Optimization problem for star covers of graphs without four cycles

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Abstract

The symmetric nonnegative trifactorization of matrices motivates the study of the SNT-rank of graphs. We focus on the family of graphs characterized by three forbidden closed 4-walks and show that the calculation of the SNT-rank of graphs from this family can be formulated as an optimization problem for star covers on graphs. Unlike traditional optimization that minimizes the number of stars, our objective is to minimize the number of components used in the cover, where each star is a join of two components.

We introduce a graph parameter $\mathrm{gap}(G)$ and show that for a graph G from the family in question it is equal to the difference between the number of vertices in the graph and the minimal number of components needed in any star cover of G. We develop an algorithm for computing $\mathrm{gap}(G)$ consisting of three main steps: (1) transformation of G to a weighted multigraph Γ, (2) reduction of Γ to a multigraph $\tau(\Gamma)$ with all edge weights equal to 0 and all vertex degrees at least three, (3) iterative process for computing $\mathrm{gap}(G)$ by removing vertices in $\tau(\Gamma)$. While steps (1) and (2) are straightforward and control the gap, step (3) is computationally complex but allows us to compute the gap for any graph in the considered family.

Mathematics Subject Classifications: 05C35, 05C50, 15A23

1 Introduction and notation

1.1 Motivation and related work

This work investigates star covers in graphs, in which we aim to minimize the number of bipartite components required to form the cover. While this setup may at first appear

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highly specialized, it arises naturally from the study of symmetric nonnegative matrix tri-factorizations.

Matrix factorizations that require entrywise nonnegativity on their latent factors can be used as a tool for discovering structures that are inherently additive and physically interpretable. Studying the support pattern of a matrix provides lower bounds on various factorization ranks defined through nonnegative factorization, [1, 13, 14]. Covering problems extend to broader matrix questions involving ranks and eigenvalues under structural constraints. For example, in the classic inverse eigenvalue problem for graphs, determining the minimum rank of a matrix family is closely connected to its path cover number, [8, 12].

The Nonnegative Matrix Factorization (NMF) is the most common such factorization. In the approximate NMF framework, a given nonnegative matrix V is decomposed into nonnegative factors W and H such that $V \approx WH$, [11], while the exact NMF requires that $V = WH$, [2]. The nonnegative rank of a matrix V is the minimal number of columns of W over all such factorizations of V. The existence of an exact NMF of rank k for a matrix V requires the existence of a Boolean factorization of rank $r \leqslant k$ for its support pattern. In particular, the Boolean rank of a matrix's zero-nonzero pattern serves as a combinatorial lower bound for its nonnegative rank. In addition, Boolean Matrix Factorization (BMF) can inform the construction of Nonnegative Matrix Factorization (NMF) models, when the approximate decomposition requires specific entries to remain zero to preserve the underlying support of the data, and is of independent interest for its capacity to extract interpretable, set-based patterns from discrete data, [3]. For an overview of all the concepts mentioned above, we refer the reader to [6].

When dealing with symmetric data, this framework is mirrored by Symmetric Nonnegative Matrix Tri-Factorization (SN-Trifactorization). SN-Trifactorization approximates a symmetric $n \times n$ input matrix A as the product of nonnegative matrices: an $n \times r$ matrix B and a symmetric $r \times r$ matrix C, such that $A = BCB^\top$ (for exact factorization), or $A \approx BCB^\top$ (for approximate factorization). This framework has applications across a variety of data science applications, such as topic modeling and community detection, [15]. The symmetric nonnegative rank of a matrix A, denoted by $\mathrm{st}_+(A)$, is defined as the minimum number of columns in B required to satisfy this equality across all such factorizations, see [9]. Analogous to the asymmetric case, a combinatorial lower bound for this rank is provided by the equivalent factorization under Boolean arithmetic.

By treating the pattern of A as a graph G, we can cast this setup as a combinatorial problem on graphs. This setup was first explored in [10], where the *SNT-rank of a graph* G is defined as follows. Let G be a graph that allows loops on n vertices with $V(G) = [n]$ and let $\mathcal{S}^+(G)$ be the set of all symmetric nonnegative $n \times n$ matrices $A = [a_{ij}]$ with $a_{ij} > 0$ if and only if $\{i, j\} \in E(G)$. We define the SNT-rank of a graph G as

$$\mathrm{st}_+(G) := \min\{\mathrm{st}_+(A) \colon A \in \mathcal{S}^+(G)\}.$$

In this paper, we consider the gap of the graph G, an equivalent parameter defined as

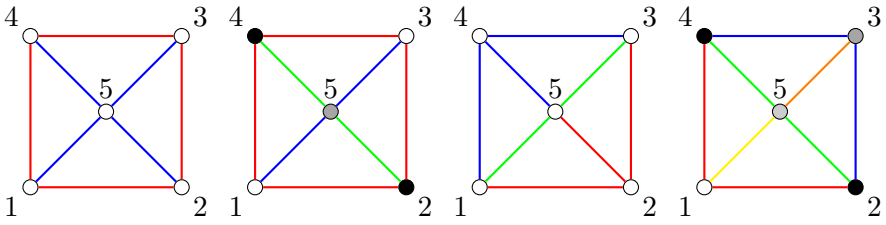
$$\mathrm{gap}(G) := |V(G)| - \mathrm{st}_+(G).$$

If G is an empty graph, then we define $\mathrm{gap}(G) = 0$. Determining the value of $\mathrm{st}_+(G)$

reduces to minimizing the number of components required in a specific cover (called set-join cover) of G. A formal definition of set-join covers and their components, skipped here for brevity, can be found at the beginning of Section 2.

This optimization problem simplifies significantly when restricted to the class of graphs that exclude a specific set of forbidden subgraphs listed in Figure 2. In such cases, the decomposition requires only the simplest form of set-joins—those corresponding to star graphs. While our approach uses star covers, we emphasize an essential difference with more common optimization problem: rather than minimizing the total number of stars, we focus on minimizing the number of components required to achieve the cover. This difference is highlighted on a small graph in our first example below.

Example 1. Let W_5 be a wheel graph on five vertices as shown in Figure 1. We highlight four edge covers of this graph, each satisfying a different optimization criterion.



(a) $\mathscr{C}_1$ (b) $\mathscr{C}_2$ (c) $\mathscr{C}_3$ (d) $\mathscr{C}_4$

Figure 1: Wheel graph W_5.

For sets of vertices $\mathcal{S}_1$ and $\mathcal{S}_2$, we use the notation $\mathcal{S}_1 \vee \mathcal{S}_2$ to denote the graph with edges $\{\{i_1, i_2\} \colon i_1 \in \mathcal{S}_1, i_2 \in \mathcal{S}_2\}$, and we say that $\mathcal{S}_1$ and $\mathcal{S}_2$ are the components of this bipartite graph.

(1) The cover $\mathscr{C}_1 = \{\{1,3\} \vee \{2,4\}, \{1,2,3,4\} \vee \{5\}\}$ consists of two complete bipartite graphs that together use four different components. This cover is constructed to minimize the number complete bipartite graphs needed to cover all edges of W_5. See Figure 1(a).

(2) The cover $\mathscr{C}_2 = \{\{1,3\} \vee \{2,4\}, \{1,3\} \vee \{5\}, \{2,4\} \vee \{5\}\}$ consists of three complete bipartite graphs and is constructed to minimize the number of components in any cover of W_5 with complete bipartite graphs. Note, the components used in $\mathscr{C}_2$ are $\{1,3\}$, $\{2,4\}$ and $\{5\}$, hence the resulting minimum value of components needed is three. See Figure 1(b).

(3) The cover $\mathscr{C}_3 = \{\{4\} \vee \{1,3,5\}, \{2\} \vee \{1,3,5\}, \{5\} \vee \{1,3\}\}$ consists of three stars that that use five components, and minimizes the number of stars in any star cover of W_5.

(4) The cover $\mathscr{C}_4 = \{\{1\} \vee \{2,4\}, \{1\} \vee \{5\}, \{2,4\} \vee \{3\}, \{2,4\} \vee \{5\}, \{3\} \vee \{5\}\}$ consists of five stars and is minimizing the number of components in any star cover of W_5. Note, the cover uses four components, $\{1\}$, $\{3\}$, $\{5\}$, and $\{2,4\}$. See Figure 1(d).

The cover $\mathscr{C}_1$ is an optimal cover of graph W_5 with bipartite graphs, see [5]. The cover $\mathscr{C}_2$ is an optimal set-join cover of graph W_5. As proven in [10, Theorem 2.12] this means $\mathrm{st}_+(W_5) = 3$. The cover $\mathscr{C}_3$ is an optimal star cover of the graph W_5, which means that $\mathrm{star}(W_5) = 3$. In this paper, we will study covers as in item (4). We will show in Proposition 8 that for a graph G that does not contain forbidden subgraphs, every set-join cover is a star cover, so minimizing the number of components in a star cover computes the SNT-rank of a graph G.

1.2 Structure of the paper

We organize the paper as follows. Sections 2 and 3 develop several observations concerning gap: Section 2 treats the general case, while Section 3 focuses on graphs that exclude certain 4-walk configurations. In particular, Lemma 14 shows that, in this restricted setting, $\mathrm{gap}(G)$ depends only on the parity of path lengths in G. Motivated by this result, Section 4 introduces a weighted multigraph $\kappa(G)$ with binomial edge weights, together with the parameter $\mathrm{gap}^*(\kappa(G))$, which coincides with $\mathrm{gap}(G)$. In Sections 5 and 6, we study general weighted multigraphs Γ with binomial weights and present several graph operations that preserve $\mathrm{gap}^*(\Gamma)$. These operations lead to an algorithm that reduces any such multigraph Γ to a weighted simple graph $\tau(\Gamma)$ (possibly with loops), in which all edge weights are zero and all vertex degrees are at least three, while maintaining control over gap^*. Since $\mathrm{gap}^*(\tau(\Gamma))$ may still be impossible to compute directly, Section 7 introduces an alternative method based on the sequential removal of vertices. The paper concludes with several illustrative examples in Section 8.

1.3 Definitions and notation

In this paper we will consider graphs with loops as well as multigraphs. We recall relevant definitions for both concepts below.

The term graph will always mean a simple graph $G = (V(G), E(G))$ that allows loops. Edges $e \in E(G)$ are subsets of $V(G)$ of the form $e = \{u, v\}$. If the edge is of the form $\{u, u\}$, it means that a vertex u has a loop. Below, we list some basic notions associated with graphs that we will depend on:

- By $G \cong H$ we denote that graphs G and H are isomorphic.
- Let P_n denote the path on n vertices, C_n a cycle on n vertices, K_n a complete graph on n vertices and $K_{m,n}$ a complete bipartite graph on $m + n$ vertices, all without loops.
- Vertices $u, v \in V(G)$ are *neighbours* if $\{u, v\} \in E(G)$. The neighbourhood of v in G, denoted by $N_G[v]$, is the set of all neighbours of v. Note that vertex $v \in V(G)$ has a loop if and only if $v \in N_G[v]$.
- Vertices u and v are *twin vertices* if $N_G[u] = N_G[v]$.

- If a vertex $v \in V(G)$ does not have a loop, then the *degree of* v in G equals the cardinality of its neighbourhood, i.e., $\deg_G(v) = |N_G[v]|$. If a vertex $v \in V(G)$ has a loop, then $\deg_G(v) = |N_G[v]| + 1$, i.e., loops count twice towards the degree.

- A vertex $v \in V(G)$ is called a *leaf* if $|N_G[u]| = 1$ and is called *a high degree vertex* if $\deg_G(v) \geqslant 3$. We denote by $V_{d\geqslant 3}(G)$ the set of all high degree vertices of G, by $V_{d=2}(G)$ the set of all degree two vertices, by $V_{d=1}(G)$ the set of all leaves, by $V_{d=0}(G)$ the set of all vertices of degree 0, and by $V_{d\neq 2}(G) = V(G) \setminus V_{d=2}(G)$. (Note that in our setting a vertex incident to a loop and another edge is not a leaf, and it is a high degree vertex.)

- Removing a set of vertices $\{v_1, \ldots, v_k\}$ from a simple graph G is a well-known operation in which we remove all vertices v_i, $i \in [k]$ together with all edges $\{v_i, u\}$, $u \in N_G[v_i]$, $i \in [k]$. We denote this graph as $G(\{v_1, \ldots, v_k\})$. When convenient, we write $G(v_1, \ldots, v_k)$ instead of $G(\{v_1, \ldots, v_k\})$. Furthermore, the subgraph of G induced on the set $\{v_1, \ldots, v_k\}$ is denoted by $G[\{v_1, \ldots, v_k\}]$.

- Let $k \in \mathbb{N}_0$. If for $u, v \in V(G)$ and for distinct vertices $x_i \in V_{d=2}(G) \setminus \{u, v\}$, $i \in [k]$, there exists a walk $u - x_1 - \ldots - x_k - v$, we abbreviate it by $u \overset{k}{\text{—}} v$.

- If G contains $w - u \overset{k}{\text{—}} v$ for $k \geqslant 0$, $w \in V_{d\geqslant 3}(G)$, $u \in V_{d=2}(G)$ and $v \in V_{d=1}(G)$, then we call $u \overset{k}{\text{—}} v$ a pendent path P_{k+2} at w.

We follow [4] and consider undirected *multigraphs* $\Gamma = (V(\Gamma), E(\Gamma))$, in which $E(\Gamma)$ is a multiset and to every $e \in E(\Gamma)$ we assign one or two endpoints from the set of vertices $V(\Gamma)$. Note that this definition allows several edges between a pair of distinct vertices and several loops at vertex u, which we denote as $\{u, u\}$. If a multigraph Γ is *weighted*, then $\Gamma = (V(\Gamma), E(\Gamma), w_\Gamma)$, where $w_\Gamma \colon E(\Gamma) \to \mathbb{R}$ is a function that assigns a weight to each edge.

If a vertex of a (weighted) multigraph is incident to s loops and t non-looped edges, then we define its *d*egree as $\deg_\Gamma(v) = t + 2s$. (For example, $\deg_\Gamma(w) = 6$ and $\deg_\Gamma(v) = 4$ for vertices $w, v \in V(\Gamma)$ in a weighted multigraph Γ shown in Figure 4.)

The *edge contraction along an edge* $e \in E(\Gamma)$ with different endpoints u and v is a graph obtained from Γ by identifying vertices u and v (into a single vertex), deleting the edge e and preserving all other incidences (and weights in the case of a weighted multigraph).

Other definitions from simple graphs to multigraphs extend in a natural way. We will consistently denote simple graphs by capital letters (G, G_1, H), and multigraphs by Greek letters (Γ, Γ_1).

2 Initial results on gap

Before limiting our analysis to graphs without forbidden subgraphs, we offer a few preliminary results on the gap of graphs satisfying a local assumption.

SNT-rank of a graph G can be described combinatorially using set-join covers of the graph. Let $\mathcal{K}$ and $\mathcal{L}$ be two nonempty subsets of $V(G)$ with possibly nonempty intersection. We define *the set-join of* $\mathcal{K}$ *and* $\mathcal{L}$, denoted by $\mathcal{K} \vee \mathcal{L}$, to be the graph with $V(\mathcal{K} \vee \mathcal{L}) = V(G)$, and $E(\mathcal{K} \vee \mathcal{L}) = \{\{i, j\} \colon i \in \mathcal{K}, j \in \mathcal{L}\}$. Note that $\mathcal{K} \vee \mathcal{L}$ has a loop on vertex i precisely when $i \in \mathcal{K} \cap \mathcal{L}$. We say that $\mathscr{C} = \{\mathcal{K}_i \vee \mathcal{L}_i \colon i \in [t]\}$ is *a set-join cover* of G if $E(G) = \bigcup_{i=1}^{t} E(\mathcal{K}_i \vee \mathcal{L}_i)$. For a set-join cover $\mathscr{C}$ of graph G the component set of $\mathscr{C}$ is

$$V(\mathscr{C}) = \{\mathcal{K}_i \colon i \in [t]\} \cup \{\mathcal{L}_i \colon i \in [t]\}$$

and we call each element of $V(\mathscr{C})$ *a component of* $\mathscr{C}$. *The order* of $\mathscr{C}$ is the number of components of $\mathscr{C}$, i.e. $|V(\mathscr{C})|$. In [10, Theorem 2.12] it is proven that

$$\mathrm{st}_+(G) = \min\{|V(\mathscr{C})| \colon \mathscr{C} \text{ a set-join cover of } G\}.$$

If $\mathscr{C}$ is a set-join cover of G of order $\mathrm{st}_+(G)$, then we say that $\mathscr{C}$ is an *optimal set-join cover* of G.

We recall selected results from [10], needed in the sequel. While these were originally stated in terms of $\mathrm{st}_+(G)$, we restate them here using $\mathrm{gap}(G)$.

Proposition 2. *Let G be a simple graph.*

(1) *[10, Lemma 3.3] If $v \in V(G)$ is a leaf and $H = G(\{v, w\})$ the graph obtained from G by deleting v and its unique neighbour w, then* $\mathrm{gap}(G) = \mathrm{gap}(H)$.

(2) *[10, Proposition 2.28] If $u, u' \in V(G)$ are twin vertices then* $\mathrm{gap}(G) = \mathrm{gap}(G(u)) + 1$.

(3) *[10, Example 3.4]* $\mathrm{gap}(P_n) = 1$ *for odd n and* $\mathrm{gap}(P_n) = 0$ *for even n.*

(4) *[10, Proposition 3.10]* $\mathrm{gap}(C_4) = 2$ *and* $\mathrm{gap}(C_n) = 0$ *for $n \neq 4$.*

Remark 3. Item (1) in Proposition 2 reduces the problem of determining the gap for graphs that contain a leaf to a smaller graph, and gives an algorithmic way to compute the gap for trees. If T is a tree with at least two vertices, then repeated application of this result can be used to prove that $\mathrm{gap}(T) = 0$ if and only if T is obtained from P_2 by consecutively adding a pendent P_2 to any vertex in the existing graph. More generally, for any tree T and $A \in \mathcal{S}^+(T)$, $\mathrm{rank}(A) = \mathrm{st}_+(A)$ [10]. Furthermore, $\mathrm{rank}(A)$ coincides with twice the matching number of T [7].

To verify this, assume first that T is obtained from P_2 by consecutively adding a pendent P_2 to a vertex. Since $\mathrm{gap}(P_2) = 0$ by Proposition 2 (3), it follows by Proposition 2 (1) that $\mathrm{gap}(T) = 0$.

Suppose that $\mathrm{gap}(T) = 0$. If $|V(T)| = 2$, then $T = P_2$ and the statement follows by Proposition 2 (3). Assume now that $|V(T)| \geqslant 3$ and that all trees on at most $|V(T)| - 1$ vertices of gap 0 are obtained from P_2 by consecutively adding a pendent P_2 to a vertex. Let v and w be two leaves of T, such that the distance between them is equal to the diameter of T, and let v_1 be the unique neighbor of v. We claim that $\deg_T(v_1) = 2$. Indeed,

$\deg_T(v_1) \geqslant 3$ and the assumption that T does not contain any cycles, implies that there exists $v' \in N_T[v_1]$, $v' \neq v$, with $d(v', w) = d(v, w) = \operatorname{diam}(T)$. Note, v' is not a leaf, since the assumptions that $\operatorname{gap}(T) = 0$ implies that T has no twin vertices, by Proposition 2 (2). Therefore there exists a vertex $v_0 \in N_T[v']$, such that $d(v_0, w) = d(v', w)+1 = \operatorname{diam}(T)+1$, a contradiction.

In particular there exists a vertex v_2 so that $N_T[v_1] = \{v, v_2\}$ and $v - v_1$ is P_2 a pendent path at v_2 in T. By item (1) in Proposition 2, $T(\{v, v_1\})$ is a tree on $|V(T)| - 2$ vertices with $\operatorname{gap}(T(\{v, v_1\})) = 0$. By the induction hypothesis, $T(\{v, v_1\})$ is obtained from P_2 by consecutively adding to a vertex a pendent P_2. Hence, T is obtained in the same way.

In the proposition below, we restrict our attention to graphs that do not contain a certain closed 4-walk. This assumption will also be adopted in subsequent sections.

Proposition 4. *Let G be a graph containing two vertex disjoint induced subgraphs G_1 and G_2 so that $V(G) = V(G_1) \cup V(G_2)$, and there do not exist vertices $u_1, v_1 \in V(G_1)$ and $u_2, v_2 \in V(G_2)$ with and $\{u_i, v_j\} \in E(G)$ for all $i, j \in \{1, 2\}$. Then*

$$\operatorname{gap}(G) \leqslant \operatorname{gap}(G_1) + \operatorname{gap}(G_2).$$

In particular, if $\operatorname{gap}(G_i) = 0$ for $i = 1, 2$, then $\operatorname{gap}(G) = 0$.

Proof. Let $\mathscr{C}$ be an optimal set-join cover of G, and let $\mathscr{C}_i \subseteq \mathscr{C}$, $i = 1, 2$, be the subset of $\mathscr{C}$ containing all the elements of $\mathscr{C}$ that cover at least one edge in G_i. Clearly, $|V(\mathscr{C}_i)| \geqslant \operatorname{st}_+(G_i)$.

We claim that $V(\mathscr{C}_1) \cap V(\mathscr{C}_2) = \emptyset$. Assuming the opposite, let $\mathcal{K} \in V(\mathscr{C}_1) \cap V(\mathscr{C}_2)$. Then for $i = 1, 2$: there exists $u_i \in \mathcal{K} \cap V(G_i)$ and a component $\mathcal{L}_i \in V(\mathscr{C}_i)$ so that $\mathcal{K} \vee \mathcal{L}_i \in \mathscr{C}_i$. In particular, there exist $v_i \in \mathcal{L}_i \cap V(G_i)$. From $\mathcal{K} \vee \mathcal{L}_i \in \mathscr{C}$ we conclude that $\{u_i, v_j\} \in E(G)$ for all $i, j = 1, 2$, a contradiction.

Since $V(\mathscr{C}_1) \cup V(\mathscr{C}_2) \subseteq V(\mathscr{C})$, this implies $\operatorname{st}_+(G) \geqslant \operatorname{st}_+(G_1) + \operatorname{st}_+(G_2)$ and thus also $\operatorname{gap}(G) \leqslant \operatorname{gap}(G_1) + \operatorname{gap}(G_2)$. □

Example 5. Note that the inequality in Proposition 4 can be strict. Let $G = C_6$ with vertices $v_1, \ldots, v_6$ in the natural cycle order. Let G_1, G_2 be induced subgraphs with $V(G_1) = \{v_1, v_2, v_3\}$ and $V(G_2) = \{v_4, v_5, v_6\}$. Then G_1 and G_2 are both isomorphic to P_3, so $\operatorname{gap}(G_1) = \operatorname{gap}(G_2) = 1$, and $\operatorname{gap}(G) = 0$.

Corollary 6. *Let G be a graph containing an induced subgraph H isomorphic to C_{2k}, $k \geqslant 3$, where vertices $V(H) = \{v_1, v_2, \ldots, v_{2k}\}$ are indexed in a natural cycle order, and $\{v_2, v_4, \ldots, v_{2k}\} \subseteq V_{d=2}(G)$. Then*

$$\operatorname{gap}(G) = \operatorname{gap}(G(V(H))).$$

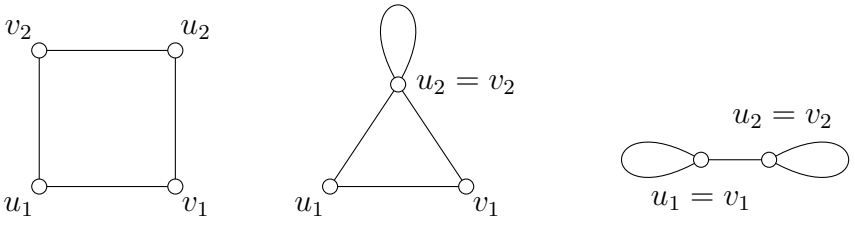
Proof. Let $W = V(H)$. Observe first that we have $\operatorname{gap}(G) \leqslant \operatorname{gap}(G(W)) + \operatorname{gap}(H) = \operatorname{gap}(G(W))$ since H and $G(W)$ satisfy the hypothesis of Proposition 4. Now, let $\mathscr{C}_W$ be an optimal set-join cover of $G(W)$, and $\mathscr{C}_1 := \{\{v_{2t-1}\} \vee N_G[v_{2t-1}] \colon t \in [k]\}$. Since $\mathscr{C} := \mathscr{C}_W \cup \mathscr{C}_1$ is a set-join cover of G with $|V(\mathscr{C})| = |V(\mathscr{C}_W)| + |V(\mathscr{C}_1)| = \operatorname{st}_+(G(W)) + 2k = |V(G)| - \operatorname{gap}(G(W))$, it follows that $\operatorname{gap}(G) \geqslant \operatorname{gap}(G(W))$. □

3 On graphs without four cycles

Now we limit our investigation to the family of graphs described in the following definition.

Definition 7. By $\mathcal{G}_{\boxtimes}$ we denote the family of all graphs G with no vertices $u_1, u_2, v_1, v_2 \in V(G)$, $u_1 \neq u_2$ and $v_1 \neq v_2$, such that $\{u_i, v_j\} \in E(G)$ for all $i, j \in \{1, 2\}$.

The family $\mathcal{G}_{\boxtimes}$ can also be characterised with the set of forbidden subgraphs given in Figure 2. Note, all three graphs are closed 4-walks in G.



(a) C_4 (b) C_3° (c) $P_2^{\circ,\circ}$

Figure 2: Forbidden subgraphs of graphs in $\mathcal{G}_{\boxtimes}$.

Set-join covers of graphs in $\mathcal{G}_{\boxtimes}$ have special properties listed in the proposition below.

Proposition 8. *Let $G \in \mathcal{G}_{\boxtimes}$. Any set-join cover $\mathscr{C}$ of G has the following properties:*

(1) *All elements in $\mathscr{C}$ are of the form $\{v\} \vee \mathcal{K}$ for some $v \in V(G)$.*

(2) *Let $\mathcal{K} \in V(\mathscr{C})$ with $|\mathcal{K}| \geqslant 2$ and $\{v_1\} \vee \mathcal{K} \in \mathscr{C}$. Then:*

   (a) *$\{v_1\} \vee \mathcal{K}$ is the only element of $\mathscr{C}$ with a component $\mathcal{K}$.*

   (b) *If $\mathscr{C}$ is an optimal set-join cover of G and $\{v_1\} \vee \mathcal{L} \in \mathscr{C}$ for $\mathcal{L} \neq \mathcal{K}$, then $|\mathcal{L}| = 1$.*

(3) *If $\mathscr{C}$ is an optimal set-join cover of G, then any two components in $V(\mathscr{C})$ intersect in at most one vertex of G.*

Proof. To prove item (1), suppose $\mathcal{K} \vee \mathcal{L} \in \mathscr{C}$, with $|\mathcal{K}| \geqslant 2$, $|\mathcal{L}| \geqslant 2$. Then there exist $u_1, u_2 \in \mathcal{K}$ and $v_1, v_2 \in \mathcal{L}$, $u_1 \neq u_2$ and $v_1 \neq v_2$, so that $\{u_i, v_j\} \in E(G)$ for all $i, j = 1, 2$, contradicting the assumption $G \in \mathcal{G}_{\boxtimes}$.

For item (2) assume that $\{v_1\} \vee \mathcal{K} \in \mathscr{C}$ with $u_1, u_2 \in \mathcal{K}$, $u_1 \neq u_2$. If $\{v_1\} \vee \mathcal{K}, \{v_2\} \vee \mathcal{K} \in \mathscr{C}$ and $v_1 \neq v_2$, then $\{u_i, v_j\} \in E(G)$ for all $i, j = 1, 2$. As this contradicts $G \in \mathcal{G}_{\boxtimes}$, (2a) follows. If $\{v_1\} \vee \mathcal{L} \in \mathscr{C}$, with $|\mathcal{L}| \geqslant 2$ and $\mathcal{L} \neq \mathcal{K}$, then $\{v_1\} \vee \mathcal{L}$ is the only element in the optimal set-join cover $\mathscr{C}$ with a component $\mathcal{L}$ by (2a). Hence, $\mathscr{C}' = \mathscr{C} \cup \{\{v_1\} \vee (\mathcal{K} \cup \mathcal{L})\} \setminus \{\{v_1\} \vee \mathcal{K}, \{v_1\} \vee \mathcal{L}\}$ is a set-join cover of G satisfying $|V(\mathscr{C}')| = |V(\mathscr{C})| - 1$, contradicting the optimality assumption for $\mathscr{C}$, and proving (2b).

For item (3) suppose that $u_1, u_2 \in \mathcal{K} \cap \mathcal{L}$ with $u_1 \neq u_2$ and $\mathcal{K} \vee \{v_1\}, \mathcal{L} \vee \{v_2\} \in \mathscr{C}$. Then $\{u_i, v_j\} \in E(G)$ for all $i, j = 1, 2$. If $v_1 = v_2$ this contradicts (2b), if $v_1 \neq v_2$ this contradicts $G \in \mathcal{G}_{\boxtimes}$. □

Corollary 9. *For $G \in \mathcal{G}_{\not\boxtimes}$, let $\mathscr{C}$ be a set-join cover of G, $\mathcal{K} \in V(\mathscr{C})$ with $|\mathcal{K}| \geqslant 2$, and $\{v\} \vee \mathcal{K} \in \mathscr{C}$. Let $\mathscr{C}_1$ be obtained from $\mathscr{C}$ by removing all elements with a component $\{v\}$, and by removing v from all remaining components. Then $\mathscr{C}_1$ is a set-join cover of $G(v)$ satisfying $|V(\mathscr{C}_1)| \leqslant |V(\mathscr{C})| - 2$. Furthermore, $\widehat{\mathscr{C}} := \mathscr{C}_1 \cup \{\{v\} \vee N_G[v]\}$ is a set-join cover of G.*

Proof. Clearly, $\mathscr{C}_1$ is a set-join cover of $G(v)$. Since $\{v\} \vee \mathcal{K} \in \mathscr{C}$ is the unique element of $\mathscr{C}$ with a component $\mathcal{K}$ by Proposition 8, $V(\mathscr{C}_1)$ does not contain $\{v\}$ nor $\mathcal{K}$, and the inequality $|V(\mathscr{C}_1)| \leqslant |V(\mathscr{C})| - 2$ follows. The second claim is straightforward. □

Consider $G \in \mathcal{G}_{\not\boxtimes}$ with a set-join cover $\mathscr{C}$. Let $V_1(\mathscr{C})$ be the set of all vertices $v \in V(G)$ with $\{v\} \in V(\mathscr{C})$, and let

$$N^1_{\mathscr{C}}[x] := N_G[x] \cap V_1(\mathscr{C}),\ N^\star_{\mathscr{C}}[x] := N_G[x] \setminus V_1(\mathscr{C})$$

for all $x \in V(G)$. Let $\mathscr{C}[x]$ be the subset of $\mathscr{C}$ that contains all the set-joins of $\mathscr{C}$ with at least one component containing x. Note that $\mathscr{C}[x]$ covers all the edges that are incident to x. We define three different types of vertices $x \in V(G)$ with respect to $\mathscr{C}$:

Type A: $|N^\star_{\mathscr{C}}[x]| \geqslant 2$ and $\mathscr{C}[x] = \{\{x\} \vee N^\star_{\mathscr{C}}[x]\} \cup \{\{x\} \vee \{v\} \colon v \in N^1_{\mathscr{C}}[x]\}$.

Type B: $N^\star_{\mathscr{C}}[x] = \emptyset$ and $\mathscr{C}[x] = \{\{x\} \vee \{v\} \colon v \in N^1_{\mathscr{C}}[x]\}$.

Type C: $x \notin V^1(\mathscr{C})$ and $\mathscr{C}[x] = \{N^\star_{\mathscr{C}}[v] \vee \{v\} \colon v \in N_G[x]\}$.

Definition 10. An optimal set-join cover $\mathscr{C}$ in which every $x \in V(G)$ is of Type A, of Type B or of Type C with respect to $\mathscr{C}$, is called an ABC *cover*.

For a general set-join cover $\mathscr{C}$, there can be vertices that are not of any of the three types (A, B or C) with respect to $\mathscr{C}$. The following proposition shows that any graph in $\mathcal{G}_{\not\boxtimes}$ has at least one ABC cover.

Proposition 11. *For every $G \in \mathcal{G}_{\not\boxtimes}$ there exists an* ABC *cover of G.*

Proof. Starting with a set-join cover $\mathscr{C}_0$ of G, we will construct a set-join cover $\mathscr{C}$ of G with the desired property and $|V(\mathscr{C})| \leqslant |V(\mathscr{C}_0)|$.

First consider $x \in V_1(\mathscr{C}_0)$ with $\mathscr{C}_0$ containing a set-join of the form $\mathcal{K} \vee \{x\}$ with $|\mathcal{K}| \geqslant 2$. Construct $\mathscr{C}_1$ by replacing all the set-joins in $\mathscr{C}_0[x]$ which are of the form $\{x\} \vee \mathcal{L}$ with $\{\{x\} \vee N^\star_{\mathscr{C}_0}[x]\} \cup \{\{x\} \vee \{v\} \colon v \in N^1_{\mathscr{C}_0}[x]\}$. Note that this replacement does not increase the number of components in the cover since $x, v \in V_1(G)$ and if $\{x\} \vee \mathcal{K}_j \in \mathscr{C}_0$ for $j \in [t]$, then $\{\mathcal{K}_j\}_{j\in[t]}$ are substituted by $N^\star_{\mathscr{C}_0}[x]$. If $|N^\star_{\mathscr{C}_0}[x]| \geqslant 2$, then x is now of Type A with respect to $\mathscr{C}_1$, otherwise it is of Type B. Furthermore, all vertices in G that are of Type B with respect to $\mathscr{C}_0$ remain of Type B with respect to $\mathscr{C}_1$, and all vertices in G that are of Type A with respect to $\mathscr{C}_0$ either remain of Type A with respect to $\mathscr{C}_1$ or become Type B with respect to $\mathscr{C}_1$.

Let $\mathscr{C}$ be the set-join that we obtain after repeatedly performing the above replacement for all $x \in G$ that are joined with a component that is not a singleton. In particular, all

$x \in V^1(\mathscr{C})$ are of Type A or of Type B, and all components of $\mathscr{C}$ that are not a singleton are of the form $N^\star_{\mathscr{C}}[v]$ for some $v \in V^1(\mathscr{C})$. Hence, all $x \in V(G)$ that are not in $V^1(\mathscr{C})$ are of Type C with respect to $\mathscr{C}$. □

Remark 12. Below we list a few straightforward observations for an ABC cover $\mathscr{C}$ of $G \in \mathcal{G}_{\boxtimes}$.

- If $x \in V_1(\mathscr{C})$, then x is not part of any component in $V(\mathscr{C})$ that contains more than one element. Hence, $V(G) \setminus V_1(\mathscr{C})$ is equal to the set of all Type C vertices with respect to $\mathscr{C}$.
- If x is of Type A, then it has at least two neighbors of Type C.
- If x is of Type C, then all its neighbors are of Type A. In particular, two vertices of Type C cannot be neighbors.

Lemma 13. *Let $H \in \mathcal{G}_{\boxtimes}$ and $\mathscr{C}$ an* ABC *cover of H. If for $u, v \in V(H)$ and $x_1, x_2 \in V_{d=2}(H) \setminus \{u, v\}$ there exists $u - x_1 - x_2 - v$ in H, then*

- *x_1 is of Type* C *and x_2 is of Type* A *with respect to $\mathscr{C}$ or vice versa, or*
- *x_1 and x_2 are both of Type* B *in $\mathscr{C}$.*

In particular, when $u = v$, both x_1 and x_2 are of Type B *in $\mathscr{C}$.*

Proof. We consider the Type of x_1 with respect to $\mathscr{C}$. If x_1 is of Type A, then $x_2 \notin V_1(\mathscr{C})$, hence it is of Type C. Note that in this case u must be of Type C and v of Type A, and so $u \neq v$. Similarly, if x_1 is of Type C, then $x_2 \in V_1(\mathscr{C}_0)$ and it must be of Type A. In this case u must be of Type A and v of Type C, so again $u \neq v$. Finally, if x_1 is of Type B, then x_2 must also be of Type B. It follows that in the case $u = v$ the only possible option is that x_1 and x_2 are both of Type B. □

The following lemma enables us to establish families of graphs that have the same gap, as detailed in Section 4.

Lemma 14. *Let $G \in \mathcal{G}_{\boxtimes}$ be a graph with a walk $u - x_1 - x_2 - v$ for $u, v \in V(G)$ and $x_1, x_2 \in V_{d=2}(G) \setminus \{u, v\}$. Let G' be a graph obtained from G by removing vertices x_1 and x_2 and adding an edge between u and v. Then* $\operatorname{gap}(G) \leqslant \operatorname{gap}(G')$. *Furthermore, if $G' \in \mathcal{G}_{\boxtimes}$, then* $\operatorname{gap}(G) = \operatorname{gap}(G')$.

Proof. Let $\mathscr{C}$ be an ABC cover of G. Using Lemma 13 and symmetry, we need to consider two cases.

Case 1: x_1 is of Type A and x_2 is of Type C. In this case we have

$$\{\{x_1\}, \{u, x_2\}, \{v\}, N^\star_{\mathscr{C}}[v]\} \subseteq V(\mathscr{C})$$

and $x_2 \in N^\star_{\mathscr{C}}[v]$. To construct a set-join cover $\mathscr{C}'$ of G' from $\mathscr{C}$ we remove the set-join $\{x_1\} \vee \{u, x_2\}$ from $\mathscr{C}$ and replace $\{v\} \vee N^\star_{\mathscr{C}}[v]$ by $\{v\} \vee (N^\star_{\mathscr{C}}[v] \setminus \{x_2\} \cup \{u\})$. Since $|V(\mathscr{C}')| = |V(\mathscr{C})| - 2$ we have $\operatorname{st}_+(G') \leqslant \operatorname{st}_+(G) - 2$ and the conclusion follows.

Case 2: Both x_1 and x_2 are of Type B. In this case $u, v, x_1, x_2 \in V_1(\mathscr{C})$. A set-join cover $\mathscr{C}'$ of G' is obtained by removing set-joins $\{u\} \vee \{x_1\}$, $\{x_1\} \vee \{x_2\}$, $\{x_2\} \vee \{v\}$ from $\mathscr{C}$ and adding a set-join $\{u\} \vee \{v\}$. Again we have $|V(\mathscr{C}')| = |V(\mathscr{C})| - 2$ and the conclusion follows.

Suppose now that $G' \in \mathcal{G}_{\boxtimes}$ and $\mathscr{C}'$ is an ABC cover of G'. If u and v are both in $V_1(\mathscr{C}')$, then $\{u\} \vee \{v\} \in \mathscr{C}'$ and

$$\mathscr{C} = \mathscr{C}' \setminus \{\{u\} \vee \{v\}\} \cup \{\{u\} \vee \{x_1\}, \{x_1\} \vee \{x_2\}, \{x_2\} \vee \{v\}\}$$

is a set-join cover of G of order $|V(\mathscr{C}')|+2$. Otherwise, either u or v is of Type C. Without loss of generality we suppose that u is of Type C and then v is of Type A. It follows that $u \in N^{\star}_{\mathscr{C}'}[v]$ and

$$\mathscr{C} = \mathscr{C}' \setminus \{\{v\} \vee N^{\star}_{\mathscr{C}'}[v]\} \cup \{\{v\} \vee (N^{\star}_{\mathscr{C}'}[v] \setminus \{u\} \cup \{x_2\}), \{x_1\} \vee \{u, x_2\}\}$$

is a set-join cover of G of order $|V(\mathscr{C}')| + 2$. This proves $\mathrm{st}_+(G) \leqslant \mathrm{st}_+(G') + 2$, or equivalently, $\mathrm{gap}(G') \leqslant \mathrm{gap}(G)$. □

Note, in Lemma 14 the assumption $G \in \mathcal{G}_{\boxtimes}$ excludes the possibility $\{u, v\} \in E(G)$. Furthermore, in the case $u = v$, G' contains a loop at u.

4 A weighted multigraph corresponding to a simple graph

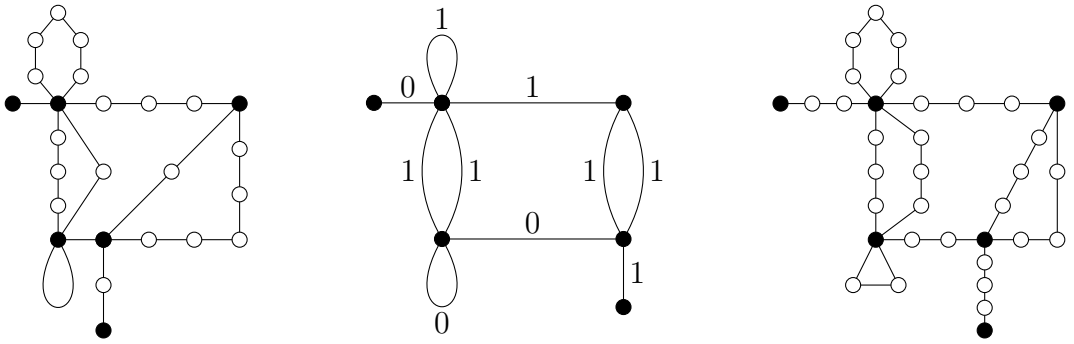
Lemma 14 enables the substitution of an induced path in a graph $G \in \mathcal{G}_{\boxtimes}$ with another path of the same length-parity, while preserving the gap. For example, the graphs in Figures 3(a) and 3(c) have the same gap. This motivates our next definition, where we introduce a multigraph $\kappa(G)$ with binomial weights (see Figure 3(b)), in which edges of weight 0 represent even paths in G and edges of weight 1 in $\kappa(G)$ represent odd paths in G.

Definition 15. For a graph $G = (V(G), E(G)) \in \mathcal{G}_{\boxtimes}$, we define $\kappa(G)$ to be a weighted multigraph $\kappa(G) = \Gamma = (V(\Gamma), E(\Gamma), w_\Gamma)$ with binomial weights $w_\Gamma \colon E(\Gamma) \to \{0, 1\}$, defined by:

- $V(\Gamma) = V_{d\neq 2}(G)$,
- For any $u, v \in V(\Gamma)$, the set of $u \overset{k}{\text{—}} v$ in G, $k \in \mathbb{N}_0$, is in bijective correspondence with edges $e \in E(\Gamma)$ with endpoints u and v in Γ and $w_\Gamma(e) = k \bmod 2$.

In a weighted multigraph Γ an edge of weight 1 is called 1-*edge* and an edge of weight 0 is called 0-*edge*; 1-*loops* and 0-*loops* are defined equivalently.

Note that $\kappa(P_n)$ is P_2 with the only edge of weight 1 if n is odd and of weight 0 if n is even. Moreover, a cycle C_n has $V(C_n) = V_{d=2}(C_n)$ and so $\kappa(C_n)$ is an empty graph, $V(\kappa(C_n)) = \emptyset$.



(a) G (b) $\kappa(G)$ (c) $\zeta(\kappa(G))$

Figure 3: An example of graph G, its weighted multigraph $\kappa(G)$ and the graph $\zeta(\kappa(G))$. All high degree vertices and leaves of simple graphs are colored black.

Definition 16. Let Γ be a weighted multigraph with binomial weights. Let $\zeta(\Gamma)$ be a simple graph (without loops) obtained from Γ in the following way:

- $V(\Gamma) \subseteq V(\zeta(\Gamma))$. The set $V(\zeta(\Gamma)) \setminus V(\Gamma)$ contains precisely the vertices, defined in the following items.
- Substitute each edge $e = \{u, v\} \in E(\Gamma)$ of weight $w_\Gamma(e) = 0$ by $u \overset{2}{—} v$. This operation adds two new vertices x_1 and x_2 to $V(\zeta(G))$, where $\deg_{\zeta(\Gamma)}(x_j) = 2$ for $j \in [2]$.
- Substitute each edge $e = \{u, v\} \in E(\Gamma)$, where $u \neq v$ and $w_\Gamma(e) = 1$, by $u \overset{3}{—} v$. This operation adds three new vertices x_1, x_2, x_3 to $V(\zeta(\Gamma))$, where $\deg_{\zeta(\Gamma)}(x_j) = 2$ for $j \in [3]$.
- Substitute each loop $e = \{u, u\} \in E(\Gamma)$ of weight $w_\Gamma(e) = 1$ by $u \overset{5}{—} u$. This operation adds five new vertices $x_1, \ldots, x_5$ to $V(\zeta(\Gamma))$ and, where $\deg_{\zeta(\Gamma)}(x_j) = 2$ for $j \in [5]$.

An illustration of mappings κ and ζ is given in Figure 3.

Observe that the graph $\zeta(\kappa(G))$ may contain more or less vertices than G. Furthermore, $\zeta(\kappa(G)) \in \mathcal{G}_{\mathbb{X}}$ and has no loops. Therefore, by Lemma 14 we have

$$\operatorname{gap}(G) = \operatorname{gap}(\zeta(\kappa(G))). \tag{1}$$

Note that the mapping κ is not injective, but ζ is. Hence, we define the gap of a multigraph Γ as the gap of its image via ζ mapping.

Definition 17. For every multigraph Γ with binomial weights we define

$$\operatorname{gap}^*(\Gamma) := \operatorname{gap}(\zeta(\Gamma)).$$

Note that the definition of gap* and (1) imply

$$\operatorname{gap}(G) = \operatorname{gap}^*(\kappa(G)) \tag{2}$$

for $G \in \mathcal{G}_{\not\boxtimes}$. We use this equality to compute the gap of a graph $G \in \mathcal{G}_{\not\boxtimes}$, and in order to do this, we present operations on Γ that preserve $\operatorname{gap}^*(\Gamma)$ in the following section.

5 Operations on Γ that preserve gap*

This section examines how the parameter gap* behaves under several fundamental operations on weighted multigraphs. In each case, we show that the gap* is preserved, and these operations will later enable us to design an algorithm for computing gap*.

5.1 Removing a vertex from Γ

To remove a vertex $v \in V(\Gamma)$ from the multigraph Γ together with all edges incident to it, we follow the notation used for simple graphs and denote the resulting graph by $\Gamma(v)$. Note that $\operatorname{gap}(\zeta(\Gamma(v)))$ is not necessarily equal to $\operatorname{gap}(\zeta(\Gamma)(v))$, so we define a different way of vertex removal from a multigraph Γ with binomial weights, which we denote by Γ_{-v}.

Definition 18. Let $\Gamma = (V(\Gamma), E(\Gamma), w_\Gamma)$ be a weighted multigraph with binomial weights and let $v \in V(\Gamma)$. Let the edges in $E(\Gamma)$ that are incident to v be:

- ℓ_j loops with weight $j \in \{0, 1\}$.
- f_j edges incident to leaves distinct from v with weight $j \in \{0, 1\}$. The leaves (other than v) incident to these edges of weight 1 are denoted by v_i for $i \in [f_1]$.
- k_j edges incident to non-leaf vertices with weight $j \in \{0, 1\}$. The edges of weight 1 we denote by $e_i = \{u_i, v\}$, $j \in [k_1]$.

We define multigraph Γ_{-v} as

$$\begin{aligned} V(\Gamma_{-v}) &= (V(\Gamma(v)) \setminus \{v_i \colon i \in [f_1]\}) \cup \{v_i' \colon i \in [\ell_1]\} \cup \{u_i' \colon i \in [k_1]\}, \\ E(\Gamma_{-v}) &= E(\Gamma(v)) \cup \{e_i' = \{u_i, u_i'\} \colon i \in [k_1]\} \text{ and} \\ w_{\Gamma_{-v}}(e) &= \begin{cases} 0, & \text{if } e \in \{\{u_i, u_i'\} \colon i \in [k_1]\}, \\ w_\Gamma(e), & \text{otherwise.} \end{cases} \end{aligned}$$

Remark 19. To construct the multigraph Γ_{-v} from Γ, we first remove the vertex v with all the incident edges, which results in the graph $\Gamma(v)$. Note that $V(\Gamma(v))$ has $f_0 + f_1$ new vertices of degree 0. To obtain Γ_{-v}:

- Remove f_1 of vertices of degree 0.
- Add vertices v_i', $i \in [\ell_1]$, to $V(\Gamma(v))$. These are kept as vertices of degree 0.

- Add vertices u'_i, $i \in [k_1]$, to $V(\Gamma(v))$. Note that a vertex $u \in V(\Gamma)$ may appear several times in the multiset $\{u_i \colon i \in [k_1]\}$, but u'_i, $i \in [k_1]$, are all distinct in $V(\Gamma_{-v})$.

- $E(\Gamma_{-v})$ consists of all the edges in $E(\Gamma(v))$ with the same weights as in Γ, together with new edges $\{e'_i = \{u_i, u'_i\} \colon i \in [k_1]\}$.

Example 20. Consider the weighted multigraph $\Gamma = \kappa(G)$ for the graph G from Figure 3 and vertices $v, w \in V(\Gamma)$ as shown in Figure 4(a). Multigraphs Γ_{-v} and Γ_{-w} are shown in Figures 4(b) and 4(c).

Remark 21. We acknowledge that the operation Γ_{-v} is technical, and we justify its inclusion by Lemma 22. However, the operation simplifies if v is incident only to edges of weight 0. In this case, $\Gamma_{-v} = \Gamma(v)$.

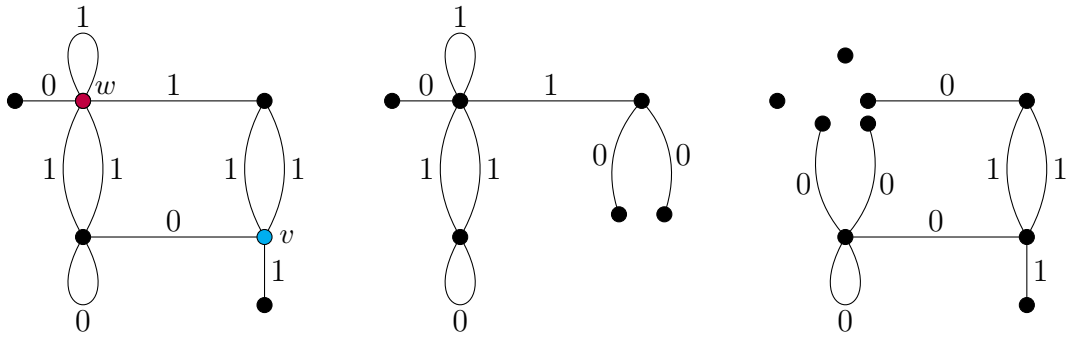


Figure 4: Two examples of removal of vertices.

Lemma 22. *Let $\Gamma = (V(\Gamma), E(\Gamma), w_\Gamma)$ be a weighted multigraph with binomial weights and $v \in V(\Gamma)$. Then*

$$\mathrm{gap}^*(\Gamma_{-v}) = \mathrm{gap}(\zeta(\Gamma_{-v})) = \mathrm{gap}(\zeta(\Gamma)(v)).$$

Proof. We consider different cases, depending on the edges that are incident to v in Γ.

If v has a loop of weight 0, it is removed in Γ_{-v}, and $\zeta(\Gamma)(v)$ contains a component isomorphic to the path P_2 with $\mathrm{gap}(P_2) = 0$. If $v \in V(\Gamma)$ has a loop of weight 1, $\zeta(\Gamma)(v)$ contains a component isomorphic to the path P_5. Graphs Γ_{-v} and $\zeta(\Gamma_{-v})$ both contain a component isomorphic to K_1, and $\mathrm{gap}(P_5) = \mathrm{gap}(K_1) = 1$.

If v is adjacent to a leaf with the edge of weight 0, $\zeta(\Gamma)(v)$ contains a component isomorphic to the path P_3, and Γ_{-v} contains a component isomorphic to K_1, and $\mathrm{gap}(P_3) = \mathrm{gap}(K_1) = 1$. If v is adjacent to a leaf with the edge of weight 1, $\zeta(\Gamma)(v)$ contains a component isomorphic to the path P_4 with $\mathrm{gap}(P_4) = 0$, and this edge together with the leaf is removed in Γ_{-v}.

If there is an edge e of weight 0 incident to v and u with $u \notin V_{d=1}(\Gamma)$, $u \neq v$, then $\zeta(\Gamma)(v)$ contains a pendent P_2 at u which can be removed by Proposition 2 (1). Also, this

edge is removed from $\zeta(\Gamma_{-v})$. Finally, if there is an edge e of weight 1 incident to v and u with $u \notin V_{d=1}(\Gamma)$, $u \neq v$, then $\zeta(\Gamma)(v)$ contains a pendent P_3 at u. In Γ_{-v}, this edge is substituted by an edge to a new leaf of weight 0, thus $\zeta(\Gamma_{-v})$ contains a pendent P_3 on vertex u as well.

Hence, $\mathrm{gap}(\zeta(\Gamma_{-v})) = \mathrm{gap}(\zeta(\Gamma)(v))$, which completes the proof. □

5.2 Elimination of low degree vertices

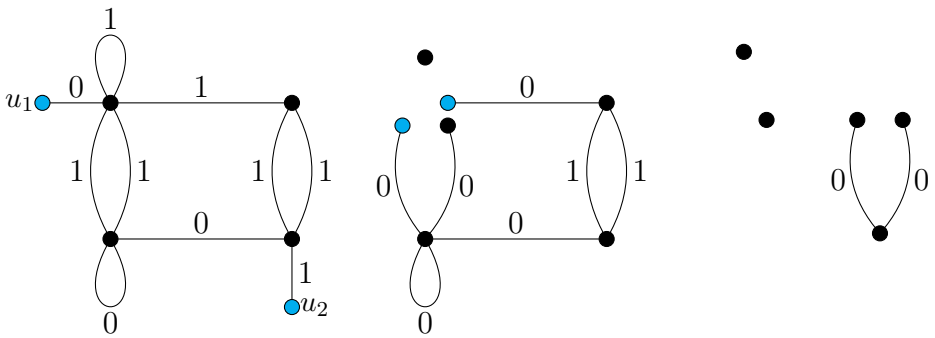
Lemma 23 (Elimination of leaves)**.** *Suppose Γ is a multigraph with binomial weights. Let v be a leaf in Γ, u its unique neighbour and $e = \{u, v\}$.*

1. *If $w_\Gamma(e) = 1$, then $\mathrm{gap}^*(\Gamma) = \mathrm{gap}^*(\Gamma(v))$.*
2. *If $w_\Gamma(e) = 0$, then $\mathrm{gap}^*(\Gamma) = \mathrm{gap}^*(\Gamma(v)_{-u})$.*

Proof. The graph $\zeta(\Gamma)$ contains either pendent P_4 (in the case $w_\Gamma(e) = 1$) or pendent P_3 (in the case $w_\Gamma(e) = 0$) at vertex u. Using Proposition 2(1) we remove a leaf twice from $\zeta(\Gamma)$. In the first case we obtain the graph $\zeta(\Gamma(v))$, so (1) follows. In the second case we obtain the graph $\zeta(\Gamma(v))(u)$ and (2) follows by Lemma 22. □

Example 24. Consider weighted multigraph $\Gamma = \kappa(G)$ from Figure 3 and let u_1 and u_2 be the leaves in it (see Figure 5(a)). By Lemma 23, we have $\mathrm{gap}^*(\Gamma) = \mathrm{gap}^*(\Gamma(u_2))$ and so the gap does not change by elimination of the leaf u_2. However, the elimination of leaf u_1 in Γ results in the removal of a high degree vertex, which will result in a graph Γ_1 with new leaves (see Figure 5(b)). So, by consequently eliminating some new leaves in Γ_1 (denoted by blue in Figure 5(b)), we obtain a graph with fewer vertices, and in some cases this reduces to a graph with a known gap^* (see Figure 5(c)). Thus

$$\mathrm{gap}^*(\Gamma) = \mathrm{gap}^*(\Gamma_1) = \mathrm{gap}^*(\Gamma_2) = 2 + \mathrm{gap}^*(P_3) = 2 + \mathrm{gap}(P_7) = 3.$$



(a) Γ (b) Γ_1 (c) Γ_2

Figure 5: Elimination of blue leaves in each step from Example 24.

For any graph $G \in \mathcal{G}_{\not\boxtimes}$ the multigraph $\Gamma = \kappa(G)$ does not contain vertices of degree 2 by construction. But it may happen that after some operations introduced in this section, the resulting multigraph also contains vertices of degree 2, see e.g. Figure 5(c). We can eliminate them as described below.

Lemma 25 (Elimination of vertices of degree 2)**.** *Suppose* Γ *is a multigraph with binomial weights and* $u \in V(\Gamma)$ *with* $\deg_\Gamma(u) = 2$*. If* u *is a vertex with one loop and incident to no other edges, let* $\Gamma' = \Gamma(u)$*. If not, denote the only incident edges to* u *by* $e_1 = \{v_1, u\}$ *and* $e_2 = \{v_2, u\}$ *and let* Γ' *be obtained from* Γ *by replacing* $v_1 - u - v_2$ *by an edge* e' *from* v_1 *to* v_2 *of weight* $w_{\Gamma'}(e') = |w_\Gamma(e_1) + w_\Gamma(e_2) - 1|$*. Then*

$$\mathrm{gap}^*(\Gamma) = \mathrm{gap}^*(\Gamma').$$

Proof. If u is a vertex with one loop and incident to no other edges, then $\zeta(\Gamma) \cong \zeta(\Gamma') \cup C_k$ where $k = 3$ or $k = 6$, so $\mathrm{gap}(\zeta(\Gamma)) = \mathrm{gap}(\zeta(\Gamma'))$ by Proposition 2 (4). If not, the edges e_1 and e_2 are not loops since $\deg(u) = 2$. It follows that $\zeta(\Gamma)$ contains $v_1 \overset{\ell}{—} v_2$, where $\ell = w_\Gamma(e_1) + w_\Gamma(e_2) + 5$. Let $e' \in E(\Gamma')$ be as in the statement. If e' is a loop with $w_{\Gamma'}(e') = 1$ then $\zeta(\Gamma')$ contains $v_1 \overset{5}{—} v_2$. In this case $w_\Gamma(e_1) + w_\Gamma(e_2)$ is even, so $\mathrm{gap}(\zeta(\Gamma')) = \mathrm{gap}(\zeta(\Gamma))$ by Theorem 14. In all other cases $\zeta(\Gamma')$ contains $v_1 \overset{m}{—} v_2$, where $m = w_{\Gamma'}(e') + 2$, so again $\mathrm{gap}(\zeta(\Gamma')) = \mathrm{gap}(\zeta(\Gamma))$ by Theorem 14. The conclusion follows. □

5.3 Elimination of 1-edges

Lemma 26 (**1**-edge contraction)**.** *Suppose* Γ *is a multigraph with binomial weights and a non-looped edge* e *of weight* 1*. Let* Γ' *be the edge contraction of* Γ *along* e*. Then*

$$\mathrm{gap}^*(\Gamma) = \mathrm{gap}^*(\Gamma').$$

Proof. Let Γ be a multigraph, and Γ' its edge contraction along the edge $e = \{u, v\}$ of weight 1 with $u \neq v$. Let $G = \zeta(\Gamma)$ and $G' = \zeta(\Gamma')$. We denote by w the vertex of Γ' obtained from the edge e and $u - s_1 - s_2 - s_3 - v$ the path in G obtained from e. We claim: $\mathrm{gap}(G) = \mathrm{gap}(G')$.

Assume first that there are k 1-edges between u and v in Γ, where $k \geqslant 2$. In this case, the path $u - s_1 - s_2 - s_3 - v$ is part of an 8-cycle $u - s_1 - s_2 - s_3 - v - t_3 - t_2 - t_1 - u$ in G, where u and v are the only vertices of this cycle that can have a degree greater than 2 in G. By Corollary 6 we have $\mathrm{gap}(G) = \mathrm{gap}(G_1)$, where $G_1 = G(\{u, s_1, s_2, s_3, v, t_3, t_2, t_1\})$. Note that $G_1 \cong G_0 \cup (k-2)P_3$ for some graph G_0. In particular, $\mathrm{gap}(G) = \mathrm{gap}(G_0) + k - 2$.

On the other hand, while the edge e is contracted in Γ', the remaining $k - 1$ 1-edges between u and v become 1-loops at w in Γ'. Those edges correspond to 6-cycles in G' that have w as the only vertex of degree greater than 2. Let $w - t'_1 - t'_2 - t'_3 - t'_4 - t'_5 - w$ be one of those cycles, and let $G'_1 = G'(\{u, t'_1, t'_2, t'_3, t'_4, t'_5\})$. By Corollary 6 we have $\mathrm{gap}(G') = \mathrm{gap}(G'_1)$. Observing that $G'_1 \cong G_0 \cup (k-2)P_5$, with the graph G_0 the same as above, we conclude $\mathrm{gap}(G') = \mathrm{gap}(G_0) + k - 2 = \mathrm{gap}(G)$, as desired.

From now on, we assume that there is only one 1-edge between u and v in Γ. In this case $V(G') = V(G) \setminus \{u, s_1, s_2, s_3, v\} \cup \{w\}$, hence $|V(G')| = |V(G)| - 4$, and to establish our claim we need to prove $\mathrm{st}_+(G) = \mathrm{st}_+(G') + 4$.

We will first prove that $\mathrm{st}_+(G') \leqslant \mathrm{st}_+(G) - 4$. Let $\mathscr{C}$ be an ABC cover of G. We consider different cases, depending on the type of vertex s_2 with respect to $\mathscr{C}$.
The vertex s_2 is of **Type A**. In this case $\{s_2\} \vee \{s_1, s_3\} \in \mathscr{C}$, hence s_1 and s_3 are of Type C, and u and v are of Type A. In particular, $\mathscr{C}$ contains the following set-joins:

$$\{u\} \vee N^\star_{\mathscr{C}}[u],\ \{v\} \vee N^\star_{\mathscr{C}}[v],\ \{s_2\} \vee \{s_1, s_3\}.$$

Let $V_R = \{\{u\}, N^\star_{\mathscr{C}}[u], \{v\}, N^\star_{\mathscr{C}}[v], \{s_2\}, \{s_1, s_3\}\}$. We define $\mathscr{C}'$ to be the set that contains all set-joins of $\mathscr{C}$ that do not involve any of the components from V_R, as well as $\{w\} \vee ((N^\star_{\mathscr{C}}[u] \cup N^\star_{\mathscr{C}}[v]) \setminus \{s_1, s_3\})$, and $\{w\} \vee \{y\}$ for $y \in N^1_{\mathscr{C}}[u] \cup N^1_{\mathscr{C}}[v]$. In particular,

$$V(\mathscr{C}') = (V(\mathscr{C}) \setminus V_R) \cup \{\{w\}, (N^\star_{\mathscr{C}}[u] \cup N^\star_{\mathscr{C}}[v]) \setminus \{s_1, s_3\}\},$$

and $|V(\mathscr{C}')| = |V(\mathscr{C})| - 4$.
The vertex s_2 is of **Type B**. In this case $\{s_2\} \vee \{s_1\}, \{s_2\} \vee \{s_3\} \in \mathscr{C}$, and s_1 and s_3 are of Type B. Hence, $\{u\} \vee \{s_1\}, \{v\} \vee \{s_3\} \in \mathscr{C}$, and u and v are either of Type A or of Type B. Let $V_R = \{\{u\}, \{v\}, \{s_1\}, \{s_2\}, \{s_3\}, N^\star_{\mathscr{C}}[u], N^\star_{\mathscr{C}}[v]\}$, where $N^\star_{\mathscr{C}}[u]$ or $N^\star_{\mathscr{C}}[v]$ may be empty. We define $\mathscr{C}'$ to be the set that contains all set-joins of $\mathscr{C}$ that do not involve any of the components from V_R, as well as $\{w\} \vee (N^\star_{\mathscr{C}}[u] \cup N^\star_{\mathscr{C}}[v])$ (provided $N^\star_{\mathscr{C}}[u] \cup N^\star_{\mathscr{C}}[v] \neq \emptyset$), and $\{w\} \vee \{y\}$ for $y \in (N^1_{\mathscr{C}}[u] \setminus \{s_1\}) \cup (N^1_{\mathscr{C}}[v] \setminus \{s_3\})$. In particular,

$$V(\mathscr{C}') = (V(\mathscr{C}) \setminus V_R) \cup \{\{w\}, N^\star_{\mathscr{C}}[u] \cup N^\star_{\mathscr{C}}[v]\},$$

and $|V(\mathscr{C}')| \leqslant |V(\mathscr{C})| - 4$.
The vertex s_2 is of **Type C**. In this case $\{s_1\} \vee \{u, s_2\}, \{s_3\} \vee \{s_2, v\} \in \mathscr{C}$, and u, v and s_2 are of Type C. Let

$$V_R = \{\{s_1\}, \{s_3\}, \{u, s_2\}, \{s_2, v\}\} \cup \{N^\star_{\mathscr{C}}[s], s \in N_G[u] \setminus \{s_1\}\} \cup \{N^\star_{\mathscr{C}}[t], t \in N_G[v] \setminus \{s_3\}\}.$$

Let $\mathscr{C}'$ be the set that contains all set-joins in $\mathscr{C}$ that do not involve any of the components in V_R, as well as

$$\{s\} \vee ((N^\star_{\mathscr{C}}[s] \setminus \{u\}) \cup \{w\}),\ s \in N_G[u] \setminus \{s_1\},\ \text{and}$$
$$\{t\} \vee ((N^\star_{\mathscr{C}}[t] \setminus \{v\}) \cup \{w\}), t \in N_G[v] \setminus \{s_3\}.$$

In particular, $|V(\mathscr{C}')| = |V(\mathscr{C})| - 4$, as desired.

Now we will prove that $\mathrm{st}_+(G) \leqslant \mathrm{st}_+(G') + 4$. Let $\mathscr{C}'$ be an ABC cover of G'. We consider different cases, depending on the type of vertex w with respect to $\mathscr{C}'$. We will construct a set-join cover $\mathscr{C}$ of G with $|V(\mathscr{C})| \leqslant |V(\mathscr{C}')| + 4$. Let

$$\widehat{\mathscr{C}'} = \{\mathcal{K} \vee \mathcal{L} \in \mathscr{C}', w \notin \mathcal{K} \cup \mathcal{L}\}.$$

Note that $\mathscr{C}' = \widehat{\mathscr{C}'} \cup \mathscr{C}'[w]$.

The vertex w is of **Type A**. In this case, let

$$\mathscr{C} = \widehat{\mathscr{C}'} \cup \{\{u\} \vee N_G[u], \{v\} \vee N_G[v], \{s_2\} \vee \{s_1, s_3\}\}.$$

Since any component of cardinality at least 2 appears only once in $\mathscr{C}'$ by Proposition 8 (2a), we have $|V(\widehat{\mathscr{C}'})| \leqslant |V(\mathscr{C}')| - 2$ ($\{w\}$ and $N^\star_{\mathscr{C}'}[w]$ do not appear in $\widehat{\mathscr{C}'}$). Since $\mathscr{C}'[w]$ cover only edges adjacent to w in G', $\widehat{\mathscr{C}'}$ covers all edges in G that are not adjacent to u, s_1, s_2, s_3, or v. Furthermore, $\{\{u\} \vee N_G[u], \{v\} \vee N_G[v], \{s_2\} \vee \{s_1, s_3\}\}$ covers all edges in G that are adjacent to u, s_1, s_2, s_3, or v. So $\mathscr{C}$ is a a set-join cover of G with

$$\begin{aligned} |V(\mathscr{C})| &\leqslant |V(\widehat{\mathscr{C}'})| + |V(\{\{u\} \vee N_G[u], \{v\} \vee N_G[v], \{s_2\} \vee \{s_1, s_3\}\})| \\ &\leqslant |V(\mathscr{C}')| - 2 + 6 = |V(\mathscr{C}')| + 4. \end{aligned}$$

The vertex w is of **Type B**. In this case $\mathscr{C}'[w] = \{\{w\} \vee \{x\}, x \in N_{G'}[w]\}$. Note that any $x \in N_{G'}[w]$ has $\deg_{G'}(x) = 2$, so it is of Type B and thus $\{x\} \in \widehat{\mathscr{C}'}$. Let

$$\mathscr{C} = \widehat{\mathscr{C}'} \cup \{\{u\} \vee \{x\}, x \in N_G[u]\} \cup \{\{v\} \vee \{x\}, x \in N_G[v]\} \cup \{\{s_1\} \vee \{s_2\}, \{s_2\} \vee \{s_3\}\}.$$

We have $|V(\widehat{\mathscr{C}'})| = |V(\mathscr{C}')| - 1$ ($\{w\}$ does not appear in $\widehat{\mathscr{C}'}$). As in the previous case, $\widehat{\mathscr{C}'}$ covers all edges in G that are not adjacent to u, s_1, s_2, s_3, or v and $\{\{u\} \vee \{x\}, x \in N_G[u]\} \cup \{\{v\} \vee \{x\}, x \in N_G[v]\} \cup \{\{s_1\} \vee \{s_2\}, \{s_2\} \vee \{s_3\}\}$ covers all edges in G that are adjacent to u, s_1, s_2, s_3, or v. Since $N_G[u] \cup N_G[v] = N_{G'}[w] \cup \{s_1, s_3\}$ we have

$$|V(\mathscr{C})| \leqslant |V(\widehat{\mathscr{C}'})| + |\{\{u\}, \{s_1\}, \{s_2\}, \{s_3\}, \{v\}\}| = |V(\mathscr{C}')| - 1 + 5 = |V(\mathscr{C}')| + 4.$$

The vertex w is of **Type C**. In this case $\mathscr{C}'[w] = \{\{x\} \vee N^\star_{\mathscr{C}'}[x], x \in N_{G'}[w]\}$. Let

$$\begin{aligned} \mathscr{C}_2 &= \{\{x\} \vee N^\star_{\mathscr{C}'}[x] \cup \{u\} \setminus \{w\}, x \in N_G[u] \setminus \{s_1\}\}, \\ \mathscr{C}_3 &= \{\{x\} \vee N^\star_{\mathscr{C}'}[x] \cup \{v\} \setminus \{w\}, x \in N_G[v] \setminus \{s_3\}\}, \text{ and} \\ \mathscr{C} &= \widehat{\mathscr{C}'} \cup \mathscr{C}_2 \cup \mathscr{C}_3 \cup \{\{s_1\} \vee \{u, s_2\}, \{s_3\} \vee \{v, s_2\}\}. \end{aligned}$$

Since any component of cardinality at least 2 appears only once in $\mathscr{C}'$, we have $|V(\widehat{\mathscr{C}'} \cup \mathscr{C}_2 \cup \mathscr{C}_3)| = |V(\mathscr{C}')|$. Furthermore, $\widehat{\mathscr{C}'} \cup \mathscr{C}_2 \cup \mathscr{C}_3$ covers all edges in G except those on the path $u - s_1 - s_2 - s_3 - v$ and $\{\{s_1\} \vee \{u, s_2\}, \{s_3\} \vee \{v, s_2\}\}$ covers all edges on the path $u - s_1 - s_2 - s_3 - v$. So $\mathscr{C}$ is a set-join cover of G with $|V(\mathscr{C})| = |V(\mathscr{C}')| + 4$, which finishes the proof. □

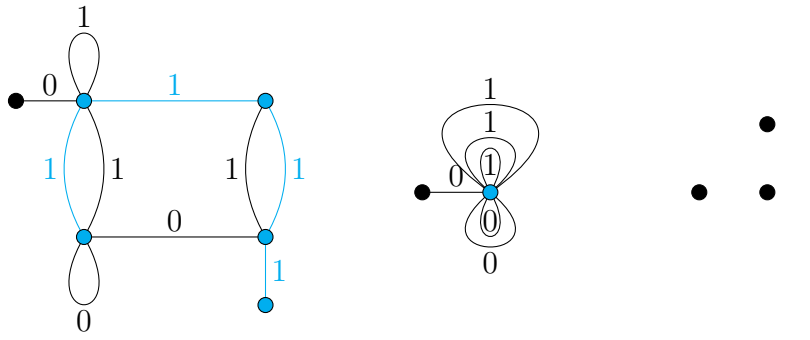
Lemma 27 (Elimination of **1**-loops)**.** *Suppose Γ is a multigraph with binomial weights and suppose there exists $v \in V(\Gamma)$ with a loop e of weight $w_\Gamma(e) = 1$. Then*

$$\operatorname{gap}^*(\Gamma) = \operatorname{gap}^*((\Gamma \setminus \{e\})_{-v}) = \operatorname{gap}^*(\Gamma_{-v}) - 1.$$

Proof. Observe that $\zeta(\Gamma)$ contains an induced subgraph G_1 isomorphic to C_6 with one of the vertices v and all other vertices of degree 2. By Corollary 6 we have $\mathrm{gap}(G) = \mathrm{gap}(G_2)$, where $G_2 = G(V(G_1))$. Furthermore, $\mathrm{gap}(G_2) = \mathrm{gap}(\zeta((\Gamma \setminus \{e\})_{-v}))$ by Lemma 22, and the first equality follows. Note that $\Gamma_{-v} \cong (\Gamma \setminus \{e\})_{-v} \cup K_1$, so we obtain also the second equality. □

Example 28. Reconsider weighted multigraph $\Gamma = \kappa(G)$ from Figure 3 (see also Figure 6(a)). After a sequence of 1-edge contractions of blue edges, we obtain multigraph graph Γ' in Figure 6(b), and by Lemma 26 we have $\mathrm{gap}^*(\Gamma) = \mathrm{gap}^*(\Gamma')$. Now the high degree vertex v of Γ' has 3 loops of weight 1, 2 loops of weight 0, and a pendent vertex. Let e be one of the loops of weight 1. Using Lemma 27 we have $\mathrm{gap}^*(\Gamma) = \mathrm{gap}^*(\Gamma' \setminus \{e\})_{-v}$, where the later graph consists of three copies of K_1, as shown in Figure 6(c). Thus,

$$\mathrm{gap}^*(\Gamma) = \mathrm{gap}^*(\Gamma') = \mathrm{gap}^*((\Gamma' \setminus \{e\})_{-v}) = \mathrm{gap}^*(3K_1) = \mathrm{gap}(3K_1) = 3.$$



(a) Γ (b) Γ' (c) $3K_1$

Figure 6: Graph Γ' is obtained from Γ by a sequence of four edge contractions (denoted by blue), and $(\Gamma' \setminus \{e\})_{-v} = 3K_1$ is obtained by further elimination of a 1-loop e of Γ'.

Observe that we used a different method than in Example 24 to obtain the same result.

5.4 Elimination of parallel 0-edges

Lemma 29 (Elimination of parallel **0**-edges)**.** *Let Γ be a multigraph with binomial weights containing $k \geqslant 2$ 0-edges between vertices u and v (where we allow $u = v$), and let Γ' be obtained from Γ by removing $k-1$ 0-edges between those two vertices. Then*

$$\mathrm{gap}^*(\Gamma) = \mathrm{gap}^*(\Gamma').$$

Proof. Let Γ_1 be the multi-graph obtained from Γ by removing one edge between u and v. It is sufficient to prove $\mathrm{gap}(\zeta(\Gamma)) = \mathrm{gap}(\zeta(\Gamma_1))$. Indeed, if there are more than two edges u between v, we can repeatedly use this result to reduce Γ to Γ'.

Let $G = \zeta(\Gamma)$, and note that $\zeta(\Gamma_1) = G(\{y_1, y_2\})$, where y_1, y_2 arise from the edge which is removed from Γ to obtain Γ_1. The graph G contains $u - x_1 - x_2 - v - y_2 - y_1 - u$, where $x_1, x_2, y_1, y_2 \in V_{d=2}(G) \setminus \{u, v\}$ are distinct vertices. Note that we allow $u = v$ here.

To prove $\mathrm{st}_+(G) \leqslant \mathrm{st}_+(G(\{y_1, y_2\}))+2$, assume that $\mathscr{C}'$ is an ABC cover of $G(\{y_1, y_2\})$. Using Lemma 13 for $H = G(\{y_1, y_2\})$ and symmetry, we need to consider two cases.
Case 1: With respect o $\mathscr{C}'$, x_1 is of Type A and x_2 is of Type C. Then u is of Type C and v of Type A. In this case we have $\{x_1\} \vee \{u, x_2\}, \{v\} \vee N^\star_{\mathscr{C}'}[v] \in \mathscr{C}'$ and $x_2 \in N^\star_{\mathscr{C}'}[v]$. To construct a cover $\mathscr{C}$ of G from $\mathscr{C}'$ we replace $\{v\} \vee N^\star_{\mathscr{C}'}[v]$ by $\{v\} \vee (N^\star_{\mathscr{C}'}[v] \cup \{y_2\})$ and add $\{y_1\} \vee \{u, y_2\}$. Since $|V(\mathscr{C})| = |V(\mathscr{C}')| + 2$ the inequality holds.
Case 2: Both x_1 and x_2 are of Type B in $\mathscr{C}'$. In this case $u, v, x_1, x_2 \in V_1(\mathscr{C}')$. A cover $\mathscr{C}$ of G is obtained by adding the set-joins $\{u\} \vee \{y_1\}$,$\{y_1\} \vee \{y_2\}$,$\{y_2\} \vee \{v\}$ to $\mathscr{C}'$. Again we have $|V(\mathscr{C})| = |V(\mathscr{C}')| + 2$.

To prove $\mathrm{st}_+(G) \geqslant \mathrm{st}_+(G(\{y_1, y_2\})) + 2$, assume that $\mathscr{C}$ is an ABC cover of G. Using Lemma 13 for $H = G$ and symmetry, we need to consider two cases.
Case 1: With respect o $\mathscr{C}$, x_1 is of Type A and x_2 is of Type C. Since u is of Type C, y_1 must be of Type A, and y_2 must be of Type C. In particular, $\mathscr{C}$ contains $\{y_1\} \vee \{y_2, u\}$. Removing this set-join from $\mathscr{C}$ reduces the number of components by 2. A set join cover $\mathscr{C}'$ of $G(\{y_1, y_2\})$ is obtained by also removing y_2 from $N^\star_G[v]$.
Case 2: Both x_1 and x_2 are of Type B in $\mathscr{C}$. In this case $u, v, x_1, x_2 \in V_1(\mathscr{C})$. This implies that y_1 and y_2 are of Type B, hence $\{y_i\}$, $i = 1, 2$, are components in $\mathscr{C}$. Removing these two components, and the three set-joins that use them, gives us a set-join cover $\mathscr{C}'$ of $G(\{y_1, y_2\})$ with $|V(\mathscr{C}')| = |V(\mathscr{C})| - 2$. □

Example 30. Let $G = \mathrm{garlic}(k_1, \ldots, k_t)$ be the graph on $|V(G)| = 2 + \sum_{i\in[t]} k_i$ vertices, $t \geqslant 2$ and $k_i \geqslant 2$, obtained as a vertex identification of all leaves of a generalised star graph $\mathrm{gstar}(k_1 + 1, \ldots, k_t + 1)$ into a single vertex. We claim that

$$\mathrm{gap}(G) = \max\{|\{i\colon k_i \text{ odd}\}| - 2, 0\}.$$

If $t = 2$ then G is a cycle with $|V(G)| \geqslant 6$, so $\mathrm{gap}(G) = 0$. Assume that $t \geqslant 3$ and let $o = |\{i\colon k_i \text{ odd}\}|$ and $e = t - o$. The multigraph $\Gamma = \kappa(G)$ contains two vertices u and v, and o edges of weight 1 and e edges of weight 0 between u and v. By Lemma 29 we may assume that e is equal to 0 or 1. If $o = 0$, then $e = 1$ and $\zeta(\Gamma) = P_4$, so $\mathrm{gap}(G) = 0$. If $o > 0$, then using 1-edge contraction we obtain a multigraph Γ' with only one vertex v and and $o - 1$ loops of weight 1 and e loops of weight 0 on it, which by Lemma 26 has $\mathrm{gap}^*(\Gamma) = \mathrm{gap}^*(\Gamma')$. If $o = 1$, then $\mathrm{gap}^*(\Gamma') = 0$, so $\mathrm{gap}(G) = 0$ again. If $o > 1$, then the graph Γ'_{-v} consists of $o - 1$ vertices of degree 0, so by Lemma 27 $\mathrm{gap}^*(\Gamma') = o - 2$, and the claim follows.

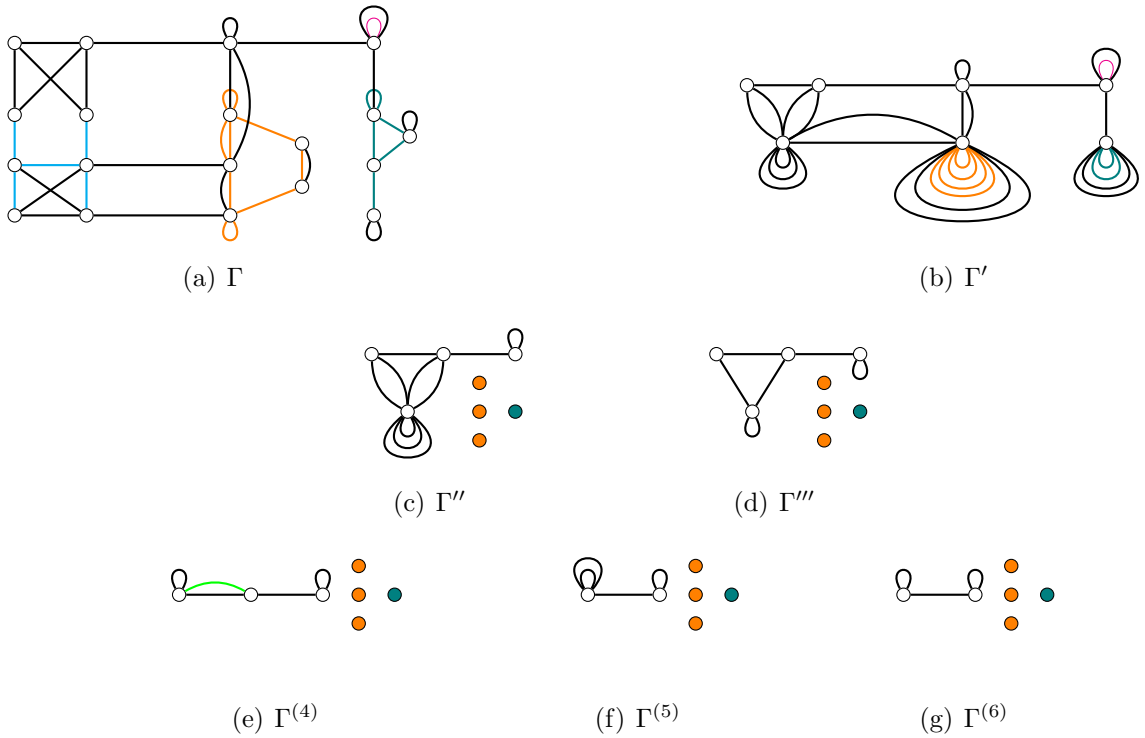
6 Reduction of a multigraph Γ to $\tau(\Gamma)$

We have found several operations that reduce the computation of $\mathrm{gap}^*(\Gamma)$ to a simplified multigraph. In Example 31 we give an example of a reduction of a multigraph Γ to a smaller graph with the same gap^*. While those operations can be performed in any order, we will later present an algorithm that makes this simplification unique and efficient.

Example 31. Consider the multigraph with binomial weights Γ in Figure 7(a). Instead of writing edge weights, we color all edges of weight 0 black and all edges of weight 1 by either blue, orange, green or pink.

To obtain $\mathrm{gap}^*(\Gamma)$ we follow reductions described in Section 5. First, we consecutively contract five blue 1-edges, four orange 1-edges and three green 1-edges. After all 12 contractions we obtain graph Γ', see Figure 7(b). Next we eliminate one orange, one green, and one pink loop of weight 1 and the resulting graph is Γ'', see Figure 7(c). After elimination of all parallel edges of weight 0 and all parallel loops of weight 0, the resulting graph Γ''' is shown in Figure 7(d). Observe that Γ''' is uniquely obtained from Γ by the described process and that it does not contain 1-edges, 1-loops or parallel 0-edges. Let us denote by Ψ the connected component of Γ''' on four vertices. Using Lemmas 26, 27 and 29 it follows that

$$\mathrm{gap}^*(\Gamma) = \mathrm{gap}^*(\Gamma') = \mathrm{gap}^*(\Gamma'') = \mathrm{gap}^*(\Gamma''') = \mathrm{gap}^*(\Psi) + 4\,\mathrm{gap}^*(K_1) = \mathrm{gap}^*(\Psi) + 4.$$



(a) Γ (b) Γ'

(c) Γ'' (d) Γ'''

(e) $\Gamma^{(4)}$ (f) $\Gamma^{(5)}$ (g) $\Gamma^{(6)}$

Figure 7: A weighted multigraph Γ and its reduction to smaller graphs, where all the blue, orange, green and pink edges and loops have weight 1 and all black edges and loops have weight 0. All seven graphs have the same gap^*.

To compute $\mathrm{gap}^*(\Gamma''')$ we can eliminate its vertex of degree 2. With this operation we obtain graph $\Gamma^{(4)}$ (Figure 7(e)), which contains an edge of weight 1. Now it is possible to continue the same reduction as described above. After 1-edge contraction in $\Gamma^{(4)}$ and elimination of a parallel 0-loop on $\Gamma^{(5)}$ we obtain graph $\Gamma^{(6)} = P_2^{\circ,\circ} \cup 4K_1$ and by

Lemmas 25, 26 and 29 we get

$$\mathrm{gap}^*(\Gamma) = \mathrm{gap}^*(\Psi) + 4 = \mathrm{gap}^*(P_2^{\circ,\circ}) + 4.$$

In the next subsection we will find a shorter way to reduce Γ to Ψ and compute $\mathrm{gap}^*(\Gamma)$.

6.1 Operation τ_1

Definition 32. Suppose that $\Gamma = (V(\Gamma), E(\Gamma), w_\Gamma)$ is a weighted multigraph with binomial weights. We denote by $\Gamma_{w=1}$ the multigraph obtained from Γ by removing all 0-edges. Let $\Gamma_1, \ldots, \Gamma_k$ be the connected components of $\Gamma_{w=1}$, ordered so that Γ_i, $i \in [\ell]$, are trees, and Γ_j, $j \in \{\ell+1, \ldots, k\}$, are not.

We denote

$$t_1(\Gamma) := \sum_{j=\ell+1}^{k} (|E(\Gamma_j)| - |V(\Gamma_j)|)$$

and define a weighted graph $\tau_1(\Gamma)$ by

- $V(\tau_1(\Gamma)) = \{w_1, \ldots, w_\ell\}$;
- for $i, j \in [\ell]$ there is a unique edge $\{w_i, w_j\} \in E(\tau_1(\Gamma))$ if and only if there exists $e = \{u, v\} \in E(\Gamma)$ of weight 0 with $u \in \Gamma_i$ and $v \in \Gamma_j$, where we allow $i = j$;
- all edges and loops in $\tau_1(\Gamma)$ are of weight 0.

If a multigraph Γ does not contain any 1-edges, parallel 0-edges or loops then $\tau_1(\Gamma) = \Gamma$ and $t_1(\Gamma) = 0$.

Example 33. For multigraph Γ in Figure 8(a) we show on Figure 8(b) multigraph $\Gamma_{w=1}$ and its components. Note that $k = 7$ and $\ell = 4$. Using Definition 32 we obtain $V(\tau_1(\Gamma)) = \{w_1, w_2, w_3, w_4\}$ and

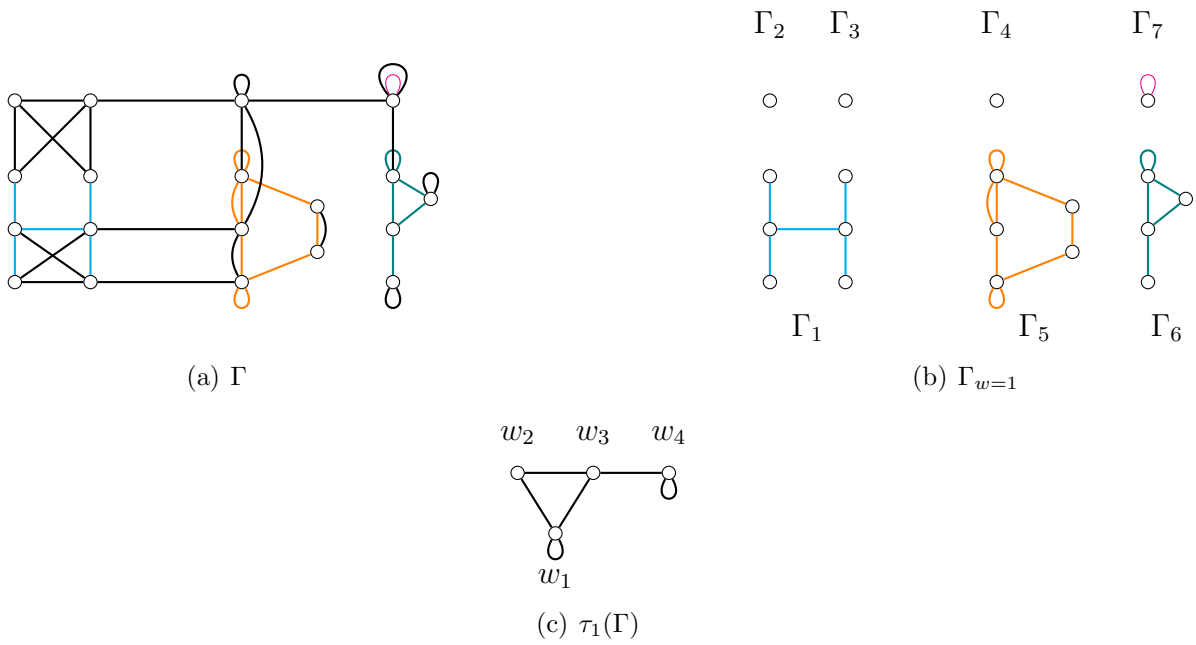
$$E(\tau_1(\Gamma)) = \{\{w_1, w_1\}, \{w_1, w_2\}, \{w_1, w_3\}, \{w_2, w_3\}, \{w_3, w_4\}, \{w_4, w_4\}\},$$

all with weights 0. This results in multigraph $\tau_1(\Gamma)$ in Figure 8(c). Note that $\tau_1(\Gamma)$ is equal to connected component Ψ of Γ''' in Figure 7(d). The other four vertices of degree 0 in Γ''' do not appear in $\tau_1(\Gamma)$, but their number is equal to

$$t_1(\Gamma) = (8-5) + (5-4) + (1-1) = 4.$$

Proposition 34. *If $\Gamma = (V(\Gamma), E(\Gamma), w_\Gamma)$ is a weighted multigraph with binomial weights, then*

$$\mathrm{gap}^*(\Gamma) = \mathrm{gap}^*(\tau_1(\Gamma)) + t_1(\Gamma).$$



(a) Γ (b) $\Gamma_{w=1}$ (c) $\tau_1(\Gamma)$

Figure 8: A weighted multigraph Γ, where all the blue, orange, green and pink edges and loops have weight 1 and all black edges and loops have weight 0, multigraph $\Gamma_{w=1}$ and $\tau_1(\Gamma)$.

Proof. The the result follows from repeated application of Lemmas 26, 27 and 29.

Let Γ' be a graph obtained from Γ using 1-edge contraction on $\sum_{i\in[k]}(|V(\Gamma_i)|-1)$ edges of weight 1 consecutively. In particular, $V(\Gamma')=\{w_1,\dots,w_k\}$, and $E(\Gamma')$ contains the following edges. Each w_i has $|E(\Gamma_i)|-|V(\Gamma_i)|+1$ loops of weight 1. In particular, vertices w_i, $i\in[\ell]$, have no loops of weight 1 in Γ' and all vertices w_j, $\ell<j\leqslant k$, have at least one loop of weight 1 in Γ'. For each $e\in E(\Gamma)$ of weight 0 incident to $u\in V(\Gamma_i)$ and $v\in V(\Gamma_j)$, there is an unique edge $e'\in E(\Gamma')$ incident to w_i and w_j with $w_{\Gamma'}(e')=0$. By Lemma 26 it follows that $\mathrm{gap}(\Gamma)=\mathrm{gap}(\Gamma')$.

Let Γ'' be a graph we obtain from Γ' by eliminating one loop of w_i with weight 1, for all $j=\ell+1,\dots,k$, together with w_j and all edges (and loops) of weight 0 incident to w_j. Graph Γ'' contains $t_1(\Gamma)$ copies of K_1 arising from remaining loops of weight 1 in Γ', and let us denote by Ψ the union of other components of Γ''. Using Lemma 27 it follows that $\mathrm{gap}^*(\Gamma')=\mathrm{gap}^*(\Gamma'')=\mathrm{gap}^*(\Psi)+t_1(\Gamma)$. Note that $V(\Psi)=\{w_1,\dots,w_\ell\}$. Finnaly, we eliminate all remaining parallel 0-edges of Ψ and we obtain the graph $\tau_1(\Gamma)$ and using Lemma 29 the proposition is proved. □

6.2 Operation τ_2

Recall that in a multigraph Γ we denote the set of all leaves by $V_{d=1}(\Gamma)$.

Definition 35. For every weighted multigraph $\Gamma=(V(\Gamma),E(\Gamma),w_\Gamma)$ with binomial

weights let

$$N_\Gamma[V_{d=1}(\Gamma)] = \bigcup_{v \in V_{d=1}(\Gamma)} N_\Gamma[v]$$

the set of all neighbours of leaves in Γ. We define

$$\tau_2(\Gamma) := \Gamma\left(V_{d=1}(\Gamma) \cup N_\Gamma[V_{d=1}(\Gamma)]\right)$$

to be the weighted multigraph obtained from Γ by eliminating all leaves and their neighbours, and

$$t_2(\Gamma) := |V_{d=1}(\Gamma)| - |N_\Gamma[V_{d=1}(\Gamma)]|.$$

To clarify the definition above, we offer some comments:

- $t_2(\Gamma) \geqslant 0$ since every leaf has a unique neighbour.
- If Γ has no leaves then $\tau_2(\Gamma) = \Gamma$ and $t_2(\Gamma) = 0$.
- The multigraph $\tau_2(\Gamma)$ may contain (new) leaves.
- The operation τ_2 eliminates any connected components of Γ that are isomorphic to P_2 with at most one looped vertex.

Proposition 36. *If a weighted multigraph $\Gamma = (V(\Gamma), E(\Gamma), w_\Gamma)$ with binomial weights contains no edges of weight 1, then*

$$\mathrm{gap}^*(\Gamma) = \mathrm{gap}^*(\tau_2(\Gamma)) + t_2(\Gamma).$$

Proof. Note, eliminating connected components of Γ that are isomorphic to P_2 with at most one looped vertex does not change gap^*. Hence, we can assume that no such components exist. In particular, $V_{d=1}(\Gamma) \cap N_\Gamma[V_{d=1}(\Gamma)] = \emptyset$.

Choose $u \in N_\Gamma[V_{d=1}(\Gamma)]$ and let $v \in V_{d=1}(\Gamma)$ be one of its neighbouring leaves. Since Γ contains no edges of weight 1 we have $\Gamma(v)_{-u} = \Gamma(\{u, v\})$. Moreover, $\Gamma(\{u, v\})$ is equal to $\Gamma\left((N_\Gamma[u] \cap V_{d=1}(\Gamma)) \cup \{u\}\right)$ together with $t_2(u) := |V_{d=1}(\Gamma) \cap N_\Gamma[u]| - 1$ vertices of degree 0, arising from leaves that are neighbours of u. By Lemma 23

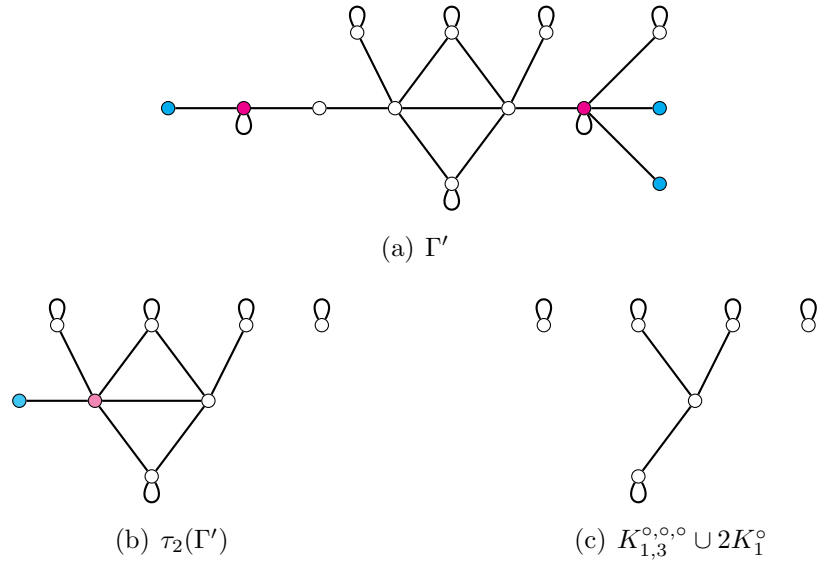
$$\mathrm{gap}^*(\Gamma) = \mathrm{gap}^*(\Gamma(v)_{-u}) = \mathrm{gap}^*(\Gamma\left((N_\Gamma[u] \cap V_{d=1}(\Gamma)) \cup \{u\}\right)) + t_2(u).$$

The result follows by repeating this process for all $u \in N_\Gamma[V_{d=1}(\Gamma)]$. □

Example 37. Consider the graph Γ' in Figure 9(a).

Observe that $\tau_2(\Gamma')$ has a vertex of degree 1 and $\tau_2(\tau_2(\Gamma')) = K_{1,3}^{\circ,\circ,\circ} \cup 2K_1^\circ$ is shown in Figure 10(b). Since $t_2(\Gamma) = 3 - 2 = 1$ and $t_2(\tau_2(\Gamma')) = 0$, Proposition 36 implies that

$$\mathrm{gap}^*(\Gamma') = \mathrm{gap}^*(\tau_2(\Gamma')) + 1 = \mathrm{gap}^*(K_{1,3}^{\circ,\circ,\circ} \cup 2K_1^\circ) + 1 = \mathrm{gap}^*(K_{1,3}^{\circ,\circ,\circ}) + 1.$$



(a) Γ'

(b) $\tau_2(\Gamma')$ (c) $K_{1,3}^{\circ,\circ,\circ} \cup 2K_1^{\circ}$

Figure 9: A multigraph Γ' and its reductions to smaller graphs using τ_2. In all three graphs all the edge weights are assumed to be equal to 0, the vertices from $V_{d=1}(\Gamma')$ and $V_{d=1}(\tau_2(\Gamma'))$ are colored blue, and the vertices from $N_\Gamma[V_{d=1}(\Gamma')]$ and $N_{\tau_2(\Gamma')}[V_{d=1}(\tau_2(\Gamma'))]$ are colored pink.

6.3 Operation τ_3

The multigraph $\Gamma = \kappa(G)$ contains no vertices of degree 2, but after operations τ_1 and τ_2 the multigraph $\tau_2(\tau_1(\Gamma))$ may contain such vertices. The multigraph obtained by eliminating all vertices of degree 2 in Γ and replacing the corresponding induced paths with edges of weight 0 or 1, using Lemma 25 consequently, is the multigraph $\kappa(\zeta(\Gamma))$. Furthermore, we count vertices of degree 0 which we obtain from parallel 1-loops and twin leaves using functions t_1 and t_2, but the multigraph $\tau_2(\tau_1(\Gamma))$ may still contain vertices of degree 0. We eliminate them now. Note that operation $\kappa(\zeta(\Gamma))$ also eliminates vertices with a single 0-loop and no other incident edges.

Recall that in a multigraph Γ we denote the set of all vertices of degree 0 by $V_{d=0}(\Gamma)$. We define

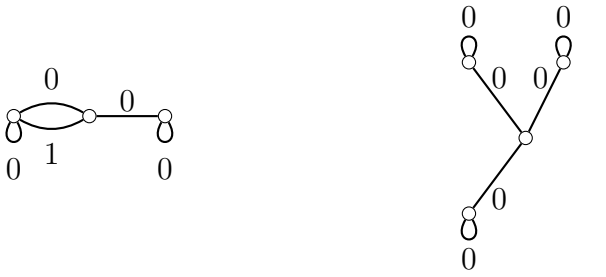
$$\tau_3(\Gamma) := \kappa(\zeta(\Gamma(V_{d=0}(\Gamma)))) \quad \text{and} \quad t_3(\Gamma) := |V_{d=0}(\Gamma)|.$$

If Γ does not contain any vertices of degree 0 or 2 then $\tau_3(\Gamma) = \Gamma$ and $t_3(\Gamma) = 0$. By equation (2) we have

$$\text{gap}^*(\Gamma) = \text{gap}^*(\tau_3(\Gamma)) + t_3(\Gamma). \tag{3}$$

Note that the multigraph $\tau_3(\Gamma)$ may contain edges of weight 1, even if Γ does not.

As an example, consider a graph $\tau_1(\Gamma)$ from Figure 8(c). The graph $\tau_3(\tau_1(\Gamma))$, shown in Figure 10(a) contains an edge of weight 1. On the other hand, observe that for $\Gamma'' = K_{1,3}^{\circ,\circ,\circ} \cup 2K_1^{\circ}$ in Figure 10(b) we have $\Gamma(V_{d=0}(\Gamma'')) = \Gamma''$, therefore $\tau_3(\Gamma) = \kappa(\zeta(\Gamma'')) = K_{1,3}^{\circ,\circ,\circ}$ and $t_3(\Gamma'') = 0$.



(a) $\tau_3(\tau_1(\Gamma))$ (b) $\tau_3(K_{1,3}^{\circ,\circ,\circ} \cup 2K_1^{\circ})$

Figure 10: Two examples of τ_3 operations.

6.4 An algorithm for computing $\tau(\Gamma)$

Using operations τ_1, τ_2 and τ_3 we can define an algorithm that reduces a multigraph $\Gamma = (V(\Gamma), E(\Gamma), w_\Gamma)$ with binomial weights to a multigraph that is invariant for those operations. That means that no further reductions developed in Section 5 are possible. This algorithm is described in Figure 11.

Definition 38. Let $\Gamma = (V(\Gamma), E(\Gamma), w_\Gamma)$ be a multigraph with binomial weights. The output of the algorithm described in Figure 11, given the input Γ, is defined as the pair $(\tau(\Gamma), t(\Gamma))$.

Note that $\tau(\Gamma)$ is a weighted simple graph with possible loops, with all edge weights equal to 0 and all vertices of degree at least 3.

Proposition 39. *Suppose that for a weighted multigraph with binomial weights Γ we have $\tau(\Gamma) = \Gamma$. Let H be the graph Γ viewed as a simple graph (i.e., we disregard the weights in Γ). If $H \in \mathcal{G}_{\not\boxtimes}$, then* $\mathrm{gap}^*(\Gamma) = \mathrm{gap}(H)$.

Proof. Note that $\kappa(H) = \Gamma$ since H contains only vertices of at least 3. So $\mathrm{gap}^*(\Gamma) = \mathrm{gap}(H)$ by equation (2). □

The next example shows that the above proposition is not true if $H \notin \mathcal{G}_{\not\boxtimes}$.

Example 40. Let $\Gamma = W_5$ be the wheel weighted multigraph with all edge weights equal to 0. Then $\tau(\Gamma) = \Gamma$ and the graph $H = W_5$ has $\mathrm{gap}(H) = 2$ by Example 1. On the other hand, we will show in Proposition 50 that $\mathrm{gap}^*(\Gamma) = 0$. To do that, we will need another tool, developed in Section 7.

We summarize the findings of this section in the following theorem.

Theorem 41. *Let $\Gamma = (V(\Gamma), E(\Gamma), w_\Gamma)$ be a weighted multigraph with binomial weights. Then*

$$\mathrm{gap}^*(\Gamma) = \mathrm{gap}^*(\tau(\Gamma)) + t(\Gamma).$$

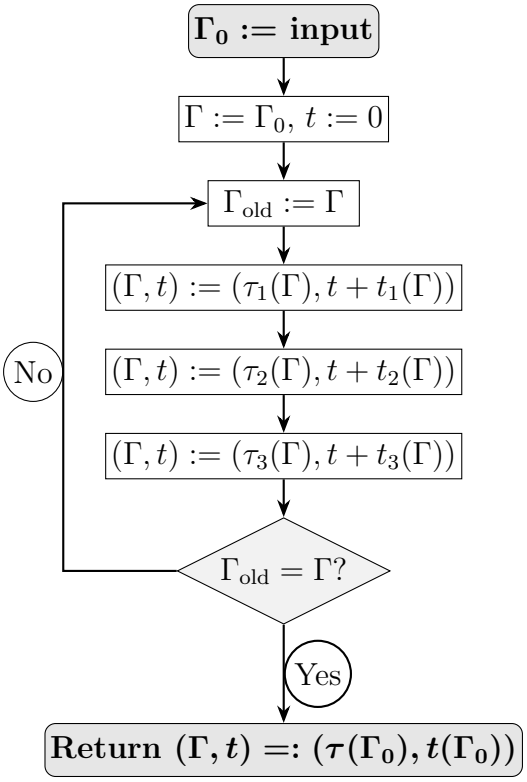


Figure 11: An algorithm for computing $\tau(\Gamma_0)$ and $t(\Gamma_0)$

Proof. The claim follows directly from Propositions 34, 36, and equation (3). Note that Proposition 36 can be applied, since in the algorithm we use operation τ_2 on the multigraph $\tau_1(\Gamma)$ which does not contain any 1-edges. □

Example 42. For a detailed example of the algorithm see Figure 12, where we present each step on the graph Γ from Figure 7(a). All black edges in graphs are of weight 0, and all colored edges are of weight 1. The input is the graph Γ shaded gray in the top left corner. For Γ the algorithm runs through three loops, and those are presented in the three columns. Since in the first two columns the top (Γ_{old}) and the bottom (Γ) graphs are not the same, the algorithm runs through another loop. In the third column the top and the bottom graph coincide ($\Gamma_{\text{old}} = \Gamma$), hence the algorithm stops. The output is the graph $\Gamma_2^{\circ,\circ} := \tau(\Gamma)$ (shaded gray in the bottom right), and $t(\Gamma) = 4$. By Theorem 41 it follows that

$$\text{gap}^*(\Gamma) = \text{gap}^*(\Gamma_2^{\circ,\circ}) + 4,$$

which gives a faster approach to Example 31. In Example 49 we will compute $\text{gap}^*(\Gamma_2^{\circ,\circ}) = 0$.

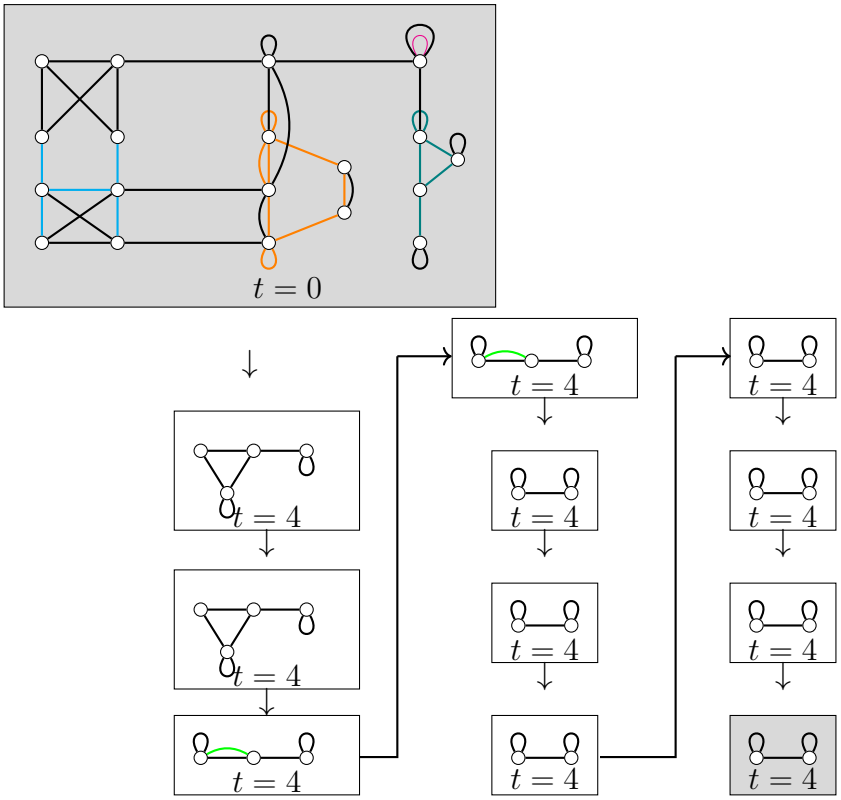


Figure 12: An illustration of an algorithm presented in Figure 11 on graph Γ from Figure 7(a). All black edges in graphs are of weight 0 and all colored edges are of weight 1.

7 Computing gap*(Γ) with vertex removal

The algorithm developed in Section 6 allows us to compute the gap in many but not all cases. In this section we develop a complementary method to compute $\mathrm{gap}^*(\Gamma)$ of a reduced multigraph $\tau(\Gamma)$ inductively by removing vertices. This method has high computational complexity but it allows us to compute the gap of for every graph $G \in \mathcal{G}_{\not\boxtimes}$.

To prove the main result of this section, Theorem 48, we need to return to simple graphs and their gaps.

7.1 Vertex removal for simple graphs

Lemma 43. *Let G be a simple graph with $|V(G)| \geqslant 2$, and $v \in V(G)$. Then*

$$\mathrm{gap}(G(v)) - 1 \leqslant \mathrm{gap}(G) \leqslant \mathrm{gap}(G(v)) + 1. \tag{4}$$

Moreover, if $G \in \mathcal{G}_{\not\boxtimes}$ and there exists an ABC *cover $\mathscr{C}$ of G such that v is of Type A with respect to $\mathscr{C}$, then the left inequality is an equality.*

Proof. Note that (4) is equivalent to

$$\mathrm{st}_+(G(v)) + 2 \geqslant \mathrm{st}_+(G) \geqslant \mathrm{st}_+(G(v)).$$

The inequality $\mathrm{st}_+(G) \geqslant \mathrm{st}_+(G(v))$ follows from the fact that $G(v)$ is an induced subgraph of G. If we add $\{v\} \vee N_G[v]$ to a set-join cover $\mathscr{C}'$ of $G(v)$, we get a set-join cover of G. As this introduces at most two new components, this implies $\mathrm{st}_+(G(v)) + 2 \geqslant \mathrm{st}_+(G)$.

Now, assume that $G \in \mathcal{G}_{\text{X}}$ and there exists an ABC cover $\mathscr{C}$ of G, where $\mathcal{K} \vee \{v\} \in \mathscr{C}$ with $|\mathcal{K}| \geqslant 2$. By Corollary 9 there exists a set-join cover $\mathscr{C}'$ of $G(v)$ with $|V(\mathscr{C}')| \leqslant |V(\mathscr{C})| - 2$. Hence, $\mathrm{st}_+(G(v)) \leqslant \mathrm{st}_+(G) - 2$, as desired. □

Remark 44. If $|V(G)| \geqslant 2$ and there exist $v, w \in V(G)$ and $k \in \mathbb{N}$ so that $\mathrm{gap}(G(v)) = k-1$ and $\mathrm{gap}(G(w)) = k + 1$, then $\mathrm{gap}(G) = k$.

Next, we express the gap of a graph G in terms of gaps of graphs $G(v)$, where v runs over a certain subset of $V(G)$. Using this lemma inductively, we can in principle compute $\mathrm{gap}(G)$ for any graph $G \in \mathcal{G}_{\text{X}}$. Since the gap of a graph is the sum of the gaps of its components, we may assume that the graph is connected.

Lemma 45. *Let $G \in \mathcal{G}_{\text{X}}$ be a connected graph with $|V(G)| \geqslant 2$. Let $\mathcal{S}$ be the set of all vertices in $v \in V(G)$ satisfying one of the following conditions:*

(A) $\deg(v) \neq 2$,

(B) $\deg(v) = 2$, *and there exists an induced walk* $w_1 - v \overset{r}{-} w_2$ *in* G, *such that* $w_i \in V_{d\neq 2}(G)$ *for* $i \in [2]$ *and* r *is even.*

Then

$$\mathrm{gap}(G) = \max\{0, \max_{v\in\mathcal{S}}\{\mathrm{gap}(G(v)) - 1\}\}. \tag{5}$$

Proof. We first prove the following claim: Let $G \in \mathcal{G}_{\text{X}}$ be a graph with $|V(G)| \geqslant 2$ that contains no vertices of degree 0. Then

$$\mathrm{gap}(G) = \max\{0, \max_{v\in V(G)}\{\mathrm{gap}(G(v)) - 1\}\}. \tag{6}$$

Note that $\mathrm{gap}(G) \geqslant \mathrm{gap}(G(v)) - 1$ for all $v \in V(G)$ by Lemma 43, so

$$\mathrm{gap}(G) \geqslant \max\{0, \max_{v\in V(G)}\{\mathrm{gap}(G(v)) - 1\}\}.$$

Let $\mathscr{C}$ be an ABC cover of G. If G contains a vertex v of Type A then by Lemma 43 $\mathrm{gap}(G) = \mathrm{gap}(G(v)) - 1$, and the claim follows. Otherwise, all vertices in G are of Type B, so all components of $\mathscr{C}$ are singletons. Since G has no vertices of degree 0, this means that $\mathrm{gap}(G) = 0$ and the claim follows again. (Note, the equation (6) already allows us to compute $\mathrm{gap}(G)$ inductively. We proceed to show that the set of vertices that need to be removed in the process can be reduced to the set $\mathcal{S}$.)

For $G = C_n$, $n \neq 4$, the set $\mathcal{S}$ is empty and $\mathrm{gap}(G) = 0$, so the equation (5) holds. From now on we assume that G is not an n-cycle. Using the above claim, it suffices to show that for every $u \in V(G) \setminus \mathcal{S}$, there exists $v \in \mathcal{S}$ so that $\mathrm{gap}(G(v)) = \mathrm{gap}(G(u))$.

Any $u \in V(G) \setminus \mathcal{S}$ satisfies $\deg(u) = 2$, and since G is not a cycle, u is a vertex of an induced walk $w_1 - u_1 - \ldots - u_{r+1} - w_2$ in G, $r \geqslant 0$, such that $\deg(w_i) \neq 2$ for $i \in [2]$,

$\deg(u_i) = 2$ for $i \in [r+1]$, and $u = u_{i_0}$ for some $i_0 \in [r+1]$. (We allow $w_1 = w_2$ or $\deg(w_i) = 1$.) Note $w_1, w_2 \in \mathcal{S}$, since they both satisfy the assumption (A). If r is even, then $u_1 \in \mathcal{S}$, since it satisfies the assumption (B).

We claim that we can choose $v \in \{w_1, w_2, u_1\}$ so that $\operatorname{gap}(G(u)) = \operatorname{gap}(G(v))$. Observe that $u \in V(G) \setminus \mathcal{S}$ implies that at least one of its neighbors is G becomes a leaf in $G(u)$. Hence, Proposition 2 (1) allows us to conclude:

$$\operatorname{gap}(G(u)) = \operatorname{gap}(G(\{w_1, u_1 \ldots, u_{i_0-1}, u\})) = \operatorname{gap}(G(w_1))$$

if i_0 is even,

$$\operatorname{gap}(G(u)) = \operatorname{gap}(G(\{u, u_{i_0+1} \ldots, u_{r+1}, w_2\})) = \operatorname{gap}(G(w_2))$$

if i_0 is odd and r is odd, and

$$\operatorname{gap}(G(u)) = \operatorname{gap}(G(\{u_1, \ldots, u_{i_0-1}, u\})) = \operatorname{gap}(G(u_1)).$$

if i_0 is odd and r is even. □

7.2 Vertex removal for multigraphs

In this section, we use Lemma 45 to prove a similar result for weighted multigraphs.

Lemma 46. *Let $\Gamma = (V(\Gamma), E(\Gamma), w_\Gamma)$ be an nonempty weighted multigraph with all weights equal to 0 and $v \in V(\Gamma)$. Then*

$$\operatorname{gap}^*(\Gamma) \geqslant \operatorname{gap}^*(\Gamma(v)) - 1.$$

Proof. Since $\operatorname{gap}^*(\Gamma_1 \cup \Gamma_2) = \operatorname{gap}^*(\Gamma_1) + \operatorname{gap}^*(\Gamma_2)$ we may assume without loss of generality that Γ is connected. If Γ contains only one vertex, then $\operatorname{gap}^*(\Gamma(v)) = 0$ and the claim is true. So suppose $|V(\Gamma)| \geqslant 2$. Since Γ contains only edges of weight 0, we have $\Gamma(v) = \Gamma_{-v}$ by Remark 21, so

$$\operatorname{gap}^*(\Gamma) = \operatorname{gap}(\zeta(\Gamma)) \geqslant \operatorname{gap}(\zeta(\Gamma)(v)) - 1 = \operatorname{gap}^*(\Gamma_{-v}) - 1 = \operatorname{gap}^*(\Gamma(v)) - 1$$

by Lemmas 43 and 22. □

Let Γ be a weighted multigraph. A set $\mathcal{I} \subseteq V(\Gamma)$ is called *independent*, if the subgraph of Γ induced on $\mathcal{I}$ contains no edges. The cardinality of the biggest independent set in Γ is called *independence number* and denoted by $\alpha(\Gamma)$. The independence number $\alpha(\Gamma)$ can be used to find a lower bound on the $\operatorname{gap}^*(\Gamma)$.

Corollary 47. *Let Γ be a weighted multigraph with all weights equal to 0. For any vertices $v_1, \ldots, v_k \in V(\Gamma)$ we have*

$$\operatorname{gap}^*(\Gamma) \geqslant \operatorname{gap}^*(\Gamma(\{v_1, \ldots, v_k\})) - k.$$

In particular,

$$\operatorname{gap}^*(\Gamma) \geqslant 2\alpha(\Gamma) - |V(\Gamma)|. \tag{7}$$

Proof. The first claim follows from Lemma 46 by induction on k. Let I be maximal independent set in Γ, let $k = |V(\Gamma)| - \alpha(\Gamma)$ and $V(\Gamma) \setminus I = \{v_1, \ldots, v_k\}$. Then

$$\mathrm{gap}^*(\Gamma) \geqslant \mathrm{gap}^*(\Gamma(\{v_1, \ldots, v_k\})) - k = \alpha(\Gamma) - (|V(\Gamma)| - \alpha(\Gamma)) = 2\alpha(\Gamma) - |V(\Gamma)|,$$

since $\Gamma(\{v_1, \ldots, v_k\})$ contains $\alpha(\Gamma)$ vertices and no edges. □

The above estimate of gap^* in terms of independence number is sharp for some families of graphs (see e.g. Example 53) while in general $\mathrm{gap}^*(\Gamma) - 2\alpha(\Gamma) + |V(\Gamma)|$ can be arbitrarily large (see e.g. Example 54).

Let $G \in \mathcal{G}_{\mathbb{X}}$, and $\Gamma = \tau(\kappa(G))$. (Γ is a simple multigraph with all weights equal to 0 and each vertex of degree at least 3.) By Theorem 41 we have:

$$\mathrm{gap}(G) = \mathrm{gap}^*(\Gamma) + t(\kappa(G)).$$

To compute $\mathrm{gap}^*(\Gamma)$ we may assume that Γ is a connected graph. In our examples, we will apply the following theorem only to multigraphs of the form $\Gamma = \tau(\kappa(G))$, but we state and prove it more generally.

Theorem 48. *Let $\Gamma = (V(\Gamma), E(\Gamma), w_\Gamma)$ be a weighted multigraph with all weights equal to 0, $|V(\Gamma)| \geqslant 2$, and no vertices of degree 0. Then*

$$\mathrm{gap}^*(\Gamma) = \max\{0, \max_{v \in V(\Gamma)} \{\mathrm{gap}^*(\Gamma(v)) - 1\}\}.$$

Proof. Suppose first that Γ is connected. Then $G := \zeta(\Gamma)$ is connected and $V(\Gamma) \subseteq V(G)$. By definition we have $\mathrm{gap}^*(\Gamma) = \mathrm{gap}(G)$. By Lemma 45 we have

$$\mathrm{gap}(G) = \max\{0, \max_{v \in \mathcal{S}} \{\mathrm{gap}(G(v)) - 1\}\},$$

where $\mathcal{S}$ is as defined in the lemma. Since all edges in Γ are of weight 0, we can describe $G(v)$ for $v \in V(\Gamma)$ in terms of $\zeta(\Gamma(v))$ as follows: the graph $G(v)$ is isomorphic to the graph $\zeta(\Gamma(v))$ with added pendent P_2 for each nonloop edge in Γ incident to v and added disjoint P_2 for each loop on v. By Proposition 2 (1) it follows that

$$\mathrm{gap}^*(\Gamma(v)) = \mathrm{gap}(\zeta(\Gamma(v))) = \mathrm{gap}(G(v)).$$

Let v be any vertex in $\mathcal{S}$. If $\deg(v) \neq 2$, then $v \in V(\Gamma)$. If $\deg(v) = 2$, then there exists an induced walk $w_1 - v \overset{r}{-} w_2$ in G with $\deg(w_i) \neq 2$ for $i \in [2]$, and $r \geqslant 0$ even. Let $v \overset{r}{-} w_2 = v - u_1 - \ldots - u_r - w_2$. By construction of G we have $r = 1$ or $u_2 \in V(\Gamma)$. Since r is even, it follows that $u_2 \in V(\Gamma)$. Now

$$\mathrm{gap}(G(v)) = \mathrm{gap}(G(\{v, u_1, u_2\})) = \mathrm{gap}(G(u_2)) = \mathrm{gap}^*(\Gamma(u_2)),$$

which implies that

$$\max_{v \in V(\Gamma)} \{\mathrm{gap}^*(\Gamma(v))\} = \max_{v \in S} \{\mathrm{gap}(G(v))\}$$

and the result follows.

Suppose now that $\Gamma = \Gamma_1 \cup \ldots \cup \Gamma_t \cup sK_1^\circ$, where the graphs $\Gamma_1, \ldots, \Gamma_t$ are connected and have at least 2 vertices. Then

$$\mathrm{gap}^*(\Gamma_i) = \max\{0, \max_{v \in V(\Gamma_i)} \{\mathrm{gap}^*(\Gamma_i(v)) - 1\}\}$$

for every $i \in [t]$, as proved above.

If $\mathrm{gap}^*(\Gamma_i) = 0$ for all $i \in [t]$, then $\mathrm{gap}^*(\Gamma(v)) \leqslant 1$ for all $v \in V(\Gamma)$ and the claim follows. Assume now there exists a j such that $\mathrm{gap}^*(\Gamma_j) = \max_{v \in V(\Gamma_j)}\{\mathrm{gap}^*(\Gamma_j(v)) - 1\}$. Then there exists $u \in \Gamma_j$ such that $\mathrm{gap}^*(\Gamma_j) = \mathrm{gap}^*(\Gamma_j(u)) - 1$. It follows that

$$\mathrm{gap}^*(\Gamma(u)) = \sum_{i=1}^{t} \mathrm{gap}^*(\Gamma_i) + 1 = \mathrm{gap}^*(\Gamma) + 1$$

and so $\mathrm{gap}^*(\Gamma) \leqslant \max_{v \in V(\Gamma)}\{\mathrm{gap}^*(\Gamma(v)) - 1\}$. On the other hand, $\mathrm{gap}^*(\Gamma) \geqslant \mathrm{gap}^*(\Gamma(v)) - 1$ for all $v \in V(\Gamma)$, by Lemma 46, so

$$\mathrm{gap}^*(\Gamma) = \max_{v \in V(\Gamma)} \{\mathrm{gap}^*(\Gamma(v)) - 1\},$$

and the claim follows. □

Example 49. To complete the computation of $\mathrm{gap}^*(\Gamma)$ considered in Example 42 we need to compute $\mathrm{gap}^*(\Gamma_0^{\circ,\circ})$, where $\Gamma_0^{\circ,\circ}$ is the output graph in Figure 12, i.e. a weighted multigraph obtained from $P_2^{\circ,\circ}$ (as in Figure 2(c)) by assigning weight 0 to all three edges. Let $V(P_2^{\circ,\circ}) = V(\Gamma_2^{\circ,\circ}) = \{u, v\}$.

Since $\Gamma_2^{\circ,\circ}(u)$ is isomorphic to a vertex with a loop of weight 0, we have $\mathrm{gap}^*(\Gamma_2^{\circ,\circ}(u)) = \mathrm{gap}(C_3) = 0$. Using symmetry and Theorem 48 it follows that

$$\mathrm{gap}^*(\Gamma_2^{\circ,\circ}) = \max\{0, -1, -1\} = 0.$$

Therefore, by Example 42 we obtain

$$\mathrm{gap}^*(\Gamma) = \mathrm{gap}^*(\Gamma_2^{\circ,\circ}) + 4 = 4.$$

Since $P_2^{\circ,\circ} \notin \mathcal{G}_{\mathbb{X}}$ (when viewed as a simple graph with loops), Lemma 45 cannot be used to compute its gap. However, the vertices u and v are twins, so by Proposition 2(2) $\mathrm{gap}(P_2^{\circ,\circ}) = \mathrm{gap}(K_1^\circ) + 1 = 1$. As $\mathrm{gap}^*(\Gamma_2^{\circ,\circ}) \neq \mathrm{gap}(P_2^{\circ,\circ})$, this example highlights the difference between the two parameters.

8 Applications

With the results of the previous two sections, we can compute gap^* of any multigraph Γ. We conclude the paper with a few applications.

We first compute gap^* for small multigraphs and then for some concrete examples.

Proposition 50. *Let* $\Gamma = (V(\Gamma), E(\Gamma), w_\Gamma)$ *be a weighted multigraph with binomial weights. If every component of* $\tau(\Gamma)$ *has at most six vertices then*

$$\mathrm{gap}^*(\Gamma) = t(\Gamma).$$

In particular, if $\tau(\Gamma) = \Gamma$ *and* $|V(\Gamma)| \leqslant 6$*, then* $\mathrm{gap}^*(\Gamma) = 0$.

Proof. Using Theorem 41 we may assume that $\tau(\Gamma) = \Gamma$ and $|V(\Gamma)| \leqslant 6$. In particular, every vertex in Γ has degree at least 3, which implies that for any pair of vertices $u, v \in V(\Gamma)$, all vertices in $\Gamma(\{u, v\})$ have degree at least 1. This allows us to apply Theorem 48 to Γ, $\Gamma(u)$, and $\Gamma(\{u, v\})$. Suppose that $|V(\Gamma)| = 6$, and let $u, v, w \in V(\Gamma)$ be three distinct vertices. Since $|V(\Gamma(\{u, v, w\}))| = 3$, we have $\mathrm{gap}^*(\Gamma(\{u, v, w\})) \leqslant 3$. By Theorem 48 $\mathrm{gap}^*(\Gamma(\{u, v\})) \leqslant 2$, $\mathrm{gap}^*(\Gamma(u)) \leqslant 1$, and $\mathrm{gap}^*(\Gamma) = 0$. Similarly, if $|V(\Gamma)| = 5$ ($|V(\Gamma)| = 4$), then $\mathrm{gap}^*(\Gamma(\{u, v, w\})) \leqslant 2$ ($\mathrm{gap}^*(\Gamma(\{u, v, w\})) \leqslant 1$), again implying $\mathrm{gap}^*(\Gamma) = 0$. Finally, assuming $|V(\Gamma)| = 3$ or $|V(\Gamma)| = 2$, we have $\mathrm{gap}^*(\Gamma(u)) \leqslant 1$ and the conclusion follows. □

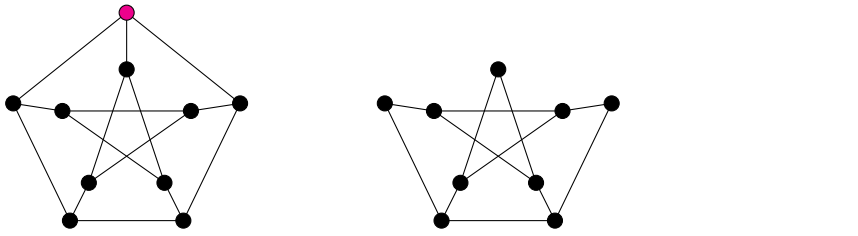
Application of Theorem 48 simplifies for graphs with a large degree of symmetry. This is illustrated in the following examples.

Example 51. Let $P_{5,2}$ be the Petersen graph. Note that the multigraph $\Gamma_{5,2} = \kappa(P_{5,2})$ is a weighted Petersen graph with all edge weights 0. Since $\tau(\Gamma_{5,2}) = \Gamma_{5,2}$, we proceed to compute $\mathrm{gap}^*(\Gamma_{5,2})$ using Theorem 48.

Using symmetries of $\Gamma_{5,2}$, we observe that removing any vertex v from $\Gamma_{5,2}$ results in a multigraph isomorphic to Γ' in Figure 13(b). From $\tau(\Gamma') = \emptyset$ and $t(\Gamma') = 0$ we get $\mathrm{gap}^*(\Gamma') = 0$ by Theorem 41. Therefore,

$$\mathrm{gap}^*(\Gamma_{5,2}) = \max\{0, \mathrm{gap}^*(\Gamma') - 1\} = 0$$

by Theorem 48, and so $\mathrm{gap}(P_{5,2}) = 0$.



(a) $\Gamma_{5,2}$ (b) Γ' (c) $\tau(\Gamma')$

Figure 13: Graph $\Gamma_{5,2} = \kappa(P_{5,2})$ is a weighted multigraph with all black edges of weight 0. Γ' is obtained by removing a pink vertex from Γ, and $\tau(\Gamma')$ is an empty graph.

Example 52. Let $n \geqslant 2$ and Γ_n be a complete graph K_n (without loops) and with all edge weights equal to 0. We prove that $\mathrm{gap}^*(\Gamma_n) = 0$ by induction on n.

If $n \leqslant 3$ we have $\mathrm{gap}^*(\Gamma_2) = \mathrm{gap}(P_4) = 0$ and $\mathrm{gap}^*(\Gamma_3) = \mathrm{gap}(C_9) = 0$. If $n \geqslant 4$ then $\tau(\Gamma_n) = \Gamma_n$ and by induction hypothesis $\mathrm{gap}^*(\Gamma_{n-1}) = 0$. For any $v \in V(\Gamma_n)$ we have $\Gamma_n(v) = \Gamma_{n-1}$, so that by Theorem 48

$$\mathrm{gap}^*(\Gamma_n) = \max\{0, \mathrm{gap}^*(\Gamma_{n-1}) - 1\} = 0.$$

The value of $\mathrm{gap}(K_n)$ can be found in [10, Lemma 4.11.].

We show in the next example that Proposition 50 is no longer true if $\tau(\Gamma)$ has a connected component with more than six vertices.

Example 53. Let $m, n \geqslant 1$, $m \geqslant n$, and $\Gamma_{m,n}$ be a complete bipartite graph $K_{m,n}$ with all edge weights equal to 0. We claim: $\mathrm{gap}^*(\Gamma_{m,n}) = m - n$, and prove this claim by induction on $m + n$.

Note, $\tau_2(\Gamma_{m,1}) = \emptyset$ and $t_2(\Gamma_{m,1}) = m - 1$, so $\mathrm{gap}^*(\Gamma_{m,1}) = m - 1$. In particular, $\mathrm{gap}^*(\Gamma_{1,1}) = \mathrm{gap}^*(\Gamma_2) = 0$.

Assume $n > 1$. The induction hypothesis asserts $\mathrm{gap}^*(\Gamma_{(m-1),n)}) = |m - 1 - n|$ and $\mathrm{gap}^*(\Gamma_{m,(n-1)}) = m - (n - 1)$. Let $V(\Gamma_{m,n}) = M \cup N$ with $|M| = m$, $|N| = n$ and $E(\Gamma_{m,n}[M]) = E(\Gamma_{m,n}[N]) = \emptyset$. If $m = n$, then $\Gamma_{m,n}(v) \cong \Gamma_{m,(m-1)}$ for any $v \in V(\Gamma_{m,n})$. Applying Theorem 48 we conclude:

$$\mathrm{gap}^*(\Gamma_{m,n}) = \max\{0, \mathrm{gap}^*(\Gamma_{m,(m-1)}) - 1\} = \max\{0, 1 - 1\} = 0 = m - n.$$

If $m > n$, then $\Gamma_{m,n}(v) \cong \Gamma_{(m-1),n}$ for any $v \in M$ and $\Gamma_{m,n}(v) \cong \Gamma_{m,(n-1)}$ for any $v \in N$. An application of Theorem 48 now completes the proof:

$$\begin{aligned}\mathrm{gap}^*(\Gamma_{m,n}) &= \max\{0, \mathrm{gap}^*(\Gamma_{(m-1),n}) - 1, \mathrm{gap}^*(\Gamma_{m,(n-1)}) - 1\} \\ &= \max\{0, (m - 1 - n) - 1, (m - (n - 1)) - 1\} = m - n.\end{aligned}$$

In particular, $|V(\Gamma_{4,3})| = 7$ and $\mathrm{gap}^*(\Gamma_{4,3}) = 1$.

Note that $\mathrm{gap}(K_{m,n}) = m + n - 2$ by Proposition 2(2).

Example 54. Let $m \geqslant 5$ be an integer, let $\Gamma_{m,3}$ be as defined in Example 53, and Γ°_{m-4} a weighted complete multigraph with all the loops on $m - 4$ vertices, and all weights equal to 0. A proof by induction on m (similar as in Example 52) establishes $\mathrm{gap}^*(\Gamma^\circ_{m-4}) = 0$.

Choose a vertex $u \in V(\Gamma_{m,3})$ of degree m, and a vertex $v \in V(\Gamma^\circ_{m-4})$, and define Γ to be a weighted multigraph on $2m - 1$ vertices with $V(\Gamma) = V(\Gamma_{m,3}) \cup V(\Gamma^\circ_{m-4})$, $E(\Gamma) = E(\Gamma_{m,3}) \cup E(\Gamma^\circ_{m-4}) \cup \{\{u, v\}\}$, and all weights equal to 0. The biggest independent set of Γ is the set of all vertices in $\Gamma_{m,3}$ of degree 3. This can be seen by noting that a vertex with a loop cannot be a member of an independent set, and that in a complete bipartite multigraph, any independent set is a subset of one of the parts. It follows that $\alpha(\Gamma) = m$ and thus $2\alpha(\Gamma) - |V(\Gamma)| = 2m - (2m - 1) = 1$. Furthermore, $\Gamma(u) \cong \Gamma_{m,2} \cup \Gamma^\circ_{m-4}$, so

$$\mathrm{gap}^*(\Gamma(u)) = \mathrm{gap}^*(\Gamma_{m,2}) + \mathrm{gap}^*(\Gamma^\circ_{m-4}) = m - 2$$

by Example 53. Hence, by Lemma 46 $\mathrm{gap}^*(\Gamma) \geqslant \mathrm{gap}^*(\Gamma(u)) - 1 = m - 3$, and

$$\mathrm{gap}^*(\Gamma) - (2\alpha(\Gamma) - |V(\Gamma)|) \geqslant m - 4.$$

as desired. This example shows that there can be an arbitrarily large difference between the left and the right-hand side in the inequality (7).

Next example is an application of Corollary 47 and offers a planar multigraph with all vertices of degree at least 3, all weights 0, and an arbitrarily large gap*.

Example 55. Let Γ be a weighed graph with $3k+2$ vertices

$$V(\Gamma) = \{x, y\} \cup \{z_{1,j}, z_{2,j}, z_{3,j} \colon j \in [k]\},$$

and $7k$ edges

$$E(\Gamma) = \{\{x,z_{i,j}\} \colon i \in [3], j \in [k]\} \cup \{\{y, z_{i,j}\} \colon i \in \{1,3\}, j \in [k]\}$$
$$\cup\ \{\{z_{1,j}, z_{2,j}\}, \{z_{2,j}, z_{3,j}\} \colon j \in [k]\},$$

all with weight 0, see Figure 14.

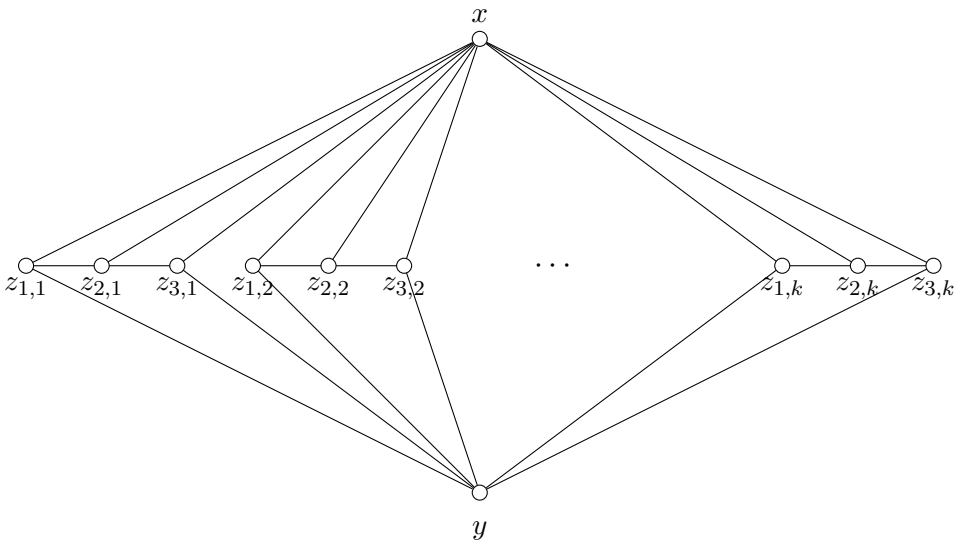


Figure 14: Planar graph Γ, $|V(\Gamma)| = 3k+2$, with $\mathrm{gap}^*(\Gamma) \geqslant k-2$.

Notice that $\{z_{1,1}, z_{3,1}, \ldots, z_{1,k}, z_{3,k}\}$ is an independent set of cardinality $2k$, so $\alpha(\Gamma) \geqslant 2k$ and thus $\mathrm{gap}^*(\Gamma) \geqslant 4k - (3k+2) = k-2$ by Corollary 47. Using techniques obtained in Sections 6 and 7 it is possible to show that $\mathrm{gap}^*(\Gamma) = k-2$.

In Sections 4–7 we developed methods that can be applied to graphs in $\mathcal{G}_{\boxtimes}$. However, in some special cases a subgraph of a graph $G \notin \mathcal{G}_{\boxtimes}$ that is forbidden in $\mathcal{G}_{\boxtimes}$ can be removed while maintaining control over the gap. In particular, we may be able to reduce the computation of $\mathrm{gap}(G)$ to a graph in $\mathcal{G}_{\boxtimes}$. This is illustrated in the following example, where pendent 4-cycles (cycles of the form $w \overset{3}{\text{—}} w$) are removed using Proposition 2(2).

A subgraph of G of the form $w \overset{k-1}{\text{—}} w$, $k \geqslant 3$, for some $w \in V_{d\geqslant 3}(G)$ is a pendent cycle C_k at w. Let G be a graph with s pendent 4-cycles G_i, $i = 1, \ldots, s$, at a vertex

$w \in V(G)$. Note that each 4-cycle G_i contains two twin vertices, both in $N_G[w]$, therefore by Proposition 2 (2), we have $\mathrm{gap}(G) = \mathrm{gap}(G_1)+s$, where G_1 is obtained by removing one of the twins in each pair. Note tha G_1 contains s pendent paths P_2 and by Proposition 2 (1) it follows that

$$\mathrm{gap}(G) = \mathrm{gap}(G') + s, \tag{8}$$

where G' is obtained from G_1 by removing all pendent paths P_2. If $G' \in \mathcal{G}_{\boxtimes}$, then we can compute $\mathrm{gap}(G)$ using the methods developed in this paper. We offer such an example below.

Example 56. Let $\mathrm{clover}(f, e, o)$, $f + e + o \geqslant 2$, be the set of graphs isomorphic to a vertex w with f pendent 4-cycles, e pendent cycles with an even number of vertices not equal to 4 and o pendent cycles with an odd number of vertices. Moreover, let $\mathrm{clover}_{\boxtimes}(e, o) := \mathrm{clover}(0, e, o) = \mathrm{clover}(f, e, o) \cap \mathcal{G}_{\boxtimes}$ denote the set of clovers without four cycles.

For every $G \in \mathrm{clover}(f, e, o)$ let G' be a graph obtained by removing all f four cycles of G and therefore $\mathrm{gap}(G) = f + \mathrm{gap}(G')$ by (8).

If $e + o \geqslant 2$ then $G' \in \mathrm{clover}_{\boxtimes}(e, o)$. The multigraph $\Gamma = \kappa(G')$ contains only one vertex v with e loops of weight 1 and o loops of weight 0. If $e \geqslant 1$, then $\tau_1(\Gamma)$ is empty and $t_1(\Gamma) = e - 1$, so

$$\mathrm{gap}(G') = \mathrm{gap}^*(\Gamma) = e - 1.$$

If $e = 0$, then $\tau_1(\Gamma)$ is a vertex with a 0-loop and $t_1(\Gamma) = 0$, so $\tau(\Gamma)$ is empty and $t(\Gamma) = 0$, thus

$$\mathrm{gap}(G') = \mathrm{gap}^*(\Gamma) = 0.$$

If $e + o = 1$ then G' is a cycle and $\mathrm{gap}(G') = 0$. Finally, if $e + o = 0$, then $G' \cong K_1$ and $\mathrm{gap}(G') = 1$.

Therefore

$$\mathrm{gap}(G) = \begin{cases} f + \max\{0, e - 1\}; & \text{if } e + o \geqslant 1, \\ f + 1; & \text{if } e + o = 0. \end{cases} \tag{9}$$

for every $G \in \mathrm{clover}(f, e, o)$. Note that (9) extends to the case $f + e + o = 1$. In this case, G is isomorphic to a cycle and so $\mathrm{gap}(G)$ coincides with (9).

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