

Enumerative Chromatic Choosability

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Submitted: May 19, 2025; Accepted: Aug 5, 2026; Published: Oct 9, 2026

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Abstract

Chromatic-choosability is a notion of fundamental importance in list coloring. A graph is *chromatic-choosable* when its chromatic number is equal to its list chromatic number. In 1990, Kostochka and Sidorenko introduced the list color function of a graph G , denoted $P_\ell(G, m)$, which is the list analogue of the chromatic polynomial of G , $P(G, m)$. It is known that for any graph G there is a positive integer k such that $P_\ell(G, m) = P(G, m)$ whenever $m \geq k$. In this paper, we study enumerative chromatic-choosability. A graph G is *enumeratively chromatic-choosable* when $P_\ell(G, m) = P(G, m)$ whenever $m \in \mathbb{N}$. We completely determine the graphs of chromatic number two that are enumeratively chromatic-choosable. We construct examples of graphs that are chromatic-choosable but fail to be enumeratively chromatic-choosable, and finally, we explore a conjecture as to whether for every graph G , there is a $p \in \mathbb{N}$ such that the join of G and K_p is enumeratively chromatic-choosable. The techniques we use to prove results are diverse and include probabilistic ideas and ideas from DP (or correspondence)-coloring.

Keywords. list coloring, chromatic polynomial, list color function, list color function threshold, enumerative chromatic-choosability

Mathematics Subject Classifications: 05C15, 05C30

1 Introduction

In this paper, all graphs are nonempty, finite, simple graphs. Generally speaking, we follow West [23] for terminology and notation. The set of natural numbers is $\mathbb{N} = \{1, 2, 3, \dots\}$. For $m \in \mathbb{N}$, we write $[m]$ for the set $\{1, \dots, m\}$. The AM-GM inequality refers to the Arithmetic Mean-Geometric Mean inequality. We write $K_{l,n}$ for complete bipartite graphs with partite sets of size l and n . For a path P , we write $P = v_1, v_2, \dots, v_n$ to mean P is the path with vertex set $\{v_1, \dots, v_n\}$ and edge set $\{v_i v_{i+1} : 1 \leq i < n\}$. If G and H are vertex disjoint graphs, we define the *join* of G and H denoted $G \vee H$ for the join of G and H to be the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$.

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1.1 List Coloring and Chromatic-Choosability

For classical vertex coloring of graphs, we wish to color the vertices of a graph G with up to m colors from $[m]$ so that adjacent vertices receive different colors, a so-called *proper m -coloring*. The *chromatic number* of a graph is a very well studied graph invariant. Denoted $\chi(G)$, the chromatic number of graph G is the smallest m such that G has a proper m -coloring. List coloring is a well-known variation on classical vertex coloring which was introduced independently by Vizing [22] and Erdős, Rubin, and Taylor [11] in the 1970s. For list coloring, we associate a *list assignment* L with a graph G such that each vertex $v \in V(G)$ is assigned a list of available colors $L(v)$ (we say L is a list assignment for G). We say G is *L -colorable* if there is a proper coloring f of G such that $f(v) \in L(v)$ for each $v \in V(G)$ (we refer to f as a *proper L -coloring* of G). A list assignment L for G is called a *m -assignment* if $|L(v)| = m$ for each $v \in V(G)$. Graph G is said to be *m -choosable* if it is L -colorable whenever L is a m -assignment for G . The *list chromatic number* of a graph G , denoted $\chi_\ell(G)$, is the smallest m for which G is m -choosable. It is easy to show that for any graph G , $\chi(G) \leq \chi_\ell(G)$. Moreover, the gap between the chromatic number and list chromatic number of a graph can be arbitrarily large since $\chi_\ell(K_{n,t}) = n + 1$ whenever $t \geq n^n$ (see e.g., [18] for further details).

A graph G is called *chromatic-choosable* if $\chi_\ell(G) = \chi(G)$ [20]. Determining whether a graph is chromatic-choosable is, in general, a challenging problem. Perhaps the most well known conjecture involving list coloring is about chromatic-choosability. Indeed, the famous Edge List Coloring Conjecture states that every line graph of a loopless multigraph is chromatic-choosable (see [12]).

It is shown in [20] that for any graph G , there exists an $N \in \mathbb{N}$ such that $K_p \vee G$ is chromatic-choosable whenever $p \geq N$. In 2015, Noel, Reed, and Wu [19] generalized this by famously proving the following.

Theorem 1 ([19]). *If G is a graph satisfying $|V(G)| \leq 2\chi(G) + 1$, then G is chromatic-choosable.*

1.2 Counting Proper Colorings and List Colorings

In 1912, Birkhoff [2] introduced the notion of the chromatic polynomial in hopes of using it to make progress on the four color problem. For $m \in \mathbb{N}$, the *chromatic polynomial* of a graph G , $P(G, m)$, is the number of proper m -colorings of G . It is easy to show that $P(G, m)$ is a polynomial in m of degree $|V(G)|$ (e.g., see [7]). For example, whenever $n \in \mathbb{N}$ it is well known that $P(K_n, m) = \prod_{i=0}^{n-1} (m - i)$, $P(C_n, m) = (m - 1)^n + (-1)^n(m - 1)$, $P(T, m) = m(m - 1)^{n-1}$ whenever T is a tree on n vertices, $P(K_{2,n}, m) = m(m - 1)^n + m(m - 1)(m - 2)^n$, and $P(K_1 \vee G, m) = mP(G, m - 1)$ for each $m \geq 2$ (see [1] and [23]). Note that the formula for the chromatic polynomial of a cycle on n vertices still holds when it is not simple (i.e., when $n = 1, 2$).

The notion of chromatic polynomial was extended to list coloring in the early 1990s by Kostochka and Sidorenko [17]. If L is a list assignment for G , let $P(G, L)$ denote the number of proper L -colorings of G . The *list color function* $P_\ell(G, m)$ is the minimum

value of $P(G, L)$ where the minimum is taken over all possible m -assignments L for G . Since an m -assignment could assign the same m colors to every vertex in a graph, it is clear that $P_\ell(G, m) \leq P(G, m)$ for each $m \in \mathbb{N}$. In general, the list color function can differ significantly from the chromatic polynomial for small values of m . For example, for any $n \geq 2$, $P_\ell(K_{n,n^n}, m) = 0$ and $P(K_{n,n^n}, m) > 1$ whenever $m \in \{2, \dots, n\}$. On the other hand, in 1992, Donner [8] showed that for any graph G there is a $k \in \mathbb{N}$ such that $P_\ell(G, m) = P(G, m)$ whenever $m \geq k$.

1.3 Enumeratively Chromatic-Choosable Graphs

With Donner's 1992 result in mind, it is natural to study the point at which the list color function of a graph becomes identical to its chromatic polynomial. Given any graph G , the *list color function number* of G , denoted $\nu(G)$, is the smallest $t \geq \chi(G)$ such that $P_\ell(G, t) = P(G, t)$. The *list color function threshold* of G , denoted $\tau(G)$, is the smallest $k \geq \chi(G)$ such that $P_\ell(G, m) = P(G, m)$ whenever $m \geq k$. Clearly, $\chi(G) \leq \chi_\ell(G) \leq \nu(G) \leq \tau(G)$.

One of the most important open questions on the list color function asks whether the list color function number of a graph can differ from its list color function threshold.

Question 2 ([16]). For every graph G , is it the case that $\nu(G) = \tau(G)$?

Much of the research on the list color function has been focused on studying the list color function threshold for a general graph G . Currently, the best known bound is as follows.

Theorem 3 ([6]). For every graph G with $|E(G)| \geq 4$, $\tau(G) \leq |E(G)| - 1$.

As far as we are aware, the notion of enumerative chromatic choosability was first formally defined in [14], even though it has been pursued since the 1990s. A graph G is said to be *enumeratively chromatic-choosable* if $\tau(G) = \chi(G)$. This means that a graph is *enumeratively chromatic-choosable* if $P_\ell(G, m) = P(G, m)$ for $m \geq \chi(G)$. Additionally, we call a graph *weakly enumeratively chromatic-choosable* if $\nu(G) = \chi(G)$; that is, when $P_\ell(G, \chi(G)) = P(G, \chi(G))$. It is worth mentioning that when a graph is enumeratively chromatic-choosable, we actually have $P_\ell(G, m) = P(G, m)$ for all $m \in \mathbb{N}$ since the list color function and chromatic polynomial of G are both zero whenever $m \in \mathbb{N}$ satisfies $m < \chi(G)$.

We now present some results on these notions. Note that *generalized theta graphs* $\Theta(l_1, \dots, l_n)$ consist of a pair of end vertices joined by n internally disjoint paths of lengths $l_1, \dots, l_n \in \mathbb{N}$. When there are three such paths we have a *theta graph*.

Theorem 4 ([4, 13, 14, 15, 16, 17]). The following statements hold.

- (i) Chordal graphs are enumeratively chromatic-choosable.
- (ii) Cycles are enumeratively chromatic-choosable
- (iii) $K_{2,3}$ is enumeratively chromatic-choosable.

- (iv) If $1 \leq l_1 \leq l_2$, $2 \leq l_2 \leq l_3$, and the parity of l_1 is different from both l_2 and l_3 , then $\Theta(l_1, l_2, l_3)$ is enumeratively chromatic-choosable.
- (v) If G is enumeratively chromatic-choosable (resp. weakly enumeratively chromatic-choosable), then $G \vee K_n$ is enumeratively chromatic-choosable (resp. weakly enumeratively chromatic-choosable) for any $n \in \mathbb{N}$.

Statement (v) is a consequence of the following more general result.

Theorem 5 ([15]). *Let G be a graph such that $P_\ell(G, m) = P(G, m)$. Then $P_\ell(G \vee K_1, m+1) = P(G \vee K_1, m+1)$.*

With this and Theorem 1 in mind we conjecture the following.

Conjecture 6. For any graph G , there is a $p \in \mathbb{N}$ such that $G \vee K_p$ is enumeratively chromatic-choosable.

Any graph that is weakly enumeratively chromatic-choosable is chromatic-choosable. Interestingly, there are few known examples of graphs that are chromatic-choosable and not weakly enumeratively chromatic-choosable. In 2009, Carsten Thomassen showed the following which is illustrated in Figure 1.

Proposition 7. [21] $\Theta(2, 2, 4)$ is chromatic-choosable, but it is not weakly enumeratively chromatic-choosable.

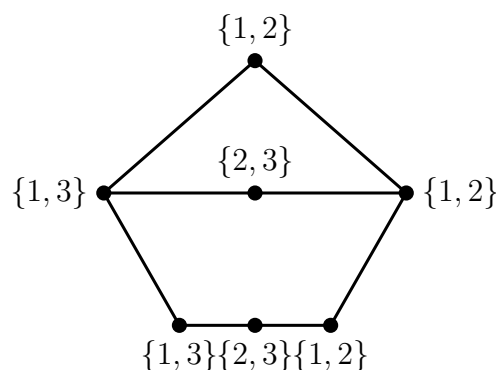


Figure 1: A 2-assignment for a copy of $\Theta(2, 2, 4)$ with exactly one proper coloring

In Theorem 13, we generalize this to $\Theta(2, 2, 2k)$, and in Theorem 30 we show that while $K_{2,2,4}$ is chromatic-choosable, it is not weakly enumeratively chromatic-choosable. In general, it is challenging to determine which graphs are enumeratively chromatic-choosable and few such graphs are known. Consequently, we will keep the following two questions in mind for the remainder of the paper.

Question 8. Which graphs are enumeratively chromatic-choosable?

Question 9. For each $k \geq 2$, does there exist a chromatic-choosable graph G such that $\chi(G) = k$ and G is not weakly enumeratively chromatic-choosable?

Theorem 13 and Theorem 30 imply that the answer to Theorem 9 is yes when $k = 2$ and $k = 3$.

1.4 Outline of Paper

We now present an outline of the paper while mentioning some open questions. In Section 2, we completely characterize the enumeratively chromatic-choosable graphs with chromatic number 2. Following [11], we say that the *core* of a connected graph G is the graph obtained from G by successively deleting vertices of degree 1. Note that the core of a tree is a K_1 .

Theorem 10. *Suppose G is a connected graph with $\chi(G) = 2$. We have that G is enumeratively chromatic-choosable if and only if the core of G is a copy of: K_1 , C_{2k+2} for $k \in \mathbb{N}$, or $K_{2,3}$.*

Interestingly, the enumeratively chromatic-choosable graphs with chromatic number 2 are the same as the chromatic-paintable graphs with chromatic number 2 (see [24]).¹

In Section 3 we consider the list color function of $G = \Theta(2, 2, 2k)$ where $k \geq 2$. Theorem 13 tells us G is chromatic-choosable, but not weakly enumeratively chromatic-choosable. Also, recall that for any graph G , there is an $N \in \mathbb{N}$ such that $G \vee K_N$ is chromatic-choosable. Consequently, it is natural to consider Theorem 6 when $G = \Theta(2, 2, 2k)$ where $k \geq 2$. We use the AM-GM Inequality and DP-coloring to prove the following theorem that such a G is close to being enumeratively chromatic-choosable.

Theorem 11. *Suppose $G = \Theta(2, 2, 2k)$ with $k \geq 2$. Then, $P_\ell(G, m) = P(G, m)$ whenever $m \geq 3$.*

Then, in Section 4, we show that $K_1 \vee \Theta(2, 2, 2k)$ is enumeratively chromatic-choosable (Theorem 29) with the help of a probabilistic lemma. This provides some support for Theorem 6. We also show that while $K_{2,2,4}$ is chromatic-choosable, it fails to be enumeratively chromatic-choosable (Theorem 30) which indicates that it would be natural to study Theorem 6 when G is a complete multipartite graph. This is a topic for further research.

2 Enumeratively Chromatic-Choosable Graphs with $\chi(G) = 2$

In this section, it is useful to keep in mind that if G is a connected graph with $\chi(G) = 2$, then G has a unique bipartition which means $P(G, 2) = 2$. In [11], Erdős, Rubin, and Taylor proved the following.

Theorem 12. [11] *Let G be a connected bipartite graph. Then G is 2-choosable if and only if the core of G belongs to K_1 , C_{2k+2} for $k \in \mathbb{N}$, or $\Theta(2, 2, 2k)$ for $k \geq 1$.*

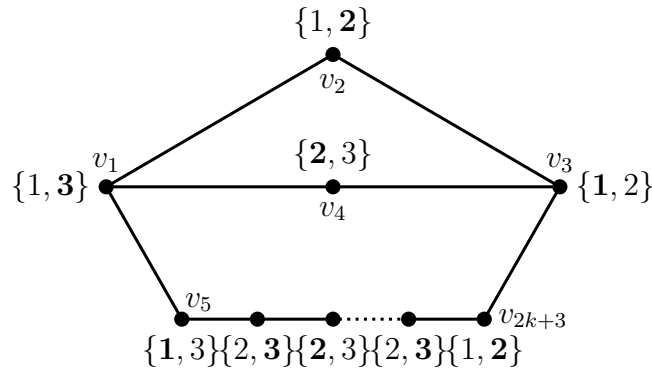


Figure 2: A 2-assignment for a copy of $\Theta(2, 2, 2k)$ with exactly one proper coloring

In Figure 2, the bold number in each list indicates the number used in the only proper coloring of $\Theta(2, 2, 2k)$ with the this list assignment.

Lemma 13. *For each $k \geq 2$, $\Theta(2, 2, 2k)$ is chromatic-choosable, but not weakly enumeratively chromatic-choosable.*

Proof. Let $G = \Theta(2, 2, 2k)$ for some $k \geq 2$. By Theorem 12, it follows that $\chi_\ell(G) = \chi(G) = 2$. So, G is chromatic-choosable. Let the three paths of G be $S_1 = v_1, v_2, v_3$, $S_2 = v_1, v_4, v_3$, and $S_3 = v_1, v_5, v_6, \dots, v_{2k+3}, v_3$. If $k = 2$, the desired result follows by Theorem 7. We may therefore assume that $k \geq 3$. Let L be the 2-assignment for G be defined by $L(v_1) = \{1, 3\}$, $L(v_2) = \{1, 2\}$, $L(v_3) = \{1, 2\}$, $L(v_4) = \{2, 3\}$, $L(v_5) = \{1, 3\}$, $L(v_i) = \{2, 3\}$ for $6 \leq i \leq 2k + 2$, and $L(v_{2k+3}) = \{1, 2\}$.

Let f be the L -coloring given by $f(v_1) = 3$, $f(v_4) = 2$, $f(v_3) = 1$, $f(v_2) = 2$, $f(v_5) = 1$, $f(v_{2k+3}) = 2$, $f(v_i) = 3$ for even $i \in \{6, 7, \dots, 2k + 2\}$, and $f(v_j) = 2$ for odd $j \in \{6, 7, \dots, 2k + 2\}$. Clearly, this is a proper L -coloring.

We claim that f is the only proper L -coloring of G . Suppose for a contradiction that there is a proper L -coloring g distinct from f . We claim that $g(v_1) = 3$. To see why, suppose for a contradiction that $g(v_1) = 1$. Then $g(v_2) = 2$, $g(v_3) = 1$, $g(v_{2k+3}) = 2$, $g(v_i) = 3$ for even $i \in \{6, 7, \dots, 2k + 2\}$ and $g(v_j) = 2$ for odd $j \in \{6, 7, \dots, 2k + 2\}$. Since $g(v_6) = 3$, $g(v_1) = 1$, and $g(v_5) \in \{1, 3\}$, it follows that g is not a proper L -coloring. So $g(v_1) = 3 = f(v_1)$, and it is easy to see this implies $g = f$ which is a contradiction.

Thus, $P_\ell(G, 2) \leq P(G, L) = 1 < 2 = P(G, 2)$. The conclusion follows. $\square$

Suppose that G is a graph and $z \in V(G)$ has degree 1. It is well-known that $P(G, m) = (m-1)P(G-z, m)$ for each $m \in \mathbb{N}$ (see [23]). We will now show that an analogous identity holds for the list color function.

¹Chromatic-paintability is an online version of list coloring that was introduced by Xuding Zhu [24].

Lemma 14. Suppose G is a graph and $m \in \mathbb{N}$. Suppose G' is the graph obtained from G by adding a new vertex z to G and connecting z to exactly one vertex in G . Then, $P_\ell(G', m) = (m - 1)P_\ell(G, m)$.

Proof. Suppose $x \in V(G)$ is the vertex to which z is adjacent. Suppose L is an m -assignment such that $P_\ell(G, m) = P(G, L)$. Let L' be the m -assignment for G' given by $L'(v) = L(v)$ for each $v \in V(G)$ and $L'(z) = L(x)$. It is clear that each proper L -coloring of G can be extended to a proper L' -coloring of G' in $(m - 1)$ ways. Consequently,

$$P_\ell(G', m) \leq P(G', L') = (m - 1)P(G, L) = (m - 1)P_\ell(G, m).$$

Now, suppose that K' is an m -assignment for G' such that $P_\ell(G', m) = P(G', K')$. Let K be the m -assignment for G obtained by restricting the domain of K' to $V(G)$. Now, each proper K -coloring of G can be extended to a proper K' -coloring of G' in m or $(m - 1)$ ways (depending on whether the color used on x is in $K'(z)$). Thus,

$$P_\ell(G', m) = P(G', K') \geq (m - 1)P(G, K) \geq (m - 1)P_\ell(G, m).$$

The result follows. $\square$

Theorem 10. Suppose G is a connected graph with $\chi(G) = 2$. We have that G is enumeratively chromatic-choosable if and only if the core of G is a copy of: K_1 , C_{2k+2} for $k \in \mathbb{N}$, or $K_{2,3}$.

Proof. Suppose the core of G is a copy of K_1 , C_{2k+2} for $k \in \mathbb{N}$, or $K_{2,3}$. We will proceed by induction on the number of vertices of G , say D , deleted to obtain the core. If G required zero deletions, then Theorem 4 implies that G is enumeratively chromatic-choosable. Now, assume that $D \geq 1$ and any graph with core K_1 , C_{2k+2} for $k \in \mathbb{N}$, or $K_{2,3}$ resulting from less than D vertex deletions is enumeratively chromatic-choosable. Let $z \in V(G)$ satisfy $\deg_G(z) = 1$, and let $G' = G - z$. By the induction hypothesis, we have that $P_\ell(G', m) = P(G', m)$ for $m \geq 2$. Notice

$$\begin{aligned} P_\ell(G, m) &= (m - 1)P_\ell(G', m) \text{ by Theorem 14} \\ &= (m - 1)P(G', m) \text{ by the induction hypothesis} \\ &= P(G, m) \end{aligned}$$

Thus, G is enumeratively chromatic-choosable.

Conversely, suppose that G is enumeratively chromatic-choosable. Since G is enumeratively chromatic-choosable, it is chromatic-choosable. Since $\chi(G) = 2$, it follows that G is bipartite and 2-choosable. Theorem 12 implies that the core of G belongs to K_1 , C_{2k+2} for $k \in \mathbb{N}$, or $\Theta(2, 2, 2k)$ for $k \geq 1$. To prove the desired we must show that the core of G can not be $\Theta(2, 2, 2k)$ for some $k \geq 2$. So, for the sake of contradiction suppose there is an enumeratively chromatic-choosable graph with core $\Theta(2, 2, 2k)$ for some $k \geq 2$.

Among all such graphs suppose that M is chosen so that the number of vertices outside the core of M is minimized. Let C be the core of M and note that $V(M) - V(C) \neq \emptyset$

by Theorem 13. This means that M has a vertex x of degree 1, and suppose that the neighbor of x in M is y . Let $M' = M - x$, and note M' has the same core as M . By the minimality of M , M' is not enumeratively chromatic-choosable which means there is an $m \geq 2$ such that $P_\ell(M', m) < P(M', m)$.

Now, since each proper m -coloring of M' can be extended to a proper m -coloring of M in exactly $(m - 1)$ ways, $P(M, m) = (m - 1)P(M', m)$. Since Theorem 14 implies $P_\ell(M, m) = (m - 1)P_\ell(M', m)$, we have $P_\ell(M, m) < P(M, m)$ which contradicts the fact that M is enumeratively chromatic-choosable. $\square$

Interestingly, the enumeratively chromatic-choosable graphs with chromatic number 2 are the same as the chromatic-paintable graphs with chromatic number 2.

Theorem 15 ([24]). *A connected graph G is 2-paintable if and only if the core of G is a copy of: K_1 , C_{2k+2} for $k \in \mathbb{N}$, or $K_{2,3}$.*

3 2-Choosable Theta Graphs

In this section we prove Theorem 11; throughout, unless otherwise noted, suppose that $G = \Theta(2, 2, 2k)$ where $k \geq 2$. Suppose that the endpoints of the paths used to construct G are u and v . Suppose the internal vertices of the paths of length 2 used to construct G are x_1 and y_1 . We call these paths S_1 and S_2 respectively. Suppose the internal vertices of the path of length $2k$ used to construct G are $z_1, \dots, z_{2k-1}$. We call this path S_3 . When L is an m -assignment for G , for each $i \in [3]$, let L_i be the m -assignment for S_i obtained by restricting the domain of L to $V(S_i)$. For each $i \in [3]$, $c \in L(u)$, and $d \in L(v)$, let $N_i(c, d)$ be the number of proper L_i -colorings of S_i where u is colored with c and v is colored with d . We begin with an important lemma.

Lemma 16. *Suppose that L is an m -assignment for G . Then,*

$$P(G, L) = \sum_{(c,d) \in L(u) \times L(v)} \prod_{i=1}^3 N_i(c, d).$$

Proof. For each $i \in [3]$, let $C_{(c,d)}^{(i)}$ be the set of proper L_i -colorings of S_i where u is colored with c and v is colored with d .

Let $M : \bigcup_{(c,d) \in L(u) \times L(v)} \left(C_{(c,d)}^{(1)} \times C_{(c,d)}^{(2)} \times C_{(c,d)}^{(3)} \right) \rightarrow \{f : f \text{ is a proper } L\text{-coloring of } G\}$ be the function defined by $M((g_1, g_2, g_3)) = T$ where $T : V(G) \rightarrow \bigcup_{w \in V(G)} L(w)$ is given by $T(u) = g_1(u)$, $T(v) = g_1(v)$, $T(x_1) = g_1(x_1)$, $T(y_1) = g_2(y_1)$, and $T(z_i) = g_3(z_i)$ for $i \in [2k - 1]$. Clearly, T is a proper L -coloring of G . It is easy to verify that M is a bijection.

Since we are considering a Cartesian product, it follows that $\left| C_{(c,d)}^{(1)} \times C_{(c,d)}^{(2)} \times C_{(c,d)}^{(3)} \right| = \prod_{i=1}^3 N_i(c, d)$. By the addition rule, we have that

$$\left| \bigcup_{(c,d) \in L(u) \times L(v)} \left(C_{(c,d)}^{(1)} \times C_{(c,d)}^{(2)} \times C_{(c,d)}^{(3)} \right) \right| = \sum_{(c,d) \in L(u) \times L(v)} \prod_{i=1}^3 N_i(c, d).$$

Since M is a bijection, this is the number of proper L -colorings of G , as required. $\square$

The AM-GM inequality immediately yields the following.

Corollary 17. *Suppose that L is an m -assignment for G . Then,*

$$P(G, L) = \sum_{(c,d) \in L(u) \times L(v)} \prod_{i=1}^3 N_i(c, d) \geq m^2 \left(\prod_{(c,d) \in L(u) \times L(v)} \prod_{i=1}^3 N_i(c, d) \right)^{1/m^2}.$$

We now prove a series of lemmas that will culminate in the proof of Theorem 11. Our general strategy is to show that for any m -assignment L for G with $m \geq 3$, $P(G, L) \geq P(G, m)$. We prove this in the case that $L(u) = L(v)$, the case that $m = 3$ and $L(u) \neq L(v)$, and the case that $m > 3$ and $L(u) \neq L(v)$. Importantly, the formula for the chromatic polynomial of theta graphs is well known. In particular, we have that for each $m \in \mathbb{N}$,

$$P(G, m) = (m-2)^2((m-1)^{2k+1} - (m-1)) + (m-1)^2((m-1)^{2k} + (m-1))$$

(see formula (2.6a) in [3]).

Lemma 18. *Suppose that L is an m -assignment for G such that $m \geq 3$ and $L(u) = L(v)$. Then, $P(G, L) \geq P(G, m)$.*

Proof. Without loss of generality assume that $L(u) = L(v) = [m]$. Suppose $(c, d) \in [m]^2$. For $i \in [2]$, we have that $N_i(c, d) \geq m-2$ when $c \neq d$, and we have that $N_i(c, d) \geq m-1$ when $c = d$.

Now, suppose that e is an edge with endpoints u and v . Let G' be the graph obtained from S_3 by adding e . Notice that a proper L_3 -coloring of S_3 that colors u and v differently is also a proper L_3 -coloring of G' . This observation along with Theorem 4 yields

$$\sum_{(c,d) \in L(u) \times L(v), c \neq d} N_3(c, d) = P(G', L_3) \geq P(C_{2k+1}, m) = (m-1)^{2k+1} - (m-1).$$

Let G'' be the graph obtained from G' by contracting e , and let w be the vertex of G'' obtained via the contraction. Let L'' be the m -assignment for G'' given by $L''(w) = [m]$ and $L''(x) = L_3(x)$ for each $x \in V(G'') - \{w\}$. There is clearly a one-to-one correspondence

between the proper L_3 -colorings of S_3 that color u and v the same color and the proper L'' -colorings of G'' . This observation along with Theorem 4 yields

$$\sum_{j=1}^m N_3(j, j) = P(G'', L'') \geq P(C_{2k}, m) = (m-1)^{2k} + (m-1).$$

So, by Theorem 16,

$$\begin{aligned} P(G, L) &= \sum_{(c,d) \in L(u) \times L(v), c \neq d} \prod_{i=1}^3 N_i(c, d) + \sum_{j=1}^m \prod_{i=1}^3 N_i(j, j) \\ &\geq \sum_{(c,d) \in L(u) \times L(v), c \neq d} (m-2)^2 N_3(c, d) + \sum_{j=1}^m (m-1)^2 N_3(j, j) \\ &\geq P(G, m). \end{aligned}$$

□

We now turn our attention to m -assignments L for G satisfying $L(u) \neq L(v)$.

Lemma 19. *Suppose P is the path of length 2 with $P = u, w, v$ and L is an m -assignment for P where $m \geq 3$. For each $c \in L(u)$ and $d \in L(v)$, let $N(c, d)$ be the number of proper L -colorings of P where u is colored with c and v is colored with d . Suppose $a = |L(u) \cap L(v)|$. Then, the following statements hold.*

(i) *For each $c \in L(u)$ and $d \in L(v)$, $m-2 \leq N(c, d) \leq m$.*

(ii) *The number of ordered pairs $(c, d) \in L(u) \times L(v)$ for which $N(c, d) = m-2$ is at most $\frac{(a+m)^2}{4} - a$.*

Proof. Since w has 2 neighbors, (i) is obvious.

Now, suppose $|L(w) \cap L(u) \cap L(v)| = k_1$, $|L(w) \cap (L(u) - L(v))| = k_2$, and $|L(w) \cap (L(v) - L(u))| = k_3$. Note that in order for $N(c, d) = m-2$: $c \in L(u) \cap L(w)$, $d \in L(v) \cap L(w)$, and $c \neq d$. Consequently, the number of ordered pairs $(c, d) \in L(u) \times L(v)$ for which $N(c, d) = m-2$ is $(k_1 + k_3)(k_1 + k_2) - k_1$. Since $k_2 + k_3 \leq m - k_1$ and $k_1 \leq a$, we have that

$$(k_1 + k_3)(k_1 + k_2) - k_1 \leq \left(k_1 + \frac{m - k_1}{2}\right)^2 - k_1 = \left(\frac{m + k_1}{2}\right)^2 - k_1 \leq \frac{(a + m)^2}{4} - a.$$

□

Our next lemma requires the DP color function which we now introduce. As it is introduced, the graph G is not necessarily a copy of $\Theta(2, 2, 2k)$ for some $k \geq 2$ but rather an arbitrary graph. The concept of DP-coloring was first put forward in 2015 by Dvořák and Postle under the name *correspondence coloring* (see [9]). Intuitively, DP-coloring generalizes list coloring by allowing the colors that are identified as the same to vary from edge to edge. Formally, for a graph G , a *DP-cover* (or simply a *cover*) of G is an ordered pair $\mathcal{H} = (L, H)$, where H is a graph and $L : V(G) \rightarrow 2^{V(H)}$ is a function satisfying the following conditions:

- $\{L(v) : v \in V(G)\}$ is a partition of $V(H)$ into $|V(G)|$ parts,
- for every pair of adjacent vertices $u, v \in V(G)$, the edges in $E_H(L(u), L(v))$ form a matching (not necessarily perfect and possibly empty), and
- $E(H) = \bigcup_{uv \in E(G)} E_H(L(u), L(v))$.

Suppose $\mathcal{H} = (L, H)$ is a cover of a graph G . A *transversal* of $\mathcal{H}$ is a set of vertices $T \subseteq V(H)$ containing exactly one vertex from each $L(v)$. A transversal T is said to be *independent* if T is an independent set in H . If $\mathcal{H}$ has an independent transversal T , then T is said to be a *proper $\mathcal{H}$ -coloring* of G , and G is said to be *$\mathcal{H}$ -colorable*. A *k -fold cover* of G is a cover $\mathcal{H} = (L, H)$ such that $|L(v)| = k$ for all $v \in V(G)$.

Suppose $\mathcal{H} = (L, H)$ is a cover of graph G . We let $P_{DP}(G, \mathcal{H})$ be the number of proper $\mathcal{H}$ -colorings of G . Then, the *DP color function*, denoted $P_{DP}(G, m)$, is the minimum value of $P_{DP}(G, \mathcal{H})$ where the minimum is taken over all possible m -fold covers $\mathcal{H}$ of G .

Now, suppose that L is an m -assignment for G . The *cover of G corresponding to L* , denoted $\mathcal{H}_L = (\Lambda_L, H_L)$, is the cover of G defined as follows. For each $v \in V(G)$, $\Lambda_L(v) = \{(v, c) : c \in L(v)\}$, and H_L is the graph with vertex set $\bigcup_{v \in V(G)} \Lambda_L(v)$ and edges created so that for any $(u, c_1), (v, c_2) \in V(H_L)$, $(u, c_1)(v, c_2) \in E(H)$ if and only if $uv \in E(G)$ and $c_1 = c_2$. Notice that if $\mathcal{C}$ is the set of proper L -colorings of G and $\mathcal{T}$ is the set of proper $\mathcal{H}$ -colorings of G , then the function $h : \mathcal{C} \rightarrow \mathcal{T}$ given by $h(f) = \{(v, f(v)) : v \in V(G)\}$ is a bijection. So, for any graph G and $m \in \mathbb{N}$,

$$P_{DP}(G, m) \leq P_\ell(G, m) \leq P(G, m).$$

Lemma 20. Let $P = x_1, x_2, \dots, x_{2k+1}$ where $k \in \mathbb{N}$ and L be an m -assignment of P with $m \geq 3$. Suppose $c \in L(x_1)$ and $d \in L(x_{2k+1})$, and let $N(c, d)$ be the number of proper L -colorings of P where x_1 is colored with c and x_{2k+1} is colored with d . For each $(c, d) \in L(x_1) \times L(x_{2k+1})$,

$$\frac{(m-1)^{2k+1} - (m-1)}{m(m-1)} \leq N(c, d).$$

Furthermore, there exist at least m elements $(y, z) \in L(x_1) \times L(x_{2k+1})$ such that

$$\frac{(m-1)^{2k+1} - (m-1)}{m(m-1)} + 1 = \frac{(m-1)^{2k} + (m-1)}{m} \leq N(y, z).$$

Proof. Let H' be the graph obtained from H_L by arbitrarily adding edges so that $E_{H'}(\Lambda_L(x_i), \Lambda_L(x_{i+1}))$ is a perfect matching for each $i \in [2k]$. Note that H' is the disjoint union of m paths. Then, let $\mathcal{H}' = (\Lambda_L, H')$. Clearly, $\mathcal{H}'$ is an m -fold cover of P . Furthermore, let $\mathcal{T}$ be the set of proper $\mathcal{H}'$ -colorings of P , and let $\mathcal{C}$ be the set of proper L -colorings of P . Then we can injectively map each $T \in \mathcal{T}$ to an $f_T \in \mathcal{C}$ by letting $f_T(v)$ be the second coordinate of the ordered pair in T with first coordinate v for each $v \in V(G)$.

Lemma 8 in [4] implies that $\mathcal{T}$ has at least $((m-1)^{2k+1} - (m-1))/(m(m-1))$ elements that contain (x_1, c) and (x_{2k+1}, d) . It further says that there are $((m-1)^{2k} + (m-1))/m$ elements of $\mathcal{T}$ that contain (x_1, c) and (x_{2k+1}, d) if and only if there is a path in H' connecting (x_1, c) and (x_{2k+1}, d) . The result immediately follows. $\square$

Lemma 21. *Suppose $G = \Theta(2, 2, 2k)$ with $k \geq 2$. Suppose that L is a 3-assignment for G with $m = 3$ such that $L(u) \neq L(v)$. Then $P(G, L) \geq P(G, 3)$.*

Proof. Note that $P(G, 3) = 4(2^{2k} + 2) + 2^{2k+1} - 2$. Let $a = |L(u) \cap L(v)|$, and notice that $a \in \{0, 1, 2\}$ since $L(u) \neq L(v)$. Theorem 19 implies that we have that $N_i(c, d) = 1$ for $i \in [2]$ for at most 4 pairs $(c, d) \in L(u) \times L(v)$. Consequently,

$$\prod_{(c,d) \in L(u) \times L(v)} N_i(c, d) \geq (1)^4(2)^5.$$

Theorem 20 implies that

$$\prod_{(c,d) \in L(u) \times L(v)} N_3(c, d) \geq \left(\frac{2^{2k} - 1}{3}\right)^6 \left(\frac{2^{2k} + 2}{3}\right)^3.$$

By Theorem 17, we have that

$$P(G, L) \geq 9 \left(1^8 \cdot 2^{10} \left(\frac{2^{2k} - 1}{3}\right)^6 \left(\frac{2^{2k} + 2}{3}\right)^3\right)^{1/9} = 6 (2(2^{2k} - 1)^6 (2^{2k} + 2)^3)^{1/9}.$$

Let $x = 2^{2k} - 1$. Then $P(G, 3) = 4(x + 3) + 2x = 6x + 12$ and

$$P(G, L) \geq 6 (2x^6(x + 3)^3)^{1/9} = 6x \left(2 \left(1 + \frac{3}{x}\right)^3\right)^{1/9}.$$

To complete the proof, we will show that $6x \left(2 \left(1 + \frac{3}{x}\right)^3\right)^{1/9} \geq 6x + 12$. When $k = 2$, $6(15) \left(2 \left(1 + \frac{3}{15}\right)^3\right)^{1/9} - 6(15) > 12$, as desired. For $k \geq 3$,

$$6x \left(2 \left(1 + \frac{3}{x}\right)^3\right)^{1/9} - 6x > 6x(2^{1/9} - 1) \geq 6(63)(2^{1/9} - 1) > 12.$$

Hence $P(G, L) \geq P(G, 3)$. $\square$

Lemma 22. *Suppose G is an arbitrary graph and L is an m -assignment for G . Suppose $uv \in E(G)$, $|L(u) - L(v)| = d \geq 1$, and for any $x \in L(u)$ and $y \in L(v)$ with $x \neq y$, there are at least C proper L -colorings of G that color u with x and v with y . Then,*

$$P(G, L) \geq P_{DP}(G, m) + Cd.$$

Proof. Let $L(u) - L(v) = \{c_1, \dots, c_d\}$. Let H' be the graph obtained from H_L by arbitrarily adding edges so that $E_{H'}(\Lambda_L(w), \Lambda_L(z))$ is a perfect matching whenever $wz \in E(G)$. Then, let $\mathcal{H}' = (\Lambda_L, H')$ so that $\mathcal{H}'$ is an m -fold cover of G . As in the proof of Theorem 20, when $\mathcal{T}$ is the set of proper $\mathcal{H}'$ -colorings of G and $\mathcal{C}$ is the set of proper L -colorings of G , we can injectively map each $T \in \mathcal{T}$ to an $f_T \in \mathcal{C}$.

Now, suppose that for each $i \in [d]$, $N_{H'}((u, c_i)) \cap \Lambda_L(v) = \{(v, d_i)\}$. It follows that no $T \in \mathcal{T}$ is mapped to an $f \in \mathcal{C}$ satisfying $f(u) = c_i$ and $f(v) = d_i$ for some $i \in [d]$. Since for each $i \in [d]$ there are at least C elements of $\mathcal{C}$ that color u with c_i and v with d_i ,

$$P(G, L) = |\mathcal{C}| \geq |\mathcal{T}| + Cd \geq P_{DP}(G, m) + Cd.$$

□

It is worth mentioning that a more general version of our next lemma has recently appeared in the literature (see Lemma 2.2 in [5]). We include a proof here for completeness.

Lemma 23. *Suppose $G = \Theta(2, 2, 2k)$ with $k \geq 2$. Suppose that L is an m -assignment for G with $m \geq 4$. If there is a $wz \in E(G)$ such that there is an $x \in L(w) - L(z)$ and a $y \in L(z) - L(w)$, then there are at least $(m-1)^{2k-1}(m-2)^2$ proper L -colorings of G that color w with x and z with y .*

Proof. By interchanging S_1 and S_2 if necessary, we may assume without loss of generality that wz is in the cycle, call it W , contained in G formed by the paths S_2 and S_3 . Suppose the vertices of W in cyclic order are $w, z, v_1, \dots, v_{2k}$. Recall that x_1 is the internal vertex of the path S_1 .

Suppose we color w with x and z with y . We can now greedily complete a proper L -coloring of G as follows. Greedily color the uncolored vertices of W with a color from each vertex's list in the following order: $v_1, \dots, v_{2k}$. Notice this can be done in at least $(m-1)^{2k-1}(m-2)$ ways. Indeed, when each of $v_1, \dots, v_{2k-1}$ is colored, it has only one colored neighbor while v_{2k} has two colored neighbors.

Finally, color x_1 with a color from its list. Since both neighbors of x_1 have already been colored, this can be done in at least $m-2$ ways. Thus, there are at least $(m-1)^{2k-1}(m-2)^2$ proper L -colorings of G that color w with x and z with y , and the result follows. □

Before proving the next lemma, we need Theorem 6 in [4].

Theorem 24 ([4]). *Let $G = \Theta(l_1, l_2, l_3)$, where $l_1 = \min\{l_1, l_2, l_3\} \geq 1$ and $l_i \geq 2$ for each $i \in \{2, 3\}$.*

(i) If the parity of l_1 is different from both l_2 and l_3 , then $P_{DP}(G, m) = P(G, m)$ for all $m \in \mathbb{N}$.

(ii) If the parity of l_1 is the same as l_3 and different from l_2 , then for $m \geq 2$:

$$P_{DP}(G, m) = \frac{1}{m} \left[(m-1)^{l_1+l_2+l_3} + (m-1)^{l_1} - (m-1)^{l_2} - (m-1)^{l_3+1} + (-1)^{l_2+1}(m-2) \right].$$

(iii) If l_1, l_2 , and l_3 all have the same parity, then for $m \geq 3$:

$$P_{DP}(G, m) = \frac{1}{m} \left[(m-1)^{l_1+l_2+l_3} - (m-1)^{l_1} - (m-1)^{l_2} - (m-1)^{l_3} + 2(-1)^{l_1+l_2+l_3} \right].$$

Note that this result implies that $P_{DP}(\Theta(2, 2, 2k), m) = ((m - 1)^{2k+4} - (m - 1)^{2k} - 2(m - 1)^2 + 2)/m$ whenever $m \geq 3$.

Lemma 25. *Suppose $G = \Theta(2, 2, 2k)$ with $k \geq 2$. Suppose that L is an m -assignment for G with $m \geq 4$ and $L(u) \neq L(v)$. Then, $P(G, L) \geq P(G, m)$.*

Proof. Since $L(u) \neq L(v)$, there is a $wz \in E(G)$ such that there is an $x \in L(w) - L(z)$ and a $y \in L(z) - L(w)$. So, by Theorems 22 and 23

$$P(G, L) \geq P_{DP}(G, m) + (m - 1)^{2k-1}(m - 2)^2.$$

Now, since $P(G, m) = (m - 2)^2((m - 1)^{2k+1} - (m - 1)) + (m - 1)^2((m - 1)^{2k} + (m - 1))$ and $P_{DP}(G, m) = ((m - 1)^{2k+4} - (m - 1)^{2k} - 2(m - 1)^2 + 2)/m$, one can calculate that $P(G, m) - P_{DP}(G, m) = (m - 1)^{2k} + 2(m - 1)^2 + (m - 1) - 2$. Now, note that since $m \geq 4$ and $k \geq 2$,

$$2(m - 1)^2 + (m - 1) - 2 \leq 3(m - 1)^2 \leq (m - 1)^3 \leq (m - 4)(m - 1)^{2k} + (m - 1)^{2k-1}.$$

This means $(m - 1)^{2k} + 2(m - 1)^2 + (m - 1) - 2 \leq (m - 1)^{2k-1}(m - 2)^2$. So, $P(G, L) \geq P(G, m)$. $\square$

Theorem 11. *Suppose $G = \Theta(2, 2, 2k)$ with $k \geq 2$. Then, $P_\ell(G, m) = P(G, m)$ whenever $m \geq 3$.*

Proof. Suppose $m \geq 3$ is fixed and that L is an m -assignment for G such that $P(G, L) = P_\ell(G, m)$. Clearly, $P(G, m) \geq P(G, L)$. If $L(u) = L(v)$, then Theorem 18 implies $P(G, L) \geq P(G, m)$ and the result follows. In the case that $m = 3$ and $L(u) \neq L(v)$, Theorem 21 implies $P(G, L) \geq P(G, m)$ and the result follows. In the case that $m > 3$ and $L(u) \neq L(v)$, Theorem 25 implies $P(G, L) \geq P(G, m)$ and the result follows. $\square$

4 Support for Theorem 6

In this section, in support of Theorem 6, we prove $K_1 \vee \Theta(2, 2, 2k)$ is enumeratively chromatic-choosable (Theorem 29). We also give an example of a graph with chromatic number 3 that is chromatic-choosable but is not weakly enumeratively chromatic-choosable which answers Theorem 9 in the affirmative for $k = 3$.

Lemma 26. *Suppose G is a graph with $V(G) = \{v_1, \dots, v_n\}$. Suppose that $m \in \mathbb{N}$, $d_1, \dots, d_n$ are nonnegative integers, and L is a list assignment for G such that $|L(v_i)| = m + d_i$. Then,*

$$P(G, L) \geq \left\lceil \frac{P_\ell(G, m) \prod_{i=1}^n (m + d_i)}{m^n} \right\rceil$$

Proof. Let $\mathcal{F}$ be the set of all proper L -colorings of G . For each $i \in [n]$ uniformly at random delete d_i colors from $L(v_i)$ and call the resulting m -assignment for G , L' . Clearly, $P_\ell(G, m) \leq P(G, L')$.

Now, for each $f \in \mathcal{F}$, let X_f be the random variable that is equal to 1 if f is a proper L' -coloring of G and is equal to 0 otherwise. Since every proper L' -coloring of G is also a proper L -coloring of G , we have that $P(G, L') = \sum_{f \in \mathcal{F}} X_f$. Notice that for each $f \in \mathcal{F}$ the probability that $f(v_i) \in L'(v_i)$ is

$$1 - \frac{d_i}{m + d_i} = \frac{m}{m + d_i}$$

for each $i \in [n]$. Consequently, the probability that $X_f = 1$ is $\prod_{i=1}^n m/(m + d_i)$. This means that

$$\mathbb{E} \left(\sum_{f \in \mathcal{F}} X_f \right) = |\mathcal{F}| \prod_{i=1}^n \frac{m}{m + d_i} = \frac{m^n P(G, L)}{\prod_{i=1}^n (m + d_i)}.$$

So, there is an m -assignment, L'' of G formed by deleting d_i colors from $L(v_i)$ for each $i \in [n]$ such that

$$P_\ell(G, m) \leq P(G, L'') \leq \frac{m^n P(G, L)}{\prod_{i=1}^n (m + d_i)}.$$

The fact that $P(G, L)$ is an integer then implies the desired result. $\square$

Before proving Theorem 29, we need an observation and proposition.

Observation 27. Suppose $G = K_{1,n}$ where $n \in \mathbb{N}$, and L is a 2-assignment for G . If L does not assign the same list of colors to each $v \in V(G)$, $P(G, L) \geq 3$.

Proposition 28. Suppose G is a connected graph with $\chi(G) = 2$ and bipartition X, Y . If $P_\ell(G, 2) \geq 1$, then $K_1 \vee G$ is weakly enumeratively chromatic-choosable.

Proof. In the case $P_\ell(G, 2) \geq 2$, Theorem 5 implies the desired result. So, we suppose that $P_\ell(G, 2) = 1$.

Let $H = K_1 \vee G$, and suppose that u is the vertex from the copy of K_1 used to form H . Clearly, $P(H, 3) = 6$. Suppose that L is a 3-assignment for H such that $P_\ell(H, 3) = P(H, L)$. We will show that $P(H, L) \geq 6$.

Suppose that $L(u) = \{c_1, c_2, c_3\}$. For each $i \in [3]$, let $o_i = |\{v \in V(G) : c_i \notin L(v)\}|$. We will show $P(H, L) \geq 6$ when: (1) $o_i = 0$ for some $i \in [3]$ and (2) $o_i \geq 1$ for each $i \in [3]$.

First, suppose without loss of generality that $o_1 = 0$. Let L_1 be the 2-assignment for $H[X \cup Y]$ given by $L_1(v) = L(v) - \{c_1\}$ for each $v \in X \cup Y$. Let L_2 be the 2-assignment for $H[Y \cup \{u\}]$ given by $L_2(v) = L(v) - \{c_1\}$ for each $v \in Y \cup \{u\}$. Let L_3 be the 2-assignment for $H[X \cup \{u\}]$ given by $L_3(v) = L(v) - \{c_1\}$ for each $v \in X \cup \{u\}$.

Notice that if L_2 and L_3 assign the same list to every vertex in $H[Y \cup \{u\}]$ and $H[X \cup \{u\}]$ respectively, then $L(v) = \{c_1, c_2, c_3\}$ for each $v \in V(G)$ which immediately implies $P(H, L) = P(H, 3) = 6$. So, we may assume without loss of generality that L_2 doesn't assign the same list to every vertex in $H[Y \cup \{u\}]$. Clearly, a proper L_1 -coloring, a proper L_2 -coloring, and a proper L_3 -coloring can be extended to a proper L -coloring of

H by coloring u with c_1 , coloring each vertex in X with c_1 , and coloring each vertex in Y with c_1 respectively. Then, Theorem 4 and Theorem 27 imply

$$\begin{aligned} P(H, L) &\geq P(H[X \cup Y], L_1) + P(H[Y \cup \{u\}], L_2) + P(H[X \cup \{u\}], L_3) \\ &\geq P_\ell(G, 2) + 3 + P_\ell(K_{1,|X|}, 2) \\ &\geq 1 + 3 + 2 = 6. \end{aligned}$$

Now, we may suppose $o_i \geq 1$ for each $i \in [3]$. For each $i \in [3]$, let $L^{(i)}$ be the list assignment for G given by $L^{(i)}(v) = L(v) - \{c_i\}$ for each $v \in V(G)$. Since $o_i \geq 1$ for each $i \in [3]$, Theorem 26 implies $P(G, L^{(i)}) \geq \lceil 1(3)/2 \rceil = 2$. Thus,

$$P(H, L) \geq \sum_{i=1}^3 P(G, L^{(i)}) \geq 6$$

as desired. $\square$

Theorem 29. *If $G = K_1 \vee \Theta(2, 2, 2k)$ where $k \in \mathbb{N}$, then G enumeratively chromatic-choosable.*

Proof. When $k = 1$, the result immediately follows from Theorem 4.

Now, suppose that $k \geq 2$. By Theorem 13 and Theorem 28, we have that $P_\ell(G, 3) = P(G, 3)$. Then, by Theorems 5 and 11, we have that $P_\ell(G, m) = P(G, m)$ whenever $m \geq 4$. Thus, G is enumeratively chromatic-choosable. $\square$

We end this section by presenting an example of a graph with chromatic number 3 that is chromatic-choosable but is not weakly enumeratively chromatic-choosable which answers Theorem 9 in the affirmative for $k = 3$.

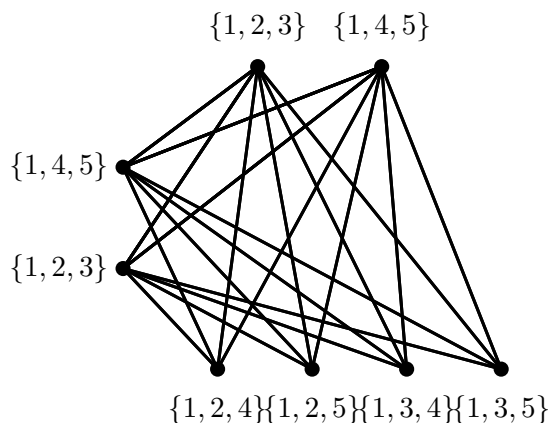


Figure 3: The graph $G = K_{2,2,4}$ with its list assignment L .

Proposition 30. *Suppose $G = K_{2,2,4}$. Then, G is chromatic-choosable, but G is not weakly enumeratively chromatic-choosable.*

Proof. Suppose $G = K_{2,2,4}$ with partite sets $X = \{x_1, x_2\}$, $Y = \{y_1, y_2\}$, and $Z = \{z_1, z_2, z_3, z_4\}$. In [10] it is shown that $\chi_\ell(G) = 3$ which implies that G is chromatic-choosable.

To show that G is not weakly enumeratively chromatic-choosable, we must show $P_\ell(G, 3) < 6 = P(G, 3)$. Suppose L is the 3-assignment for G given by: $L(x_1) = L(y_1) = [3]$, $L(x_2) = L(y_2) = \{1, 4, 5\}$, $L(z_1) = \{1, 2, 4\}$, $L(z_2) = \{1, 2, 5\}$, $L(z_3) = \{1, 3, 4\}$, and $L(z_4) = \{1, 3, 5\}$ (see Figure 3).

Now, suppose that f is a proper L -coloring of G . We first observe that $f(v) \neq 1$ for each $v \in X \cup Y$. Indeed, suppose that some vertex of X is colored with 1. Then, y_1 must be colored with some $a \in \{2, 3\}$ and y_2 must be colored with some $b \in \{4, 5\}$. By the definition of L , there is a vertex $z \in Z$ such that $L(z) = \{1, a, b\}$. Since z is adjacent to all the vertices in $X \cup Y$, there is z can't be colored with an element of $L(z)$ that maintains properness which is a contradiction. By symmetry, no vertex of Y can be colored with 1.

Consequently, we know that $\{2, 3\} \not\subseteq f(Z)$ and $\{4, 5\} \not\subseteq f(Z)$. Since $x_1y_1, x_2y_2 \in E(G)$, we have that $\{1, z\} \not\subseteq f(Z)$ whenever $z \in \{2, 3, 4, 5\}$.

Now, we claim that $1 \in f(Z)$. Suppose that $1 \notin f(Z)$. Since $\{2, 3\} \not\subseteq f(Z)$ and $\{4, 5\} \not\subseteq f(Z)$, there are $a \in \{2, 3\}$ and $b \in \{4, 5\}$ such that $f(Z) \subseteq \{a, b\}$. If a' is the element of $\{2, 3\} - \{a\}$ and b' is the element of $\{4, 5\} - \{b\}$. By the construction of L , there is a vertex $z \in Z$ such that $L(z) = \{1, a', b'\}$ which is impossible since $1 \notin f(Z)$ and $f(Z) \subseteq \{a, b\}$. Thus, $1 \in f(Z)$. Since $\{1, c\} \not\subseteq f(Z)$ for every $c \in \{2, 3, 4, 5\}$, it follows that $f(Z) = \{1\}$.

So, we have that $f(\{x_1, y_1\}) = \{2, 3\}$ and $f(\{x_2, y_2\}) = \{4, 5\}$. It follows that there are only four possible proper L -colorings of G . This means $4 = P(G, L) \geq P_\ell(G, 3)$ as desired. $\square$

Acknowledgment. The authors would like to thank Doug West for suggesting the study of enumeratively chromatic-choosable graphs. The authors would also like to thank the anonymous reviewers of this paper whose comments improved its readability.

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