

The two-variable hypergraph Tutte polynomial via embedding activities and computation-tree activities

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Abstract

We generalize Bernardi's embedding activity definition for the Tutte polynomial from graphs to hypergraphs, and Gordon-Traldi-McMahon's computation-tree activities from matroids to polymatroids. Furthermore, we introduce a new variant of computation-tree activities, and show that they also yield a definition for the polymatroid Tutte polynomial. In all three cases, the key is to show that the given activity notion yields a Crapo decomposition. We also examine the relationship of these activity notions, and pose some questions about them.

Mathematics Subject Classifications: 05C31; 05C65; 52B40

1 Introduction

The Tutte polynomial is one of the most important and well-studied polynomials associated to graphs, and more generally, to matroids, whose specializations count numerous interesting graph-parameters. One of the nice properties of the Tutte polynomial is that it has many equivalent definitions, in particular, using various activity notions. Recently, Bernardi, Kálmán and Postnikov [2] generalized the Tutte polynomial to hypergraphs, and polymatroids. In this paper, we show that various alternative activity-definitions of the graph/matroid Tutte polynomial can be generalized for the case of hypergraphs or polymatroids, and we investigate the relationship of these definitions.

The class of definitions that we are interested in gives the Tutte polynomial as a generating function of various types of activities of spanning trees. The original definition of Tutte used activities with respect to an arbitrary fixed ordering of the edges [13]. A remarkable fact about that definition is that although the activities of individual spanning trees depend on the chosen edge ordering, the generating function does not. A different definition, due to Bernardi [1], defines activities via an embedding of the graph into an

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orientable surface, and in this sense, replaces the edge ordering with a more natural auxiliary structure. Gordon and Traldi [8], and Gordon and McMahon [7] defined activities using a decision tree, called computation-tree activities. Later, these activities were rediscovered, and called Δ -activities by Courtiel. It was shown about each of these activities that they yield an interval decomposition of the Boolean lattice (also called Crapo decomposition) [5, 1, 7, 8]. This is a property that implies that the given activities yield the Tutte polynomial.

Kálmán [9] generalized Tutte's activity definition of $T(x, 1)$ and $T(1, y)$ to hypergraphs, and more generally, to polymatroids. He called these polynomials the *interior polynomial*, and the *exterior polynomial*, respectively. Later, Bernardi, Kálmán and Postnikov [2] gave a two-variable polynomial $\mathcal{T}_P(x, y)$ for a polymatroid P using activities (again, with respect to a fixed ordering of the ground set). This polynomial generalizes the Tutte polynomial $T_M(x, y)$ of matroids (subject to a change of variables). A different two-variable polymatroid Tutte polynomial was defined by Fink and Cameron [3] using lattice point counts instead of activities. Their polynomial is also a transformation of $\mathcal{T}_P(x, y)$. We note that the main technical tool for Bernardi, Kálmán and Postnikov was proving the existence of a Crapo decomposition of $\mathbb{Z}^E$ with respect to their activity notion, and proving that the existence of a Crapo decomposition implies the well-definedness of $\mathcal{T}_P(x, y)$.

In this paper, we show that the embedding activity definition and also the computation-tree activity definition can be generalized (in the case of embedding activities to hypergraphs and in the case of computation-tree activities, to polymatroids), and they all yield Crapo decompositions in their respective settings. Moreover, we propose a modified version of the computation-tree activities, and show that the alternative computation-tree activities also yield a Crapo decomposition. We also investigate whether one of these activity notions is a special case of the other in the more general setting.

It is not a priori obvious if Bernardi's embedding activity definition can be generalized further than the class of graphs. However, in [10], Kálmán and the author of the current paper showed that embedding activities can be generalized to hypergraphs (that can be thought of as a class of polymatroids). They proved that a formula can be given for $\mathcal{T}_H(x, 1)$ of a hypergraph using embedding activities. They also conjectured an analogous formula for $\mathcal{T}_H(1, y)$ using embedding activities, however, their method of proof could not handle external embedding activities. As a consequence, it also remained open whether a formula can be given for a two-variable hypergraph Tutte polynomial using embedding activities. We note that Bernardi's arguments for the graph case cannot be easily generalized to hypergraphs.

In this paper, we resolve this question, and give a Bernardi-style definition for the two-variable hypergraph Tutte polynomial. In the mean time, we confirm that the conjectured Bernardi-style definition for $\mathcal{T}_H(1, y)$ from [10] indeed gives the exterior polynomial.

We note that [10] also proposed an alternative way to define embedding activities. We conjecture that these alternative embedding activities also yield the polynomial $\mathcal{T}_H(x, y)$, but this remains a conjecture (see Section 5).

We also point out that computation-tree activities can be naturally generalized to polymatroids, and they give the Tutte polynomial also in this case. For computation-

tree activities, defining activity using the minimum or the maximum rule gives different notions. We call these, respectively $(ct, \min)$ and $(ct, \max)$ -activities. The computation-tree activities of Gordon-Traldi-McMahon are the $(ct, \max)$ activities, while it seems that $(ct, \min)$ activities were not investigated so far.

Concerning the relationship of these activity notions, we show that there are $(ct, \min)$ -activities that cannot be obtained as a $(ct, \max)$ -activity, and conversely, there are $(ct, \max)$ -activities that cannot be obtained as a $(ct, \min)$ -activity. There are also embedding activities that cannot be obtained as a $(ct, \max)$ -activity. It remains an open question if embedding activities form a subclass of $(ct, \min)$ -activities.

Outline of the paper: In Section 2, we introduce the necessary background on activities, hypergraphs and polymatroids, and we recall how the existence of a Crapo decomposition for a notion of polymatroid-activities implies that the given activity gives the Tutte polynomial as a generating function. In Section 3, we define $(ct, \min)$ and $(ct, \max)$ activities for polymatroids, and prove that both activity notions yield a Crapo decomposition. In Section 4, we recall the definition of embedding activities for hypergraphs, and prove that they yield a Crapo-decomposition. The proof for embedding activities follows similar ideas as the proofs for computation-tree activities, but it needs significantly more technical lemmas. In Section 5, we recall an alternative definition for embedding activities suggested in [10], and pose it as an open problem if those activities also yield $\mathcal{T}_H$. Finally, in Section 6, we investigate the relationship of different activity notions.

2 Preliminaries

In this section we introduce the necessary background on graphs, hypergraphs and activities.

2.1 Graphs, matroids, and their activities

Let G be a graph. As the Tutte polynomial is multiplicative on connected components, we will assume throughout the paper that our graphs are connected. Let T be a spanning tree of G .

If $e \notin T$, then $T \cup e$ has a unique cycle, which is called the *fundamental cycle* of e in T , and is denoted by $C(T, e)$.

If $e \in T$, then $T - e$ has two connected components, and the edges of G connecting the two components form a cut, which is called the *fundamental cut* of e in T , and is denoted by $C^*(T, e)$.

2.1.1 Activities with respect to an edge ordering

Activities with respect to an edge ordering were introduced by Tutte.

Definition 1 (activity (with respect to a fixed edge ordering)). Let G be a graph, and $<$ an arbitrary ordering of the edges of G . Let T be a spanning tree of G .

An edge $e \in T$ is *internally active* if there is no edge $f < e$ such that $T - e \cup f$ is also a spanning tree. $i(T)$ denotes the number of internally active edges for T .

An edge $e \notin T$ is *externally active* if there is no edge $f < e$ such that $T \cup e - f$ is also a spanning tree. $e(T)$ denotes the number of externally active edges for T .

For shortness, we will refer to activities with respect to a fixed edge ordering as *Tutte activities*.

Definition 2 (Tutte polynomial [13]). Let G be a graph. The *Tutte polynomial* of G is defined as

$$T_G(x, y) = \sum_{T \text{ spanning tree}} x^{i(T)} y^{e(T)}.$$

This is well-defined, that is, $T_G(x, y)$ does not depend on the ordering $<$ used to define the activities [13].

Activities can also be defined for matroids, by simply substituting the word “spanning tree” by “basis”, and the word “edge” by “element”.

2.1.2 Embedding activities

Next, we introduce embedding activities, that rely on an embedding instead of a fixed edge ordering. Here, it is important that we have a graph (and not just a matroid).

Let G be a connected graph. A *ribbon structure* of G is a family of cyclic permutations: for each vertex x of G , a cyclic permutation of the edges incident to x is given. For an edge xy of G , we use the notation xy_G^+ for the edge following xy at vertex x . If G is clear from the context, we omit the subscript.

In addition to a ribbon structure, we also need to fix a *basis*, which means a pair (b_0, b_0b_1) , where b_0 is an arbitrary node of the graph and b_0b_1 is an arbitrary edge incident to b_0 .

Suppose that a ribbon structure and a basis are fixed. Then any spanning tree T of G gives a “walk” in the graph. This was defined by Bernardi [1], and following him we call it the *tour* of T .

Definition 3 (Tour of a spanning tree). Let G be a ribbon graph with a basis (b_0, b_0b_1) , and let T be a spanning tree of G . The *tour* of T is a sequence of node-edge pairs, starting with (b_0, b_0b_1) . If the current node-edge pair is (x, xy) and $xy \notin T$, then the current node-edge pair of the next step is (x, xy^+) . If the current node-edge pair is (x, xy) and $xy \in T$, then the current node-edge pair of the next step is (y, yx^+) . The tour stops right before when (b_0, b_0b_1) would once again become the current node-edge pair.

See Figure 1 for an example. Bernardi proved [1, Lemma 4.2 (Lemma 5 in the arxiv version)] that in the tour of a spanning tree T , each edge xy of G becomes current edge twice, in one case with x as current vertex, and in the other case with y as current vertex. This naturally orders the edges of the graph in the order that they first become current.

Bernardi defined the internal and external *embedding activities* of a tree T as the internal and external activities with respect to the order induced by the tour of T . He

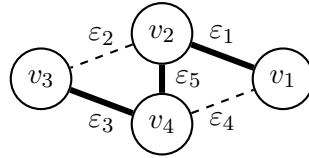


Figure 1: The tour of a spanning tree. Let the ribbon structure be the one induced by the positive orientation of the plane. The edges of the tree are drawn by thick lines, the non-edges by dashed lines. With $b_0 = v_1, b_1 = v_2$, we get the tour $(v_1, \varepsilon_1), (v_2, \varepsilon_2), (v_2, \varepsilon_5), (v_4, \varepsilon_3), (v_3, \varepsilon_2), (v_3, \varepsilon_3), (v_4, \varepsilon_4), (v_4, \varepsilon_5), (v_2, \varepsilon_1), (v_1, \varepsilon_4)$.

then showed that the two-variable generating function of internal and external embedding activities also gives the Tutte polynomial (in particular, the generating function does not depend on the chosen ribbon structure and basis).

Embedding activities were generalized to hypergraphs in [10], where also their connections with geometry were explored. (They are related to dissections of root polytopes of bipartite graphs.) We give these definitions Section 4. In [10], a formula for $\mathcal{T}_{\mathcal{H}}(x, 1)$ was given using embedding activities. In this paper, we show that $\mathcal{T}_{\mathcal{H}}(x, y)$ can also be defined via embedding activities.

2.1.3 Computation-tree activities for matroids

Gordon and Traldi [8], and Gordon and McMahon [7] generalized Tutte-activities by substituting fixed edge orderings with a decision tree. Courtiel [4] later rediscovered these activities, and called them Δ -activities, moreover, he showed that graph embedding activities are a special case.

Take a matroid $M = (E, \mathcal{B})$. Let Δ be a rooted binary decision tree whose vertices are labeled by elements of E , moreover, each branch from the root to a leaf has exactly $|E|$ vertices, and each $e \in E$ occurs once as a label on the branch. (See Figure 2.)

Then, based on Δ , the following two activity notions can be defined [8, 7, 4]. Let $B \in \mathcal{B}$ be a basis. We associate an ordering $\leq_B$ to B the following way. Take the element labelling the root node. If that element is in B , then take the left child of the root, if it is not in B , then take the right child of the root, and continue similarly. The ordering $\leq_B$ is defined as the order in which we see the labels. As on each branch, each vertex appears once as a label, this is a well-defined ordering of E . An element $e \in B$ is then defined *internally* $(ct, \max)$ -active ($e \notin B$ is defined *externally* $(ct, \max)$ -active) in B , if there exists no $f >_B e$ such that $B - e + f$ ($B + e - f$) is a basis. (This is the activity notion defined in [8, 7, 4].) See Figure 2 for an example.

Similarly, one can define $e \in B$ to be *internally* $(ct, \min)$ -active ($e \notin B$ to be *externally* $(ct, \min)$ -active) in B , if there exists no $f <_B e$ such that $B - e + f$ ($B + e - f$) is a basis. This activity notion seems to be new.

Remark 4. Note that for Tutte activities one can reverse the order of the edges and hence activities defined with the min rule are equivalent to activities defined with the max rule.

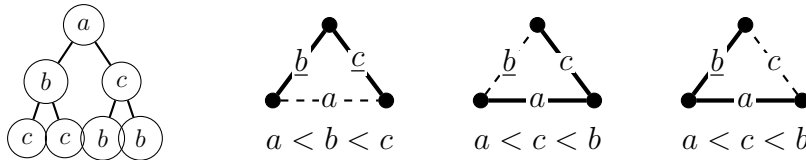


Figure 2: An example for a $(ct, \max)$ -activity. Underneath each spanning tree, we have its corresponding ordering. (Internally or externally) $(ct, \max)$ -active elements are underlined.

However, the definition of the orderings $<_B$ is not symmetric, hence one cannot do this for computation-tree activities. Indeed, in Section 6, we will show examples that there are indeed cases where a $(ct, \max)$ -activity cannot be obtained as a $(ct, \min)$ -activity and vice versa.

2.2 Polymatroids, hypergraphs and their activities

2.2.1 Polymatroids and hypergraphs

Polymatroids were introduced by Edmonds [6], as a generalization of matroids. Here, by polymatroids, we will mean what is commonly called an integer polymatroid.

Definition 5. For a set E , a set $P \subseteq \mathbb{Z}^E$ is called a *polymatroid* if for any $b, b' \in P$, and $e \in E$, if we have $b(e) < b'(e)$, then there exists some $f \in E$ with $b(f) > b'(f)$ such that $b + \mathbf{1}_e - \mathbf{1}_f$ and $b' - \mathbf{1}_e + \mathbf{1}_f$ are both in P .

We call E the *ground set* of the polymatroid. We refer to a vector $b \in P$ as a *basis* of P .

We note that Edmonds also requires that $P \subseteq \mathbb{Z}_{\geq 0}^E$. We follow [2] in omitting this requirement, as the definition of the polymatroid Tutte polynomial is translation invariant.

An important example of a polymatroid is a matroid: If one takes $P \subseteq \mathbb{Z}^E$ to be the set of characteristic vectors of bases, then the symmetric exchange theorem implies that we get a polymatroid in the sense of Definition 5.

Another important example of polymatroids is hypergraphs. A *hypergraph* is an ordered pair $\mathcal{H} = (V, E)$, where V is a finite set and E is a finite multiset of subsets of V . We refer to elements of V as *vertices* and to elements of E as *hyperedges*. It is convenient to represent a hypergraph by a bipartite graph. For a hypergraph $\mathcal{H} = (V, E)$, let the *underlying bipartite graph* $\text{Bip } \mathcal{H}$ be the bipartite graph with vertex classes V and E , where $v \in V$ is connected to $e \in E$ if $v \in e$ in $\mathcal{H}$. We will mostly think of hypergraphs as bipartite graphs. When talking about $\text{Bip } \mathcal{H}$, we will call the elements of $V \cup E$ *nodes*. Specifically, the elements of V will be called *violet* nodes and elements of E *emerald* nodes. We use the words hyperedge and emerald node interchangeably. We use greek letters to denote the edges of $\text{Bip } \mathcal{H}$.

Throughout the paper, we assume that $\mathcal{H}$ is *connected*, by which we mean that $\text{Bip } \mathcal{H}$ is connected.

Spanning trees of graphs are the bases of a matroid: the graphic matroid. For hypergraphs, Kálmán [9] defined hypertrees, that generalize (characteristic vectors of) spanning trees from graphs to hypergraphs.

Definition 6. Let $\mathcal{H} = (V, E)$ be a hypergraph with underlying bipartite graph $\text{Bip } \mathcal{H}$. We say that the vector $h \in \mathbb{Z}^E$ is a *hypertree* if there exists a spanning tree T of $\text{Bip } \mathcal{H}$ that has degree $d_T(e) = h(e) + 1$ at each node $e \in E$. In this case we say that T represents h .

We denote the set of all hypertrees of $\mathcal{H}$ by $H(\mathcal{H})$. If $\mathcal{H}$ is clear from the context, we simply write H . For an example, see Figure 5. Most importantly for us, hypertrees form the bases of a polymatroid [9].

Proposition 7. [9] *Let h and h' be hypertrees and e be a hyperedge with $h(e) < h'(e)$. Then there exists a hyperedge f with $h(f) > h'(f)$ such that $h + \mathbf{1}_e - \mathbf{1}_f$ and $h' - \mathbf{1}_e + \mathbf{1}_f$ are both hypertrees.*

Note that a hypertree might have many different representing spanning trees. In general, we do not care about the representing spanning trees, though in some cases, it will be useful to fix one for each hypertree (see Section 4).

It is an easy exercise to check that if G is a connected graph, then the hypertrees of $\text{Bip } G$ are exactly the characteristic vectors of spanning trees, hence hypertrees indeed generalize spanning trees.

2.2.2 Polymatroid activities with respect to a fixed ordering

Let us recall how [9] and [2] associates activities to bases of a polymatroid. We will refer to these as Tutte activities, since they generalize activities with respect to a fixed ordering of the ground set.

Definition 8. Let P be a polymatroid on the ground set E with an order $<$ on E .

An element $e \in E$ is *internally active* for the basis b , with respect to $<$, if $b - \mathbf{1}_e + \mathbf{1}_f$ is not a basis for any $f < e$.

An element $e \in E$ is *externally active* for the basis b , with respect to $<$, if $b + \mathbf{1}_e - \mathbf{1}_f$ is not a basis for any $f < e$.

Remark 9. We note that activities for graphs (Definition 1) are not a direct special case of this definition, as Definition 1 required that $e \in T$ for an internally active edge, and $e \notin T$ for an externally active edge. As polymatroids can take values other than 0 and 1, those conditions cannot really be generalized to the polymatroid case, however, by Definition 8 an edge $e \notin T$ is always internally active, and an edge $e \in T$ is always externally active. Hence compared to the graph activity definition, Definition 8 shifts the internal activity of each spanning tree by $|V| - 1$ and the external activity of each spanning tree by $|E| - |V| + 1$, modifying the generating function in a trivial way.

Let $Int_{<}(b)$ denote the set of internally active elements of b with respect to the order $<$, and let $Ext_{<}(b)$ denote the set of externally active elements of b with respect to $<$.

Let $i_{<}(b) = |Int_{<}(b)|$ denote the number of internally active elements in b and call this value the *internal activity* of b . Let $e_{<}(b) = |Ext_{<}(b)|$ denote the number of externally active elements in b and call this value the *external activity* of b .

Moreover, let $oi_{<}(b) = |Int_{<}(b) - Ext_{<}(b)|$, $oe_{<}(b) = |Ext_{<}(b) - Int_{<}(b)|$ and $ie_{<}(b) = |Int_{<}(b) \cap Ext_{<}(b)|$ be respectively the number of only internally active, only externally active, and both internally and externally active elements.

Bernardi, Kálmán and Postnikov defined the polymatroid Tutte polynomial the following way:

Definition 10. [2] Let P be a polymatroid. Its Tutte polynomial is

$$\mathcal{T}_P(x, y) = \sum_{b \in P} x^{oi_{<}(b)} y^{oe_{<}(b)} (x + y - 1)^{ie_{<}(b)}.$$

In [2], they show that this polynomial is well-defined, that is, it does not depend on the chosen ordering of the ground set E . Moreover, it has many nice properties. The polymatroid Tutte polynomial does not agree with the usual graphic Tutte polynomial for graphic matroids, but they have the following simple relationship (see [2, Theorem 5.2]): $T_G(x, y) = x^{|E|-|V|+1} y^{|V|-1} \mathcal{T}_G(\frac{x+y-1}{y}, \frac{x+y-1}{x})$.

Two interesting specializations of $\mathcal{T}_P(x, y)$ are

$$\mathcal{T}_P(x, 1) = \sum_{b \in P} x^{i_{<}(b)} \quad \text{and} \quad \mathcal{T}_P(1, y) = \sum_{b \in P} y^{e_{<}(b)},$$

that we call the *interior* and *exterior* polynomial, respectively. We note that in [9], $x^{|E|} \mathcal{T}_P(\frac{1}{x}, 1)$ is called the interior polynomial, and $y^{|E|} \mathcal{T}_P(1, \frac{1}{y})$ is called the exterior polynomial.

2.3 Crapo intervals and Tutte polynomials

The main tool of Bernardi, Kálmán and Postnikov [2] for establishing the well-definedness of the polymatroid Tutte polynomial is defining the corank-nullity polynomial for polymatroids. The corank-nullity polynomial does not depend on any activities, and its well-definedness is immediate. Then, they establish a Crapo interval decomposition for (Tutte-activities of) polymatroids, and prove that the existence of a Crapo decomposition implies that the Tutte polynomial is a certain transformation of the corank-nullity polynomial, which implies the well-definedness of the Tutte polynomial. We will also rely on this idea for all three activity notions investigated in this paper.

In this section, we define the Crapo decomposition of a polymatroid with respect to a general activity notion, and show that the existence of a Crapo decomposition implies that the given activity gives a Tutte polynomial that agrees with the Tutte polynomial defined via Tutte activities (Definition 10). We include the proof for completeness, because the argument is a direct analogue of the corresponding results in [2].

Suppose that we have a notion of activities for a polymatroid $P \subset \mathbb{Z}^E$. By this, we mean that for each basis $b \in P$, we have a set $Int(b) \subseteq E$ of internally active elements, and a set $Ext(b) \subseteq E$ of externally active elements. We define $oi(b) = |Int(b) - Ext(b)|$, $oe(b) = |Ext(b) - Int(b)|$ and $ie(b) = |Int(b) \cap Ext(b)|$.

Definition 11. The *two-variable polymatroid Tutte polynomial* with respect to the activity notion Int, Ext is

$$\mathcal{T}_P^{Int, Ext}(x, y) = \sum_{b \in P} x^{oi(b)} y^{oe(b)} (x + y - 1)^{ie(b)}.$$

The goal is to see when this agrees with $\mathcal{T}_P$ defined via Tutte-activities.

We recall the corank-nullity polynomial of Bernardi, Kálmán and Postnikov [2]. Firstly, one needs a generalization of corank and nullity.

Definition 12. [2] Let $S \subseteq \mathbb{Z}^E$ and $c \in \mathbb{Z}^E$. We denote

$$d_1^<(S, c) = \min\{\sum_{e \in E} \max\{0, c(e) - s(e)\} \mid s \in S\},$$

$$d_1^>(S, c) = \min\{\sum_{e \in E} \max\{0, s(e) - c(e)\} \mid s \in S\}.$$

Here $d_1^<(P, c)$ is a generalization of nullity. Indeed, if we take a graph, that is, the set of hypertrees P is the set of characteristic vectors of spanning trees, then for a $c \in \{0, 1\}^E$, $d_1^<(P, c)$ is exactly the nullity of c . Similarly, $d_1^>(P, c)$ is the generalization of corank.

Using these notions, Bernardi, Kálmán and Postnikov define the following corank-nullity polynomial (understood as a formal power series):

$$\tilde{\mathcal{T}}_P(u, v) = \sum_{c \in \mathbb{Z}^E} u^{d_1^>(P, c)} v^{d_1^<(P, c)}$$

It is clear that $\tilde{\mathcal{T}}_H$ is well-defined. Bernardi, Kálmán and Postnikov [2] shows the well-definedness of $\mathcal{T}_P$ by showing that

$$\tilde{\mathcal{T}}_P(u, v) = \mathcal{T}_P\left(\frac{1}{1-u}, \frac{1}{1-v}\right)$$

which is understood as an identity of formal power series, where $\frac{1}{1-u} = \sum_{i=0}^{\infty} u^i$. In fact, their argument works for an arbitrary activity notion that yields a Crapo decomposition. Let us now define those objects.

Definition 13 (Crapo interval). Let P be a polymatroid, with $b \in P$ a basis. The *Crapo interval* of b with respect to the activities Int, Ext is

$$C^{Int, Ext}(b) = \{y \in \mathbb{Z}^E : (y(e) > b(e) \Rightarrow e \in Ext(b)), (y(e) < b(e) \Rightarrow e \in Int(b))\}$$

For $c \in \mathbb{Z}^E$ and $S \subseteq \mathbb{Z}^E$ let us denote $d_1(S, c) = \min\{\sum_{e \in E} |s(e) - c(e)| \mid s \in S\}$. This is commonly called the Manhattan distance.

Definition 14. We say that the activity notion Int, Ext induces a *Crapo decomposition*, if the following two properties are satisfied for their Crapo intervals.

1. The Crapo intervals $\{C^{Int,Ext}(b) \mid b \in P\}$ partition $\mathbb{Z}^E$.
2. For each $c \in \mathbb{Z}^E$, if $c \in C^{Int,Ext}(b)$, then $d_1(P, c) = d_1(b, c)$.

With exactly the same proof as in [2], we obtain the following result.

Theorem 15. *Let P be a polymatroid with activity notions Int, Ext . If the activities Int, Ext induce a Crapo decomposition, then $\tilde{\mathcal{T}}_P(u, v) = \mathcal{T}_P^{Int,Ext}\left(\frac{1}{1-u}, \frac{1}{1-v}\right)$, hence $\mathcal{T}_P^{Int,Ext} = \mathcal{T}_P$.*

A key Lemma for [2] (and also for us) is the following.

Lemma 16. *[2, Lemma 10.3] For any $c \in \mathbb{Z}^E$ there is at least one $b \in P$ such that*

$$d_1^<(P, c) = d_1^<(b, c) \text{ and } d_1^>(P, c) = d_1^>(b, c).$$

Hence $d_1(P, c) = d_1(b, c) = d_1^<(b, c) + d_1^>(b, c)$, and for any b' with $d_1(P, c) = d_1(b', c)$, we have $d_1^<(P, c) = d_1^<(b', c)$ and $d_1^>(P, c) = d_1^>(b', c)$.

Proof of Theorem 15. We show that $\tilde{\mathcal{T}}_P(u, v) = \mathcal{T}_P^{Int,Ext}\left(\frac{1}{1-u}, \frac{1}{1-v}\right)$ as formal power series, where $\frac{1}{1-u} = \sum_{i=0}^{\infty} u^i$. As by [2], $\tilde{\mathcal{T}}_P(u, v) = \mathcal{T}_P\left(\frac{1}{1-u}, \frac{1}{1-v}\right)$, this means that $\mathcal{T}_P^{Int,Ext}$ agrees with $\mathcal{T}_P$.

By the definition of $\tilde{\mathcal{T}}_P$ and from property (1) of the Crapo decomposition, we have

$$\tilde{\mathcal{T}}_P(u, v) = \sum_{c \in \mathbb{Z}^E} u^{d_1^>(P, c)} v^{d_1^<(P, c)} = \sum_{b \in P} \sum_{c \in C^{Int,Ext}(b)} u^{d_1^>(P, c)} v^{d_1^<(P, c)}$$

By property (2) of Crapo decompositions, and Lemma 16 this further equals

$$= \sum_{b \in P} \sum_{c \in C^{Int,Ext}(b)} u^{d_1^>(b, c)} v^{d_1^<(b, c)}$$

Now, by the definition of $C^{Int,Ext}(b)$, this further equals

$$\begin{aligned} &= \sum_{b \in P} \left(\frac{1}{1-u}\right)^{oi(b)} \left(\frac{1}{1-v}\right)^{oe(b)} \left(\frac{1}{1-u} + \frac{1}{1-v} - 1\right)^{ie(b)} = \\ &= \mathcal{T}_P^{Int,Ext}\left(\frac{1}{1-u}, \frac{1}{1-v}\right) \end{aligned}$$

where the equalities are understood as equalities of formal power series, with $\frac{1}{1-u} = \sum_{i=0}^{\infty} u^i$. $\square$

3 Computation-tree activities for polymatroids

In this section, we show that $(ct, \max)$ and $(ct, \min)$ activities can be naturally generalized to polymatroids, moreover, they both induce Crapo decompositions.

3.1 Definitions

Let P be a polymatroid on ground set E . In this section, for ease of notation, we suppose that $P \subset \mathbb{Z}_{\geq 0}^E$, and for $e \in E$, let $r(e) = \max\{b(e) : b \in P\}$. Let Δ be a rooted tree such that each branch from the root to a leaf has exactly $|E|$ vertices and each $e \in E$ occurs exactly once as a label. Furthermore, if a vertex is labeled by e , then it has $r(e) + 1$ children.

We associate an ordering $<_b$ to a basis $b \in P$ the following way. Let $x \in E$ be the element labeling the root of Δ . If $b(x) = i$, choose the $(i + 1)^{th}$ child of the root, and continue similarly with the element labeling that vertex. (As $0 \leq b(x) \leq r(x)$, this makes sense.) $<_b$ is defined as the ordering of E in which we see the elements on the branch corresponding to b .

Definition 17. An element $e \in E$ is *internally* $(ct, \max)$ -active (resp. *externally* $(ct, \max)$ -active) in b if there exists no $f >_b e$ such that $b - \mathbf{1}_e + \mathbf{1}_f$ (resp. $b + \mathbf{1}_e - \mathbf{1}_f$) is a basis.

An element $e \in E$ is *internally* $(ct, \min)$ -active (resp. *externally* $(ct, \min)$ -active) in b if there exists no $f <_b e$ such that $b - \mathbf{1}_e + \mathbf{1}_f$ (resp. $b + \mathbf{1}_e - \mathbf{1}_f$) is a basis.

Note that because of the asymmetry in the definition of $<_b$, $(ct, \min)$ -activities are formally different from $(ct, \min)$ -activities. We will see in Section 6 (Propositions 60 and 58) that they indeed are different notions.

Let us denote the Crapo interval of a basis b with respect to $(ct, \max)$ -activities (respectively $(ct, \min)$ -activities) by $C_{ct, \max}(b)$ (resp. $C_{ct, \min}(b)$).

We show that both of these activity notions induce a Crapo-decomposition (and hence they define the same Tutte polynomial as activities with respect to a fixed ordering of E). Because of the nonsymmetric definition of the orderings $<_b$, the proofs for $(ct, \max)$ - and for $(ct, \min)$ -activities will be slightly different.

Let us start with a construction that we need in both proofs (and also for embedding activities in Section 4).

For a given vector $c \in \mathbb{Z}^E$, let $P_c = \{b \in P \mid d_1(P, c) = d_1(b, c)\}$. When proving the existence of a Crapo decomposition, to satisfy (2), for each $c \in \mathbb{Z}^E$, we need to find some basis $b \in P_c$ such that c is in the Crapo interval of b .

To achieve this, for all three activity notions investigated in this paper, we will define some complete ordering $<_c$ of P_c . Then, we show that if some $b \in P_c$ does not have c in its Crapo interval, then there is another basis $b' \in P_c$ with $b' <_c b$. As P_c is a finite set, this implies that at some point, we need to find a basis in P_c containing c in its Crapo interval.

The definitions of the orderings $<_c$ will depend on the type of activity, but in all three cases, the key will be a decomposition of E (with respect to c).

Proposition 18. *If for an edge e , there exists $b \in P_c$ with $b(e) > c(e)$, then for each $b' \in P_c$, we have $b'(e) \geq c(e)$, and symmetrically, if there exists $b \in P_c$ with $b(e) < c(e)$, then for each $b' \in P_c$, we have $b'(e) \leq c(e)$.*

Proof. Suppose for a contradiction that there is an element e such that there are $b, b' \in P_c$ with $b(e) < c(e) < b'(e)$. Then, by Proposition 7, there exists an element f with $b(f) > b'(f)$ such that $b + \mathbf{1}_e - \mathbf{1}_f$ and $b' - \mathbf{1}_e + \mathbf{1}_f$ are both bases.

Then $d_1(b + \mathbf{1}_e - \mathbf{1}_f, c) \leq d_1(b, c)$. As $d_1(b, c)$ is minimal among all bases, we need to have $d_1(b + \mathbf{1}_e - \mathbf{1}_f, c) = d_1(b, c)$. This implies that $c(f) \geq b(f)$. Similarly, $d_1(b' - \mathbf{1}_e + \mathbf{1}_f, c) \leq d_1(b', c)$. As $d_1(b', c)$ is also minimal among all bases, we need to have $d_1(b' - \mathbf{1}_e + \mathbf{1}_f, c) = d_1(b', c)$. Hence $c(f) \leq b'(f)$. These together imply that $b(f) \leq b'(f)$, which is a contradiction. $\square$

We say that an element e is *external* with respect to c if there is a basis $b \in P_c$ such that $b(e) < c(e)$. We say that an element e is *internal* with respect to c if there exists a basis $b \in P_c$ with $b(e) > c(e)$. Then by Proposition 18, each element is either internal or external, or neither of them, in which case each $b \in P_c$ has $b(e) = c(e)$. Moreover, for an internal hyperedge, each $b \in P_c$ has $b(e) \geq c(e)$, and for an external hyperedge, each $b \in P_c$ has $b(e) \leq c(e)$.

Now we are ready to prove the Crapo decomposition property for $(\Delta, \max)$ and for $(\Delta, \min)$ -activities.

Theorem 19. *For a polymatroid P and any choice of decision tree Δ ,*

1. *the Crapo intervals $\{C_{\Delta, \max}(b) \mid b \in P\}$ partition $\mathbb{Z}^E$,*
2. *for each $c \in \mathbb{Z}^E$, if $c \in C_{\Delta, \max}(b)$, then $d_1(P, c) = d_1(b, c)$.*

Proof. First, we need to show that if b_1 and b_2 are distinct bases, then $C_{\Delta, \max}(b_1) \cap C_{\Delta, \max}(b_2) = \emptyset$. This can be proved similarly to the Tutte activity case [2, Theorem 9.3]. Take the branch of Δ corresponding to b_1 and the branch corresponding to b_2 . Suppose that the last common vertex of the two branches has label e . Then (since from this point the branches of b_1 and b_2 part) $b_1(e) \neq b_2(e)$. By symmetry, we may assume $b_1(e) > b_2(e)$. Then, by the definition of a polymatroid, there exists $f \in E$ such that $b_1(f) < b_2(f)$ and $b_1 - \mathbf{1}_e + \mathbf{1}_f$ and $b_2 + \mathbf{1}_e - \mathbf{1}_f$ are both bases of P . As the branches of b_1 and b_2 agree until the vertex labeled by e , the elements preceding e in $<_{b_1}$ and in $<_{b_2}$ agree, and for these elements g , we have $b_1(g) = b_2(g)$. Hence $f >_{b_1} e$ and $f >_{b_2} e$, thus, e is internally $(ct, \max)$ -passive in b_1 and externally $(ct, \max)$ -passive in b_2 . Thus, for $x \in C_{ct, \max}(b_1)$ and $y \in C_{ct, \max}(b_2)$ we have a $x(e) \geq b_1(e) > b_2(e) \geq y(e)$, which means $x \neq y$.

Next, we show that for each $c \in \mathbb{Z}^n$, there exists a basis b such that $c \in C_{ct, \max}(b)$.

We define an ordering of the bases in P_c the following way. Let $b, b' \in P_c$, and suppose that the branches of Δ corresponding to them part at a vertex labeled e . Then we have $b(e) \neq b'(e)$. We define $b' <_c b$ if either

- e is external with respect to c and $b'(e) > b(e)$, or
- e is internal with respect to c and $b'(e) < b(e)$.

Otherwise we define $b <_c b'$. It cannot happen that e is neither external nor internal, because in that case $b(e) = b'(e) = c(e)$. Note that $<_c$ is a complete ordering on P_c .

Take an arbitrary basis $b \in P_c$. We show that if $c \notin C_{ct,\max}(b)$, then we can find some $b' \in P_c$ with $b' <_c b$. As P_c is finite, this means that we eventually find a basis $b'' \in P_c$ with $c \in C_{ct,\max}(b'')$.

Suppose that $c \notin C_{ct,\max}(b)$. This means that either there are some element(s) e such that e is externally $(ct, \max)$ -passive in b but $c(e) > b(e)$, or some element(s) e such that e is internally $(ct, \max)$ -passive in b but $c(e) < b(e)$ (or both). Take the first such bad element e according to $<_b$.

Suppose that e is externally $(ct, \max)$ -passive in b but $c(e) > b(e)$. Then, e is an external element with respect to c , and by the definition of external $(ct, \max)$ -passivity, there exists some $e' >_b e$ such that $b' = b + \mathbf{1}_e - \mathbf{1}_{e'}$ is also a basis. Since $b'(e) > b(e)$ and $c(e) > b(e)$, we have $d_1(b', c) \leq d_1(b, c)$, hence $b' \in B_c$. As the branch of b contains the label e before label e' , the branches of b and b' part at e . We conclude that $b' <_c b$.

The case when the first bad element e is internal is completely analogous. $\square$

Proposition 20. *For a polymatroid P and any choice of decision tree Δ ,*

1. *the Crapo intervals $\{C_{ct,\min}(b) \mid b \in P\}$ partition $\mathbb{Z}^E$,*
2. *for each $c \in \mathbb{Z}^E$, if $c \in C_{ct,\min}(b)$, then $d_1(P, c) = d_1(b, c)$.*

Proof. First, we need to show that if b_1 and b_2 are distinct bases, then $C_{ct,\min}(b_1) \cap C_{ct,\min}(b_2) = \emptyset$. Take the branch of Δ corresponding to b_1 and the branch corresponding to b_2 . Suppose that the last common vertex of the two branches has label e . Then (since from this point the branches of b_1 and b_2 part) $b_1(e) \neq b_2(e)$. By symmetry, we may assume $b_1(e) > b_2(e)$. Then, by the definition of a polymatroid, there exists $f \in E$ such that $b_1(f) < b_2(f)$ and $b_1 - \mathbf{1}_e + \mathbf{1}_f$ and $b_2 + \mathbf{1}_e - \mathbf{1}_f$ are both bases of P . As the branches of b_1 and b_2 agree until the vertex labeled by e , the elements preceding e in $<_{b_1}$ and in $<_{b_2}$ agree, and for these elements g , we have $b_1(g) = b_2(g)$. Hence $f >_{b_1} e$ and $f >_{b_2} e$, thus, f is internally $(ct, \min)$ -passive in b_2 and externally $(ct, \min)$ -passive in b_1 . Thus, for $x \in C_{ct,\min}(b_1)$ and $y \in C_{ct,\min}(b_2)$ we have a $x(f) \leq b_1(f) < b_2(f) \leq y(f)$, which means $x \neq y$.

Next, we show that for each $c \in \mathbb{Z}^n$, there exists a basis b such that $c \in C_{ct,\min}(b)$.

We define an ordering of the bases in P_c the following way. Let $b, b' \in P_c$, and suppose that the branches of Δ corresponding to them part at a vertex labeled e . Then we have $b(e) \neq b'(e)$. We define $b' <_c b$ if either

- e is external with respect to c and $b'(e) < b(e)$, or
- e is internal with respect to c and $b'(e) > b(e)$.

Otherwise we define $b <_c b'$. (Note that the ordering $<_c$ was defined differently here than for $(ct, \max)$ -activities.) Again, it cannot happen that e is neither external nor internal, because in that case $b(e) = b'(e) = c(e)$. Note also that $<_c$ is a complete ordering on P_c .

Take an arbitrary basis $b \in P_c$. We show that if $c \notin C_{ct,\min}(b)$, then we can find some $b' \in P_c$ with $b' <_c b$. As P_c is finite, this means that we eventually find a basis $b'' \in P_c$ with $c \in C_{ct,\min}(b'')$.

Suppose that $c \notin C_{ct, \min}(b)$. This means that either there are some element(s) e such that e is externally $(ct, \min)$ -passive in b but $c(e) > b(e)$, or some element(s) e such that e is internally $(ct, \min)$ -passive in b but $c(e) < b(e)$ (or both). Take the first such bad element e according to $<_b$.

Suppose that e is externally $(ct, \min)$ -passive in b but $c(e) > b(e)$. Then, by the definition of external $(ct, \min)$ -passivity, there exists some $e' <_b e$ such that $b' = b + \mathbf{1}_e - \mathbf{1}_{e'}$ is also a basis. Since $b(e) < b'(e)$ and $b(e) < c(e)$, we have $d_1(b', c) \leq d_1(b, c)$. Since $d_1(b, c) = d_1(P, c)$, we need to have $d_1(b', c) = d_1(b, c)$ (in other words, $b' \in P_c$), and $|b'(e') - c(e')| = |b(e') - c(e')| + 1$. The latter implies $b'(e') < b(e') \leq c(e')$.

As the branch of b contains the label e' before label e , the branches of b and b' part at e' . As we have $b'(e') < b(e') \leq c(e')$, we conclude that e' is an external element with respect to c , and $b' <_c b$.

The case when the first bad element e is internal is completely analogous. $\square$

Corollary 21. *For any nonnegative polymatroid P and decision tree Δ , the polynomials $\mathcal{T}_P^{(ct, \min)}(x, y)$ and $\mathcal{T}_P^{(ct, \max)}(x, y)$ are well-defined and both agree with $\mathcal{T}_P(x, y)$.*

Proof. This follows from Theorems 20, 19 and 15. $\square$

4 Embedding activities for hypergraphs

In this section, we show that hypergraph embedding activities induce a Crapo decomposition, hence their two-variable generating function also gives the hypergraph Tutte polynomial. The main idea of the proof will be similar to its counterparts about $(ct, \min)$ and $(ct, \max)$ activities, but it needs more preparation, and some technical lemmas.

4.1 Definitions and statements

We start by recalling how embedding activities can be generalized from graphs to hypergraphs. This was done in [10], but in that paper, it was only proved that the generating function of internal embedding activities gives $\mathcal{T}(x, 1)$.

To define embedding activities for hypergraphs, we assume that $\text{Bip } \mathcal{H}$ has a ribbon graph structure, and a basis $(b_0, b_0 b_1)$ is fixed (where b_0 is a node of $\text{Bip } \mathcal{H}$ (either from V or from E) and $b_0 b_1$ is an edge of $\text{Bip } \mathcal{H}$ incident to b_0).

To generalize Bernardi's embedding activities, one needs to generalize the tour of a spanning tree to hypertrees. To do this, we will make use of a nice representation of hypertrees.

Recall that hypertrees are exactly those vectors $h \in \mathbb{Z}^E$ such that $h + \mathbf{1}_E$ is the degree sequence of some spanning tree of G . It is very useful to have a system of representing spanning trees at hand. Fortunately, ribbon structures provide us such trees.

Definition 22 (Jaeger tree [10]). In a bipartite graph with a ribbon structure and basis, we call a spanning tree T a *Jaeger tree* if for each edge $ev \notin T$ with $e \in E$ and $v \in V$, the tour of T has (e, ev) as a current node-edge pair before (v, ev) . In other words, in the tour of a Jaeger tree, each non-edge is first seen at its emerald endpoint.

Example 23. See Figure 3 for examples of Jaeger trees. The last panel shows the labels of the nodes. In particular, the tour of the tree on the first panel is $(v_0, v_0e_0), (e_0, e_0v_1), (e_0, e_0v_0), (v_0, v_0e_1), (e_1, e_1v_1), (e_1, e_1v_2), (e_1, e_1v_0), (v_0, v_0e_3), (e_3, e_3v_2), (v_2, v_2e_1), (v_2, v_2e_2), (e_2, e_2v_1), (v_1, v_1e_1), (v_1, v_1e_0), (v_1, v_1e_2), (e_2, e_2v_2), (v_2, v_2e_3), (e_3, e_3v_0)$. This is a Jaeger tree since (e_0, e_0v_1) precedes (v_1, v_1e_0) , (e_1, e_1v_1) precedes (v_1, v_1e_1) and (e_1, e_1v_2) precedes (v_2, v_2e_1) .

Theorem 24. [10] *Let $\mathcal{H}$ be a connected hypergraph, and fix an arbitrary ribbon structure and basis. Then for each hypertree h , there is exactly one Jaeger tree representing h .*

Remark 25. Of course one could also define Jaeger trees with the rule that each non-tree edge is first seen at its violet endpoint. In [10] both types of trees are investigated, and are called emerald, and violet Jaeger trees, respectively. In this paper, we will define embedding activities using emerald Jaeger trees, and hence we drop the word “emerald”. One could also try to define embedding activities based on violet Jaeger trees, but it is yet open if this can be done successfully, see Section 5.

We note that Jaeger trees also have a nice geometric interpretation as a dissection of the root polytope of $\text{Bip } \mathcal{H}$ (see [10, Section 6]). While this interpretation is often very useful, it will not play a role in this paper.

Since given a ribbon structure and basis, we now have a well-defined set of representing spanning trees for the hypertrees, we can associate to hypertrees the tours of their Jaeger trees. Then, we can use the tour to define an ordering of the hyperedges, as we now explain.

Definition 26 (ordering corresponding to a hypertree, $<_h$). Let $\mathcal{H} = (V, E)$ be a hypergraph and fix a ribbon structure on $\text{Bip } \mathcal{H}$ and a basis (b_0, b_0b_1) . Let h be a hypertree. Then $<_h$ is a complete ordering of E defined the following way.

Let T be the unique Jaeger tree representing h . For $e, f \in E$, we define $e <_h f$, if in the tour of T , the first time that e is reached is before the first time that f is reached. We denote $e \leq_h f$ if either $e = f$ or $e <_h f$.

Example 27. The first panel of Figure 3 shows the hypertree $h(e_0) = 0, h(e_1) = 0, h(e_2) = 1, h(e_3) = 1$. The ordering $<_h$ corresponding to this hypertree is $e_0 < e_1 < e_3 < e_2$.

Definition 28 (Embedding activities). Let $\mathcal{H} = (V, E)$ be a hypergraph and fix a ribbon structure on $\text{Bip } \mathcal{H}$ and a basis (b_0, b_0b_1) .

A hyperedge $e \in E$ is *internally embedding active* for the hypertree h (with respect to the ribbon structure and basis), if $h - \mathbf{1}_e + \mathbf{1}_f$ is not a hypertree for any $f <_h e$.

A hyperedge $e \in E$ is *externally embedding active* for the hypertree h (with respect to the ribbon structure and basis), if $h + \mathbf{1}_e - \mathbf{1}_f$ is not a hypertree for any $f <_h e$.

We denote by $\text{Int}_{em}(h)$ the set of internally embedding active elements of h , and by $\text{Ext}_{em}(h)$ the set of externally embedding active elements of h .

We denote $oi_{em}(h) = |\text{Int}_{em}(h) - \text{Ext}_{em}(h)|$, $oe_{em}(h) = |\text{Ext}_{em}(h) - \text{Int}_{em}(h)|$, $ie_{em}(h) = |\text{Int}_{em}(h) \cap \text{Ext}_{em}(h)|$.

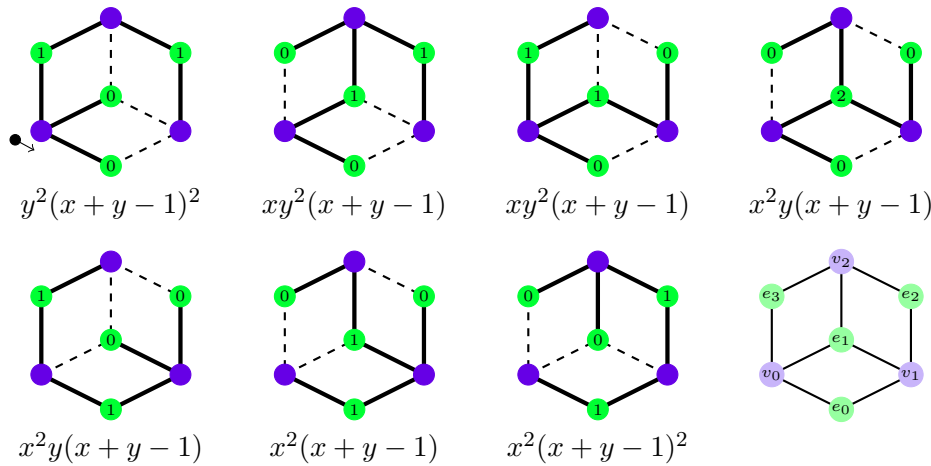


Figure 3: A bipartite graph corresponding to a hypergraph with 3 vertices (violet nodes) and 4 hyperedges (emerald nodes). Take the ribbon structure induced by the positive orientation of the plane, and basis (v_0, v_0e_0) (see the last panel for the notations). The first 7 panels show the 7 hypertrees of the hypergraph, together with their representing Jaeger trees, and the corresponding terms of $\mathcal{T}_{\mathcal{H}}^{em}$.

Example 29. Take the graph $\text{Bip } \mathcal{H}$ of Figure 3 with the indicated ribbon structure and basis. (For the node names, see the last panel.) The 6th panel shows the Jaeger tree of the hypertree h with $h(e_0) = h(e_1) = 1$, $h(e_2) = h(e_3) = 0$. We have $e_0 <_h e_2 <_h e_1 <_h e_3$. For the hypertree h , hyperedge e_0 is both internally and externally active (since it is the smallest element with respect to $<_h$). Hyperedge e_2 is internally active, since a hypertree is nonnegative, however, it is externally passive since $h + \mathbf{1}_{e_2} - \mathbf{1}_{e_0}$ is a hypertree. Hyperedge e_1 is both internally and externally passive, since $h + \mathbf{1}_{e_1} - \mathbf{1}_{e_0}$ and $h - \mathbf{1}_{e_1} + \mathbf{1}_{e_2}$ are both hypertrees. Finally, e_3 is internally active since $h(e_3) = 0$, but externally passive since $h + \mathbf{1}_{e_3} - \mathbf{1}_{e_0}$ is a hypertree.

In accordance with the general definition (Definition 11), we define:

Definition 30. For a connected hypergraph $\mathcal{H}$, the *two variable Tutte polynomial via embedding activities* is

$$\mathcal{T}_{\mathcal{H}}^{em}(x, y) = \sum_{h \in H} x^{oi_{em}(h)} y^{oe_{em}(h)} (x + y - 1)^{ie_{em}(h)}.$$

Example 31. For the hypergraph of Figure 3, we have

$$\mathcal{T}_{\mathcal{H}}^{em}(x, y) = \begin{matrix} & & y^2 & -2y^3 & +y^4 \\ & & -4xy^2 & +4xy^3 & \\ -3x^2y & +6x^2y^2 & & & \\ -x^3 & +4x^3y & & & \\ +x^4 & & & & \end{matrix}.$$

The well-definedness of $\mathcal{T}^{em}$ follows from Theorem 15, and the following theorem, that will be proved in Subsection 4.5. Let us denote the Crapo interval of a hypertree h with respect to embedding activities by $C_{em}(h)$.

Theorem 32. *Let $\mathcal{H}$ be an embedded hypergraph, and let us denote $H(\mathcal{H}) = H$.*

1. *The Crapo intervals $\{C_{em}(h) \mid h \in H\}$ partition $\mathbb{Z}^E$.*
2. *For each $c \in \mathbb{Z}^E$, if $c \in C_{em}(h)$, then $d_1(H, c) = d_1(h, c)$.*

By Theorem 15, we obtain:

Corollary 33. *For any embedded hypergraph $\mathcal{H}$, $\mathcal{T}_{\mathcal{H}}^{em}(x, y)$ is well-defined and it agrees with $\mathcal{T}_{\mathcal{H}}(x, y)$.*

Note the following two specializations:

$$\mathcal{T}_{\mathcal{H}}^{em}(x, 1) = \sum_{h \in H} x^{oi_{em}(h)} x^{ie_{em}(h)} = x^{|Int_{em}(h)|}$$

and

$$\mathcal{T}_{\mathcal{H}}^{em}(1, y) = \sum_{h \in H} y^{oe_{em}(h)} y^{ie_{em}(h)} = y^{|Ext(h)|}.$$

In [10], Kálmán and the author proved that $\mathcal{T}_{\mathcal{H}}^{em}(x, 1)$ is well-defined and agrees with $\mathcal{T}_{\mathcal{H}}(x, 1)$. Theorem 32 also implies the following, which was stated as a conjecture in [10].

Corollary 34. *$\mathcal{T}_{\mathcal{H}}^{em}(1, y)$ is well-defined and it agrees with $\mathcal{T}_{\mathcal{H}}(1, y)$.*

The proof of Theorem 32 will follow the same ideas as the proofs of Theorems 19 and 20 for $(ct, \max)$ and $(ct, \min)$ -activities, but there are significantly more subtleties that we need to address. Because of these, we need to prove some lemmas first. Before getting started with the lemmas, let us give some hints that proving the Crapo decomposition property for embedding activities is indeed a more subtle business than for computation-tree activities.

Recall that computation-tree activities were defined in two variants, as $(ct, \max)$, and as $(ct, \min)$ -activities, and both notions induced Crapo decompositions. On the other hand, we only defined embedding activities with the minimum rule. As we next point out, embedding activities with the maximum rule do not induce a Crapo decomposition.

Definition 35 (Max-embedding activities). Let $\mathcal{H} = (V, E)$ be a hypergraph and fix a ribbon structure on $\text{Bip } \mathcal{H}$ and a basis $(b_0, b_0 b_1)$.

A hyperedge $e \in E$ is *internally $(em, \max)$ -active* for the hypertree h (with respect to the ribbon structure and basis), if $h - \mathbf{1}_e + \mathbf{1}_f$ is not a hypertree for any $f >_h e$.

A hyperedge $e \in E$ is *externally $(em, \max)$ -active* for the hypertree h (with respect to the ribbon structure and basis), if $h + \mathbf{1}_e - \mathbf{1}_f$ is not a hypertree for any $f >_h e$.

Proposition 36. *$(em, \max)$ -activities do not induce a Crapo-decomposition.*

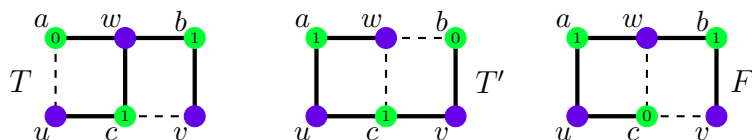


Figure 4: Three Jaeger trees for the ribbon structure which is induced by the positive orientation of the plane around emerald vertices, and by the negative orientation around violet vertices, and the basis (u, uc) .

Proof. Consider the graph on Figure 4. Take the ribbon structure which is induced by the positive orientation of the plane around emerald vertices, and by the negative orientation around violet vertices, and let the basis be (u, uc) . Then the trees T and T' on first two panels are Jaeger trees, representing respectively the hypertrees $h = (0, 1, 1)$ and $h' = (1, 0, 1)$ (on a, b and c).

Moreover, we have $c <_h a <_h b$ and $c <_{h'} b <_{h'} a$.

Note that b is internally $(em, \max)$ -active in h , since b is the maximal element of $\{a, b, c\}$ with respect to $<_h$.

Also, note that a is internally $(em, \max)$ -active in h' since a is the maximal element of $\{a, b, c\}$ with respect to $<_{h'}$.

Hence the vector $(0, 0, 1)$ is in the Crapo-interval of both h and h' . $\square$

Remark 37. Let us compare the situation of $(em, \max)$ -activities to the proof that $(ct, \max)$ -activities induce a Crapo decomposition (Theorem 19).

For $(ct, \max)$ -activities, when proving that $C_{ct, \max}(b) \cap C_{ct, \max}(b') = \emptyset$, one took the first node of the decision tree Δ where the branches of b and b' diverged. The label of that node was an element e such that $b(e) \neq b'(e)$. Moreover, according to both $<_b$ and $<_{b'}$, e was the first element where b and b' differed.

On the other hand, we can see that for embedding activities, the first difference in the Jaeger trees can be at a hyperedge e such that $h(e) = h'(e)$ (as in the case of Figure 4 with $e = c$). Hence it can happen (and we again see it on Figure 4) that there is no well-defined “first difference” between h and h' .

Remark 38. The fact that $(em, \max)$ -activities do not induce a Crapo decomposition makes it even more surprising that embedding activities (defined with the minimum rule) do induce a Crapo decomposition, and perhaps explains the long and technical proof.

As we will later see in Section 6, embedding activities do not fit into the framework of $(ct, \max)$ -activities. It is an open problem whether they are a subclass of $(ct, \min)$ -activities. (Obviously, the orderings $<_h$ are not coming from a decision tree, as seen for example for the hypertrees of Figure 4. However, there might still be orderings coming from a decision tree that induce the same activities as the orderings $<_h$.)

4.2 Lemmas on Jaeger trees and embedding orders

We collect some properties of Jaeger trees that we will need. Throughout this section, fix a ribbon structure and basis for $\text{Bip } \mathcal{H}$.

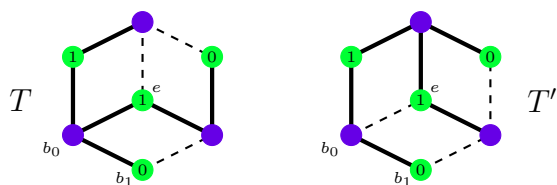


Figure 5: A hypertree and two representations.

We first recall an alternative characterization of Jaeger trees from [11] that uses an ordering of all spanning trees of $\text{Bip } \mathcal{H}$.

For any two spanning trees T and T' , their tours (Definition 3)) have some initial segment that coincides (this segment might be empty), and in this initial segment the next node-edge pair is chosen the same way for the two trees. Hence, the first difference between the tours of T and T' needs to be that some node-edge pair (x, xy) is treated differently for the two trees: $xy \in T - T'$ or $xy \in T' - T$. This enables us to define an ordering $\prec$ of spanning trees of $\text{Bip } \mathcal{H}$.

Definition 39 ($\prec$). For two spanning trees T and T' of $\text{Bip } \mathcal{H}$, we set $T \prec T'$ if for the first difference (x, xy) of their respective tours, either x is an emerald node and $xy \in T' - T$ or x is a violet node and $xy \in T - T'$.

The following is the key property of Jaeger trees.

Theorem 40. [11, Theorems 7.1 and 7.6] *For each hypertree h , there is a unique representing Jaeger tree T . Moreover, T is the minimal tree according to $\prec$ among all trees representing h .*

We note that the minimality in the above theorem implies that for a given hypertree, the unique representing Jaeger tree can be computed greedily. This is explained in more detail in [11, Section 7].

Example 41. Take Figure 5. Let the ribbon structure be the one induced by the positive orientation of the plane, and take the basis (b_0, b_0b_1) according to the figure. T (on the left panel) and T' (on the right panel) are two spanning trees representing the same hypertree. T is Jaeger, while T' is not, since $b_0e \notin T'$ is such that (b_0, b_0e) precedes (e, b_0e) in the tour of T' . The first difference between the tours of T and T' is at (b_0, b_0e) , where b_0 is violet and $b_0b_1 \in T - T'$. Hence indeed $T \prec T'$, cf. Theorem 40.

For a spanning tree T of $\text{Bip } \mathcal{H}$ and edge $\varepsilon \in T$, we call the component of $T - \varepsilon$ containing b_0 the *base component*. The following property of fundamental cuts of Jaeger trees (an updated version of [10, Lemma 6.13]) makes them very useful to us.

Lemma 42. [10, Lemma 6.13] *Let T be a Jaeger tree of $\text{Bip } \mathcal{H}$, and $ev \in T$. If $e'v' \in C^*(T, ev) - T$ is an edge such that its emerald endpoint e' is in the base component, then $(e', e'v')$ comes earlier in the tour of T than (e, ev) .*

Lemma 43. Let T be a Jaeger tree of $\text{Bip } \mathcal{H}$, representing hypertree h , and let $ev \in T$ such that e is in the base component of $T - ev$. Let E_1 be the set of emerald nodes not in the base component of $T - ev$. Then for any $f \in E_1$, we have $f >_h e$.

Proof. The tour of T only reaches the nodes of E_1 after traversing ev . $\square$

4.3 Lemmas on exchanges

The following lemma is a special case of a well-known lemma in matroid theory, see for example [12, Theorem 39.13]. We sketch its proof for completeness.

Lemma 44. [12, Theorem 39.13] Let $\eta_1, \dots, \eta_t, \varepsilon_1, \dots, \varepsilon_t$ be edges in the graph G , and let T be a spanning tree such that $\eta_1, \dots, \eta_t \in T$, $\varepsilon_1, \dots, \varepsilon_t \notin T$, $\eta_i \in C(T, \varepsilon_i) \forall i$ and $\eta_i \notin C(T, \varepsilon_j)$ for $j < i$. Then $T - \{\eta_1, \dots, \eta_t\} + \{\varepsilon_1, \dots, \varepsilon_t\}$ is also a spanning tree.

Proof. Let $T_t = T - \eta_t + \varepsilon_t$. Then T_t is a spanning tree since $\eta_t \in C(T, \varepsilon_t)$. Moreover, since $\eta_t \notin C(T, \varepsilon_j)$ for $j < t$, we have $C(T_t, \varepsilon_j) = C(T, \varepsilon_j)$ for all $j < t$. This implies that $T_{t-1} = T_t - \eta_{t-1} + \varepsilon_{t-1}$ is also a spanning tree, moreover, since $\eta_t, \eta_{t-1} \notin C(T, \varepsilon_j)$ for $j < t-1$, $C(T_{t-1}, \varepsilon_j) = C(T, \varepsilon_j)$ for all $j < t-1$. We may continue switching edges like this until we arrive at $T_1 = T - \{\eta_1, \dots, \eta_t\} + \{\varepsilon_1, \dots, \varepsilon_t\}$. $\square$

Let us state two variations of Lemma 44, since these will be the formulations convenient to us.

Lemma 45. Let $e_0v_0, \dots, e_{t-1}v_{t-1}$ and $e_1u_1, \dots, e_tu_t$ be edges in the graph G , and let T be a spanning tree such that $e_0v_0, \dots, e_{t-1}v_{t-1} \in T$, $e_1u_1, \dots, e_tu_t \notin T$ and $e_iv_i \in C(T, e_{i+1}u_{i+1})$ for $i = 0, \dots, t-1$. Then there exist indices $i_0 = 0 < i_1 < \dots < i_s = t$ such that

$$T - \{e_{i_0}v_{i_0}, \dots, e_{i_{s-1}}v_{i_{s-1}}\} + \{e_{i_1}u_{i_1}, \dots, e_{i_s}u_{i_s}\}$$

is a spanning tree.

Proof. Take the following auxiliary digraph D : Let the vertex set be $\{0, \dots, t\}$. Draw an edge from i to j if $e_iv_i \in C(T, e_ju_j)$. Then, by our assumptions, an edge points from i to $i+1$ for each $0 \leq i \leq t-1$, and there might also be other edges in the digraph. In any case, there is a directed path in D from 0 to t . Choose a shortest directed path from 0 to t in D , and let the vertices of this path be $i_0 = 0, i_1, \dots, i_s = t$. Then $e_{i_j}v_{i_j} \in C(T, e_{i_{j+1}}u_{i_{j+1}})$ for each $0 \leq j < s$ by construction, and $e_{i_j}v_{i_j} \notin C(T, e_{i_{j+1}+r}u_{i_{j+1}+r})$ for $r > 0$ because otherwise we could pick a shorter path from 0 to t in D .

Now we can apply Lemma 44 for T with $\eta_j = e_{i_{s-j}}v_{i_{s-j}}$ and $\varepsilon_j = e_{i_{s-j+1}}u_{i_{s-j+1}}$. We get that $T - \{e_{i_0}v_{i_0}, \dots, e_{i_{s-1}}v_{i_{s-1}}\} + \{e_{i_1}u_{i_1}, \dots, e_{i_s}u_{i_s}\}$ is indeed a spanning tree. $\square$

Lemma 46. Let $e_0v_0, \dots, e_{t-1}v_{t-1}$ and $e_1u_1, \dots, e_tu_t$ be edges in the graph G , and let T be a spanning tree such that $e_0v_0, \dots, e_{t-1}v_{t-1} \notin T$, $e_1u_1, \dots, e_tu_t \in T$ and $e_{i+1}u_{i+1} \in C(T, e_iv_i)$ for $i = 0, \dots, t-1$. Then there exist indices $i_0 = 0 < i_1 < \dots < i_s = t$ such that

$$T + \{e_{i_0}v_{i_0}, \dots, e_{i_{s-1}}v_{i_{s-1}}\} - \{e_{i_1}u_{i_1}, \dots, e_{i_s}u_{i_s}\}$$

is a spanning tree.

Proof. Let us again define an auxiliary digraph D on the vertex set $\{0, \dots, t\}$. Draw an edge from i to j if $e_j u_j \in C(T, e_i v_i)$. Then, by our assumptions, an edge points from i to $i+1$ for each $0 \leq i \leq t-1$, and there might also be other edges in the digraph. In any case, there is a directed path in D from 0 to t . Choose a shortest directed path from 0 to t in D , and let the vertices of this path be $i_0 = 0, i_1, \dots, i_s = t$. Then $e_{i_{j+1}} u_{i_{j+1}} \in C(T, e_{i_j} v_{i_j})$ for each $0 \leq j < s$, and $e_{i_{j+1}+r} u_{i_{j+1}+r} \notin C(T, e_{i_j} v_{i_j})$ for $r > 0$ because otherwise we could pick a shorter path from 0 to t in D . Now we can apply Lemma 44 for T with $\eta_j = e_{i_{k+j}} u_{i_{k+j}}$ and $\varepsilon_j = e_{i_{k+j-1}} v_{i_{k+j-1}}$. We get that $T + \{e_{i_0} v_{i_0}, \dots, e_{i_{s-1}} v_{i_{s-1}}\} - \{e_{i_1} u_{i_1}, \dots, e_{i_s} u_{i_s}\}$ is indeed a spanning tree. $\square$

4.4 Three technical lemmas on first differences of Jaeger trees

In this section, we collect the 3 main technical lemmas needed to prove Theorem 32.

Lemma 47. *Let h and h' be hypertrees with respective Jaeger trees T and T' . Suppose that the first difference between the tours of T and T' is at (e, ε) , where $\varepsilon \in T' - T$.*

Then there exists a hyperedge f such that $h(f) > h'(f)$, moreover, $h - \mathbf{1}_f + \mathbf{1}_e$ and $h' + \mathbf{1}_f - \mathbf{1}_e$ are both hypertrees, and $e <_h f$ and $e <_{h'} f$.

Moreover, for the Jaeger tree F of $h' + \mathbf{1}_f - \mathbf{1}_e$, we have $\varepsilon \notin F$, and the tours of T and F agree up to (and including) (e, ε) .

For an example of such a situation, see the first two panels of Figure 4, where $e = c$, $\varepsilon = cv$ and $f = b$. The Jaeger tree F representing $h' + \mathbf{1}_f - \mathbf{1}_e$ can be seen on the third panel.

Proof. Recall that $<_h$ is defined as the order in which the tour of T reaches the nodes in E . We first prove that there exists f with $h(f) > h'(f)$ where $h - \mathbf{1}_f + \mathbf{1}_e$ and $h' + \mathbf{1}_f - \mathbf{1}_e$ are both hypertrees, and $e <_h f$ and $e <_{h'} f$. Then, at the end we prove that the Jaeger tree F of $h' + \mathbf{1}_f - \mathbf{1}_e$ also satisfies the required properties.

Take the fundamental cut $C^*(T', \varepsilon)$. We call the connected component of $T' - \varepsilon$ containing b_0 the *base component*, moreover, we denote its set of violet nodes by V_0 and its set of emerald nodes by E_0 . Also, we denote $V_1 = V - V_0$ and $E_1 = E - E_0$.

Apply Lemma 42 to the Jaeger tree T' . This implies that each edge of $C^*(T', \varepsilon) - \varepsilon$ that has its emerald endpoint in the base component became current in the tour of T' before (e, ε) . Since the tours of T and T' coincide until (e, ε) , these edges also became current in the tour of T , and as they are not in T' , they are also not in T . As also $\varepsilon \notin T$, we conclude:

$$T \text{ has no edge from } C^*(T', \varepsilon) \text{ that has an endpoint in } E_0. \quad (1)$$

We claim that for any $g \in E_1$, we have $e <_h g$ and $e <_{h'} g$. Indeed, by Lemma 43, we have $e <_{h'} g$. As the tours of T and T' agree until reaching e , this implies that e is also reached before g in T , hence also $e <_h g$. We will look for our hyperedge f within E_1 .

Take the fundamental cycle $C(T, \varepsilon)$. A cycle and a cut always meet in an even number of edges, and $\varepsilon \in C^*(T', \varepsilon) \cap C(T, \varepsilon)$, hence the cycle $C(T, \varepsilon)$ contains at least one edge

$\eta = e_1 u_1$ of $T \cap C^*(T', \varepsilon)$. For this η , we also have $\varepsilon \in C(T', \eta)$, as η has one endpoint in $E_0 \cup V_0$ and one endpoint in $E_1 \cup V_1$, and the only edge of T' connecting these two sets is ε . By (1), e_1 cannot be in E_0 , thus, $e_1 \in E_1$.

Since $\eta \in C(T, \varepsilon)$, the graph $T - \eta + \varepsilon$ is a spanning tree. Moreover, it represents $h - \mathbf{1}_{e_1} + \mathbf{1}_e$ where $e <_h e_1$. Also, $\varepsilon \in C(T', \eta)$, hence $T' - \varepsilon + \eta$ is a spanning tree, and it represents $h' + \mathbf{1}_{e_1} - \mathbf{1}_e$ where $e <_{h'} e_1$.

Thus, if $h(e_1) > h'(e_1)$, then e_1 is a suitable choice for f . If $h(e_1) \leq h'(e_1)$, then we have to continue looking for an appropriate hyperedge f . We describe the general step of this process. In a general step, we will have a spanning tree $\tilde{T}$ satisfying the following properties:

$$\begin{aligned} \tilde{T} \subseteq T \cup T' \text{ is a spanning tree representing } h, \\ \tilde{T} \Delta T' \subseteq T \Delta T', \text{ and } \varepsilon \notin \tilde{T}. \end{aligned} \quad (2)$$

At the beginning, we have $\tilde{T} = T$, that satisfies all requirements. In a general step, $\tilde{T}$ is typically not a Jaeger tree (as T is the only Jaeger tree representing h). However, since $\tilde{T} \subseteq T \cup T'$ and $\varepsilon \notin \tilde{T}$, we still have the following property:

$$\tilde{T} \text{ has no edge from } C^*(T', \varepsilon) \text{ that has an endpoint in } E_0. \quad (3)$$

Let $\varepsilon_0 = \varepsilon$, and suppose that for some $t \geq 1$, we have a sequence of edges

$$\varepsilon_0, \eta_1, \varepsilon_1, \dots, \eta_{t-1}, \varepsilon_{t-1}, \eta_t$$

with the following properties:

$$\begin{aligned} \forall i \in [0, t-1] : \quad \varepsilon_i = e_i v_i \in T' - \tilde{T}, \\ \forall i \in [1, t] : \quad \eta_i = e_i u_i \in \tilde{T} - T', \\ \forall i \in [1, t] : \quad \eta_{i+1} \in C^*(T', \varepsilon_i) \cap C(\tilde{T}, \varepsilon_i) - \varepsilon_i, \\ \forall i \in [0, t-1] : \quad \varepsilon_i \in C(T', \eta_{i+1}), \\ \forall i \in [1, t] : \quad e_i \in E_1, \end{aligned} \quad (4)$$

the edges $\varepsilon_0, \eta_1, \varepsilon_1, \dots, \varepsilon_{t-1}, \eta_t$ are all distinct,

$h' + \mathbf{1}_{e_t} - \mathbf{1}_e$ is a hypertree,

$h - \mathbf{1}_{e_t} + \mathbf{1}_e$ is a hypertree.

For $\tilde{T} = T$, we have just proved that such a set of edges can be chosen for $t = 1$. Also, note that in that argument, we did not use the Jaeger property of T , only (1). The analogue of that, (3) is true for an arbitrary $\tilde{T}$. Hence for any $\tilde{T}$ satisfying (2), we can choose η_1 such that (4) holds (with $t = 1$).

Suppose that we have $\varepsilon_0, \eta_1, \varepsilon_1, \dots, \eta_{t-1}, \varepsilon_{t-1}, \eta_t$ for some $t \geq 1$, satisfying (4). If $h(e_t) > h'(e_t)$, then $f = e_t$ satisfies all properties required by the lemma, and the proof is complete (except for the statement on F).

If $h(e_t) \leq h'(e_t)$, then we show that either we can add a new pair of edges $\varepsilon_t, \eta_{t+1}$ and still satisfy (4), or we can find a new tree $\tilde{T}$ satisfying (2) that has a smaller symmetric difference to T' than the current $\tilde{T}$ (and start anew with this new $\tilde{T}$). If we prove this, then as $|T\Delta T'|$ is finite, moreover, $\{\varepsilon_0, \dots, \varepsilon_{t-1}\}$ is a subset of $T' - \tilde{T} \subseteq T' - T$ which is a finite set, the process cannot go on indefinitely. Hence there must be a moment when we conclude that $h(e_t) > h'(e_t)$, and this gives us an f satisfying the requirements of the lemma.

Hence let us suppose that $h(e_t) \leq h'(e_t)$. Note that $e_t \neq e$, since $e \in E_0$ but $e_t \in E_1$ by our assumption. There might be more than one indices i such that $e_i = e_t$, but in any case, since $e_t \neq e$, the number of edges η_i incident to e_t (from $\tilde{T} - T'$) is one larger than the number of edges ε_i (from $T' - \tilde{T}$). By (4), these edges are all different. Since $h(e_t) = d_{\tilde{T}}(e_t) \leq h'(e_t) = d_{T'}(e_t)$, the degree of e_t in $\tilde{T} - T'$ is at most the degree of e_t in $T' - \tilde{T}$. Thus, there is at least one so far unchosen edge $\varepsilon_t = e_t v_t$ incident to e_t that is in $T' - \tilde{T}$.

Take $C^*(T', \varepsilon_t) \cap C(\tilde{T}, \varepsilon_t)$. As ε_t is in this intersection, there needs to be at least one more edge $\eta_{t+1} \in C^*(T', \varepsilon_t) \cap C(\tilde{T}, \varepsilon_t)$ that is necessarily in $\tilde{T} - T'$. Let $\eta_{t+1} = e_{t+1} u_{t+1}$.

This η_{t+1} might or might not agree with some previously chosen η_i , but before addressing this issue, let us note two things.

Note that $\varepsilon_t \in C(T', \eta_{t+1})$ as ε_t is the only edge of T' connecting the two components of $T' - \varepsilon_t$.

Also, let us show that $e_{t+1} \in E_1$. Recall that $e_t \in E_1$. As $(E_0 \cup V_0, E_1 \cup V_1)$ is a fundamental cut of T' , one of the components of $T' - \varepsilon_t$ only contains vertices from $E_1 \cup V_1$. Hence $e_{t+1} \in E_0$ would imply that η_{t+1} has endpoints in E_0 and V_1 , thus, $\eta_{t+1} \in \tilde{T} \cap C^*(T', \varepsilon)$. However, by (3) such an edge cannot have an endpoint in E_0 , hence we conclude that $e_{t+1} \in E_1$.

From here, we distinguish 2 cases:

Case 1: η_{t+1} does not agree with any of the edges $\eta_1, \dots, \eta_t$. In this case we show that $\varepsilon_0, \eta_1, \varepsilon_1, \dots, \eta_{t-1}, \varepsilon_t, \eta_{t+1}$ satisfies (4). To prove this, it is left to show that $h' + \mathbf{1}_{e_{t+1}} - \mathbf{1}_e$ is a hypertree, and $h - \mathbf{1}_{e_{t+1}} + \mathbf{1}_e$ is a hypertree.

We know that $\varepsilon_i \in C(T', \eta_{i+1})$, for each $0 \leq i \leq t$. Apply Lemma 45 for T' , and $\varepsilon_0, \eta_1, \dots, \varepsilon_t, \eta_{t+1}$. By the Lemma, $h' + \mathbf{1}_{e_{t+1}} - \mathbf{1}_e$ is indeed a hypertree. At the end of the proof, we will prove some properties of the representing Jaeger tree of $h' + \mathbf{1}_{e_{t+1}} - \mathbf{1}_e$. For those arguments, note that Lemma 45 implies the existence of a representing spanning tree S for $h' + \mathbf{1}_{e_{t+1}} - \mathbf{1}_e$ that has the properties $S \subset T \cup T'$, $\varepsilon \notin S$, and moreover, each edge of $T' \Delta S - \varepsilon$ is incident to a node in E_1 .

For each $0 \leq i \leq t$ we also have $\eta_{i+1} \in C(\tilde{T}, \varepsilon_i)$. Hence we can apply Lemma 46 for $\tilde{T}$ and the edges $\varepsilon_0, \eta_1, \dots, \varepsilon_t, \eta_{t+1}$, and obtain that $h - \mathbf{1}_{e_{t+1}} + \mathbf{1}_e$ is a hypertree.

Case 2: $\eta_{t+1} = \eta_i$ for some $1 \leq i \leq t$. In this case we show that we can choose a new $\tilde{T}$ satisfying (2) that has a strictly smaller symmetric difference to T' than the current one.

Note that for each $0 \leq i \leq t$, $\eta_{i+1} \in C(\tilde{T}, \varepsilon_i)$. Hence we can apply Lemma 46 for $\tilde{T}$ and the edges $\varepsilon_i, \eta_{i+1}, \dots, \varepsilon_t, \eta_{t+1} = \eta_i$. The Lemma gives us a new tree $\tilde{T}_1 = \tilde{T} + \varepsilon_{i_0} - \eta_{i_1} + \varepsilon_{i_1} - \eta_{i_2} + \dots + \varepsilon_{i_{s-1}} - \eta_{i_s}$ where $\varepsilon_{i_0} = \varepsilon_i$ and $\eta_{i_s} = \eta_{t+1} = \eta_i$. Hence $\tilde{T}_1$ is a spanning tree that again represents h , moreover, $\tilde{T}_1 \subseteq \tilde{T} \cup T' \subseteq T \cup T'$. As neither of the

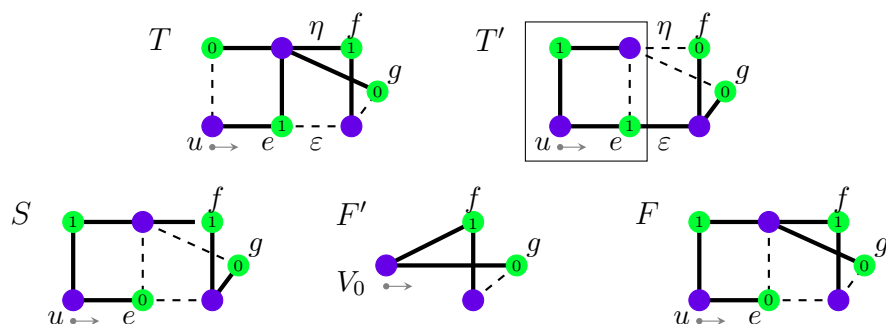


Figure 6: Illustration for the proof of Lemma 47. The ribbon structure is the positive orientation of the plane, the basis is (u, ue) , except for the 4th panel, where the basis is (V_0, V_0g) .

$\eta_{i+1}, \dots, \varepsilon_t, \eta_{t+1}$ are incident to e , we also have $\varepsilon \notin \tilde{T}_1$. Also, since $\tilde{T}_1 \Delta T' \subseteq \tilde{T} \Delta T' - \varepsilon_i$, by choosing $\tilde{T}_1$ as our new $\tilde{T}$, we obtain a new tree satisfying (2), that has a strictly smaller symmetric difference to T' than the previous one.

With this, we have proved that there exists f with $h(f) > h'(f)$ where $h - \mathbf{1}_f + \mathbf{1}_e$ and $h' + \mathbf{1}_f - \mathbf{1}_e$ are both hypertrees, and $e <_h f$ and $e <_{h'} f$. Now let us prove that for the Jaeger tree F of $h' + \mathbf{1}_f - \mathbf{1}_e$, we have $\varepsilon \notin F$, and the tours of T and F agree up to (and including) (e, ε) . We proved above that $h' + \mathbf{1}_f - \mathbf{1}_e$ has a representing spanning tree S with $S \subset T \cup T'$, $\varepsilon \notin S$, moreover, each edge of $T' \Delta S - \varepsilon$ is incident to a node in E_1 .

This implies that restricted to $E_0 \cup V_0$, the trees S and T' agree. Moreover, by (1) and $S \subset T \cup T' - \varepsilon$, all edges of S in $C^*(T', \varepsilon)$ have their endpoint in V_0 .

Now let us show that F agrees with S restricted to $E_0 \cup V_0$, and $\varepsilon \notin F$. Take the graph G' , obtained by deleting those edges of $C^*(T', \varepsilon)$ that have an endpoint in E_0 , and contracting $V_0 \cup E_0$ to a point. (See Figure 6 for an example.) Note that G' is bipartite, and it inherits a ribbon structure from G . Moreover, let S' be obtained from S by contracting $V_0 \cup E_0$ to a point. Then, as S spans $V_0 \cup E_0$ (since T' spans it, too), and we did not delete edges of S from $C^*(T', \varepsilon)$, the obtained S' is a spanning tree in G' .

Our next step is to choose a Jaeger tree in G' that represents the same hypertree as S' , but for this, we need a basis for G' . Take the contracted V_0 as the base point of G' . We will choose a base edge the following way: Take the tour of T' in G , but act as if ε were not an edge of T' . Let ε' be the first edge of $C^*(T', \varepsilon)$ taken after (e, ε) . This edge necessarily has its endpoints in V_0 and E_1 by the Jaeger property of T' and Lemma 42, hence $\varepsilon' \in G'$. Let ε' be the base edge of G' . Let F' be the Jaeger tree (of G' , with basis (V_0, ε')) that represents the same hypertree of G' as S' . Then, undo the contraction, and take $F'' = F' \cup S|_{V_0 \cup E_0}$. Note that since $\varepsilon \notin G'$, we have $\varepsilon \notin F''$. Also, F'' represents the same hypertree as S , that is, $h' + \mathbf{1}_f - \mathbf{1}_e$.

We prove that F'' is a Jaeger tree, that is $F = F''$, then, we show that the tours of F'' and T agree up to (and including) (e, ε) .

To see that F'' is a Jaeger tree, note that restricted to $E_0 \cup V_0$, the tour of F agrees with the tour of T' . For an edge of $C^*(T', \varepsilon)$, if the emerald endpoint is in E_0 , then since

$F = T'$ on $E_0 \cup V_0$, and by Lemma 42 applied to T' , each such edge is seen no later than (e, ε) . Hence in F , we see these edges at their emerald endpoint before even arriving to $V_1 \cup E_1$. Thus, these edges cannot spoil the Jaeger property. The rest of the edges are edges of G' . Note that among them, the tour of F'' reaches ε' first. Hence restricted to G' , the tour of F'' agrees with the tour of F' with basis (V_0, ε') . As F' was a Jaeger tree for this basis, no edge is cut at its violet endpoint among these edges, either. Hence, $F = F''$.

The tours of F and T' agree with each other up to reaching (e, ε) . This is true because until reaching (e, ε) , the tour of T' is in the subgraph spanned by $E_0 \cup V_0$, and cutting some edges from $C^*(T', \varepsilon)$, but the latter edges are all incident to E_0 because of the Jaeger property of T' . As S does not have any edge in $C^*(T', \varepsilon)$ with endpoint in E_0 , these edges are also cut in the tour of S . Because $S = T'$ restricted to $E_0 \cup V_0$, edges spanned by $E_0 \cup V_0$ are also treated the same way in the tours of S and T' . Thus, the two tours indeed agree until reaching (e, ε) . Hence, also the tours of F and T agree up to reaching (e, ε) , and as $\varepsilon \notin T$ and $\varepsilon \notin F$, they also agree in (e, ε) . $\square$

Lemma 48. *Let J and J' be Jaeger trees representing hypertrees x and $x' = x + \mathbf{1}_a - \mathbf{1}_b$. Suppose that the first difference between the tours of J and J' is at (g, gv) with $gv \in J' - J$. Then*

- (i) $g \neq b$,
- (ii) $x + \mathbf{1}_a - \mathbf{1}_g$ is a hypertree,
- (iii) $x + \mathbf{1}_g - \mathbf{1}_b$ is a hypertree.

Note that $a = g$ is possible, and in that case (ii) and (iii) are true by definition.

Proof. Apply Lemma 47 with $h = x$, $h' = x'$, $T = J$, $T' = J'$, and $e = g$. We get that there exists some hyperedge f such that $x(f) > x'(f)$, moreover, $x - \mathbf{1}_f + \mathbf{1}_g$ and $x' + \mathbf{1}_f - \mathbf{1}_g$ are both hypertrees, and $g <_h f$, $g <_{h'} f$.

As $x' = x + \mathbf{1}_a - \mathbf{1}_b$, the only hyperedge with $x(f) > x'(f)$ is $f = b$. Hence $x - \mathbf{1}_b + \mathbf{1}_g$ and $x' + \mathbf{1}_b - \mathbf{1}_g = x + \mathbf{1}_a - \mathbf{1}_g$ are both hypertrees. Finally, as $g <_h b$, we cannot have $g = b$. $\square$

Lemma 49. *Let x and $x' = x + \mathbf{1}_a - \mathbf{1}_b$ be two hypertrees with respective Jaeger trees J and J' . Suppose that the first difference in the tours of J and J' is at (g, gv) . Then $g \leq_x a$, and $g \leq_x b$.*

Note by definition $<_x$ compares emerald nodes, but by the Jaeger property g is an emerald node, hence $g \leq_x a$ and $g \leq_x b$ indeed makes sense.

We can have $g <_x a$ and $g <_x b$ in Lemma 49. An example for this can be seen on the first two panels of Figure 4 with $g = c$.

Proof of Lemma 49. It is enough to prove the statement if $gv \in J' - J$. Indeed, if we know the statement for $gv \in J' - J$, but we have $gv \in J - J'$, then we can apply the Lemma for x' and $x = x' + \mathbf{1}_b - \mathbf{1}_a$, and obtain that $g \leq_{x'} b$ and $g \leq_{x'} a$. However, as

the tours of J and J' agree until reaching g , $g \leq_{x'} b$ implies $g \leq_x b$ and $g \leq_{x'} a$ implies $g \leq_x a$.

Hence let us suppose from now on that $gv \in J' - J$. Apply Lemma 47 with $T = J$, $T' = J'$, $h = x$, $h' = x'$ and $e = g$. We get that there exists a hyperedge f such that $g <_x f$ and $x(f) > x'(f)$. As $x' = x + \mathbf{1}_a - \mathbf{1}_b$, we can only have $f = b$, and hence $g <_x b$.

We also get from Lemma 47 that $x' + \mathbf{1}_b - \mathbf{1}_g = x + \mathbf{1}_a - \mathbf{1}_g$ is a hypertree, and for its Jaeger tree F , the tours of J and F agree up to (and including) (g, gv) . It is possible that $g = a$, but then $g \leq_x a$ and we are done.

If $g \neq a$, then J and F are different trees. In this case, let (c, cu) be the first difference between the tours of J and F . We claim that $g \leq_x c$.

Indeed, if $c <_x g$, and the tours of J and F agree up to (and including) (g, gv) , moreover, (c, cu) is the first difference between the tours of J and F , then both tours need to go to c at some point, then later get to g to take (g, gv) , and then return to c to take (c, cu) . However, this means that by the time the tours of J and F get to (c, cu) , the vertex g is finished. As x and $x + \mathbf{1}_a - \mathbf{1}_g$ do not have the same value at g , the two tours have to differ by the time they finish g , in contradiction with the assumption that (c, cu) is the first difference.

We finish our argument with applying Lemma 47 again, but we need to look at two cases. If $cu \in F - J$, we can apply Lemma 47 with $h = x$, $T = J$, $h' = x + \mathbf{1}_a - \mathbf{1}_g$, $T' = F$, and $e = c$. We get that there exist some f such that $x(f) > (x + \mathbf{1}_a - \mathbf{1}_g)(f)$, hence $f = g$. The lemma then implies $c <_x g$, which is not possible, as we have shown above.

We conclude that $cu \in J - F$. Hence we can apply Lemma 47 with $h = x + \mathbf{1}_a - \mathbf{1}_g$, $T = F$, $h' = x$, $T' = J$, and $e = c$. We get that there exist some f such that $(x + \mathbf{1}_a - \mathbf{1}_g)(f) > x(f)$, hence $f = a$. Also, we get that $c <_x a$. As we also have $g \leq_x c$, we conclude that $g <_x a$, finishing the proof. $\square$

Remark 50. In hindsight, in the case where $gv \in J' - J$ and $a \neq g$, by applying Lemma 49 to $x + \mathbf{1}_a - \mathbf{1}_g$ of the proof, we see that in this case $c \leq_x g$, hence $c = g$. Thus, the first difference between x and $x + \mathbf{1}_a - \mathbf{1}_g$ is at g .

4.5 Proof of Theorem 32

Theorem 51. *Fix an arbitrary ribbon structure and basis. If h and h' are two hypertrees of a hypergraph $\mathcal{H}$, then $C_{em}(h) \cap C_{em}(h') = \emptyset$.*

Proof. Let T and T' be the respective Jaeger trees of h and h' . By symmetry, we may suppose that for the first difference (e, ε) between the tours of T and T' , we have $\varepsilon \in T - T'$. (As both T and T' are Jaeger trees, e needs to be a hyperedge.)

Apply Lemma 47. Notice that the hyperedge f given by the Lemma has the following properties:

$$\begin{aligned} h(f) &> h'(f), \\ f &\text{ is not internally embedding active in } h, \\ f &\text{ is not externally embedding active in } h'. \end{aligned} \tag{5}$$

This implies that for each $x \in C_{em}(h)$ and $y \in C_{em}(h')$ we have $x(f) \geq h(f) > h'(f) \geq y(f)$, hence x and y cannot be the same. $\square$

Theorem 52. *For any $c \in \mathbb{Z}^E$, there is at least one hypertree h such that $c \in C_{em}(h)$, moreover, $d_1(h, c) = d_1(H, c)$.*

For a given vector $c \in \mathbb{Z}^E$, let $H_c = \{h \in H \mid d_1(H, c) = d_1(h, c)\}$. We are looking for a hypertree $h \in H_c$ such that $c \in C_{em}(h)$.

Recall the definition of external and internal elements with respect to c from Section 3.

Once again, our method is to define a complete ordering $<_c$ on H_c , then show that if a hypertree $h \in H_c$ does not have $c \in C_{em}(h)$, then we can find another hypertree $h' \in H_c$ with $h' <_c h$. For embedding activities, we use the following definition for $<_c$:

Definition 53 ($<_c$). Let h and h' be two hypertrees in H_c with respective Jaeger trees T and T' . Suppose that the first difference between the tours of T and T' is at (e, ev) , where $ev \in T - T'$. Then

1. if e is an external hyperedge with respect to c , we define $h' <_c h$
2. if e is an internal hyperedge with respect to c , we define $h <_c h'$
3. if e is neither an external, nor an internal hyperedge with respect to c , we define $h <_c h'$.

It is easy to check that for any $c \in \mathbb{Z}^E$, the above defined $<_c$ is a complete ordering of H_c .

Proof of Theorem 52. For some $h \in H_c$, we have $c \in C_{em}(h)$ if and only if the following two properties hold:

$$\begin{aligned} \text{If } e \text{ is an external edge with respect to } c \text{ and } c(e) > h(e), \\ \text{then } e \text{ is externally embedding active in } h, \end{aligned} \tag{6}$$

$$\begin{aligned} \text{if } e \text{ is an internal edge with respect to } c \text{ and } c(e) < h(e), \\ \text{then } e \text{ is internally embedding active in } h. \end{aligned} \tag{7}$$

Take an arbitrary hypertree $h \in H_c$. If h satisfies (6) and (7) then we are done. If not, then we will modify h to get a new hypertree $h' \in H_c$ with $h' <_c h$. As $<_c$ is a complete ordering on H_c , which is a finite set, at some point we need to have a hypertree $h \in H_c$ satisfying (6) and (7).

If h does not satisfy either (6) or (7), then there is either an external hyperedge e not satisfying (6) or an internal hyperedge e not satisfying (7) (or both). We next show how to modify h in these two cases.

Suppose that e is an external hyperedge such that $c(e) > h(e)$ but e is not externally embedding active in h . By the definition of external embedding activity, this means that there exists a hyperedge $f <_h e$ such that $h' = h + \mathbf{1}_e - \mathbf{1}_f$ is also a hypertree. Note

that such an f is necessarily an external hyperedge, and $h' \in H_c$. Indeed, $h \in H_c$ implies $d_1(h, c) \leq d_1(h', c)$. As $|c(e) - h'(e)| < |c(e) - h(e)|$ we conclude that $c(f) > h'(f)$, thus, f is also external, and $h' \in H_c$.

We show that we can choose f such that $h' <_c h$. More precisely, we show that if we choose f to be the earliest emerald node according to $<_h$ such that $h + \mathbf{1}_e - \mathbf{1}_f$ is a hypertree, then for the obtained $h' = h + \mathbf{1}_e - \mathbf{1}_f$ and its Jaeger tree T' , the first difference between the tours of T and T' is at a pair (f, fv) , where $fv \in T - T'$. As f is an external hyperedge (with respect to c), this means that $h' <_c h$.

Suppose for a contradiction that f is chosen to be the earliest emerald node according to $<_h$ such that $h + \mathbf{1}_e - \mathbf{1}_f$ is a hypertree, but the above properties do not hold for the Jaeger trees T and T' . There are 3 ways in which the property can be violated.

Case 1: The first difference between T and T' is at (g, gv) where $g \neq f$, and $gv \in T - T'$.

By Lemma 49 applied to $x = h$, $x' = h'$, $a = e$, $b = f$, in this case $g \leq_h f$ and $g \leq_h e$. By part (ii) of Lemma 48, applied with $J = T'$, $J' = T$, $x = h'$, $a = f$, $b = e$, we get that $h' + \mathbf{1}_f - \mathbf{1}_g = h + \mathbf{1}_e - \mathbf{1}_g$ is also a hypertree. This contradicts the fact that f was the earliest emerald node according to $<_h$ such that $h + \mathbf{1}_e - \mathbf{1}_f$ is a hypertree.

Case 2: The first difference between T and T' is at (g, gv) where $g \neq f$, and $gv \in T' - T$.

By Lemma 49 in this case $g \leq_h f$ and $g \leq_h e$. By part (ii) of Lemma 48, applied for $J = T$, $J' = T'$, $x = h$, $a = e$, $b = f$, we get that $h + \mathbf{1}_e - \mathbf{1}_g$ is also a hypertree. This contradicts the fact that f was the earliest emerald node according to $<_h$ such that $h + \mathbf{1}_e - \mathbf{1}_f$ is a hypertree.

Case 3: The first difference between T and T' is at (f, fv) , but we have $fv \in T' - T$. This is impossible by (i) of Lemma 48 (applied for $J = T$, $J' = T'$, $a = e$, $b = f$).

Now let us look at the case if there is an internal hyperedge violating (7).

Let e be an internal hyperedge such that $c(e) < h(e)$ and e is not internally embedding active in h . By the definition of internal embedding activity, this means that there exists a hyperedge $f <_h e$ such that $h' = h - \mathbf{1}_e + \mathbf{1}_f$ is also a hypertree. Note that f is necessarily an internal hyperedge, and $h' \in H_c$. Indeed, $h \in H_c$ implies $d_1(h, c) \leq d_1(h', c)$. As $|c(e) - h'(e)| < |c(e) - h(e)|$ we conclude that $c(f) < h'(f)$, thus, f is also internal, and $h' \in H_c$.

We show that we can choose f such that $h' <_c h$. More precisely, we will show that if we choose f to be the earliest emerald node according to $<_h$ such that $h - \mathbf{1}_e + \mathbf{1}_f$ is a hypertree, then for the obtained $h' = h - \mathbf{1}_e + \mathbf{1}_f$ and its Jaeger tree T' , the first difference between the tours of T and T' is at a pair (f, fv) , where $fv \in T' - T$. As f is an internal hyperedge (with respect to c), this means that $h' <_c h$.

Suppose for a contradiction that f is chosen to be the earliest emerald node according to $<_h$ such that $h - \mathbf{1}_e + \mathbf{1}_f$ is a hypertree, but the above properties do not hold for the Jaeger trees. There are 3 ways in which the property can be violated.

Case 1: The first difference between T and T' is at (g, gv) where $g \neq f$, and $gv \in T - T'$.

By Lemma 49 applied with $x = h$, $x' = h'$, $a = f$ and $b = e$, in this case $g \leq_h f$ and $g \leq_h e$.

By (iii) of Lemma 48 applied with $J = T'$, $J' = T$, $x = h'$, $x' = h$, $a = e$ and $b = f$, we get that $h' + \mathbf{1}_g - \mathbf{1}_f = h - \mathbf{1}_e + \mathbf{1}_g$ is a hypertree. This contradicts the fact that f

was the earliest emerald node according to $<_h$ such that $h - \mathbf{1}_e + \mathbf{1}_f$ is a hypertree.

Case 2: The first difference between T and T' is at (g, gv) where $g \neq f$, and $gv \in T' - T$. By Lemma 49 in this case $g \leq_h f$ and $g \leq_h e$.

By (iii) of Lemma 48 applied with $J = T$, $J' = T'$, $x = h$, $x' = h'$, $a = f$ and $b = e$, we get that $h + \mathbf{1}_g - \mathbf{1}_e$ is a hypertree. This again contradicts the fact that f was the earliest emerald node according to $<_h$ such that $h - \mathbf{1}_e + \mathbf{1}_f$ is a hypertree.

Case 3: The first difference between T and T' is at (f, fv) , but we have $fv \in T - T'$. This is impossible by (i) of Lemma 48 applied for $J = T'$, $J' = T$, $x = h'$, $x' = h$, $a = e$ and $b = f$. $\square$

Proof of Theorem 32. The statement follows from Theorems 51 and 52. $\square$

5 Open questions concerning variants of embedding activities

Let us recall an alternative definition for embedding activities from [10]. In this paper, Jaeger trees are defined via the rule that each non-tree edge is first seen at its emerald endpoint. However, as mentioned in Remark 25, one could also define Jaeger trees with the rule that each non-tree edge is first seen at its violet endpoint (see Figure 7 for an example).

In [10], both types of trees are investigated, and are called emerald, and violet Jaeger trees, respectively.

It is also true that for any hypertree h of $\mathcal{H}$, there is a unique violet Jaeger tree representing h . Hence one could potentially associate an ordering of E to h using the violet Jaeger tree. There are in fact two natural ways to do this.

One possibility is to define the ordering $<_h^{\text{violet}}$ as the ordering in which emerald nodes are reached (that is, become current node) in the tour of T , where T is the violet Jaeger tree representing h .

Another possibility is to take the ordering $<_h^{\text{violet}'}$ which is the ordering in which emerald nodes appear as an endpoint of the current edge in the tour of T , where T is the violet Jaeger tree representing h .

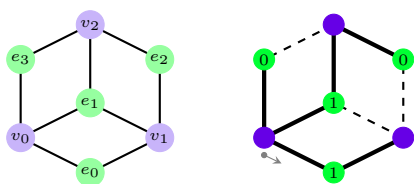


Figure 7: A violet Jaeger tree for the ribbon structure induced by the positive orientation of the plane, with basis (v_0, v_0e_0) .

Example 54. Figure 7 shows the unique violet Jaeger tree T for the hypertree h with $h(e_0) = h(e_1) = 1$, $h(e_2) = h(e_3) = 0$. We have $e_0 <_h^{\text{violet}} e_1 <_h^{\text{violet}} e_2 <_h^{\text{violet}} e_3$ since this is the order in which emerald nodes become current in the tour of T . However, we have

$e_0 <_h^{\text{violet}'} e_2 <_h^{\text{violet}'} e_1 <_h^{\text{violet}'} e_1 <_h^{\text{violet}'} e_3$, since $(v_1, v_1 e_2)$ precedes $(v_1, v_1 e_1)$ in the tour of T .

Note that for an emerald Jaeger tree, these two orders coincide, since if an emerald endpoint e first appears as an endpoint of the current edge such that the current node-edge pair is (v, ve) , then in the next step, the current node needs to be e . This is not true, however, if T is a violet Jaeger tree, as also the previous example shows.

In some sense, $<_h^{\text{violet}}$ seems to be the more natural ordering, however, it is easy to come up with examples where it does not give $\mathcal{T}_{\mathcal{H}}$.

However, $<_h^{\text{violet}'}$ seems to yield $\mathcal{T}_{\mathcal{H}}$.

Conjecture 55. Embedding activities defined via $<_h^{\text{violet}'}$ produce the polynomial $\mathcal{T}_{\mathcal{H}}$ for hypergraphs.

6 Relationship of different activities

In this section, we explore the relationship of the activity notions considered in this paper.

We note that for Courtiél [4], the motivation of studying computation-tree activities (more precisely, $(ct, \max)$ -activities) was that $(ct, \max)$ -activities include many notions of graph activities. Tutte activities are trivially $(ct, \max)$ -activities, but more surprisingly, embedding activities of graphs are also $(ct, \max)$ -activities (for an appropriately chosen decision tree Δ) [4].

In this section, we show that the latter does not remain true for hypergraphs: We show that there are hypergraph embedding activities that are not obtainable as $(ct, \max)$ -activity.

We leave it as an open problem if embedding activities of hypergraphs are a subclass of $(ct, \min)$ -activities.

Also, we show that there are $(ct, \max)$ -activities that are not obtainable as $(ct, \min)$ -activity (even for graphs), and there are $(ct, \min)$ -activities that are not obtainable as $(ct, \max)$ -activity (even for graphs).

We need some preparations. We call an element e nontrivially internally $(ct, \max)$ -active ($(ct, \min)$ -active) for $b \in P$ if there exists a basis $b' \in P$ with $b'(e) < b(e)$ and e is internally $(ct, \max)$ -active ($(ct, \min)$ -active) for b . Similarly, we call an element e nontrivially externally $(ct, \max)$ -active ($(ct, \min)$ -active) for $b \in P$ if there exists a basis $b' \in P$ with $b'(e) > b(e)$ and e is externally $(ct, \max)$ -active ($(ct, \min)$ -active) for b .

We will use the following observation, whose matroid version was noted by Courtiél [4]. (We note that Courtiél's Δ -activities actually correspond to what we call nontrivial $(ct, \max)$ -activities.)

Proposition 56. *For any polymatroid $(ct, \max)$ -activity, there exists an element of the ground set that is neither nontrivially internally $(ct, \max)$ -active, nor nontrivially externally $(ct, \max)$ -active for any basis.*

Proof. The element e in the root of the decision tree will be the first element for each ordering $<_b$. It follows directly from the definition of a polymatroid that this element is never nontrivially internally active, nor nontrivially externally active for any basis. $\square$

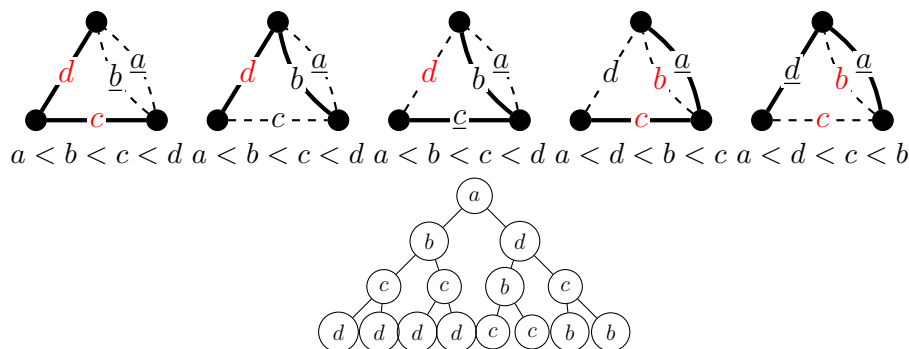


Figure 8: A graph with 5 spanning trees, a decision tree Δ , and the edge orderings corresponding to the spanning trees.

We can state a similar counterpart for $(ct, \min)$ -activities.

Proposition 57. *Suppose that $P \subset \mathbb{Z}^E$ is a polymatroid where for each $e \in E$, there exist $b, b' \in P$ with $b(e) \neq b'(e)$. Then for any $(ct, \min)$ -activity of P , there exists an element of the ground set that is either nontrivially internally $(ct, \min)$ -active or nontrivially externally $(ct, \min)$ -active for any basis.*

Proof. The element e in the root of the decision tree will be the first element for each ordering $<_b$, hence e is both internally and externally $(ct, \min)$ -active in any basis. If there exists $b' \in P$ with $b'(e) < b(e)$, then e will be nontrivially internally $(ct, \min)$ -active in b . If there exists $b' \in P$ with $b'(e) > b(e)$ then e will be nontrivially externally $(ct, \min)$ -active in b . At least one of the two cases holds because of our assumption. $\square$

Proposition 58. *There exists a graph with a $(ct, \min)$ -activity that cannot be obtained as a $(ct, \max)$ -activity.*

Remark 59. In [4], Courtiel conjectures that each activity notion of graphs inducing a Crapo-decomposition is a $(ct, \max)$ -activity. The above proposition refutes this conjecture.

Proof. Take the graph on Figure 8. The 5 panels in the first row show the 5 spanning trees cd, bd, bc, ac and ad of this graph. If we take the decision tree shown below, we get the orders shown below the spanning trees. (That is, the orders corresponding to cd, bd and bc are $a < b < c < d$, the order corresponding to ac is $a < d < b < c$ and the order corresponding to ad is $a < d < c < b$.)

For each tree, the underlined labels are those edges that are nontrivially (internally or externally) $(ct, \min)$ -active for the corresponding tree.

We can see that each edge is nontrivially active for some spanning tree, hence by Proposition 56, this activity notion cannot be obtained as a $(ct, \max)$ -activity. $\square$

Proposition 60. *There exists a graph with a $(ct, \max)$ -activity that cannot be obtained as a $(ct, \min)$ -activity.*

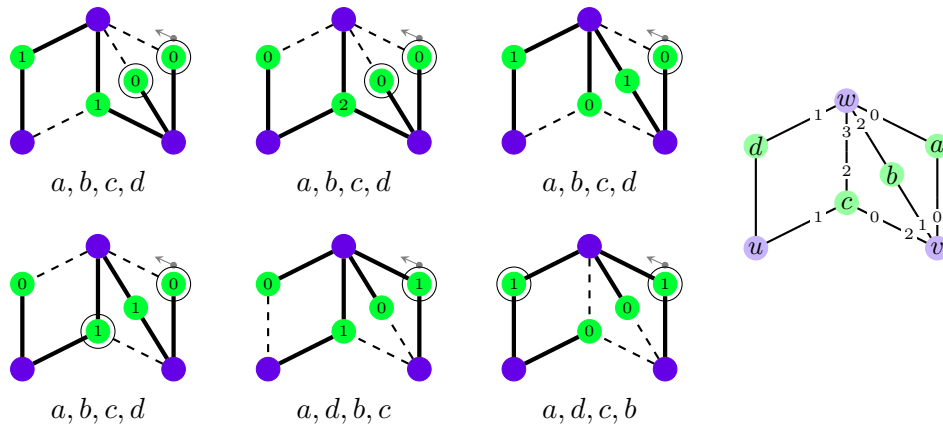


Figure 9: An embedding activity that is not a $(ct, \max)$ -activity. See the proof of Proposition 61 for the notations and more explanation.

Proof. Again the graph on Figure 8, with the same decision tree shows this.

On each panel, the red labels are those edges that are nontrivially (internally or externally) $(ct, \max)$ -active for the corresponding tree.

We can see that none of the edges are nontrivially active for all spanning trees, hence by Proposition 57, this activity notion cannot be obtained as a $(ct, \min)$ -activity. $\square$

Proposition 61. *There exists a hypergraph with a ribbon structure and basis such that the embedding activities cannot be obtained as and a $(ct, \max)$ -activity.*

Proof. Take the hypergraph represented by the bipartite graph on Figure 9, with 3 vertices (violet nodes) and 4 hyperedges (emerald nodes). The labelings of the nodes and the ribbon structure are depicted on the rightmost panel. (The cyclic ordering of the edges incident to a node is indicated by the numbers written on the edges near the node. For clarity of the picture, the (trivial) information is omitted for degree 2 nodes.) Let the basis be (a, aw) .

The 6 left panels of Figure 9 show the 6 hypertrees of the hypergraph, together with their representing Jaeger trees, and the corresponding orderings $\leq_h$ on the hyperedges (that is, emerald nodes). For each hypertree, nontrivially (internally or externally) embedding active emerald nodes are circled.

By Proposition 56, for a $(ct, \max)$ -activity, there must exist a hyperedge that is neither nontrivially internally active, nor nontrivially externally active for any basis.

However, we see on Figure 9 that for the constructed example, each emerald node a, b, c and d is nontrivially embedding-active for some hypertree, hence we conclude that no decision tree can give us this set of $(ct, \max)$ -active hyperedges. $\square$

Problem 62. Can embedding activities of hypergraphs be obtained as $(ct, \min)$ -activities for a suitably chosen decision tree?

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