

A note on vertex-critical induced subgraphs of shift graphs

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Abstract

Shift graphs, introduced by Erdős and Hajnal in 1964, form one of the simplest known non-recursive constructions of triangle-free graphs with arbitrarily large chromatic number. In this note, we identify a surprising property: for each integer $k \geq 1$, the smallest k -chromatic shift graph contains a *unique* induced k -vertex-critical subgraph. We give an explicit description of this subgraph and prove its uniqueness. This provides a new family of vertex-critical triangle-free graphs of arbitrarily large chromatic number.

Mathematics Subject Classifications: 05C15

1 Introduction

The study of graphs that are simultaneously highly chromatic and sparse—typically having large (odd) girth—has had a profound influence on graph theory. It has played a central role in the development of both the probabilistic method [3] and the topological method [8]. For background on this rich line of research, see the surveys [6, 11, 13, 15, 16].

The earliest constructions of triangle-free graphs with arbitrarily large chromatic number, due to Tutte (writing under an alias) [1, 2], Zykov [17], Kelly and Kelly [7], and Mycielski [10], were all recursive. The first non-recursive constructions were shift graphs [4], which are the focus of this note, and Kneser graphs [8].

For $N \geq 5$, the *shift graph* $G_{N,2}$ is defined as follows. The vertices are ordered pairs (x, y) with integers $1 \leq x < y \leq N$, and two vertices (x, y) and (y, z) are adjacent

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whenever $x < y < z$. Erdős and Hajnal [4] and, independently, Harner and Entringer [5] proved that $G_{N,2}$ is triangle-free and

$$\chi(G_{N,2}) = \lceil \log_2 N \rceil.$$

See also [9, Problem 9.26] and [14].

A vertex of a graph is *critical* if its removal decreases the chromatic number. A graph is *k-vertex-critical* if it is *k*-chromatic and all of its vertices are critical. Vertex-critical graphs are central objects in graph colouring, as structural questions about all *k*-chromatic graphs often reduce to the vertex-critical case.

Our main result shows that each shift graph $G_{2^n+1,2}$ contains a *unique* induced $(n+1)$ -vertex-critical subgraph, which we describe explicitly. This phenomenon is in sharp contrast with Kneser graphs $KG(n, k)$, which contain $(n-1)!/2$ isomorphic copies of the vertex-critical Schrijver graph $SG(n, k)$ [12], along with additional non-isomorphic vertex-critical subgraphs.

For integers $p \leq q$, let $[p, q]$ denote the interval $\{p, p+1, \dots, q\}$. For $n \geq 2$, define intervals $I_0, \dots, I_n \subseteq [1, 2^n+1]$ by

$$I_\ell = [2^\ell, 2^n - 2^{n-\ell} + 2], \quad 0 \leq \ell \leq n.$$

Thus the interval $[1, 2^n+1]$ contains $2^\ell - 1$ elements to the left of I_ℓ and $2^{n-\ell} - 1$ elements to the right of I_ℓ . Let

$$W = \{(x, y) : 1 \leq x < y \leq 2^n + 1 \text{ and } \{x, y\} \subseteq I_\ell \text{ for some } \ell\}.$$

A geometric depiction of W for $n = 4$ is given in Figure 1.

We are now ready to state the main theorem.

Theorem 1. *For every integer $n \geq 2$, the shift graph $G_{2^n+1,2}$ contains a unique induced $(n+1)$ -vertex-critical subgraph, namely $G_{2^n+1,2}[W]$.*

2 Proof of Theorem 1

Fix an integer $n \geq 2$, and let us denote the shift graph $G_{2^n+1,2}$ by G . Thus, G has chromatic number $\chi(G) = n+1$. Theorem 1 will follow from Theorems 3 and 4 below.

Let $a = (a_1, \dots, a_{2^n+1})$ be a sequence of elements of the set of all subsets of $[1, n]$, and let X be a subset of the vertex set $V(G)$. Recall that the elements of X are ordered pairs (x, y) of integers satisfying $1 \leq x < y \leq 2^n + 1$. We say that a is *X-good* if for each i, j such that $1 \leq i < j \leq 2^n + 1$,

$$a_i \not\subseteq a_j \text{ whenever } (i, j) \in X.$$

The subgraph of G induced by the set X is denoted by $G[X]$.

The following proposition characterises induced n -colourable subgraphs of G in terms of *X-good* sequences.

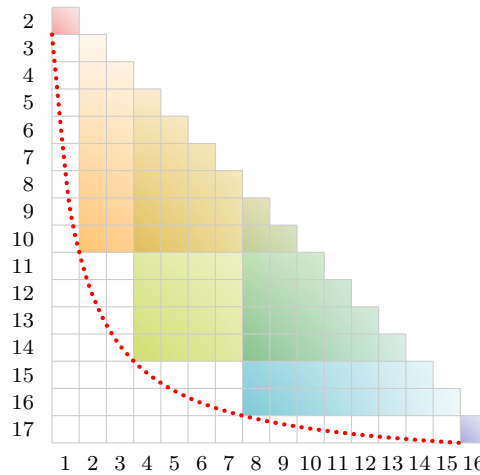


Figure 1: A geometric representation of the vertex set W of the shift graph $G_{17,2}$. Each box with coordinates (x, y) corresponds to a vertex (x, y) of $G_{17,2}$. Triangular regions of the same colour correspond to pairs (x, y) lying in the same interval I_ℓ ; the lower-left corners of each region lie on the dotted red hyperbola.

Proposition 2. For $X \subseteq V(G)$, $\chi(G[X]) \leq n$ if and only if there exists an X -good sequence.

Proof. This is essentially the Erdős–Hajnal argument. Suppose that there exists an n -colouring c of $G[X]$. For $i \in [1, 2^n + 1]$, let

$$c_i = \{c((i, j)) : i < j \leq 2^n + 1 \text{ and } (i, j) \in X\}.$$

Note that $c_{2^n+1} = \emptyset$. Furthermore, $c_1, \dots, c_{2^n+1}$ is X -good, since if $(i, j) \in X$, then $c((i, j)) \in c_i$ while $c((i, j)) \notin c_j$ because of the adjacency between (i, j) and the vertex (j, k) for each k such that $j < k \leq 2^n + 1$. Thus, $c((i, j))$ testifies that $c_i \not\subseteq c_j$.

On the other hand, given an X -good sequence $(a_1, \dots, a_{2^n+1})$, we colour each vertex $(i, j) \in X$ with an arbitrary colour from the nonempty set $a_i \setminus a_j$. $\square$

The following theorem shows that every vertex in W is critical.

Theorem 3. For $v \in W$, $\chi(G - v) < \chi(G)$.

Proof. Let $v = (i, j) \in W$ — say, $i, j \in I_r$ for some r , $0 \leq r \leq n$. We construct a $(V(G) - v)$ -good sequence which will show that $\chi(G - v) = n$. Let $A = \{1, \dots, n - r\}$. We set $a_i = a_j = A$. Since $i \geq 2^r$, we can position all proper supersets of A at the beginning of the subsequence $a_1, \dots, a_{i-1}$. To eventually obtain a $(V(G) - v)$ -good sequence, we do it in such a way that the size of the sets of the sequence is non-increasing. Since $j \leq 2^n - 2^{n-r} + 2$, all proper subsets of A can be placed at the end of $a_{j+1}, \dots, a_{2^n+1}$ (in a similar non-increasing way). Finally, we put the subsets of $[1, n]$ which do not yet appear in the sequence in the places left free so far. The only required property is that,

again, the size of these sets is non-increasing. It is not hard to check that for any ordering satisfying the above requirements, we obtain a $(V(G) - v)$ -good sequence, which implies that $\chi(G - v) = n$. $\square$

To prove Theorem 1, it just remains to show that W induces an $(n + 1)$ -chromatic subgraph of G .

Theorem 4. $\chi(G[W]) = \chi(G) = n + 1$.

Proof. By contradiction. Consider a W -good sequence $a = (a_1, \dots, a_{2^n+1})$. We first transform the sequence to another W -good sequence using the following procedure. If there is an index r such that some proper subset a'_r of a_r is not included among $a_{r+1}, \dots, a_{2^n+1}$, let s be the largest such index and replace a_s (at position s) in the sequence with a'_s . We prove that the resulting sequence is W -good. Suppose not. Since $a'_s \subseteq a_s$, there cannot be $i < s$ such that $(i, s) \in W$ and $a_i \subseteq a'_s$ as this would contradict the W -goodness of the original sequence. Thus, let us consider $j > s$ such that $(s, j) \in W$. In this case, a'_s cannot be a subset of a_j , because it would be a proper subset (by the choice of s), so by the maximality of s , it would be included among $a_{j+1}, \dots, a_{2^n+1}$, contradicting the choice of s . Thus, the resulting sequence is W -good, as claimed.

Relabel the new element a'_s as a_s , and repeat the above step until this is no longer possible. Clearly, the algorithm only takes a finite number of steps and terminates with a W -good sequence $a_1, \dots, a_{2^n+1}$ with the following property:

For each $i \in [1, 2^n + 1]$ and every proper subset $b \subset a_i$, there is $j \in [i + 1, 2^n + 1]$ such that $a_j = b$.

Observe that, as a consequence, $a_{2^n+1} = \emptyset$. Furthermore, for each $j \in [1, 2^n + 1]$,

$$\text{if } |a_j| = k, \text{ then } j \leq 2^n - 2^k + 2, \quad (1)$$

as there has to be space to the right of a_j for $2^k - 1$ proper subsets of a_j .

Since the number of subsets of $[1, n]$ is 2^n while the length of the sequence a is $2^n + 1$, one of the subsets needs to occur multiple times in a . In particular, there exists a pair $i, j \in [1, 2^n + 1]$ such that:

$i < j$ and $a_i \subseteq a_j$; moreover, we choose this pair such that i is as large as possible and, subject to this condition, j is as small as possible.

Let us write $k = |a_j|$.

By the maximality of i , the subsequence $(a_{i+1}, \dots, a_{2^n+1})$ of a consists of pairwise distinct subsets of $[1, n]$. In addition, it is easy to see that by the choice of i and j , the only superset of a_j in this subsequence is a_j itself. The fact that the remaining $2^{n-k} - 1$ supersets are missing implies an upper bound on the length of the subsequence:

$$2^n - i + 1 \leq 2^n - (2^{n-k} - 1),$$

from which we find $2^{n-k} \leq i$. In combination with (1), we obtain

$$2^{n-k} \leq i < j \leq 2^n - 2^k + 2. \quad (2)$$

This just says that $i, j \in I_{n-k}$. As a consequence, $(i, j) \in W$, which contradicts the W -goodness of a and concludes the proof of Theorem 4. $\square$

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