

Ultra log-concavity and real-rootedness of dependence polynomials

Yan-Ting Xie Shou-Jun Xu

Submitted: Aug 20, 2024; Accepted: Sep 22, 2026; Published: Oct 9, 2026

© The authors. Released under the CC BY-ND license (International 4.0).

Abstract

For some positive integer m , a real polynomial $P(x) = \sum_{k=0}^m a_k x^k$ with $a_k \geq 0$ is called log-concave (resp. ultra log-concave) if $a_k^2 \geq a_{k-1}a_{k+1}$ (resp. $a_k^2 \geq (1 + \frac{1}{k})(1 + \frac{1}{m-k}) \cdot a_{k-1}a_{k+1}$) for all $1 \leq k \leq m-1$. If $P(x)$ has only real roots, then it is called real-rooted. It is well-known that the conditions of log-concavity, ultra log-concavity and real-rootedness are ever-stronger.

For a graph G , a dependent set is a set of vertices which is not independent, i.e., the set of vertices whose induced subgraph contains at least one edge. The dependence polynomial of G is defined as $D(G, x) := \sum_{k \geq 0} d_k(G)x^k$, where $d_k(G)$ is the number of dependent sets of size k in G . Horrocks proved that $D(G, x)$ is log-concave for every graph G [J. Combin. Theory, Ser. B, 84 (2002) 180–185]. In the present paper, we prove that, for a graph G , $D(G, x)$ is ultra log-concave if G is $(K_2 \cup 2K_1)$ -free or contains an independent set of size $|V(G)| - 2$, and give the characterization of graphs whose dependence polynomials are real-rooted. Finally, we focus more attention to the problems of log-concavity about independence systems and pose several conjectures closely related the famous Mason's Conjecture.

Mathematics Subject Classifications: 05C31, 05E45, 05B35

1 Introduction

Throughout this paper, except where stated otherwise, all graphs we consider are undirected, finite and simple.

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. An *independent set* of G is a subset of $V(G)$ whose induced subgraph in G contains no edges. Let $i_k(G)$ be

School of Mathematics and Statistics, Gansu Center for Applied Mathematics, Lanzhou University, Lanzhou, Gansu 730000, China (1976916367@qq.com (Yan-Ting Xie), shjxu@lzu.edu.cn (Shou-Jun Xu)).

the number of independent sets of size k in G . In particular, $i_0(G) = 1$. The generating polynomial of $\{i_k(G)\}$,

$$I(G, x) := \sum_{k \geq 0} i_k(G) x^k,$$

is called the *independence polynomial* of G [13].

If a subset of $V(G)$ is not independent, we call it a *dependent set*, i.e., a dependent set is a subset of $V(G)$ whose induced subgraph in G contains at least one edge. Let $d_k(G)$ be the number of dependent sets of size k in G . In particular, $d_0(G) = d_1(G) = 0$. Similarly, the generating polynomial of $\{d_k(G)\}$,

$$D(G, x) := \sum_{k \geq 0} d_k(G) x^k,$$

is called the *dependence polynomial* of G .

More generally, a graph polynomial is a graph invariant with values in a polynomial ring, usually a subring of real polynomial ring $\mathbb{R}[x]$. Like independence polynomials and dependence polynomials, many of graph polynomials are the generating polynomials for the sequences of numbers of subsets of vertex or edge set with some given properties. A *graph property* $\mathcal{A}$ is defined as a family of graphs closed under isomorphism. Let G be a graph. For $S \subseteq V(G)$, $G[S]$ denotes the subgraph induced by S . Let $\mathcal{A}$ be a graph property and $a_k(G)$ the number of induced subgraphs of order k of G which are in $\mathcal{A}$. The graph polynomial associated with the graph G and the graph property $\mathcal{A}$, denoted by

$P_{\mathcal{A}}(G, x)$, is defined as: $P_{\mathcal{A}}(G, x) := \sum_{S \subseteq V(G), G[S] \in \mathcal{A}} x^{|S|} = \sum_{k=0}^{|V(G)|} a_k(G) x^k$. Correspondingly,

when $\mathcal{A}$ consists of all graphs with no edges, $P_{\mathcal{A}}(G, x)$ is the independence polynomial of G ; when $\mathcal{A}$ consists of all graphs containing at least one edge, $P_{\mathcal{A}}(G, x)$ is the dependence polynomial of G . The two more examples are: when $\mathcal{A}$ consists of all complete graphs, $P_{\mathcal{A}}(G, x)$ is the clique polynomial of G [14]; when $\mathcal{A}$ consists of all forests, $P_{\mathcal{A}}(G, x)$ is the acyclic polynomial of G [5].

Let G be a graph and $\mathcal{A}$ a graph property. The *complement* of $\mathcal{A}$, denoted by $\bar{\mathcal{A}}$, is the graph property satisfying that $G \in \bar{\mathcal{A}} \iff G \notin \mathcal{A}$. About the graph polynomials associated with the graph properties $\mathcal{A}$ and $\bar{\mathcal{A}}$, we have

Proposition 1. For any graph G and graph property $\mathcal{A}$, $P_{\mathcal{A}}(G, x) + P_{\bar{\mathcal{A}}}(G, x) = (x + 1)^{|V(G)|}$.

Proof. For any integer $0 \leq k \leq |V(G)|$, denote $\mathcal{A}_k(G) := \{S \subseteq V(G) : G[S] \in \mathcal{A}, |S| = k\}$ (resp. $\bar{\mathcal{A}}_k(G) := \{S \subseteq V(G) : G[S] \in \bar{\mathcal{A}}, |S| = k\}$) and $a_k(G) := |\mathcal{A}_k(G)|$ (resp. $\bar{a}_k(G) := |\bar{\mathcal{A}}_k(G)|$). Since $\mathcal{A}_k(G)$ and $\bar{\mathcal{A}}_k(G)$ are disjoint, $a_k(G) + \bar{a}_k(G) = |\mathcal{A}_k(G) \cup \bar{\mathcal{A}}_k(G)| = |\{S \subseteq V(G) : |S| = k\}| = \binom{|V(G)|}{k}$. Combined with $P_{\mathcal{A}}(G, x) = \sum_{k=0}^{|V(G)|} a_k(G) x^k$

and $P_{\bar{\mathcal{A}}}(G, x) = \sum_{k=0}^{|V(G)|} \bar{a}_k(G) x^k$, $P_{\mathcal{A}}(G, x) + P_{\bar{\mathcal{A}}}(G, x) = \sum_{k=0}^{|V(G)|} a_k(G) x^k + \sum_{k=0}^{|V(G)|} \bar{a}_k(G) x^k = \sum_{k=0}^{|V(G)|} \binom{|V(G)|}{k} x^k = (x + 1)^{|V(G)|}$. □

We call the graph property $\mathcal{A}$ *hereditary* if it is closed under taking induced subgraphs, i.e., for any graphs G, H , if $G \in \mathcal{A}$ and H is an induced subgraph of G , then $H \in \mathcal{A}$. In contrast, replacing the condition ‘ H is an induced subgraph of G ’ by ‘ G is an induced subgraph of H ’, we say that $\mathcal{A}$ is *co-hereditary*. It is obvious that $\mathcal{A}$ is co-hereditary if and only if $\bar{\mathcal{A}}$ is hereditary.

For example, if $\mathcal{A}$ is the graph property of containing at least one edge, then $\bar{\mathcal{A}}$ is the graph property of containing no edges. We can see that $\mathcal{A}$ is co-hereditary and $\bar{\mathcal{A}}$ is hereditary. Moreover, by Proposition 1 and the definitions of dependence and independence polynomials, for any graph G , $D(G, x) + I(G, x) = (x + 1)^{|V(G)|}$.

Now, we introduce the concepts of unimodality, log-concavity, real-rootedness, and so on. Let $\{a_k\}_{k=0}^m$ be a sequence with real numbers. We define:

- **Unimodality:** There exists an integer k ($0 \leq k \leq m$) satisfying that $a_0 \leq a_1 \leq \dots \leq a_k \geq a_{k+1} \geq \dots \geq a_m$.
- **Logarithmically concavity (or log-concavity for short):** $a_k^2 \geq a_{k-1}a_{k+1}$ for $1 \leq k \leq m-1$.
- **Ordered log-concavity:** $a_k^2 \geq (1 + \frac{1}{k}) a_{k-1}a_{k+1}$ for $1 \leq k \leq m-1$ (equivalent to the log-concavity of the sequence $\{k!a_k\}_{k=0}^m$).
- **Ultra log-concavity:** $a_k^2 \geq (1 + \frac{1}{k}) (1 + \frac{1}{m-k}) a_{k-1}a_{k+1}$ for $1 \leq k \leq m-1$ (equivalent to the log-concavity of the sequence $\{a_k / \binom{m}{k}\}_{k=0}^m$).
- **Real-rootedness:** The polynomial $\sum_{k=0}^m a_k x^k$ has all real roots.

For the non-negative sequences, it is easy to see that these conditions become ever-stronger starting from the second one (see [6]). Moreover, adding the condition of having no internal zeros (that is, for any $0 \leq i < j < k \leq m$, $a_i, a_k > 0 \implies a_j > 0$), the log-concavity implies the unimodality. A polynomial is called *unimodal* (resp. *log-concave*, *ordered log-concave*, *ultra log-concave*) if the sequence of its coefficients is unimodal (resp. log-concave, ordered log-concave, ultra log-concave).

Let G be a graph and $\mathcal{A}$ a graph property. Recently, Makowsky and Rakita [19] obtained the following results about $P_{\mathcal{A}}(G, x)$:

1. If $\mathcal{A}$ is co-hereditary, $P_{\mathcal{A}}(G, x)$ is unimodal for almost every graph $G \in \mathcal{G}(n, p)$ for every edge probability $p \in (0, 1)$.
2. If $\mathcal{A}$ is hereditary with a graph $G \in \mathcal{A}$ which is neither a clique nor an edgeless graph, then $P_{\mathcal{A}}(G, x)$ is real-rooted if and only if $G \in \mathcal{A}$.

It is a natural problem what we can say about the log-concavity or the real-rootedness of $P_{\mathcal{A}}(G, x)$ when $\mathcal{A}$ is co-hereditary. In this paper, we consider this problem on dependence polynomials. As mentioned above, dependence polynomials and independence polynomials are complementary, but for the problem of unimodality, log-concavity or

real-rootedness, the independence polynomials are more difficult than the dependence polynomials. The independence polynomials are not necessarily log-concave or unimodal [2]. Even for bipartite graphs [7] and well-covered graphs [21], there are examples that the independence polynomials are not unimodal. More recently, the examples of trees with non-log-concave independence polynomials were constructed [17, 18]. For the real-rootedness, only very few independence polynomials of graph classes can be proved having only real roots, like claw-free graphs [12]. In contrast, for dependence polynomials, Horrocks [15] showed that $D(G, x)$ is log-concave for every graph G . However, the stronger properties of dependence polynomials, like ordered log-concavity, ultra log-concavity and real-rootedness, have never been studied.

If G contains no edges, there are no dependent sets in G , i.e., $D(G, x) \equiv 0$. Thus, we only consider the graphs which contains at least one edge in what follows.

We denote the *complete graph*, the *complete bipartite graph*, the *path* and the *cycle* by K_n , $K_{m,n}$, P_n and C_n , respectively. In particular, $K_{1,n}$ is called a *star*. Let G and H be two disjoint graphs. The union $G \cup H$ is the graph with the vertex set $V(G \cup H) = V(G) \cup V(H)$ and the edge set $E(G \cup H) = E(G) \cup E(H)$. For a graph G and $n \in \mathbb{N}^*$, nG denotes the disjoint union of n copies of G . For a graph H , a graph G is called *H -free* if no induced subgraphs of G are isomorphic to H . About the ultra log-concavity of dependence polynomials, we have

Theorem 2. *Let G be a graph of order n containing at least one edge. If G is a $K_2 \cup 2K_1$ -free graph or has an independent set of size $n-2$, then the dependence polynomial $D(G, x)$ is ultra log-concave.*

Remark 3. The complement of $K_2 \cup 2K_1$ is $K_4 - e$, which is called a *diamond*, so the $K_2 \cup 2K_1$ -free graphs are equivalent to the complements of the diamond-free graphs.

Let G_1, G_2 be the two graphs illustrated in Fig. 1. We give the characterization of graphs with real-rooted dependence polynomials as follows:

Theorem 4. *Let G be a graph of order n which contains at least one edge. The dependence polynomial $D(G, x)$ is real-rooted if and only if $G \in \{K_2 \cup (n-2)K_1, P_3 \cup (n-3)K_1, K_3 \cup (n-3)K_1, 2K_2 \cup (n-4)K_1, P_4 \cup (n-4)K_1, C_4 \cup (n-4)K_1, G_1 \cup (n-4)K_1, G_2 \cup (n-5)K_1\}$.*

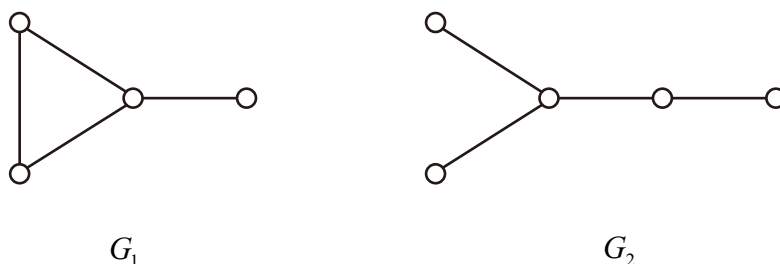


Figure 1: G_1 and G_2 .

For the graphs without isolated vertices, we have the following corollary:

Corollary 5. *Let G be a graph without isolated vertices. The dependence polynomial $D(G, x)$ is real-rooted if and only if $G \in \{K_2, P_3, K_3, 2K_2, P_4, C_4, G_1, G_2\}$.*

The paper is organized as follows. We will prove Theorem 2 in Section 2 and Theorem 4 in Section 3. Finally, in Section 4, we conclude the paper and give some discussion on the log-concave problems of independence systems, which generalize Mason's Conjecture.

2 Proof of Theorem 2

Let G be a graph of order n . For $S \subseteq V(G)$, let's denote $G - S := G[V(G) \setminus S]$. S is called a *co-dependent set* of G if $V(G) \setminus S$ is a dependent set of G . The number of co-dependent sets of size k in G is denoted by $\bar{d}_k(G)$. It is obvious that $\bar{d}_k(G) = d_{n-k}(G)$ and $\bar{d}_n(G) = \bar{d}_{n-1}(G) = 0$. By the definition of ultra log-concavity, $D(G, x)$ is ultra log-concave if and only if the sequence $\{\bar{d}_k(G)\}_{k=0}^n$ is ultra log-concave. Thus, we consider the co-dependent sets of G in what follows. We will write $\bar{d}_k(G)$ as $\bar{d}_k$ shortly without causing confusion.

Let $\bar{\mathcal{D}}(G)$ ($\bar{\mathcal{D}}$ without causing confusion) be the family of co-dependent sets in G . We can see that $(V(G), \bar{\mathcal{D}})$ is an *independence system* (or a *simplicial complex* in a view of geometry), i.e., for any $S, T \subseteq V(G)$, if $S \subseteq T$ and $T \in \bar{\mathcal{D}}$, then $S \in \bar{\mathcal{D}}$. And further, co-dependent sets satisfy the following property:

Observation 6. *Let S be a co-dependent set of a graph G and T a subset of $V(G) \setminus S$. T is a co-dependent set of $G - S$ if and only if $S \cup T$ is a co-dependent set of G .*

The main idea of the proof of Theorem 2 is analogous to that presented in [3]. Define the bivariate function:

$$f_G(y, z) := \sum_{k=0}^{n-2} \bar{d}_k y^{n-k} z^k.$$

Let ∂_y (resp. $\partial_z, \partial_{z_i}$, etc.) denote the partial derivative operator that maps a multivariate polynomial to its partial derivative with respect to y (resp. z, z_i , etc.). Denote the quadratic form

$$\begin{aligned} q_k(y, z) &:= \frac{\partial_y^{n-k-1} \partial_z^{k-1} f_G}{(k-1)!} \\ &= \frac{(n-k+1)!}{2} \bar{d}_{k-1} y^2 + (n-k)! k \bar{d}_k y z + \frac{(n-k-1)! k(k+1)}{2} \bar{d}_{k+1} z^2 \end{aligned} \quad (1)$$

for $1 \leq k \leq n-3$.

For a function $f \in \mathbb{R}[z_1, z_2, \dots, z_n]$, the *Hessian matrix* (or shortly, *Hessian*), denoted by $\nabla^2 f$, is an $n \times n$ symmetric matrix whose (i, j) -th entry is $(\nabla^2 f)_{i,j} := \partial_{z_i} \partial_{z_j} f$. In particular, if f is a quadratic form, then its matrix representation is $f = \frac{1}{2} \mathbf{z}^T \nabla^2 f \mathbf{z}$, where $\mathbf{z} := (z_1, z_2, \dots, z_n)$. Now, we prove the following lemma.

Lemma 7. *Let G be a graph of order n containing at least one edge. If G is a $K_2 \cup 2K_1$ -free graph or has an independent set of size $n-2$, then the determinant of the Hessian $\det(\nabla^2 q_k) \leq 0$ for $1 \leq k \leq n-3$.*

Proof. W.l.o.g., denote $V(G) = [n] = \{1, 2, \dots, n\}$. We define the homogeneous polynomial $g_G \in \mathbb{R}[y, z_1, z_2, \dots, z_n]$ as follows:

$$g_G(y, z_1, z_2, \dots, z_n) := \sum_{S \in \overline{\mathcal{D}}} y^{n-|S|} \prod_{i \in S} z_i.$$

For $S \subseteq V(G)$, denote the differential operator $\partial^S := \prod_{i \in S} \partial_{z_i}$. Let S be a co-dependent set of size $k-1$ in G . Denote $V(G) \setminus S := \{l_1, l_2, \dots, l_{n-k+1}\}$. We consider $q_k^S := \partial_y^{n-k-1} \partial^S g_G$, which is a quadratic form in variables $y, z_{l_1}, z_{l_2}, \dots, z_{l_{n-k+1}}$. Let $\overline{\mathcal{D}}_i(G)$ denote the family of the co-dependent sets of size i in G . By Observation 6, $S' \supset S$ and $S' \in \overline{\mathcal{D}}_k(G)$ (resp. $S' \in \overline{\mathcal{D}}_{k+1}(G)$) if and only if $S' \setminus S \in \overline{\mathcal{D}}_1(G-S)$ (resp. $S' \setminus S \in \overline{\mathcal{D}}_2(G-S)$). Then

$$\begin{aligned} q_k^S(y, z_{l_1}, z_{l_2}, \dots, z_{l_{n-k+1}}) &:= \partial_y^{n-k-1} \partial^S g_G \\ &= \frac{(n-k+1)!}{2} y^2 + (n-k)! \sum_{\{i\} \in \overline{\mathcal{D}}_1(G-S)} y z_i \\ &\quad + (n-k-1)! \sum_{\{i,j\} \in \overline{\mathcal{D}}_2(G-S)} z_i z_j \\ &= \frac{1}{2} \hat{\mathbf{z}}_S^\top (\nabla^2 q_k^S) \hat{\mathbf{z}}_S, \end{aligned} \quad (2)$$

where $\hat{\mathbf{z}}_S := (y, z_{l_1}, z_{l_2}, \dots, z_{l_{n-k+1}})^\top$. Let $\mathbf{a} = (a_1, a_2, \dots, a_{n-k+1})^\top$ be a $(0, 1)$ -vector where $a_i = 1$ if $\{l_i\} \in \overline{\mathcal{D}}_1(G-S)$ and $a_i = 0$ otherwise, $B = (b_{ij})$ an $(n-k+1) \times (n-k+1)$ symmetric $(0, 1)$ -matrix where $b_{ij} = 1$ if $\{l_i, l_j\} \in \overline{\mathcal{D}}_2(G-S)$ and $b_{ij} = 0$ otherwise. Then

$$\nabla^2 q_k^S = (n-k-1)! \begin{bmatrix} (n-k)(n-k+1) & (n-k)\mathbf{a}^\top \\ (n-k)\mathbf{a} & B \end{bmatrix}.$$

Combined with (1) and (2), we can obtain:

$$q_k(y, z) = \sum_{S \in \overline{\mathcal{D}}_{k-1}} q_k^S(y, z, z, \dots, z). \quad (3)$$

Substituting $z_{l_1} = z, z_{l_2} = z, \dots, z_{l_{n-k+1}} = z$ into (2), we can obtain:

$$\begin{aligned} q_k^S(y, z, z, \dots, z) &= \\ \frac{(n-k-1)!}{2} [y \ z] &\begin{bmatrix} (n-k)(n-k+1) & (n-k) \sum_{1 \leq i \leq n-k+1} a_i \\ (n-k) \sum_{1 \leq i \leq n-k+1} a_i & \sum_{1 \leq i, j \leq n-k+1} b_{ij} \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix}. \end{aligned} \quad (4)$$

Denote the 2×2 matrix $Q_S := \begin{bmatrix} (n-k)(n-k+1) & (n-k) \sum_{1 \leq i \leq n-k+1} a_i \\ (n-k) \sum_{1 \leq i \leq n-k+1} a_i & \sum_{1 \leq i, j \leq n-k+1} b_{ij} \end{bmatrix}$. Com-

bined with (3), (4) and the fact that $q_k(y, z) = \frac{1}{2} \cdot [y \ z] \nabla^2 q_k \begin{bmatrix} y \\ z \end{bmatrix}$, we obtain:

$$\nabla^2 q_k = (n - k - 1)! \sum_{S \in \overline{\mathcal{D}}_{k-1}} Q_S. \quad (5)$$

For $S \in \overline{\mathcal{D}}_{k-1}(G)$, let's consider the subgraph $G - S$. Since S is a co-dependent set of G , $G - S$ must contain at least one edge. We call a vertex i in $V(G) \setminus S$ a *co-dependent vertex* in $G - S$ if $\{i\} \in \overline{\mathcal{D}}_1(G - S)$, and a pair of vertices $\{i, j\}$ a *co-dependent pair* if $\{i, j\} \in \overline{\mathcal{D}}_2(G - S)$. By the definitions, $\sum_{1 \leq i \leq n-k+1} a_i$ is the number of co-dependent vertices in $G - S$ and $\sum_{1 \leq i, j \leq n-k+1} b_{ij}$ is twice of the number of co-dependent pairs in $G - S$. If a vertex $i \in V(G) \setminus S$ is not a co-dependent vertex in $G - S$, we call i a *co-independent vertex* in $G - S$. The definition of *co-independent pairs* is similar.

Claim. The number of co-independent vertices in $G - S$ is at most 2.

By contradiction, w.l.o.g., assume that l_1, l_2 and l_3 are three co-independent vertices in $G - S$. By the definition, all of $\{l_2, l_3, l_4, \dots, l_{n-k+1}\}$, $\{l_1, l_3, l_4, \dots, l_{n-k+1}\}$ and $\{l_1, l_2, l_4, \dots, l_{n-k+1}\}$ are independent sets in $G - S$. This deduces that $G - S$ contains no edges, a contradiction.

Now, we distinguish cases to discuss Q_S .

Case 1. There are two co-independent vertices in $G - S$.

W.l.o.g., assume that l_1 and l_2 are co-independent vertices. Then $\{l_2, l_3, l_4, \dots, l_{n-k+1}\}$ and $\{l_1, l_3, l_4, \dots, l_{n-k+1}\}$ are independent sets and further $G - S \cong K_2 \cup (n - k - 1)K_1$, which only l_1 and l_2 are adjacent. In this case, $\sum_{1 \leq i \leq n-k+1} a_i = n - k - 1$, $\sum_{1 \leq i, j \leq n-k+1} b_{ij} = 2 \cdot \binom{n-k-1}{2} = (n - k - 1)(n - k - 2)$. So

$$Q_S = \begin{bmatrix} (n - k)(n - k + 1) & (n - k)(n - k - 1) \\ (n - k)(n - k - 1) & (n - k - 1)(n - k - 2) \end{bmatrix}.$$

For convenience, we denote

$$H_1 = \begin{bmatrix} (n - k)(n - k + 1) & (n - k)(n - k - 1) \\ (n - k)(n - k - 1) & (n - k - 1)(n - k - 2) \end{bmatrix} := \begin{bmatrix} \alpha_1 & \beta_1 \\ \beta_1 & \gamma_1 \end{bmatrix}.$$

Case 2. There is exactly one co-independent vertex in $G - S$.

In this case, $G - S \cong K_{1,p} \cup (n - k - p)K_1$ ($p \geq 2$) and $\sum_{1 \leq i \leq n-k+1} a_i = n - k$. W.l.o.g., assume l_1 is the center of $K_{1,p}$.

Subcase 2.1. $p = 2$.

Assume l_2, l_3 are the leaves of the $K_{1,2}$. In this subcase, the co-independent pairs of $G - S$ are $\{l_2, l_3\}$ and $\{l_1, l_i\}$ ($2 \leq i \leq n - k + 1$). Thus,

$$\sum_{1 \leq i, j \leq n-k+1} b_{ij} = 2 \left(\binom{n-k+1}{2} - (n - k) - 1 \right) = (n - k)(n - k - 1) - 2.$$

So

$$Q_S = \begin{bmatrix} (n-k)(n-k+1) & (n-k)^2 \\ (n-k)^2 & (n-k)(n-k-1)-2 \end{bmatrix}.$$

Let's denote

$$H_2 = \begin{bmatrix} (n-k)(n-k+1) & (n-k)^2 \\ (n-k)^2 & (n-k)(n-k-1)-2 \end{bmatrix} := \begin{bmatrix} \alpha_2 & \beta_2 \\ \beta_2 & \gamma_2 \end{bmatrix}.$$

Subcase 2.2. $3 \leq p \leq n-k$.

In this subcase, the co-independent pairs are $\{l_1, l_i\}$ ($2 \leq i \leq n-k+1$). Thus, $\sum_{1 \leq i, j \leq n-k+1} b_{ij} = 2 \cdot \left(\binom{n-k+1}{2} - (n-k) \right) = (n-k)(n-k-1)$. So

$$Q_S = \begin{bmatrix} (n-k)(n-k+1) & (n-k)^2 \\ (n-k)^2 & (n-k)(n-k-1) \end{bmatrix}.$$

Let's denote

$$H_3 = \begin{bmatrix} (n-k)(n-k+1) & (n-k)^2 \\ (n-k)^2 & (n-k)(n-k-1) \end{bmatrix} := \begin{bmatrix} \alpha_3 & \beta_3 \\ \beta_3 & \gamma_3 \end{bmatrix}.$$

Case 3. There are no co-independent vertices in $G-S$.

In this case, $\sum_{1 \leq i \leq n-k+1} a_i = n-k+1$. Combined with the fact that $G-S$ contains at least one edge and $G-S \not\cong K_2 \cup (n-k-1)K_1$, it contains at most $n-k-2$ isolated vertices. We distinguish three subcases in what follows.

Subcase 3.1. There are $n-k-2$ isolated vertices in $G-S$.

In this subcase, $G-S \cong K_3 \cup (n-k-2)K_1$. W.l.o.g., assume that l_1, l_2, l_3 induce the K_3 . Then the co-independent pairs of $G-S$ are $\{l_1, l_2\}$, $\{l_1, l_3\}$ and $\{l_2, l_3\}$ and $\sum_{1 \leq i, j \leq n-k+1} b_{ij} = 2 \cdot \left(\binom{n-k+1}{2} - 3 \right) = (n-k)(n-k+1) - 6$. So

$$Q_S = \begin{bmatrix} (n-k)(n-k+1) & (n-k)(n-k+1) \\ (n-k)(n-k+1) & (n-k)(n-k+1)-6 \end{bmatrix}.$$

Subcase 3.2. There are $n-k-3$ isolated vertices in $G-S$.

In this subcase, $G-S \cong G^* \cup (n-k-3)K_1$, where G^* is a graph of order 4 containing no isolated vertices and not a star. Since G^* is a graph of order 4, the number of co-independent pairs $G-S$ is equal to the one of edges in the complement of G^* , i.e., $|E(\overline{G^*})|$. We can see that $|E(\overline{G^*})| \in \{0, 1, 2, 3, 4\}$ (see Fig. 2), then $\sum_{1 \leq i, j \leq n-k+1} b_{ij} = (n-k)(n-k+1) - 8$, $(n-k)(n-k+1) - 6$, $(n-k)(n-k+1) - 4$, $(n-k)(n-k+1) - 2$ or $(n-k)(n-k+1)$. Thus,

$$Q_S = \begin{bmatrix} (n-k)(n-k+1) & (n-k)(n-k+1) \\ (n-k)(n-k+1) & (n-k)(n-k+1)-8 \end{bmatrix},$$

$$\begin{bmatrix} (n-k)(n-k+1) & (n-k)(n-k+1) \\ (n-k)(n-k+1) & (n-k)(n-k+1)-6 \end{bmatrix},$$

$$\begin{bmatrix} (n-k)(n-k+1) & (n-k)(n-k+1) \\ (n-k)(n-k+1) & (n-k)(n-k+1) - 4 \end{bmatrix},$$

$$\begin{bmatrix} (n-k)(n-k+1) & (n-k)(n-k+1) \\ (n-k)(n-k+1) & (n-k)(n-k+1) - 2 \end{bmatrix}$$

or

$$\begin{bmatrix} (n-k)(n-k+1) & (n-k)(n-k+1) \\ (n-k)(n-k+1) & (n-k)(n-k+1) \end{bmatrix}.$$

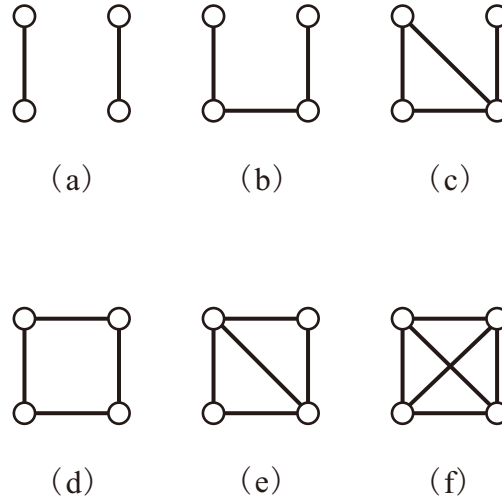


Figure 2: All graphs of order 4 containing no isolated vertices and not a star: (a) contains 4 co-independent pairs, (b) contains 3 co-independent pairs, (c) and (d) contain 2 co-independent pairs, (e) contains 1 co-independent pair and (f) contains no co-independent pairs.

Subcase 3.3. There are at most $n - k - 4$ isolated vertices in $G - S$.

If $G - S$ contains no co-independent pairs,

$$Q_S = \begin{bmatrix} (n-k)(n-k+1) & (n-k)(n-k+1) \\ (n-k)(n-k+1) & (n-k)(n-k+1) \end{bmatrix}.$$

Now, we assume $G - S$ contains a co-independent pair. W.l.o.g., let $\{l_1, l_2\}$ be a co-independent pair. Since $G - S$ contains no co-independent vertices, $N_{G-S}(l_1) \cap \{l_3, l_4, \dots, l_{n-k+1}\} \neq \emptyset$ and $N_{G-S}(l_2) \cap \{l_3, l_4, \dots, l_{n-k+1}\} \neq \emptyset$, where $N_{G-S}(l_i)$ is the neighborhood of l_i in $G - S$. Since there are at most $n - k - 4$ isolated vertices in $G - S$, $|(N_{G-S}(l_1) \cup N_{G-S}(l_2)) \cap \{l_3, l_4, \dots, l_{n-k+1}\}| \geq 3$. If $|N_{G-S}(l_1) \cap \{l_3, l_4, \dots, l_{n-k+1}\}| \geq 2$ and $|N_{G-S}(l_2) \cap \{l_3, l_4, \dots, l_{n-k+1}\}| \geq 2$, then $\{l_1, l_2\}$ is the unique co-independent pair in $G - S$ (see Fig. 3) and further $\sum_{1 \leq i, j \leq n-k+1} b_{ij} = (n-k)(n-k+1) - 2$. Otherwise, w.l.o.g., assume l_2 has exactly one neighbor l_3 in $\{l_3, l_4, \dots, l_{n-k+1}\}$ in $G - S$. On this

assumption, l_1 must have two neighbors other than l_3 in $\{l_3, l_4, \dots, l_{n-k+1}\}$. We can see that the co-independent pairs in $G - S$ are $\{l_1, l_2\}$ and $\{l_1, l_3\}$ (see Fig. 4), then $\sum_{1 \leq i, j \leq n-k+1} b_{ij} = (n-k)(n-k+1) - 4$. So, in this subcase,

$$Q_S = \begin{bmatrix} (n-k)(n-k+1) & (n-k)(n-k+1) \\ (n-k)(n-k+1) & (n-k)(n-k+1) \end{bmatrix},$$

$$\begin{bmatrix} (n-k)(n-k+1) & (n-k)(n-k+1) \\ (n-k)(n-k+1) & (n-k)(n-k+1) - 2 \end{bmatrix}$$

or

$$\begin{bmatrix} (n-k)(n-k+1) & (n-k)(n-k+1) \\ (n-k)(n-k+1) & (n-k)(n-k+1) - 4 \end{bmatrix}.$$

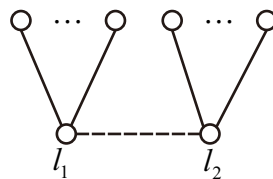


Figure 3: The case that $|N_{G-S}(l_1) \cap \{l_3, l_4, \dots, l_{n-k+1}\}| \geq 2$ and $|N_{G-S}(l_2) \cap \{l_3, l_4, \dots, l_{n-k+1}\}| \geq 2$: dashed edge represent that l_1, l_2 are adjacent or not.

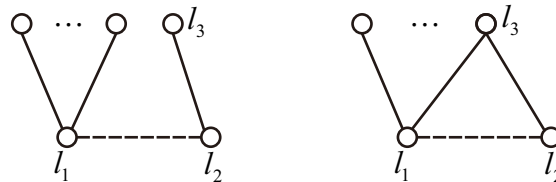


Figure 4: The case that l_2 has exactly one neighbor l_3 in $\{l_3, l_4, \dots, l_{n-k+1}\}$: dashed edge represent that l_1, l_2 are adjacent or not.

Let's denote

$$H_4 = \begin{bmatrix} (n-k)(n-k+1) & (n-k)(n-k+1) \\ (n-k)(n-k+1) & (n-k)(n-k+1) - 8 \end{bmatrix} := \begin{bmatrix} \alpha_4 & \beta_4 \\ \beta_4 & \gamma_4 \end{bmatrix},$$

$$H_5 = \begin{bmatrix} (n-k)(n-k+1) & (n-k)(n-k+1) \\ (n-k)(n-k+1) & (n-k)(n-k+1) - 6 \end{bmatrix} := \begin{bmatrix} \alpha_5 & \beta_5 \\ \beta_5 & \gamma_5 \end{bmatrix},$$

$$H_6 = \begin{bmatrix} (n-k)(n-k+1) & (n-k)(n-k+1) \\ (n-k)(n-k+1) & (n-k)(n-k+1) - 4 \end{bmatrix} := \begin{bmatrix} \alpha_6 & \beta_6 \\ \beta_6 & \gamma_6 \end{bmatrix},$$

$$H_7 = \begin{bmatrix} (n-k)(n-k+1) & (n-k)(n-k+1) \\ (n-k)(n-k+1) & (n-k)(n-k+1) - 2 \end{bmatrix} := \begin{bmatrix} \alpha_7 & \beta_7 \\ \beta_7 & \gamma_7 \end{bmatrix}$$

and

$$H_8 = \begin{bmatrix} (n-k)(n-k+1) & (n-k)(n-k+1) \\ (n-k)(n-k+1) & (n-k)(n-k+1) \end{bmatrix} := \begin{bmatrix} \alpha_8 & \beta_8 \\ \beta_8 & \gamma_8 \end{bmatrix}.$$

Now, it has been proved that, for any co-dependent set $S \in \overline{\mathcal{D}}_{k-1}(G)$, the matrix $Q_S = H_i$ for some $i \in \{1, 2, \dots, 8\}$. Denote $t_i := |\{S \in \overline{\mathcal{D}}_{k-1}(G) | Q_S = H_i\}|$ ($i = 1, 2, \dots, 8$). By (5), we obtain:

$$\det(\nabla^2 q_k) = ((n-k-1)!)^2 \det\left(\sum_{i=1}^8 t_i H_i\right) = ((n-k-1)!)^2 \begin{vmatrix} \sum_{i=1}^8 \alpha_i t_i & \sum_{i=1}^8 \beta_i t_i \\ \sum_{i=1}^8 \beta_i t_i & \sum_{i=1}^8 \gamma_i t_i \end{vmatrix}. \quad (6)$$

The determinant $\begin{vmatrix} \sum_{i=1}^8 \alpha_i t_i & \sum_{i=1}^8 \beta_i t_i \\ \sum_{i=1}^8 \beta_i t_i & \sum_{i=1}^8 \gamma_i t_i \end{vmatrix}$ is a quadratic form in $t_1, t_2, \dots, t_8$. Let's denote its matrix representation:

$$\begin{vmatrix} \sum_{i=1}^8 \alpha_i t_i & \sum_{i=1}^8 \beta_i t_i \\ \sum_{i=1}^8 \beta_i t_i & \sum_{i=1}^8 \gamma_i t_i \end{vmatrix} := \mathbf{t}^T A \mathbf{t}, \quad (7)$$

where $\mathbf{t} = (t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8)^T$, A is an 8×8 symmetric matrix and $A_{ij} = \frac{1}{2}(\alpha_i \gamma_j + \alpha_j \gamma_i - 2\beta_i \beta_j)$ for $1 \leq i, j \leq 8$. For convenience, we denote $r := n - k + 1$. $r \geq 4$ since $k \leq n - 3$. By the calculation, we obtain:

$$A = (r-1) \cdot \begin{bmatrix} -2(r-2) & -2(r-1) & -(r-2) & -3r & -2r & -r & 0 & r \\ -2(r-1) & -(3r-1) & -(2r-1) & -5r & -4r & -3r & -2r & -r \\ -(r-2) & -(2r-1) & -(r-1) & -4r & -3r & -2r & -r & 0 \\ -3r & -5r & -4r & -8r & -7r & -6r & -5r & -4r \\ -2r & -4r & -3r & -7r & -6r & -5r & -4r & -3r \\ -r & -3r & -2r & -6r & -5r & -4r & -3r & -2r \\ 0 & -2r & -r & -5r & -4r & -3r & -2r & -r \\ r & -r & 0 & -4r & -3r & -2r & -r & 0 \end{bmatrix}. \quad (8)$$

If G is $K_2 \cup 2K_1$ -free, for any co-dependent set S of size $k-1$, $G-S \not\cong K_2 \cup (n-k-1)K_1$, that is, $t_1 = 0$. If G has an independent set of size $n-2$, for any co-dependent set S of size $k-1$, $G-S$ has an independent set of size $n-k-1$. Then $G-S$ must contain co-independent pairs. It deduces that $t_8 = 0$.

Since $t_i \geq 0$ for all $1 \leq i \leq 8$, combined with (6), (7) and (8), we can see that $\det(\nabla^2 q_k) \leq 0$ if $t_1 = 0$ or $t_8 = 0$. This completes the proof. $\square$

Now, we prove Theorem 2.

Proof of Theorem 2. By (1),

$$\begin{aligned}\det(\nabla^2 q_k) &= \begin{vmatrix} (n-k+1)!\bar{d}_{k-1} & (n-k)!k\bar{d}_k \\ (n-k)!k\bar{d}_k & (n-k-1)!k(k+1)\bar{d}_{k+1} \end{vmatrix} \\ &= k^2((n-k)!)^2 \left(\left(1 + \frac{1}{k}\right) \left(1 + \frac{1}{n-k}\right) \bar{d}_{k-1}\bar{d}_{k+1} - \bar{d}_k^2 \right).\end{aligned}$$

By Lemma 7, if G is $K_2 \cup 2K_1$ -free or has an independent set of size $n-2$, $\det(\nabla^2 q_k) \leq 0$ for $1 \leq k \leq n-3$, that is, $\left(1 + \frac{1}{k}\right) \left(1 + \frac{1}{n-k}\right) \bar{d}_{k-1}\bar{d}_{k+1} - \bar{d}_k^2 \leq 0$ for $1 \leq k \leq n-3$. Combined with $\bar{d}_n = \bar{d}_{n-1} = 0$, we obtain that $\{\bar{d}_k\}_{k=0}^n$ is ultra log-concave. Thus, $D(G, x)$ is ultra log-concave. $\square$

3 Proof of Theorem 4

Before the proof of Theorem 4, we will introduce some lemmas and terminology in what follows. Firstly, we have the following lemma:

Lemma 8. *For any graph G and $m \in \mathbb{N}$, $D(G \cup mK_1, x) = (x+1)^m D(G, x)$.*

Proof. Assume that $|V(G)| = n$. It is known that $I(G \cup mK_1, x) = (I(K_1, x))^m I(G, x) = (x+1)^m I(G, x)$ (see [13]). Then, by Proposition 1, $D(G \cup mK_1, x) = (x+1)^{m+n} - I(G \cup mK_1, x) = (x+1)^{m+n} - (x+1)^m I(G, x) = (x+1)^m [(x+1)^n - I(G, x)] = (x+1)^m D(G, x)$. $\square$

Let $P(x) = \sum_{k=0}^m a_k x^k$ be a polynomial. $P'(x) := \sum_{k=0}^{m-1} (k+1)a_{k+1}x^k$ denotes the *derivative* of $P(x)$. It is known that

Proposition 9 (Folklore). *Let $P(x)$ be a real polynomial with degree at least 2. If $P(x)$ is real-rooted, so is $P'(x)$.*

The following proposition will be useful in our proof.

Lemma 10. *Let $P(x) = \sum_{k=0}^m a_k x^k$ be a real polynomial with $a_k \geq 0$ for $1 \leq k \leq m$. If there exists an integer j for $1 \leq j \leq m-1$ such that $a_{j-1} = a_j = a_{j+1} > 0$, then $P(x)$ is not real-rooted.*

Proof. **Case 1.** $a_0 > 0$.

In this case, if there exists an integer j such that $a_{j-1} = a_j = a_{j+1} > 0$ for $1 \leq j \leq m-1$, then

$$a_j^2 < \left(1 + \frac{1}{j}\right) a_j^2 = \left(1 + \frac{1}{j}\right) a_{j-1} a_{j+1}.$$

That is, $P(x)$ is not ordered log-concave, so it is not real-rooted.

Case 2. $a_0 \leq 0$.

In this case, if there exists an integer j such that $a_{j-1} = a_j = a_{j+1} > 0$, then $2 \leq j \leq m-1$. By contradiction assume $P(x)$ is real-rooted. Then $P'(x)$ is real-rooted by Proposition 9. The coefficient sequence of $P'(x)$, $\{(k+1)a_{k+1}\}_{k=0}^{m-1}$, is a non-negative sequence. Let's consider the coefficients of the terms x^j , x^{j-1} and x^{j-2} of $P'(x)$. We obtain

$$\left(1 + \frac{1}{j-1}\right) (j+1)a_{j+1} \cdot (j-1)a_{j-1} = j(j+1)a_j^2 > (ja_j)^2.$$

Thus, $P'(x)$ is not ordered log-concave, a contradiction with real-rootedness of $P'(x)$. $\square$

Now, we introduce a necessary and sufficient condition of real-rootedness of real polynomials: Sturm's Theorem. Firstly, we give the definition of Sturm sequence:

Definition 11. [4, 10, 19] Let $P(x)$ be a real polynomial with degree at least 1 and positive leading coefficient. The polynomial sequence

$$P_0(x), P_1(x), \dots, P_k(x)$$

is called a *Sturm sequence* of $P(x)$ if it satisfies

1. $P_0(x) := P(x)$,
2. $P_1(x) := P'(x)$,
3. $P_i(x) := -R_{i-2,i-1}(x)$ for $i \geq 2$, where

$$P_{i-2}(x) = Q_i(x)P_{i-1}(x) + R_{i-2,i-1}(x)$$

and the degree of $R_{i-2,i-1}(x)$ is less than $P_{i-1}(x)$, i.e., $R_{i-2,i-1}(x)$ is the remainder of the division of $P_{i-2}(x)$ by $P_{i-1}(x)$, and

4. $P_k(x)$ is the polynomial with the smallest possible non-negative degree such that $P_k(x) \neq 0$.

Sturm's Theorem is shown in what follows:

Lemma 12 (Sturm's Theorem). [4, 10, 19] Let $P(x)$ be a real polynomial with degree at least 1 and positive leading coefficient. Let $P_0(x), P_1(x), \dots, P_k(x)$ be Sturm sequence of $P(x)$, where the degree of $P_i(x)$ is d_i for $0 \leq i \leq k$. $P(x)$ is real-rooted if and only if $P_i(x)$ has a positive leading coefficient for $0 \leq i \leq k$ and $d_i - d_{i+1} = 1$ for $0 \leq i \leq k-1$.

By using Lemma 12, we obtain the following lemma:

Lemma 13. Let G be a graph of order n and $\mathcal{A}$ a graph property. If $P_{\mathcal{A}}(G, x)$ is real-rooted and $P_{\mathcal{A}}(G, x) \neq (x+1)^n$, then there exists a subset S of $V(G)$ such that $|S| \geq n-2$ and $G[S] \notin \mathcal{A}$. In particular, if G contains at least one edge and $D(G, x)$ is real-rooted, then G has an independent set of size $n-2$.

Proof. By contradiction assume that $G[S] \in \mathcal{A}$ for any $S \subseteq V(G)$ where $|S| \geq n - 2$. By the definition, the degree of $P_{\bar{\mathcal{A}}}(G, x)$ is at most $n - 3$. Combined with $P_{\mathcal{A}}(G, x) \neq (x + 1)^n$ and Proposition 1, $P_{\bar{\mathcal{A}}}(G, x) \neq 0$ or $(x + 1)^n$. Let $P_0(x), P_1(x), \dots, P_k(x)$ be Sturm sequence of $P_{\mathcal{A}}(G, x)$, where the degree of $P_i(x)$ is d_i for $0 \leq i \leq k$. By the definition of Sturm sequence, $P_0(x) := P_{\mathcal{A}}(G, x) = (x + 1)^n - P_{\bar{\mathcal{A}}}(G, x)$, $P_1(x) := P'_{\mathcal{A}}(G, x) = n(x + 1)^{n-1} - P'_{\bar{\mathcal{A}}}(G, x)$. Since the degree of $P_{\bar{\mathcal{A}}}(G, x)$ is at most $n - 3$, the degree of $P'_{\bar{\mathcal{A}}}(G, x)$ is at most $n - 4$ and $d_1 = n - 1$. Since $P_{\bar{\mathcal{A}}}(G, x) \neq 0$ or $(x + 1)^n$, $P_2(x) = \frac{1}{n}(x + 1)P_1(x) - P_0(x) = P_{\bar{\mathcal{A}}}(G, x) - \frac{1}{n}(x + 1)P'_{\bar{\mathcal{A}}}(G, x) \neq 0$. We can see that $d_2 \leq n - 3$, then $d_1 - d_2 \geq 2$, a contradiction with Lemma 12. $\square$

Now, we can prove Theorem 4.

Proof of Theorem 4. Sufficiency. By Lemma 8, we only need to prove the real-rootedness of the dependence polynomials of $K_2, P_3, K_3, 2K_2, P_4, C_4, G_1$ and G_2 . Let's calculate them one by one: $D(K_2, x) = x^2$; $D(P_3, x) = x^3 + 2x^2 = x^2(x + 2)$; $D(K_3, x) = x^3 + 3x^2 = x^2(x + 3)$; $D(2K_2, x) = x^4 + 4x^3 + 2x^2 = x^2(x + 2 - \sqrt{2})(x + 2 + \sqrt{2})$; $D(P_4, x) = x^4 + 4x^3 + 3x^2 = x^2(x + 1)(x + 3)$; $D(C_4, x) = x^4 + 4x^3 + 4x^2 = x^2(x + 2)^2$; $D(G_1, x) = x^4 + 4x^3 + 4x^2 = x^2(x + 2)^2$; $D(G_2, x) = x^5 + 5x^4 + 8x^3 + 4x^2 = x^2(x + 1)(x + 2)^2$. They are all real-rooted.

Necessity. Let G be a graph of order n which contains at least one edge. Assume that $D(G, x)$ is real-rooted. By Lemma 13, G has an independent set of size $n - 2$, denoted by X . Set $\{u, v\} := V(G) \setminus X$. Let's denote $A := N(u) \cap X$, $B := N(v) \cap X$ and $C := A \cup B$. Set $a := |A|$, $b := |B|$, $c := |C|$. W.l.o.g., we assume $a \leq b$. If u, v are not adjacent, we denote G by $G_{a,b,c}$. Otherwise, we denote it by $G_{a,b,c}^+$ (see Fig. 5).

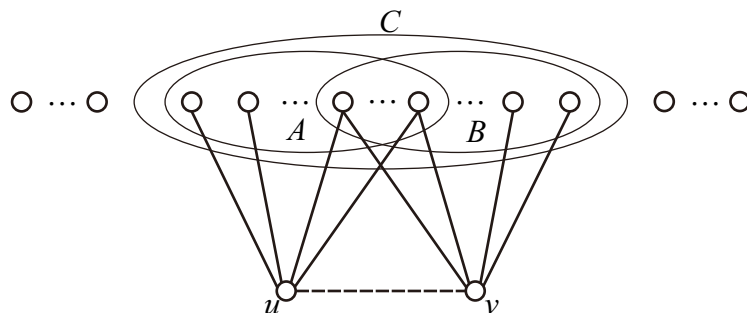


Figure 5: $G_{a,b,c}^+$ or $G_{a,b,c}$: dashed edge represent that u, v are adjacent or not.

Now, let's calculate $D(G_{a,b,c}, x)$ and $D(G_{a,b,c}^+, x)$. For $G \in \{G_{a,b,c}, G_{a,b,c}^+\}$, let $d_k^{(1)}(G)$ (resp. $d_k^{(2)}(G)$, or $d_k^{(3)}(G)$) be the number of dependence sets of size k which contains u but not v (resp. v but not u , or both u and v). Denote $D_1(G, x) := \sum_{k \geq 0} d_k^{(1)}(G)x^k$ (resp. $D_2(G, x) := \sum_{k \geq 0} d_k^{(2)}(G)x^k$, $D_3(G, x) := \sum_{k \geq 0} d_k^{(3)}(G)x^k$). Since X is an independent set, a

dependent set of G must contain u or v . Thus, $D(G, x) = D_1(G, x) + D_2(G, x) + D_3(G, x)$. Now, we calculate $D_1(G, x)$, $D_2(G, x)$ and $D_3(G, x)$. The Product formula of generating function in what follows will be used.

The Product formula. (see [8], p. 156, Theorem 8.5) Let $k \geq 2$ be an integer. Let $a_m^{(1)}$ be the number of ways to build a certain structure on an m -element set, $a_m^{(2)}$ the number of ways to build the second structure on an m -element set, $\dots$, $a_m^{(k)}$ the number of ways to build the k -th structure on an m -element set. Let a_m be the number of ways to separate an m -element set into the k disjoint subsets $S_1, S_2, \dots, S_k$ ($S_1, S_2, \dots, S_k$ are allowed to be empty), and then to build a structure of the first kind on S_1 , a structure of the second kind on S_2 , $\dots$, a structure of the k -th kind on S_k . Let $P_1(x), P_2(x), \dots, P_k(x), P(x)$ be the respective generating functions of the sequences $\{a_m^{(1)}\}, \{a_m^{(2)}\}, \dots, \{a_m^{(k)}\}, \{a_m\}$. Then $P(x) = P_1(x)P_2(x) \cdots P_k(x)$.

For $G \in \{G_{a,b,c}, G_{a,b,c}^+\}$, a dependent set S containing u but not v can be separated into three disjoint parts: $\{u\}$, $S \cap A$ and $S \cap (X \setminus A)$. Firstly, since u must be chosen in S , the generating function for the sequence of numbers of ways to choose vertices in $\{u\}$ is x . Then, to ensure that $G[S]$ contains at least one edge, $S \cap A \neq \emptyset$, so the generating function for the sequence of numbers of ways to choose vertices in A is $\sum_{k=1}^a \binom{a}{k} x^k = (x+1)^a - 1$. Finally, the generating function for the sequence of numbers of ways to choose vertices in $X \setminus A$ is $\sum_{k=0}^{n-a-2} \binom{n-a-2}{k} x^k = (x+1)^{n-a-2}$. By the Product formula,

$$D_1(G, x) = x(x+1)^{n-a-2}[(x+1)^a - 1]. \text{ Similarly, } D_2(G, x) = x(x+1)^{n-b-2}[(x+1)^b - 1].$$

Now let's consider $D_3(G, x)$. When $G = G_{a,b,c}$, for every dependent set S containing both u and v , $S \cap C \neq \emptyset$ since $\{u, v\}$ is an independent set and there must exist at least one vertex in $S \cap C$ to ensure that $G[S]$ contains edges. Similar to the analysis of $D_1(G, x)$, we obtain that $D_3(G_{a,b,c}, x) = x^2(x+1)^{n-c-2}[(x+1)^c - 1]$. When $G = G_{a,b,c}^+$, since $\{u, v\}$ is a dependent set, the union of $\{u, v\}$ and every subset of X is also a dependent set. Thus, $D_3(G_{a,b,c}^+, x) = x^2(x+1)^{n-2}$.

Then

$$\begin{aligned} D(G_{a,b,c}, x) &= D_1(G_{a,b,c}, x) + D_2(G_{a,b,c}, x) + D_3(G_{a,b,c}, x) \\ &= x(x+1)^{n-a-2}[(x+1)^a - 1] + x(x+1)^{n-b-2}[(x+1)^b - 1] \\ &\quad + x^2(x+1)^{n-c-2}[(x+1)^c - 1]; \\ D(G_{a,b,c}^+, x) &= D_1(G_{a,b,c}^+, x) + D_2(G_{a,b,c}^+, x) + D_3(G_{a,b,c}^+, x) \\ &= x(x+1)^{n-a-2}[(x+1)^a - 1] + x(x+1)^{n-b-2}[(x+1)^b - 1] \\ &\quad + x^2(x+1)^{n-2}. \end{aligned}$$

It is obvious that a polynomial $P(x)$ is real-rooted if and only if $P(x-1)$ is real-rooted. Let's consider $D(G_{a,b,c}, x-1)$ and $D(G_{a,b,c}^+, x-1)$. Noting that $0 \leq a \leq b \leq c$, we have

$$\begin{aligned} D(G_{a,b,c}, x-1) &= (x-1)x^{n-c-2}(x^{c+1} + x^c - x^{c-a} - x^{c-b} - x + 1); \\ D(G_{a,b,c}^+, x-1) &= (x-1)x^{n-b-2}(x^{b+1} + x^b - x^{b-a} - 1). \end{aligned}$$

Let's denote $Q_1(x) := x^{c+1} + x^c - x^{c-a} - x^{c-b} - x + 1$ and $Q_2(x) := x^{b+1} + x^b - x^{b-a} - 1$. Then $D(G_{a,b,c}, x)$ is real-rooted if and only if $Q_1(x)$ is real-rooted; $D(G_{a,b,c}^+, x)$ is real-rooted if and only if $Q_2(x)$ is real-rooted. Now, we discuss the real-rootedness of $Q_1(x)$ and $Q_2(x)$.

Case 1. The real-rootedness of $Q_1(x)$.

We distinguish two subcases: $b = c$ or not.

Subcase 1.1. $b = c$.

When $a = b = c$, $Q_1(x) = x^{c+1} + x^c - x - 1 = (x+1)(x^c - 1)$. It is real-rooted if and only if $c \leq 2$. Since $G_{a,b,c}$ contains at least one edge, $c \neq 0$. When $a = b = c = 1$, $G_{a,b,c} \cong P_3 \cup (n-3)K_1$. When $a = b = c = 2$, $G_{a,b,c} \cong C_4 \cup (n-4)K_1$.

When $a < b = c$, $Q_1(x) = x^{c+1} + x^c - x^{c-a} - x$. Since $Q_1(0) = Q_1(1) = 0$, we can assume $Q_1(x) := x(x-1)Q_3(x)$. Then $Q_1(x)$ is real-rooted if and only if either $Q_3(x)$ is real-rooted, or $Q_3(x) \neq 0$ is a constant function. When $a = 0$, $Q_3(x) = x^{c-1} + x^{c-2} + \dots + x + 1$. If it is real-rooted, then $c = 2$ and further $G_{a,b,c} \cong P_3 \cup (n-3)K_1$; if it is a non-zero constant function, then $c = 1$ and further $G_{a,b,c} \cong K_2 \cup (n-2)K_1$. When $a \geq 1$, $Q_3(x) = x^{c-1} + 2x^{c-2} + 2x^{c-3} + \dots + 2x^{c-a-1} + x^{c-a-2} + x^{c-a-3} + \dots + x + 1$. If $Q_3(x)$ is a non-zero constant function, then $c = 1$ and it contradicts with $a < c$. Now assume that $Q_3(x)$ is real-rooted. By Lemma 10, $c-3 \leq c-a-1$, i.e., $a \leq 2$. If $c-a-3 \geq 0$, considering the coefficients of the terms x^{c-a-1} , x^{c-a-2} , x^{c-a-3} of $Q_3(x)$, we obtain $2 \cdot 1 > 1^2$. Therefore, $Q_3(x)$ is not log-concave, a contradiction. Thus, $c \leq a+2$. When $a = 1$, $b = c = 2$, $G_{a,b,c} \cong P_4 \cup (n-4)K_1$. When $a = 1$, $b = c = 3$, $G_{a,b,c} \cong G_2 \cup (n-5)K_1$. When $a = 2$, $b = c = 3$, $Q_3(x) = x^2 + 2x + 2$, which is not real-rooted, a contradiction. When $a = 2$, $b = c = 4$, $Q_3(x) = x^3 + 2x^2 + 2x + 1 = (x+1)(x^2 + x + 1)$, which is not real-rooted, also a contradiction.

Subcase 1.2. $b < c$.

In this subcase, since $b < c \leq a+b$, it can be deduced that $a \geq 1$. We can obtain that $Q_1(x) = x^{c+1} + x^c - x^{c-a} - x^{c-b} - x + 1 = (x-1)(x^c + 2x^{c-1} + 2x^{c-2} + \dots + 2x^{c-a} + x^{c-a-1} + x^{c-a-2} + \dots + x^{c-b} - 1)$. Let's denote $Q_4(x) := x^c + 2x^{c-1} + 2x^{c-2} + \dots + 2x^{c-a} + x^{c-a-1} + x^{c-a-2} + \dots + x^{c-b} - 1$. Since $c \neq 0$, $Q_1(x)$ is real-rooted if and only if $Q_4(x)$ is real-rooted. Assume $Q_4(x)$ is real-rooted. Similar to Subcase 1.1, by Lemma 10, $c-2 \leq c-a$, i.e., $a \leq 2$. Now, we prove that $b \leq a+1$. By contradiction assume $b \geq a+2$, then $c-a-3 \geq 0$ since $c > b$. The derivative $Q'_4(x) = cx^{c-1} + 2(c-1)x^{c-2} + \dots + 2(c-a)x^{c-a-1} + (c-a-1)x^{c-a-2} + (c-a-2)x^{c-a-3} + \dots + (c-b)x^{c-b-1}$. By Proposition 9, $Q'_4(x)$ is real-rooted. Considering the coefficients of the terms x^{c-a-1} , x^{c-a-2} , x^{c-a-3} of $Q'_4(x)$, we obtain $2(c-a)(c-a-2) = 2(c-a-1)^2 - 2 \geq (c-a-1)^2 + 2 > (c-a-1)^2$. Therefore, $Q'_4(x)$ is not log-concave, a contradiction. Thus, $b \leq a+1$. When $a = 1$, $b = 1$, $c = 2$, $G_{a,b,c} \cong 2K_2 \cup (n-4)K_1$. When $a = 1$, $b = 2$, $c = 3$, $Q_4(x) = x^3 + 2x^2 + x - 1$. By analyzing the cubic function, $Q_4(x)$ has only one real root, a contradiction. When $a = 2$, $b = 2$, $c = 3$, $Q_4(x) = x^3 + 2x^2 + 2x - 1$. $Q'_4(x) = 3x^2 + 4x + 2$, which is not real-rooted, a contradiction with Proposition 9. When $a = 2$, $b = 2$, $c = 4$, $Q_4(x) = x^4 + 2x^3 + 2x^2 - 1 = (x+1)(x^3 + x^2 + x - 1)$. $(x^3 + x^2 + x - 1)' = 3x^2 + 2x + 1$ is not real-rooted, neither is $x^3 + x^2 + x - 1$, a contradiction with the real-rootedness of $Q_4(x)$. When $a = 2$, $b = 3$, $c = 4$, $Q_4(x) = x^4 + 2x^3 + 2x^2 + x - 1$. $Q'_4(x) = 4x^3 + 6x^2 + 4x + 1 = (2x+1)(2x^2 + 2x + 1)$,

which is not real-rooted, a contradiction with Proposition 9. When $a = 2, b = 3, c = 5$, $Q_4(x) = x^5 + 2x^4 + 2x^3 + x^2 - 1$ and $Q'_4(x) = 5x^4 + 8x^3 + 6x^2 + 2x = x(5x^3 + 8x^2 + 6x + 2)$. Since $6^2 < (1 + \frac{1}{1}) (1 + \frac{1}{3-1}) \cdot 2 \cdot 8$, the polynomial $5x^3 + 8x^2 + 6x + 2$ is not real-rooted, neither is $Q'_4(x)$, a contradiction with Proposition 9.

Case 2. The real-rootedness of $Q_2(x)$.

We distinguish two subcases: $a = 0$ or $a \geq 1$.

Subcase 2.1. $a = 0$.

In this subcase, $Q_2(x) = x^{b+1} - 1$. It is real-rooted if and only if $b = 0$ or 1 . When $b = 0$, $G_{a,b,c}^+ \cong K_2 \cup (n-2)K_1$. When $b = 1$, $G_{a,b,c}^+ \cong P_3 \cup (n-3)K_1$.

Subcase 2.2. $a \geq 1$.

In this subcase, $Q_2(x) = x^{b+1} + x^b - x^{b-a} - 1 = (x-1)(x^b + 2x^{b-1} + 2x^{b-2} + \dots + 2x^{b-a} + x^{b-a-1} + x^{b-a-2} + \dots + x + 1)$. Let's denote $Q_5(x) := x^b + 2x^{b-1} + 2x^{b-2} + \dots + 2x^{b-a} + x^{b-a-1} + x^{b-a-2} + \dots + x + 1$. Since $b \geq a \geq 1$, $Q_2(x)$ is real-rooted if and only if $Q_5(x)$ is real-rooted. Assume $Q_5(x)$ is real-rooted. Similarly, we can deduce that $a \leq 2$ and $b-a = 0$ or 1 . When $a = 1, b = 1$, $G_{a,b,c}^+ \cong K_3 \cup (n-3)K_1$ if $c = 1$, and $G_{a,b,c}^+ \cong P_4 \cup (n-4)K_1$ if $c = 2$. When $a = 1, b = 2$, $G_{a,b,c}^+ \cong G_1 \cup (n-4)K_1$ if $c = 2$, and $G_{a,b,c}^+ \cong G_2 \cup (n-5)K_1$ if $c = 3$. When $a = 2, b = 2$, $Q_5(x) = x^2 + 2x + 2$, which is not real-rooted, a contradiction. When $a = 2, b = 3$, $Q_5(x) = x^3 + 2x^2 + 2x + 1 = (x+1)(x^2 + x + 1)$, which is not real-rooted, a contradiction.

In conclusion, the proof is completed. $\square$

4 Conclusions and future work

In the present paper, we proved that, for a graph G of order n , the dependence polynomial $D(G, x)$ is ultra log-concave if G is a $K_2 \cup 2K_1$ -free graph or has an independent set of size $n-2$. And further, the graphs of order n with real-rooted dependence polynomials are characterized, which is exactly one of the following eight graphs: $K_2 \cup (n-2)K_1$, $P_3 \cup (n-3)K_1$, $K_3 \cup (n-3)K_1$, $2K_2 \cup (n-4)K_1$, $P_4 \cup (n-4)K_1$, $C_4 \cup (n-4)K_1$, $G_1 \cup (n-4)K_1$, $G_2 \cup (n-5)K_1$.

Combined with the proof of Theorem 2, we obtain that, for a graph G , $D(G, x)$ is ultra log-concave if and only if $\det(\nabla^2 q_k) \leq 0$. Observing (6), (7) and (8), we notice that all the coefficients of the quadratic form (7) is nonpositive except the one of $t_1 t_8$. Thus, if $\det(\nabla^2 q_k) > 0$, t_i ($i = 2, 3, \dots, 7$) must be much less than t_1 and t_8 . We believe that there are no graphs satisfying such condition, so we conjecture:

Conjecture 14. For every graph G , $D(G, x)$ is ultra log-concave.

For the graph polynomials associated with co-hereditary graph properties, Makowsky and Rakita [19] showed that these polynomials are unimodal for almost all graphs. Meanwhile, they asked: Under what conditions can almost unimodality be improved to unimodality or log-concavity?

Let G be a graph with $V(G) = [n]$ and $\mathcal{A}$ a co-hereditary graph property. Let $\mathcal{I}$ denote the family of sets of vertices S satisfying $G-S \in \mathcal{A}$, i.e., $\mathcal{I} := \{S \subseteq V(G) | G-S \in \mathcal{A}\}$. Since $\mathcal{A}$ is co-hereditary, it is easy to see that $([n], \mathcal{I})$ is an independence system.

Let's denote $\mathcal{A}_{\min} := \{H \in \mathcal{A} \mid \text{any induced subgraph } H' \neq H \text{ of } H \text{ is not in } \mathcal{A}\}$. Assume $\max_{H \in \mathcal{A}_{\min}} |V(H)|$ is finite and denote $l := \max_{H \in \mathcal{A}_{\min}} |V(H)|$. We have

Proposition 15. *$G, \mathcal{A}, \mathcal{I}$ and l are defined above. Then the independence system $([n], \mathcal{I})$ satisfies the following condition:*

For any $S, T \in \mathcal{I}$ where $|T| \geq |S| + l$, there exists $x \in T \setminus S$ such that $S \cup \{x\} \in \mathcal{I}$. ()*

Proof. If $S \subset T$, since T is an independence set, $S \cup \{x\} \in \mathcal{I}$ for any $x \in T \setminus S$. Otherwise, $|T \setminus S| \geq l + 1$, so $|S| \leq |S \cup T| - (l + 1) \leq |V(G)| - (l + 1)$. Since $S \in \mathcal{I}$, by the definition, $G - S \in \mathcal{A}$, that is, there is an induced subgraph of $G - S$ in $\mathcal{A}_{\min}$, denoted by H . By the definition of l , $|V(H)| \leq l$. Recall that $|T \setminus S| \geq l + 1$, so there must exist a vertex $x \in T \setminus S$ and $x \notin V(H)$. For such x , H is still an induced subgraph of $G - (S \cup \{x\})$, so $S \cup \{x\} \in \mathcal{I}$. $\square$

More abstractly, for $l \in \mathbb{N}^*$, we call an independence system $\mathcal{M} := ([n], \mathcal{I})$ satisfying (*) an l -matroid. By this definition, we can see that if $\mathcal{M}$ is an l_1 -matroid, $\mathcal{M}$ is also an l_2 -matroid for all $l_2 > l_1$. In particular, 1-matroids are commonly referred to as *matroids*. For matroids, the following famous conjecture was posed by Mason:

Conjecture 16 (Mason's Conjecture). [20] Let $\mathcal{M}$ be an n -element matroid and i_k the number of independent sets of size k . Then

- (i) the sequence $\{i_k\}_{k=0}^n$ is log-concave;
- (ii) the sequence $\{i_k\}_{k=0}^n$ is ordered log-concave;
- (iii) the sequence $\{i_k\}_{k=0}^n$ is ultra log-concave.

Adiprasito, Huh, and Katz [1] proved (i) using techniques from Hodge theory and algebraic geometry in 2018. (ii) was proved by Huh, Schröter and Wang [16] in 2022. More recently, (iii) was proved by Anari et al. [3], Brändén and Huh [9], and Chan and Pak [11] in different methods independently. Motivated by the discussions above, our question is: under what conditions can the analogous version of Mason's Conjecture hold true for l -matroids? By Proposition 15, it is a generalization of Makowsky and Rakita's question. For the case of $l = 2$, we conjecture:

Conjecture 17. Let $\mathcal{M}$ be an n -element 2-matroid, i_k the number of independent sets of size k in $\mathcal{M}$. Then

- (i) the sequence $\{i_k\}_{k=0}^n$ is log-concave;
- (ii) the sequence $\{i_k\}_{k=0}^n$ is ordered log-concave;
- (iii) the sequence $\{i_k\}_{k=0}^n$ is ultra log-concave.

Recall $\overline{\mathcal{D}}$ the family of co-dependent sets in a graph G . Then $(V(G), \overline{\mathcal{D}})$ is a 2-matroid. Thus, Conjecture 17 is a stronger version of Conjecture 14.

It is remarkable that for 3-matroids, the analogous conjecture is not true. Let's give a counterexample:

Example 18. Assume $\mathcal{M}$ is a 3-matroid on a 10-element ground set, where $\mathcal{I} := \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$. We can see that $i_1 = 10$, $i_2 = 3$ and $i_3 = 1$. $i_2^2 < i_1 i_3$, so the sequence $\{i_k\}_{k=0}^{10}$ is not log-concave.

Acknowledgements

This work is partially supported by National Natural Science Foundation of China (Grants No. 12671410).

References

- [1] K. Adiprasito, J. Huh, and E. Katz, Hodge theory for combinatorial geometries, *Ann. of Math.* 188: 381–452, 2018.
- [2] Y. Alavi, P. J. Malde, A. J. Schwenk, and P. Erdős, The vertex independence sequence of a graph is not constrained, *Congr. Numer.* 58: 15–23, 1987.
- [3] N. Anari, K. Liu, S. O. Gharan, and C. Vintzant, Log-concave polynomials III: Mason's ultra-log-concavity conjecture for independent sets of matroids, *Proc. Amer. Math. Soc.* 152 (5): 1969–1981, 2024.
- [4] E. J. Barbeau, *Polynomials*, Springer Science & Business Media, 2003.
- [5] C. Barton, J. I. Brown, and D. Pike, Acyclic polynomials of graphs, *Austr. J. Combin.* 82 (2): 146–181, 2022.
- [6] A. Basit and D. Galvin, On the independent set sequence of a tree, *Electron. J. Combin.* 28 (3): Paper No. 3.23, 2021.
- [7] A. Bhattacharyya and J. Kahn, A bipartite graph with non-unimodal independent set sequence, *Electron. J. Combin.* 20: #P11, 2013.
- [8] M. Bóna, *A Walk through Combinatorics: An Introduction to Enumeration and Graph Theory*, World Scientific Publishing Co. Inc., 2011.
- [9] P. Brändén and J. Huh, Hodge–Riemann relations for Potts model partition functions, [arXiv:1811.01696v2](https://arxiv.org/abs/1811.01696v2), 2018.
- [10] J. I. Brown and C. A. Hickman, On chromatic roots with negative real part, *Ars Combin.* 62: 211–221, 2002.
- [11] S. H. Chan and I. Pak, Introduction to the combinatorial atlas, *Expo. Math.* 40: 1014–1048, 2022.
- [12] M. Chudnovsky and P. Seymour, The roots of the independence polynomial of a clawfree graph, *J. Combin. Theory Ser. B* 97 (3): 350–357, 2007.

- [13] I. Gutman and F. Harary, Generalizations of the matching polynomial, *Util. Math.* 24: 97–106, 1983.
- [14] C. Hoede and X. Li, Clique polynomials and independent set polynomials of graphs, *Discrete Math.* 125: 219–228, 1994.
- [15] D. G. C. Horrocks, The numbers of dependent k -sets in a graph is log concave, *J. Combin. Theory, Ser. B* 84: 180–185, 2002.
- [16] J. Huh, B. Schröter, and B. Wang, Correlation bounds for fields and matroids, *J. Eur. Math. Soc.* 24: 1335–1351, 2022.
- [17] O. Kadrawi and V. E. Levit, The independence polynomial of trees is not always log-concave starting from order 26, *ARS Mathematica Contemporanea* 25: #P4.03, 2025.
- [18] O. Kadrawi, V. E. Levit, R. Yosef, and M. Mizrachi, On computing of independence polynomials of trees, A chapter in Recent Research in Polynomials (F. Özger, ed.), IntechOpen 2023. [doi:10.5772/intechopen.1001130](https://doi.org/10.5772/intechopen.1001130).
- [19] J. A. Makowsky and V. Rakita, Almost unimodal and real-rooted graph polynomials, *European J. Combin.* 108: 103637, 2023.
- [20] J. Mason, Matroids: unimodal conjectures and motzkin’s theorem, in: Proc. Conf. Combin. Math., Inst. Math. Appl, Southend-on-Sea, UK, 1972, pp. 207–220, available at tinyurl.com/7w7wjz6v.
- [21] T. S. Michael and W. M. Traves, Independence sequences of well-covered graphs: Non-unimodality and the Roller-Coaster Conjecture, *Graphs Combin.* 19: 403–411, 2023.