

Bounds on treewidth via excluding disjoint unions of cycles

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Abstract

One of the fundamental results in graph minor theory is that for every planar graph H , there is a minimum integer $f(H)$ such that graphs with no minor isomorphic to H have treewidth at most $f(H)$. The best bound known for an arbitrary planar H is $O(|V(H)|^9 \text{poly log}|V(H)|)$. We show that if H is the disjoint union of cycles, then $f(H)$ is $O(|V(H)| \log^2 |V(H)|)$, which is a $\log |V(H)|$ factor away from being optimal.

Mathematics Subject Classifications: 05C83, 05C38

1 Introduction

Treewidth measures the similarity of a given graph to a tree. It is of both algorithmic and structural importance and was independently discovered multiple times [BB72, Hal76, RS84]. Many algorithmically hard problems are fixed-parameter tractable with treewidth as the parameter; in particular, Courcelle's theorem [Cou90] states that every property expressible in monadic second-order logic can be determined in linear time for graphs of bounded treewidth. More precisely, the running time of those algorithms is of the form $g(w)n^{O(1)}$, where g is a function and w is the treewidth of the input graph. Hence, obtaining quantitatively good bounds for the maximum treewidth of graphs in specific graph classes is of interest.

Robertson and Seymour [RS91] proved that for every planar graph H , there is an integer n_H such that graphs that do not contain H as a minor have treewidth at most n_H . We define $f(H)$ as the smallest n_H for which this is true. The best upper bound known on $f(H)$ for arbitrary H , obtained by Chuzhoy and Tan [CT21], is $O(|V(H)|^9 \text{poly log}|V(H)|)$

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as a combined consequence of their result and a result of Robertson, Seymour, and Thomas [RST94] that every planar H is a minor of a $(k \times k)$ -grid for $k = 2|V(H)|$.

It is natural to ask for a better bound on $f(H)$ for H in various classes of planar graphs. A trivial lower bound for $f(H)$ is $\Omega(|V(H)|)$ since the complete graph on $|V(H)| - 1$ vertices has treewidth $|V(H)| - 2$ and does not contain H as a minor. Another lower bound comes from counting the number of disjoint cycles. It is well-known (for example, see [RS86]) that there are n -vertex graphs with girth $\Omega(\log n)$ and treewidth $\Omega(n)$ for infinitely many integers n . Such graphs have $O(n/\log n)$ disjoint cycles and hence cannot contain any graph H with more than $O(n/\log n)$ disjoint cycles as a minor. This implies that if H contains a spanning subgraph that is a disjoint union of cycles of length $O(1)$, then $f(H)$ is $\Omega(|V(H)| \log |V(H)|)$. In fact, if H is a disjoint union of cycles of length at most a fixed constant, then $f(H)$ is $\Theta(|V(H)| \log |V(H)|)$, where the upper bound follows from the Erdős-Pósa theorem for cycles of length at least a fixed constant [FH14, MNŠW17] since every graph that can be made a graph with no cycle of length at least ℓ , which has treewidth at most $\ell - 2$ [Bir03, FL89], by deleting at most w vertices has treewidth at most $w + \ell - 2$.

This paper focuses on $f(H)$ for graphs H that are disjoint unions of cycles without the $O(1)$ upper bound on the length of the cycles.

Fellows and Langston [FL89] and Birmele [Bir03] showed that if H is a cycle, then $f(H) = |V(H)| - 2$. Mousset, Noever, Škorić, and Weissenberger [MNŠW17] proved the tight bound of $\Theta(r\ell + r \log r)$ in the case that all r cycles are of length ℓ . Gollin, Hendrey, Oum, and Reed [GHOR25] proved that if H is the disjoint union of two cycles, then $f(H) = (1 + o(1))|V(H)|$ while if H is the disjoint union of $o(\frac{\sqrt{|V(H)|}}{\sqrt{\log |V(H)|}})$ cycles, then $f(H)$ is at most $\frac{3|V(H)|}{2} + o(|V(H)|)$.

Theorem 1 ([GHOR25]). *There is an absolute constant c such that for every $r \geq 3$, if H is the disjoint union of r cycles, then*

$$f(H) \leq \frac{3|V(H)|}{2} + cr^2 \log r.$$

If H is the disjoint union of two cycles, then

$$f(H) < |V(H)| + \frac{9}{2} \lceil \sqrt{4 + |V(H)|} \rceil + 2.$$

With the trivial upper bound of $r \leq \frac{1}{3}|V(H)|$, Theorem 1 implies that $f(H)$ is $O(|V(H)|^2 \log |V(H)|)$ for general unions of cycles H . An improvement to $f(H) = O(|V(H)|^2)$ can be obtained from the aforementioned result of [MNŠW17] by taking $r = \ell = |V(H)|$. In this paper, we further improve this bound to $O(|V(H)| \log^2 |V(H)|)$; this is a consequence of the following more fine-grained bound that takes into account not only the number r of cycles whose union make up the graph H but also an upper bound ℓ on the length of these cycles.

Theorem 2. *There is an absolute constant c such that if H is the disjoint union of r cycles of length at most ℓ , then*

$$f(H) \leq c|V(H)| \log(r+1) + cr \log r \log \ell.$$

Note that ℓ can be dependent on $|V(H)|$. If H is a disjoint union of cycles of the same length, then $r = \frac{|V(H)|}{\ell}$ and Theorem 2 give $f(H) \leq c|V(H)| \log(r+1) + c|V(H)| \log r \cdot \frac{\log \ell}{\ell} = O(|V(H)| \log r)$. When $\ell = O(1)$, Theorem 2 recovers the bound $O(|V(H)| \log |V(H)|)$ stated above.

2 Preliminaries

All graphs in this paper are simple and finite, and all logarithms are considered base 2. In proving our result, we focus on a concept dual to treewidth: the *bramble number*. A *bramble* $\mathcal{B}$ in a graph G is a family of connected subgraphs of G , every two of which intersect or are joined by an edge. A *hitting set* for a bramble is a set of vertices intersecting all of its elements. The *order* of a bramble $\mathcal{B}$, denoted $\text{ord}(\mathcal{B})$, is the minimum size of a hitting set for $\mathcal{B}$. The *bramble number* of G is the largest order of a bramble in G . Any subset of a bramble $\mathcal{B}$ is a bramble, called a *subbramble* of $\mathcal{B}$. Seymour and Thomas [ST93] showed the following, see [Ree97].

Theorem 3 ([ST93]). *The treewidth of a graph is exactly one less than its bramble number.*

We observe that for every vertex v in G , the treewidth of $G - v$ is at least the treewidth of G minus 1 because given a bramble of order b in G , deleting the bramble elements containing v yields a bramble of order at least $b - 1$ in $G - v$. More generally, for every subset X of $V(G)$ and maximum order bramble $\mathcal{B}$ of G if the subbramble of $\mathcal{B}$ consisting of its elements which intersect X has order a , then the treewidth of G is at most a more than the treewidth of $G - X$.

For a bramble $\mathcal{B}$ in a graph G and a subset X of $V(G)$, we define $\mathcal{B}_X$ as the set consisting of the elements of $\mathcal{B}$ intersecting X . Hence, the above observation implies the following lemma.

Lemma 4. *Let G be a graph. Let $\mathcal{B}$ be a bramble in G of maximum order. If $X \subseteq V(G)$ such that $\mathcal{B}_X$ has order at most k , then $\text{tw}(G) \leq \text{tw}(G - X) + k$.*

We also need results about brambles hit by cycles (Theorem 5) and paths (Theorem 7).

Lemma 5 ([BBR07, Theorem 2.4]). *Let G be a graph having a bramble $\mathcal{B}$ of order at least three. Then, there is a cycle C meeting every element of $\mathcal{B}$.*

Lemma 6. *Let G be a graph with a bramble $\mathcal{B}$. Let S and T be two subsets of $V(G)$ such that $\mathcal{B}_S$ and $\mathcal{B}_T$ have order at least ℓ . Then there are ℓ disjoint paths in G from S to T .*

Proof. Suppose towards a contradiction that the desired ℓ disjoint paths between S and T do not exist. By Menger's Theorem, there is a cutset X of size less than ℓ separating S from T . Now, there exists an element B of $\mathcal{B}$ disjoint from X because the order of $\mathcal{B}$ is at least ℓ . Since $G[B]$ is connected, one of S or T does not intersect the component of $G - X$ containing B . By symmetry, we may assume that S does not intersect the component of $G - X$ containing B . Since all elements of $\mathcal{B}_S$ either intersect B or are joined by an edge to B , they all intersect X . But then X is a hitting set for $\mathcal{B}_S$ and therefore $\mathcal{B}_S$ has order at most $|X| < \ell$, which is a contradiction. $\square$

Combining these two lemmas, we obtain the following result, which is similar to the $\ell = 2$ case of [RW12, Lemma 3.2]:

Lemma 7. *Let $t \geq 2$ be a positive integer and $\mathcal{B}$ a bramble of order at least $2t$ in a graph G . Then there exist*

1. *two disjoint paths P_1 and P_2 in G such that both $\mathcal{B}_{V(P_1)}$ and $\mathcal{B}_{V(P_2)}$ have order exactly t , and*
2. *disjoint paths $Q_1, \dots, Q_t$ each of which has an endpoint on each P_i and is internally disjoint from $P_1 \cup P_2$.*

Proof. We apply Theorem 5 to obtain a cycle C intersecting every element of $\mathcal{B}$. We let P_1 be a subpath of C such that $\mathcal{B}_{V(P_1)}$ has order at least t and is minimal with this property. So, $\mathcal{B}_{V(P_1)}$ has order exactly t , and hence $C - V(P_1)$ is a path such that $\mathcal{B}_{V(C - V(P_1))}$ has order at least t . We let P_2 be a subpath of $C - V(P_1)$ such that $\mathcal{B}_{V(P_2)}$ has order at least t and is minimal with this property. So, $\mathcal{B}_{V(P_2)}$ has order exactly t . Theorem 6 implies there are t disjoint paths from $V(P_1)$ to $V(P_2)$. We choose these to minimise their total length so each is internally disjoint from $P_1 \cup P_2$. $\square$

We will use the paths in Theorem 7 to construct disjoint cycles (Theorem 10), which relies on the following famous theorem of Erdős and Pósa [EP65].

Theorem 8 ([EP65]). *There is a constant $c^* \geq 1$ such that, for every positive integer k , every graph contains either k disjoint cycles or a set of vertices of size at most $c^* k \log k$ which hits every cycle.*

Lemma 9. *Let G be a graph of maximum degree at most three. Let k be a positive integer. If $|E(G)| \geq |V(G)| + 3c^* k \log k$, where c^* is the constant in Theorem 8, then G contains at least k disjoint cycles.*

Proof. Suppose, to the contrary, that G contains at most $k - 1$ disjoint cycles. Then, a set of at most $c^* k \log k$ vertices in G hitting all cycles in G exists by Theorem 8. Since G has maximum degree at most three, there is a set S of at most $3c^* k \log k$ edges hitting all cycles in G . So $G - S$ is a graph with $|E(G - S)| \geq |V(G)| - |V(G - S)|$ with no cycle, a contradiction. $\square$

Lemma 10. *Let k be a positive integer and G be a graph. Let P_1 and P_2 be disjoint paths in G . Let $Q_1, \dots, Q_\ell$ be disjoint paths in G between $V(P_1)$ and $V(P_2)$ internally disjoint from $V(P_1 \cup P_2)$. If $\ell \geq 2 + 3c^*k \log k$, where c^* is the constant in Theorem 8, then $P_1 \cup P_2 \cup \bigcup_{i=1}^\ell Q_i$ contains at least k disjoint cycles, each containing at least two paths in $\{Q_1, \dots, Q_\ell\}$.*

Proof. We obtain an auxiliary graph J from $P_1 \cup P_2 \cup \bigcup_{i=1}^\ell Q_i$ by contracting each Q_i into a single edge. Then J is a graph of maximum degree at most three and $|E(J)| = |E(P_1)| + |E(P_2)| + \ell = |V(J)| - 2 + \ell \geq |V(J)| + 3c^*k \log k$. By Theorem 9, J contains at least k disjoint cycles. Note that each of those cycles contains at least two edges not in $E(P_1 \cup P_2)$. By replacing the contracted edges with the original paths, this yields at least k disjoint cycles in $P_1 \cup P_2 \cup \bigcup_{i=1}^\ell Q_i$, each containing at least two paths in $\{Q_1, \dots, Q_\ell\}$. $\square$

3 The proof of Theorem 2

We now have everything in place to prove our main theorem.

We proceed by induction, with three different induction steps. Which step we apply depends on the length ℓ_1 of the longest cycle C_1 of H and its relationship to the number r of cycles in H .

If there is a cycle C in G of length at least ℓ_1 such that $\mathcal{B}_{V(C)} < 6\ell_1$ (Case 1), then we delete it and apply induction on $H - C_1$ in $G - C$. We note that if we could always apply this step, we would get that $f(H) < 6|V(H)|$; but it is not true in general because $f(H)$ is $\Omega(|V(H)| \log |V(H)|)$ as we discussed in the introduction.

Otherwise, if ℓ_1 is large in terms of r (Case 2.1), then a straightforward argument shows that we can find all the cycles with one application of Theorem 10. Indeed, this case could also be handled by applying Theorem 1 from [GHOR25].

The remaining case (Case 2.2) is more delicate. It includes the case that there are more than $\frac{\sqrt{n}}{\log n}$ cycles, and a more nuanced application of Theorem 10 is required.

Proof of Theorem 2. We set $c = 68c^* + 12$, where c^* is the constant in Theorem 8. Let r, h, ℓ be positive integers. Let H be an h -vertex graph that is a union of r disjoint cycles of length at most ℓ . We show that every graph G with treewidth at least $ch \log(r+1) + cr \log r \log \ell$ contains H as a minor. We let $\mathcal{B}$ be a bramble of maximum order in G . By Theorem 3, $\text{ord}(\mathcal{B}) \geq 1 + ch \log(r+1) + cr \log r \log \ell$.

We proceed by induction on r . If $r = 1$, then Theorem 5 implies that there exists a cycle C hitting all elements of $\mathcal{B}$, so $|V(C)| \geq \text{ord}(\mathcal{B}) \geq h$, and hence C contains H as a minor.

So, we can assume $r \geq 2$. We enumerate the components of H as $C_1, \dots, C_r$ so that letting $\ell_i = |V(C_i)|$ we have $\ell \geq \ell_1 \geq \ell_2 \geq \dots \geq \ell_r$.

Case 1: G contains a cycle C with length at least ℓ_1 such that the order of $\mathcal{B}_{V(C)}$ is at most $6\ell_1$.

Proof of Case 1: By Theorem 4 we have, $\text{tw}(G - V(C)) \geq \text{tw}(G) - 6\ell_1$. Since $c \log r \geq 6$,

$$\text{tw}(G) - 6\ell_1 \geq ch \log r + cr \log r \log \ell - 6\ell_1 \geq c(h - \ell_1) \log r + c(r - 1) \log(r - 1) \log \ell.$$

So, by the induction hypothesis, $H - V(C_1)$ is a minor of $G - V(C)$. Since the length of C is at least ℓ_1 , C contains C_1 as a minor. So H is a minor of G . ■

Case 2: G contains no cycle C of length at least ℓ_1 such that the order of $\mathcal{B}_{V(C)}$ is at most $6\ell_1$.

We break **Case 2** into two subcases depending on the relation between ℓ_1 and r .

Case 2.1: $\ell_1 \geq 1 + 3c^*r \log r$.

Proof of Case 2.1: Note that $ch \log r \geq 4\ell_1$. Apply Theorem 7 with $t = 2\ell_1$ to obtain the paths $P_1, P_2, Q_1, \dots, Q_{2\ell_1}$. If at least ℓ_1 of those Q_i 's have $|V(Q_i)| \leq \ell_1$, then there are $1 \leq i_1 < i_2 \leq 2\ell_1$ such that the subpath P'_1 of P_1 between the endpoints of Q_{i_1} and Q_{i_2} has length at least $\ell_1 - 1$. So, the union $P'_1 \cup P_2 \cup Q_{i_1} \cup Q_{i_2}$ contains a cycle C containing P'_1 (which has length at least ℓ_1) with $\text{ord}(\mathcal{B}_{V(C)}) \leq |V(Q_1)| + |V(Q_2)| + \text{ord}(\mathcal{B}_{V(P_1)}) + \text{ord}(\mathcal{B}_{V(P_2)}) \leq 6\ell_1$, contradicting the assumption for **Case 2**.

So at least $\ell_1 + 1$ paths among $Q_1, \dots, Q_{2\ell_1}$ have at least $\ell_1 + 1$ vertices. By symmetry, we may assume that $Q_1, \dots, Q_{\ell_1+1}$ have at least $\ell_1 + 1$ vertices. By Theorem 10, $P_1 \cup P_2 \cup \bigcup_{i=1}^{\ell_1+1} Q_i$ contains at least r disjoint cycles of length at least $\min_{1 \leq i \leq \ell_1} |V(Q_i)| \geq \ell_1$. So G contains H as a minor. ■

Case 2.2: $\ell_1 < 1 + 3c^*r \log r$.

Proof of Case 2.2: Let $a = \lfloor \frac{cr \log r}{4} \rfloor$. By Theorem 7, there exist two disjoint paths P_1 and P_2 in G such that both $\mathcal{B}_{V(P_1)}$ and $\mathcal{B}_{V(P_2)}$ have order exactly a , and there exist disjoint paths $Q_1, \dots, Q_a$ each of which has an endpoint on each P_i and is internally disjoint from $P_1 \cup P_2$.

If at least $2 + 3c^*r \log r$ of those Q_i 's have $|V(Q_i)| \geq \ell_1$, then there are at least r disjoint cycles of length at least ℓ_1 contained in the union of $P_1 \cup P_2$ and those Q_i 's with $|V(Q_i)| \geq \ell_1$ by Theorem 10, which gives an H minor in G .

Hence we may assume that there are at most $2 + 3c^*r \log r$ indices $i \in [a]$ such that $|V(Q_i)| \geq \ell_1$.

Let b be the largest integer i with $1 \leq i \leq \lceil 2^{3/4} \frac{3r}{4\ell_1} \rceil$ such that $\ell_i \geq 2^{-3/4} \ell_1$. We claim that

$$c \cdot \sum_{i=1}^b \ell_i \cdot \log r + cr \log r \log \ell - cr \log r \log \ell_{b+1} \geq \frac{3cr \log r}{4}. \quad (1)$$

By the definition of b , we know $\sum_{i=1}^b \ell_i \geq b 2^{-3/4} \ell_1$, so if $b = \lceil 2^{3/4} \frac{3r}{4\ell_1} \rceil$, then $\sum_{i=1}^b \ell_i \geq b 2^{-3/4} \ell_1 \geq \frac{3r}{4}$ and (1) holds. If $b \leq \lceil 2^{3/4} \frac{3r}{4\ell_1} \rceil - 1$, then since $\ell_{b+1} < 2^{-3/4} \ell_1$, we have

$$cr \log r \log \ell - cr \log r \log \ell_{b+1} \geq cr \log r \log \frac{\ell_1}{\ell_{b+1}} \geq \frac{3cr \log r}{4}.$$

This proves (1).

Recall that there are at most $2 + 3c^*r \log r$ indices $i \in [a]$ such that $|V(Q_i)| \geq \ell_1$. So at least $a - (2 + 3c^*r \log r)$ of those paths $Q_1, \dots, Q_a$ have less than ℓ_1 vertices. Now we claim that $a - (2 + 3c^*r \log r) > \ell_1(2 + 3c^*b \log b)$. If $\frac{r}{\ell_1} < 0.79$, then $b \leq \lceil 2^{3/4} \frac{3r}{4\ell_1} \rceil \leq 1$, so $\ell_1(2 + 3c^*b \log b) \leq 2\ell_1$; if $\frac{r}{\ell_1} \geq 0.79$, then $b \leq \lceil 2^{3/4} \frac{3r}{4\ell_1} \rceil \leq \frac{2.6r}{\ell_1}$, so $\ell_1(2 + 3c^*b \log b) \leq$

$2\ell_1 + 7.8c^*r \log r$. Since $\ell_1 < 1 + 3c^*r \log r$, we have $\ell_1(2 + 3c^*b \log b) \leq 2\ell_1 + 7.8c^*r \log r < 2 + 13.8c^*r \log r \leq a - (2 + 3c^*r \log r)$.

Hence, we can choose a set S of $\lceil 2 + 3c^*b \log b \rceil$ paths such that every path in S equals Q_i for some $i \in [a]$ with $|V(Q_i)| < \ell_1$ and the distance between the endpoints in P_1 of any two paths in S is at least ℓ_1 . By Theorem 10, $P_1 \cup P_2 \cup \bigcup_{P \in S} P$ contains b disjoint cycles $F_1, F_2, \dots, F_b$ each containing at least two paths in S . Since the distance between any of the endpoints in P_1 of any two distinct paths in S is at least ℓ_1 , each F_i has a length of at least ℓ_1 . So $\bigcup_{i=1}^b F_i$ contains $\bigcup_{i=1}^b C_i$ as a minor.

Moreover, since each path in S has at most ℓ_1 vertices,

$$\begin{aligned} \text{ord}(\mathcal{B}_{\bigcup_{i=1}^b V(F_i)}) &\leq \text{ord}(\mathcal{B}_{V(P_1 \cup P_2)}) + |S| \cdot \ell_1 \\ &\leq 2a + (2 + 3c^*b \log b)\ell_1 + \ell_1 \\ &< 2a + (a - (2 + 3c^*r \log r)) + (1 + 3c^*r \log r) \\ &= 3a - 1. \end{aligned}$$

This implies

$$\text{ord}(\mathcal{B}_{\bigcup_{i=1}^b V(F_i)}) \leq 3a \leq \frac{3cr \log r}{4} \leq c \cdot \sum_{i=1}^b \ell_i \cdot \log r + cr \log r \log \ell - cr \log r \log \ell_{b+1}$$

by (1). Hence, by Theorem 4,

$$\begin{aligned} \text{tw}(G - \bigcup_{i=1}^b V(F_i)) &\geq \text{tw}(G) - (c \cdot \sum_{i=1}^b \ell_i \cdot \log r + cr \log r \log \ell - cr \log r \log \ell_{b+1}) \\ &\geq c(h - |V(\bigcup_{i=1}^b C_i)|) \cdot \log(r + 1) + cr \log r \log \ell_{b+1} \\ &= c \cdot |V(\bigcup_{i=b+1}^r C_i)| \cdot \log(r + 1) + cr \log r \log \ell_{b+1}. \end{aligned}$$

Since the longest cycle in $\bigcup_{i=b+1}^r C_i$ has length at most ℓ_{b+1} , the induction hypothesis implies that $G - \bigcup_{i=1}^b V(F_i)$ contains $\bigcup_{i=b+1}^r C_i$ as a minor. Recall that $\bigcup_{i=1}^b F_i$ contains $\bigcup_{i=1}^b C_i$ as a minor. Hence, H is a minor of G . $\square$

Update

While a version of this paper was under review, Joret and Micek [JM26] improved Theorem 2 by showing $f(H) = O(|V(H)| + r \log r)$.

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