have all rank r. This means that the only generators $y_{i,j}$ of the rank-nullity ring $R^*(U_{r,n})$ that are nonzero are

$$y_{1,0}, y_{2,0}, \ldots, y_{r,0}, y_{r,1}, y_{r,2}, \ldots, y_{r,n-r}.$$

To simplify the notation, in this case we will index the generators using just one index $l \in \{1, \ldots, n\}$, as

$$z_l := \sum_{S \in \binom{[n]}{l}} x_S = \begin{cases} y_{l,0} & \text{if } l \leqslant r \\ y_{r,l-r} & \text{if } l > r \end{cases} \in A^*(U_{n,n}).$$

With this notation, we have

$$R^*(U_{r,n}) = \mathbb{Z}[z_1, z_2, \ldots, z_n] \subseteq A^*(U_{n,n}).$$

Note that the rank-nullity ring $R^*(U_{r,n})$ of the uniform matroid $U_{r,n}$ does not depend on the rank r but only on the size of the ground set n.

Proposition 13. *The rank-nullity ring of the uniform matroid $U_{r,n}$ is equal to the S_n-invariant subring of the Chow ring $A^*(U_{n,n})$:*

$$R^*(U_{r,n}) = A^*(U_{n,n})^{S_n}.$$

Proof. The inclusion $R^*(U_{r,n}) \subseteq A^*(U_{n,n})^{S_n}$ follows from Theorem 11. For the reverse inclusion, let $f \in A^*(U_{n,n})^{S_n}$. We can write

$$f = \sum \alpha_f(S_1, \ldots, S_t, p_1, \ldots, p_t)\, x_{S_1}^{p_1} x_{S_2}^{p_2} \ldots x_{S_t}^{p_t}, \tag{3.9}$$

where the sum is over all sequences of subsets $\emptyset \subsetneq S_1 \subsetneq \cdots \subsetneq S_t \subseteq [n]$ and all positive integers $p_1, \ldots, p_k$ satisfying $p_1 + \cdots + p_t \leqslant n-1$. Since f is invariant under the action of S_n, the coefficients $\alpha_f(S_1, \ldots, S_t, p_1, \ldots, p_t)$ depend only on the sequence of sizes $|S_1|, \ldots, |S_t|$ and the exponents $p_1, \ldots, p_t$. Letting $s_i = |S_i|$, we thus denote

$$\alpha_f(s_1, \ldots, s_t, p_1, \ldots, p_t) := \alpha_f(S_1, \ldots, S_t, p_1, \ldots, p_t).$$

We can group together the terms in Equation (3.9) corresponding to the same sequence of sizes $1 \leqslant s_1 < s_2 < \cdots < s_t \leqslant n$ to obtain

$$f = \sum_{\substack{1 \leqslant s_1 < \cdots < s_t \leqslant n \\ p_1 + \cdots + p_t \leqslant n-1}} \left(\alpha_f(s_1, \ldots, s_t, p_1, \ldots, p_t) \sum_{\substack{S_1 \subsetneq \cdots \subsetneq S_t \\ |S_j| = s_j}} x_{S_1}^{p_1} x_{S_2}^{p_2} \ldots x_{S_t}^{p_t} \right).$$

As in the proof of Theorem 9, we can rewrite this as

$$\begin{aligned} f &= \sum_{\substack{1\leqslant s_1<\cdots<s_t\leqslant n\\ p_1+\cdots+p_t\leqslant n-1}} \alpha_f(s_1,\ldots,s_t,p_1,\ldots p_t)\left(\sum_{|S_1|=s_1} x_{S_1}^{p_1}\right)\left(\sum_{|S_2|=s_2} x_{S_2}^{p_2}\right)\cdots\left(\sum_{|S_t|=s_t} x_{S_t}^{p_t}\right)\\ &= \sum_{\substack{1\leqslant s_1<\cdots<s_t\leqslant n\\ p_1+\cdots+p_t\leqslant n-1}} \alpha_f(s_1,\ldots,s_t,p_1,\ldots p_t)\, z_{s_1}^{p_1} z_{s_2}^{p_2}\cdots z_{s_t}^{p_t}, \end{aligned}$$

which shows that $f \in R^*(U_{r,n})$. □

In view of the result above, the study of the rank-nullity ring $R^*(U_{r,n}) = A^*(U_{n,n})^{\mathrm{Aut}(M)}$ of the uniform matroids is of independent interest. In the remainder of this subsection we provide a $\mathbb{Z}$-basis for it and also a Gröbner basis for the relations among its generators.

Proposition 14. *The set*

$$\mathcal{B}_n = \{z_{s_1}^{p_1} z_{s_2}^{p_2}\cdots z_{s_l}^{p_l} : 1 \leqslant p_i < s_i - s_{i-1} \text{ and } 1 \leqslant s_1 < s_2 < \cdots < s_l \leqslant n\}$$

(where by convention $s_0 = 0$) is a $\mathbb{Z}$-basis for the rank-nullity ring $R^*(U_{r,n})$ of the uniform matroid $U_{r,n}$.

Proof. Applying [9, Corollary 1] to the free matroid $U_{n,n}$, we get that the set

$$\mathcal{FY}_n = \{x_{S_1}^{p_1} x_{S_2}^{p_2}\cdots x_{S_l}^{p_l} : \emptyset \subsetneq S_1 \subsetneq S_2 \subsetneq \cdots \subsetneq S_l \subseteq [n] \text{ and } p_i < |S_i| - |S_{i-1}|\}$$

(where by convention $S_0 = \emptyset$) is a $\mathbb{Z}$-basis for the Chow ring $A^*(U_{n,n})$. Note that the action of S_n on $A^*(U_{n,n})$ permutes the elements of $\mathcal{FY}_n$.

To show that the set $\mathcal{B}_n$ generates $R^*(U_{r,n})$, take $f \in R^*(U_{r,n}) = A^*(U_{n,n})^{S_n}$ and write it as a linear combination of the elements of $\mathcal{FY}_n$:

$$f = \sum_{\substack{\emptyset\subsetneq S_1\subsetneq\cdots\subsetneq S_l\subseteq[n]\\ p_i<|S_i|-|S_{i-1}|}} \alpha_f(S_1,\ldots,S_l,p_1,\ldots,p_l)\, x_{S_1}^{p_1} x_{S_2}^{p_2}\ldots x_{S_l}^{p_l}.$$

Since f is invariant under the action of S_n, the coefficients $\alpha_f(S_1,\ldots,S_l,p_1,\ldots,p_l)$ depend only on the p_i's and the sizes $|S_1|,|S_2|,\ldots,|S_l|$. Hence, we can group the expression above for f in the same way as in the proof of Theorem 13 and obtain an expression for f as a linear combination of the elements of $\mathcal{B}_n$.

Finally, the set $\mathcal{B}_n$ is also linearly independent since its elements are finite sums of elements of $\mathcal{FY}_n$ without repetitions, and thus a linear dependence among the elements of $\mathcal{B}_n$ would imply a linear dependence among the elements of $\mathcal{FY}_n$. □

Example 15. The $\mathbb{Z}$-basis given in Theorem 14 for the rank-nullity ring $R^*(U_{r,5})$ of the uniform matroid $U_{r,5}$ is

$$\mathcal{B}_5 = \{1,\ z_2,\ z_3,\ z_4,\ z_5,\ z_2z_4,\ z_2z_5,\ z_3^2,\ z_3z_5,\ z_4^2,\ z_5^2,\ z_2z_5^2,\ z_3^2z_5,\ z_4^3,\ z_5^3,\ z_5^4\}.$$

In Theorem 19, we use the basis $\mathcal{B}_n$ to compute simple expressions for the Hilbert function and Hilbert series of the rank-nullity ring of the uniform matroid.

We conclude this subsection with a description of all the relations among the generators z_i of the rank-nullity ring of the uniform matroid. In fact, we give a Gröbner basis for the ideal of all such relations.

Theorem 16. *The rank-nullity ring $R^*(U_{r,n}) = A^*(U_{n,n})^{S_n}$ of the uniform matroids is isomorphic to the quotient of the free polynomial ring $\mathbb{Z}[Z_1, \ldots, Z_n]$ by the ideal $\mathcal{I}$ generated by the polynomials*

$$Z_a \cdot \left(\sum_{i=b}^{n} \binom{i-a-1}{b-a-1} (Z_i + Z_{i+1} + \cdots + Z_n)^{b-a} \right) \qquad \text{for } 0 \leqslant a < b \leqslant n \tag{3.10}$$

(with the convention that $Z_0 = 1$), where the class $[Z_i]$ in this quotient corresponds to the element $z_i \in R^*(U_{r,n})$.

Furthermore, the polynomials in Equation (3.10) *are a Gröbner basis for the ideal $\mathcal{I}$ with respect to any monomial order in which $Z_1 \succ Z_2 \succ \cdots \succ Z_n$.*

Proof. We first show that the elements $z_i \in R^*(U_{r,n})$ satisfy the relations in Equation (3.10). Following [9, Theorem 1], the generators of the Chow ring $A^*(U_{n,n})$ satisfy the relations

$$0 = x_S \cdot \left(\sum_{T' \supseteq T} x_{T'} \right)^{|T| \setminus |S|} \qquad \text{for } \emptyset \subseteq S \subsetneq T \subseteq [n],$$

with the convention that $x_\emptyset = 1$. Since $x_{T_1} x_{T_2} = 0$ whenever $T_1 \not\subseteq T_2$ and $T_2 \not\subseteq T_1$, using the multinomial theorem we can expand this as

$$0 = x_S \cdot \sum \binom{|T| \setminus |S|}{d_1, d_2, \ldots, d_k} x_{T_1}^{d_1} \cdots x_{T_k}^{d_k}$$

where the sum is over all tuples $(T_1, \ldots, T_k)$ of subsets of $[n]$ such that $T \subseteq T_1 \subsetneq T_2 \subsetneq \cdots \subsetneq T_k$ and all sequences of positive integers $(d_1, \ldots, d_k)$ with $d_1 + \cdots + d_k = |T| \setminus |S|$.

Now, fix $0 \leqslant a < b \leqslant n$. Adding the relations above over all pairs $S \subsetneq T$ satisfying $|S| = a$ and $|T| = b$, we obtain

$$0 = \sum_{\substack{S \subsetneq T \\ |S|=a,\ |T|=b}} \left(x_S \cdot \sum_{\substack{T \subseteq T_1 \subsetneq T_2 \subsetneq \cdots \subsetneq T_k \\ d_1 + \cdots + d_k = b-a}} \binom{b-a}{d_1, d_2, \ldots, d_k} x_{T_1}^{d_1} \cdots x_{T_k}^{d_k} \right).$$

In this double summation, a term $\binom{b-a}{d_1,d_2,\ldots,d_k} x_S\, x_{T_1}^{d_1} \cdots x_{T_k}^{d_k}$ appears as many times as there are subsets T satisfying $S \subsetneq T \subseteq T_1$, which is $\binom{|T_1|-a}{b-a}$ times. We can thus write the previous equation as

$$0 = \sum_{\substack{S \subsetneq T_1 \subsetneq T_2 \subsetneq \cdots \subsetneq T_k \\ |S|=a,\ |T_1| \geqslant b,\ d_1 + \cdots + d_k = b-a}} \binom{|T_1|-a}{b-a} \binom{b-a}{d_1, d_2, \ldots, d_k} x_S\, x_{T_1}^{d_1} \cdots x_{T_k}^{d_k}.$$

Since variables corresponding to incomparable subsets have product equal to zero in the Chow ring $A^*(U_{n,n})$, we can further rewrite this as

$$\begin{aligned}0 &= \sum_{\substack{b\leqslant t_1<t_2<\cdots<t_k\leqslant n\\ d_1+\cdots+d_k=b-a}} \binom{t_1-a}{b-a}\binom{b-a}{d_1,d_2,\ldots,d_k}\left(\sum_{|S|=a} x_S\right)\left(\sum_{|T_1|=t_1} x_{T_1}^{d_1}\right)\cdots\left(\sum_{|T_k|=t_k} x_{T_k}^{d_k}\right)\\ &= \sum_{\substack{b\leqslant t_1<t_2<\cdots<t_k\leqslant n\\ d_1+\cdots+d_k=b-a}} \binom{t_1-a}{b-a}\binom{b-a}{d_1,d_2,\ldots,d_k} z_a\, z_{t_1}^{d_1}\cdots z_{t_k}^{d_k}.\end{aligned}$$

Grouping together terms with the same value of t_1 and d_1, and then using the identity $\binom{b-a}{d_1,d_2,\ldots,d_k} = \binom{b-a}{d_1}\binom{b-a-d_1}{d_2,\ldots,d_k}$, we get

$$\begin{aligned}0 &= \sum_{\substack{t_1\geqslant b\\ 1\leqslant d_1\leqslant b-a}} \left(\binom{t_1-a}{b-a} z_a\, z_{t_1}^{d_1}\cdot \sum_{\substack{t_1+1\leqslant t_2<\cdots<t_k\leqslant n\\ d_2+\cdots+d_k=b-a-d_1}} \binom{b-a}{d_1,d_2,\ldots,d_k} z_{t_2}^{d_2}\cdots z_{t_k}^{d_k}\right)\\ &= z_a\cdot \sum_{\substack{t_1\geqslant b\\ 1\leqslant d_1\leqslant b-a}} \left(\binom{t_1-a}{b-a}\binom{b-a}{d_1} z_{t_1}^{d_1}\cdot \sum_{\substack{t_1+1\leqslant t_2<\cdots<t_k\leqslant n\\ d_2+\cdots+d_k=b-a-d_1}} \binom{b-a-d_1}{d_2,\ldots,d_k} z_{t_2}^{d_2}\cdots z_{t_k}^{d_k}\right)\\ &= z_a\cdot \sum_{\substack{t_1\geqslant b\\ 1\leqslant d_1\leqslant b-a}} \binom{t_1-a}{b-a}\binom{b-a}{d_1} z_{t_1}^{d_1}\cdot (z_{t_1+1}+z_{t_1+2}+\cdots+z_n)^{b-a-d_1}.\end{aligned}$$

This can be further rewritten as

$$\begin{aligned}0 &= z_a\cdot \sum_{t_1\geqslant b}\binom{t_1-a}{b-a}\cdot\left(\sum_{d_1=1}^{b-a}\binom{b-a}{d_1} z_{t_1}^{d_1}(z_{t_1+1}+z_{t_1+2}+\cdots+z_n)^{b-a-d_1}\right)\\ &= z_a\cdot \sum_{t_1\geqslant b}\binom{t_1-a}{b-a}\left((z_{t_1}+z_{t_1+1}+z_{t_1+2}+\cdots+z_n)^{b-a}-(z_{t_1+1}+z_{t_1+2}+\cdots+z_n)^{b-a}\right).\end{aligned}$$

Finally, reordering terms yields

$$\begin{aligned}0 &= z_a\cdot \sum_{t_1\geqslant b}\left(\binom{t_1-a}{b-a}-\binom{t_1-1-a}{b-a}\right)(z_{t_1}+z_{t_1+1}+z_{t_1+2}+\cdots+z_n)^{b-a}\\ &= z_a\cdot \sum_{t_1\geqslant b}\binom{t_1-1-a}{b-a-1}(z_{t_1}+z_{t_1+1}+z_{t_1+2}+\cdots+z_n)^{b-a},\end{aligned}$$

which is the desired relation.

To show that these relations form a Gröbner basis, we note that the leading term of the polynomial in Equation (3.10) is $Z_a Z_b^{b-a}$. The ideal generated by these monomials, with $0\leqslant a<b\leqslant n$, consists precisely of the monomials not in the basis $\mathcal{B}_n$ described in Theorem 14, proving the statement. □

Remark 17. A slightly different presentation of the S_n-invariant subring $A^*(U_{n,n})^{S_n}$ of the Chow ring of the permutahedral variety appears in [14, Section 3.1], following the work of Klyachko [11] on permutahedral varieties of more general Coxeter types. Working with $\mathbb{Q}$-coefficients, the subring $A^*(U_{n,n})^{S_n}$ is equal to the polynomial ring on $n-1$ variables $\mathbb{Q}[u_1, \ldots, u_{n-1}]$ modulo the relations $2u_i^2 = u_i u_{i-1} + u_i u_{i+1}$ (with the convention that $u_0 = u_n = 0$). The changes of variables linking the two presentations are

$$u_i = z_{n-i+1} + 2z_{n-i+2} + 3z_{n-i+3} + \cdots + iz_n,$$
$$z_i = u_{n-i+1} - 2u_{n-i} + u_{n-i-1}.$$

Question 18. It would be interesting to classify the matroids M for which the containment $R^*(M) \subseteq A^*(U_{n,n})^{\mathrm{Aut}(M)}$ is an equality. As stated in Theorem 13, this class includes all uniform matroids. Considering elements of degree 1 shows that this equality implies that M satisfies:

(*) For any $0 \leqslant i, j \leqslant n$, the automorphism group $\mathrm{Aut}(M)$ acts transitively on the collection of sets $\mathcal{S}_{i,j} := \{S \subseteq [n] : \mathrm{r_M}(S) = i \text{ and } \mathrm{n_M}(S) = j\}$.

In particular, taking $i = \mathrm{r_M}(M)$ and $j = 0$, this means that $\mathrm{Aut}(M)$ acts transitively on the set of bases of M. Similarly, taking $i = \mathrm{r_M}(M)$ and $j = 1$ we see that $\mathrm{Aut}(M)$ acts transitively on the set of circuits of M, since subsets of rank $\mathrm{r_M}(M)$ and nullity 1 contain exactly one circuit, and all circuits are contained in such a subset. However, it is unclear to us exactly what matroids satisfy the very restrictive property (*).

3.4 The Hilbert function of the rank-nullity ring

We now present simple expressions for the Hilbert series of the rank-nullity ring of the uniform matroid, and provide examples of the general behavior for other matroids.

Proposition 19. *The Hilbert function of the rank-nullity ring $R^*(U_{r,n}) = A^*(U_{n,n})^{S_n}$ of the uniform matroids is given by*

$$\mathrm{HF}(d) := \mathrm{rank}_{\mathbb{Z}}(R^d(U_{r,n})) = \binom{n-1}{d}.$$

Consequently, its Hilbert series is equal to

$$\mathrm{HS}(t) := \sum_{d \geqslant 0} \mathrm{HF}(d)\, t^d = (1+t)^{n-1}.$$

Proof. By Theorem 14, the following set of monomials is a $\mathbb{Z}$-basis for the degree-d part of $R^*(U_{r,n})$:

$$\mathcal{B}_n^d = \{z_{s_1}^{p_1} z_{s_2}^{p_2} \ldots z_{s_l}^{p_l} : \textstyle\sum p_i = d \text{ and } 1 \leqslant p_i < s_i - s_{i-1}\}.$$

We show that there are exactly $\binom{n-1}{d}$ monomials in $\mathcal{B}_n^d$ by describing a concrete bijection with the set $\binom{[n-1]}{d} = \{S \subseteq [n-1] : |S| = d\}$. Any subset $S \in \binom{[n-1]}{d}$ naturally decomposes as a disjoint union $S = I_1 \sqcup I_2 \sqcup \cdots \sqcup I_l$ of intervals of consecutive numbers;

more precisely, the subsets I_j have the form $I_j = \{a_j, a_j+1, \ldots, b_j\}$ and $b_j+1 < a_{j+1}$. The map $f : \binom{[n-1]}{d} \to \mathcal{B}_n^d$ sending the subset S to the monomial $z_{s_1}^{p_1} z_{s_2}^{p_2} \cdots z_{s_l}^{p_l}$ where $s_j = b_j+1$ and $p_j = |I_j|$ is a bijection. Indeed, the inverse map $f^{-1} : \mathcal{B}_n^d \to \binom{[n-1]}{d}$ sends a monomial $z_{s_1}^{p_1} z_{s_2}^{p_2} \cdots z_{s_l}^{p_l}$ to the subset $S = I_1 \sqcup I_2 \sqcup \cdots \sqcup I_l$ where $I_j = \{s_j - p_j, s_j - p_j + 1, \ldots, s_j - 1\}$. □

Example 20. Let $n = 5$. The bijection f described in the proof of Theorem 19 between subsets of $\{1, 2, 3, 4\}$ and monomials in the basis $\mathcal{B}_5$ is (ignoring brackets and commas in our notation for subsets):

$$\begin{array}{cccccccccccc} \emptyset & \mapsto & 1 & & 12 & \mapsto & z_3^2 & & & & & \\ & & & 1 & \mapsto & z_2 & 1\ 3 & \mapsto & z_2z_4 & 123 & \mapsto & z_4^3 \\ & & & 2 & \mapsto & z_3 & 1\ 4 & \mapsto & z_2z_5 & 12\ 4 & \mapsto & z_3^2z_5 \\ & & & 3 & \mapsto & z_4 & 23 & \mapsto & z_4^2 & 1\ 34 & \mapsto & z_2z_5^2 \\ & & & 4 & \mapsto & z_5 & 2\ 4 & \mapsto & z_3z_5 & 234 & \mapsto & z_5^3 \\ & & & & & & 34 & \mapsto & z_5^2 & & & \end{array} \qquad 1234 \mapsto z_5^4 \ .$$

In the following example we see that, for general matroids, the Hilbert function of the rank-nullity ring is not necessarily symmetric.

Example 21. Table 1 shows the Hilbert function of the rank-nullity ring $R^*(M)$ of some matroids M, computed using Macaulay2. These are displayed as the vector $\mathrm{HF}_{R^*(M)} := (\mathrm{rank}_{\mathbb{Z}}(R^0(M)), \ldots, \mathrm{rank}_{\mathbb{Z}}(R^{n-1}(M)))$. Recall that the rank-nullity ring of the uniform matroids $R^*(U_{r,n})$ depends only on the size n of the ground set, hence in this case we do not specify the rank. Our list includes all non-isomorphic matroids of size at most 4, but only some of the matroids of higher ranks. For reference, we also list the Hilbert function $\mathrm{HF}_{A^*(U_{n,n})}$ of the Chow ring $A^*(U_{n,n})$ of the free matroid on the same ground set, of which these rings are subrings.

In the table we use the following notation:

- M_1 is the rank-2 matroid on 4 elements with only one non-basis.
- M_2 is the rank-3 matroid on 5 elements with two non-bases $\{1, 2, 3\}, \{3, 4, 5\}$. This is a graphic matroid, associated to the graph K_4 minus an edge.
- M_3 is the rank-2 matroid on 6 elements with three non-bases $\{1, 2\}, \{3, 4\}, \{5, 6\}$. This is the graphic matroid associated to a triangle graph with double edges.
- $M(K_4)$ is the rank-3 graphic matroid on 6 elements associated to the graph K_4.
- M_4 is the rank-4 matroid on 6 elements obtained by taking $U_{4,5}$ and replacing an element by 2 parallel elements. This is the graphic matroid associated to a pentagon graph where one of the sides is a double edge.
- F_7^- is the non-Fano matroid.

As we saw in the previous examples, the top non-zero degree of the rank-nullity ring of a matroid is always equal to that of the Chow ring $A^*(U_{n,n})$.

M	$\mathrm{HF}_{R^*(M)}$	$\mathrm{HF}_{A^*(U_{n,n})}$
$U_{r,1}$	(1)	(1)
$U_{r,2}$	$(1,1)$	$(1,1)$
$U_{0,1} \oplus U_{1,1}$	$(1,1)$	
$U_{r,3}$	$(1,2,1)$	$(1,4,1)$
$U_{0,1} \oplus U_{1,2}$	$(1,3,1)$	
$U_{0,1} \oplus U_{0,1} \oplus U_{1,1}$	$(1,3,1)$	
$U_{0,1} \oplus U_{1,1} \oplus U_{1,1}$	$(1,3,1)$	
$U_{1,1} \oplus U_{1,2}$	$(1,3,1)$	
$U_{r,4}$	$(1,3,3,1)$	$(1,11,11,1)$
$U_{0,1} \oplus U_{1,3}$	$(1,4,5,1)$	
$U_{0,1} \oplus U_{0,1} \oplus U_{1,2}$	$(1,5,6,1)$	
$U_{0,1} \oplus U_{0,1} \oplus U_{0,1} \oplus U_{1,1}$	$(1,5,5,1)$	
M_1	$(1,4,5,1)$	
$U_{1,2} \oplus U_{1,2}$	$(1,4,4,1)$	
$U_{0,1} \oplus U_{2,3}$	$(1,5,5,1)$	
$U_{1,1} \oplus U_{1,3}$	$(1,5,5,1)$	
$U_{0,1} \oplus U_{1,1} \oplus U_{1,2}$	$(1,6,8,1)$	
$U_{0,1} \oplus U_{0,1} \oplus U_{1,1} \oplus U_{1,1}$	$(1,6,6,1)$	
$U_{1,1} \oplus U_{2,3}$	$(1,4,5,1)$	
$U_{1,1} \oplus U_{1,1} \oplus U_{1,2}$	$(1,5,6,1)$	
$U_{0,1} \oplus U_{1,1} \oplus U_{1,1} \oplus U_{1,1}$	$(1,5,5,1)$	
$U_{r,5}$	$(1,4,6,4,1)$	$(1,26,66,26,1)$
$U_{0,1} \oplus U_{0,1} \oplus U_{1,3}$	$(1,6,12,9,1)$	
$U_{0,1} \oplus U_{0,1} \oplus U_{0,1} \oplus U_{1,2}$	$(1,7,14,9,1)$	
$U_{0,1} \oplus U_{0,1} \oplus U_{0,1} \oplus U_{0,1} \oplus U_{1,1}$	$(1,7,12,7,1)$	
M_2	$(1,5,9,7,1)$	
$U_{0,1} \oplus U_{2,4}$	$(1,6,11,7,1)$	
$U_{0,1} \oplus U_{1,1} \oplus U_{2,3}$	$(1,8,18,12,1)$	
$U_{1,1} \oplus U_{3,4}$	$(1,5,9,7,1)$	
$U_{r,6}$	$(1,5,10,10,5,1)$	$(1,57,302,302,57,1)$
$U_{0,1} \oplus U_{0,1} \oplus U_{0,1} \oplus U_{0,1} \oplus U_{1,2}$	$(1,9,25,28,12)$	
M_3	$(1,6,14,16,9,1)$	
$M(K_4)$	$(1,6,14,16,8,1)$	
$U_{0,1} \oplus U_{1,1} \oplus U_{2,4}$	$(1,10,32,39,16,1)$	
M_4	$(1,8,22,25,11,1)$	
$U_{2,3} \oplus U_{2,3}$	$(1,7,18,20,8,1)$	
$U_{1,1} \oplus U_{4,5}$	$(1,6,14,16,9,1)$	
F_7^-	$(1,7,20,30,25,11,1)$	$(1,120,1191,2416,1191,120,1)$

Table 1: Hilbert functions of some rank-nullity rings

Proposition 22. *The rank-nullity ring $R^*(M)$ of a matroid M on the ground set $[n]$ has highest non-zero degree equal to $n-1$. In other words,*

$$\operatorname{rank}_{\mathbb{Z}}(R^{n-1}(M)) = 1.$$

Proof. Since $R^{n-1}(M) \subseteq A^{n-1}(U_{n,n}) \cong \mathbb{Z}$, it is enough to show that $R^{n-1}(M) \neq \{0\}$. For this, consider, for instance, the element

$$f := \prod_{k=1}^{n-1} \left(\sum_{i+j=k} y_{i,j} \right) = \prod_{k=1}^{n-1} \left(\sum_{|S|=k} x_S \right) \quad \in R^{n-1}(M).$$

Expanding out the product on the right hand side gives an expression for f as the sum of all monomials $x_{S_1} x_{S_2} \dots x_{S_{n-1}}$ where $\emptyset \subsetneq S_1 \subsetneq S_2 \subsetneq \dots \subsetneq S_{n-1} \subsetneq [n]$ is a maximal chain of subsets of $[n]$. Each of these monomials is equal to 1 under the isomorphism $A^{n-1}(U_{n,n}) \cong \mathbb{Z}$ [1, Proposition 5.8], and thus in particular, f is nonzero (in fact, it is equal to the number of maximal chains of subsets of $[n]$, which is $n!$). □

We note that the Hilbert functions of the rank-nullity rings displayed in Table 1 are all unimodal, and even log-concave. The fact that these Hilbert functions increase up to its middle value follows from the "Kähler package" satisfied by the Chow ring $A^*(U_{n,n})$, as we now explain.

Following [1, Section 1.2], we call a function $f : 2^E \to \mathbb{R}$ *strictly submodular* if

$$f(S_1) + f(S_2) > f(S_1 \cup S_2) + f(S_1 \cap S_2)$$

for every pair of incomparable subsets $S_1, S_2 \subseteq E$, with the convention that $f(\emptyset) = f(E) = 0$. An example of such a function is

$$f(S) = |S| \cdot |E \setminus S|. \tag{3.11}$$

Proposition 23. *The Hilbert function of the rank-nullity ring $R^*(M)$ of a matroid M on the ground set $[n]$ satisfies*

$$\operatorname{rank}_{\mathbb{Z}}(R^0(M)) \leqslant \operatorname{rank}_{\mathbb{Z}}(R^1(M)) \leqslant \dots \leqslant \operatorname{rank}_{\mathbb{Z}}(R^{\lfloor \frac{n}{2} \rfloor}(M)).$$

Proof. Any subset $S \subseteq [n]$ of rank i and nullity j must have size $i+j$. This implies that the strictly submodular function f in Equation (3.11) is constant on subsets of the same rank and nullity. As in [1, Theorem 1.4], the element

$$\ell = \sum_{\emptyset \subsetneq S \subsetneq [n]} f(S)\, x_S = \sum_{0 \leqslant i,j \leqslant n} (i+j)(n-i-j)\, y_{i,j} \in R^*(M)$$

is an element of $A^1(U_{n,n})$ defining an ample class on the permutahedral toric variety, which in particular implies that, for any $q \leqslant (n-1)/2$, the multiplication map $L_\ell^q : A^q(U_{n,n}) \to A^{q+1}(U_{n,n})$ sending $a \mapsto \ell \cdot a$ is injective. Since $\ell \in R^1(M)$, this multiplication map restricts to an injection from $R^q(M)$ into $R^{q+1}(M)$, which implies the desired result. □

We have not pursued the following questions much further beyond the examples in Table 1.

Question 24. Is the Hilbert function of the rank-nullity ring of a matroid M always unimodal? Is it log-concave? Is it the sequence of coefficients of a real-rooted polynomial?

Question 25. For which matroids is the Hilbert function of its rank-nullity ring symmetric?

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