Maximum subsets of $(0, 1]$ with no solutions to $x + y = kz$

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Abstract

If k is a positive real number, we say that a set S of real numbers is k-sum-free if there do not exist x, y, z in S such that $x+y = kz$. For k greater than or equal to 4 we find the essentially unique measurable k-sum-free subset of $(0, 1]$ of maximum size.

1 Introduction

We say that a set S of real numbers is sum-free if there do not exist x, y, z is S such that $x + y = z$. If k is a positive real number, we say that a set S of real numbers is k-sum-free if there do not exist x, y, z in S such that $x + y = kz$ (we require that not all x, y, and z be equal to each other to avoid a meaningless problem when $k = 2$).

Let $f(n, k)$ denote the maximum size of a k-sum-free subset of $\{1, 2, \ldots ,n\}$. It is easy to show [1, 2] that

$$f(n, 1) = \left\lceil \frac{n}{2} \right\rceil .$$

For $k = 1$ and n odd there are precisely two such maximum sets: the odd integers and the "top half." For n even and greater than 9 there are precisely three such sets (see [1]): the two maximum sets for the odd number $n - 1$, and the top half.

The problem of determining $f(n, 2)$ is unsolved. Roth [4] proved that a subset of the positive integers with positive upper density contains three-term arithmetic progressions. The current best bounds for $f(n, 2)$ were established by Salem and Spencer [5] and Heath-Brown and Szeméredi [3].

Chung and Goldwasser [1] proved a conjecture of Erdös that $f(n,3)$ is roughly $\frac{n}{2}$. They showed that $f(n,3)=\left\lceil\frac{n}{2}\right\rceil$ for $n\neq 4$ and that for $n\geq 23$ the set of odd integers less than or equal to n is the unique maximum set.

Loosely speaking, the set of odd numbers less than or equal to n qualifies as a k-sum-free set for odd k because of "parity" considerations while the top half maximum sum-free set qualifies because of "magnitude" considerations: the sum of two numbers in the top half is too big. There is an obvious way to take a "magnitude" k-sum-free subset of $\{1,2,\dots,n\}$ and get an analogue k-sum-free subset of the interval $(0,1]$. The top half maximum sum-free subset of $\{1,2,\dots n\}$ becomes $\left(\frac{1}{2},1\right]$ and the "size" seems to be preserved. On the other hand it is not so obvious how to get the analogue on $(0,1]$ for the odd numbers maximum sum-free subset of $\{1,2,\dots n\}$. One could try to "fatten up" each odd integral point on $[0,n]$ by as much as possible while keeping it sum-free and then normalize. It turns out one can fatten each odd integer j to $\left(j-\frac{1}{3},j+\frac{1}{3}\right)$ and, after normalization, one ends up with a subset of $(0,1]$ of size roughly $\frac{1}{3}$.

Chung and Goldwasser have conjectured that if $k\geq 4$, n is sufficiently large, and S is a k-sum-free subset of $\{1,2,\dots,n\}$ of size $f(n,k)$, then S is the union of three strings of consecutive integers. Such a set has an analogue k-sum-free subset of $(0,1]$ of the same "size," so we can learn someting about k-sum-free subsets of $\{1,2,\dots,n\}$ by studying k-sum-free subsets of $(0,1]$.

We say that a (Lebesgue) measurable subset S of $(0,1]$ is a *maximum* k-sum-free-set if S is k-sum-free, has maximum size among all measurable k-sum-free subsets of $(0,1]$, and is not a proper subset of any k-sum-free subset of $(0,1]$. So S is a maximum k-sum-free set if both S and $\mu(S)$ are maximal where $\mu(S)$ denotes the measure of S. In this paper, for each real number k greater than or equal to 3 we will construct a family of k-sum-free subsets $(0,1]$, each of which is the union of finitely many intervals (Lemma 1). We will find which set in the family has maximum size (Theorem 1). Then we will show that for $k\geq 4$ any maximum k-sum-free subset of $(0,1]$ must be in the family (Section 3). This also gives us a lower bound for $\lim_{n\to\infty}\frac{f(n,k)}{n}$, and we conjecture that the bound is the actual value.

2 A family of k-sum-free sets.

Let k be a positive integer greater than or equal to 3. (In fact, the construction works for any real number k greater than 2.) Let m be a positive integer, and a_1 and c be real numbers such that

$$0 < c < \frac{k}{2}a_1. \tag{2.1}$$

We define sequences $\{a_i\}$ and $\{b_i\}$ by

$$\begin{aligned} b_i = & \quad \frac{k}{2}a_i & \quad i = 1, 2, \ldots, m \\ a_{i+1} = & \quad kb_i - c & \quad i = 1, 2, \ldots, m-1. \end{aligned} \tag{2.2}$$

We normalize to get sequences $\{e_1\}$ and $\{f_i\}$ defined by

$$\begin{aligned} e_1 = & \quad \frac{1}{b_m}\max\{a_1, c\} & \\ e_i = & \quad \frac{a_i}{b_m} & \quad i = 2, 3, \ldots, m \\ f_i = & \quad \frac{b_i}{b_m} & \quad i = 1, 2, \ldots, m \end{aligned} \tag{2.3}$$

It is easy to show that $e_1 < f_1 < e_2 < f_2 < \cdots < e_m < f_m$, so the set $W = \cup_{i=1}^{m} [e_i, f_i)$ is the union of m disjoint intervals and is a subset of $(0, 1]$. Furthermore, W is k-sum-free because if $x \in [e_i, f_i), y \in [e_j, f_j)$, $z \in [e_r, f_r)$ and if $r = \max\{i, j, r\}$ then $x + y < kz$, while if $r < \max\{i, j, r\}$ then $x + y > kz$. In fact it is not hard to show that W is a maximal k-sum-free set (*i.e.* it is not a proper subset of any k-sum free subset of $(0, 1]$).

The parameter c controls the spacing of the intervals and the size of $[e_1, f_1)$. If $c = a_1$ then the set S can be constructed by a greedy procedure. We first put e_1 into S and then, moving to the right from e_1 we put in anything we can as long as the set remains k-sum-free. So $f_1 = \sup\{x \in [e_1, 1] \mid [e_1, x]$ is k-sum-free$\}$. But f_1 cannot be in S, so we have $[e_1, f_1)$ so far. Then let $e_2 = \inf\{x \in [f_1, 1] \mid [e_1, f_1) \cup \{x\}$ is k-sum-free$\}$, and so on. A lengthy calculation (Lemma 1) is required to determine e_1 so that the value of f_m turns out to be 1. An alternative procedure would be to let $a_1 = 1$, perform the greedy procedure to get m intervals, and then normalize. In Section 3 we will show that if $c = a_1, m = 3$, and $k \geq 4$, then S is a maximum k-sum-free set.

If $c \in (a_1, \frac{k}{2}a_1)$ then the greedy procedure would produce $f_1 = \frac{k}{2}e_1$, a larger value of f_1 than produced by equations (2.3). However, the greedy procedure does produce S if you start with $[e_1, f_1) \cup \{e_2\}$ and then work to the right from e_2. If $c \in (0, a_1)$ then the greedy procedure would produce a smaller value of e_i than that produced by equations (2.2) and (2.3) for $i = 2, 3, 4, \cdots, m$.

Now we calculate $\mu(S)$. From equations (2.2) we get

$$a_{i+1} - a_i = \frac{k^2}{2}(a_i - a_{i-1}) \qquad i = 2, 3, \ldots, m-1$$

$$a_2 - a_1 = \frac{k^2 - 2}{2}a_1 - c$$

which has solution

$$a_i = c_1 \left(\frac{k^2}{2}\right)^i + c_2 \qquad i = 1, 2, \ldots, m$$

where

$$\begin{aligned} c_1 &= \frac{2}{k^2}a_1 - \frac{4}{k^2(k^2-2)}c \\ c_2 &= \frac{2c}{k^2-2}. \end{aligned} \tag{2.4}$$

If $d = \max\{0, c - a_1\}$ then

$$\begin{aligned}
\mu(W) &= \frac{1}{b_m}\sum_{i=1}^{m}(b_i - a_i) - \frac{d}{b_m} \\
&= \frac{k-2}{2b_m}\left[\frac{k^2 c_1}{2}\cdot\frac{\left(\frac{k^2}{2}\right)^m - 1}{\frac{k^2}{2} - 1} + c_2 m\right] - \frac{d}{b_m} \\
&= \frac{k(k-2)}{k^2-2}\left[1 + \frac{\frac{k^2-2}{k^2}\cdot\frac{c_2}{c_1}\cdot m - \frac{2(k^2-2)}{k^2(k-2)}\cdot\frac{d}{c_1} - \frac{c_2}{c_1} - 1}{\left(\frac{k^2}{2}\right)^m + \frac{c_2}{c_1}}\right]
\end{aligned}$$

where we have summed the geometric series and simplified. Now we let $y = \dfrac{c}{a_1}$ so that

$$0 < y < \frac{k}{2}$$

by equation (2.1). Then from equations (2.4) we get

$$\begin{aligned}
c_1 &= \frac{2a_1}{k^2(k^2-2)}[k^2 - 4 - 2(y-1)] \\
c_2 &= \frac{2a_1}{k^2-2}y
\end{aligned}$$

and

$$\frac{c_2}{c_1} = \frac{k^2 y}{k^2 - 2 - 2y}.$$

So now we substitute and simplify to get

$$\mu(W) = \frac{k(k-2)}{k^2-2}\left[1 + \frac{k^2-2}{k^2}\cdot\frac{2y(m-1) - 2 - \max\left\{0, \frac{2(k^2-2)(y-1)}{k-2}\right\}}{(k^2-2)\left(\frac{k^2}{2}\right)^{m-1} - 2y\left[\left(\frac{k^2}{2}\right)^{m-1} - 1\right]}\right] \tag{2.5}$$

With k fixed we note that because of the normalization, $\mu(W)$ is a function of m and y alone. So we have the following result.

Lemma 1 *Let m be a positive integer, k a positive integer greater than or equal to 3, a_1 and c real numbers such that $0 < c < \frac{k}{2}a_1, y = \frac{c}{a_1}$, and let $S_k(m, y) = \cup_{i=1}^{m}(e_i, f_i)$ where $\{e_i\}$ and $\{f_i\}$ are defined by (2.2) and (2.3). Then $S_k(m, y)$ is a k-sum-free set. If $c \leq a_1$, then $0 < y \leq 1$ and*

$$\mu(S_k(m,y)) = \frac{k(k-2)}{k^2-2} + \frac{2}{k} \cdot \frac{[y(m-1)-1](k-2)}{(k^2-2)\left(\frac{k^2}{2}\right)^{m-1} - 2y\left[\left(\frac{k^2}{2}\right)^{m-1} - 1\right]} \tag{2.6}$$

while if $c \geq a_1$ then $1 \leq y < \frac{k}{2}$ and

$$\mu(S_k(m,y)) = \frac{k(k-2)}{k^2-2} + \frac{2}{k} \cdot \frac{k(k-1) - y[(k^2+k-4) - m(k-2)]}{(k^2-2)\left(\frac{k^2}{2}\right)^{m-1} - 2y\left[\left(\frac{k^2}{2}\right)^{m-1} - 1\right]}. \tag{2.7}$$

For any positive integer k greater than 2 we define the set $S_k(\infty)$ by

$$S_k(\infty) = \cup_{i=1}^{\infty}\left(\frac{2}{k}\left(\frac{2}{k^2}\right)^{i-1}, \left(\frac{2}{k^2}\right)^{i-1}\right).$$

If $P_k(\infty)$ is formed from $S_k(\infty)$ by including one end-point of each interval then it is easy to see that $P_k(\infty)$ is a maximal k-sum-free set and

$$\mu(P_k(\infty)) = \mu(S_k(\infty)) = \frac{k-2}{k}\sum_{i=1}^{\infty}\left(\frac{2}{k^2}\right)^{i-1} = \frac{k(k-2)}{k^2-2}.$$

We remark that $\mu(S_k(\infty)) = \mu(S_k(2,1))$ and that $S_k(\infty) = \lim_{m\to\infty} S_k(m,y)$ for any $y \in \left(0, \frac{k}{2}\right)$ in the following sense. For m fixed, let $v_{im} = e_{m-i}$ and $w_{im} = f_{m-i}$ for $i = 1, 2, \cdots, m$, so that (v_{im}, w_{im}) is the i-th interval from the right in $S_k(m, y)$. Then for any fixed positive integer i, $\lim_{m\to\infty} v_{im} = \frac{2}{k}(\frac{2}{k^2})^{i-1}$ and $\lim_{m\to\infty} w_{im} = (\frac{2}{k^2})^{i-1}$.

If m is fixed, the expression in (2.6) is clearly an increasing function of y on $(0,1]$, so to maximize $\mu(S_k(m,y))$ we need only consider $y \in \left[1, \frac{k}{2}\right)$ and use (2.7). For fixed k we define the functions

$$\begin{aligned}
f(m,y) &= k(k-1) - y[(k^2+k-4) - m(k-2)] \\
g(m,y) &= (k^2-2)\left(\frac{k^2}{2}\right)^{m-1} - 2y\left[\left(\frac{k^2}{2}\right)^{m-1} - 1\right] \\
h(m,y) &= \frac{f(m,y)}{g(m,y)}
\end{aligned}$$

where m is a positive integer and $y \in \left[1, \frac{k}{2}\right)$. With y fixed, the function $F_y(m) = f(m,y)$ is an increasing linear function of m with root $m(y)$ given by

$$m(y) = k + 3 - \frac{k(k-1) - 2y}{y(k-2)}.$$

So the root $m(y)$ of $F_y(m)$ is an increasing function of y for $y \in \left[1, \frac{k}{2}\right)$ and hence

$$2 = m(1) \le m(y) < m\left(\frac{k}{2}\right) = k + 1$$

This means that for each $y \in \left[1, \frac{k}{2}\right)$ there is a positive integer $m_c(y) \in \{2, 3, \ldots, k+1\}$ such that $f(m,y) < 0$ for $m < m_c(y)$ and $f(m,y) \ge 0$ for $m \ge m_c(y)$. It is easy to show that if y is fixed then $h(m,y) > h(m+1,y)$ for all m greater than $m_c(y)$ (because $g(m,y)$ is positive and exponential in m). So for fixed y the maximum value of $h(m,y)$ occurs when $m \in \{m_c(y), m_c(y)+1\}$, which means for some m satisfying

$$2 \le m \le k + 2. \tag{2.8}$$

Now with m fixed and satisfying (2.8) we let $H_m(y) = h(m,y)$. We differentiate to get

$$H'_m(y) = \frac{A}{[g(m,y)]^2}$$

where

$$\begin{aligned} A &= 2k(k-1)\left[\left(\frac{k^2}{2}\right)^{m-1} - 1\right] \\ &\quad -(k^2-2)\left(\frac{k^2}{2}\right)^{m-1}[(k^2+k-4)-m(k-2)] \\ &\leq -k^2(k-2)\left(\frac{k^2}{2}\right)^{m-1} - 2k(k-1) \end{aligned}$$

by (2.8). So $H_m(y)$ is strictly decreasing on $\left[1, \frac{k}{2}\right)$ for any m satisfying (2.8). And hence $\mu\left(S_k(m,y)\right)$ is a maximum if and only if $y = 1$ and $R(m) = h(m,1)$ is a maximum over $\{2, 3, \ldots, k+2\}$. We have

$$\begin{aligned} R(m) &= \frac{k(k-1)-(k^2+k-4)+m(k-2)}{(k^2-4)\left(\frac{k^2}{2}\right)^{m-1} + 2} \\ &= \frac{1}{k+2} \cdot \frac{m-2}{\left(\frac{k^2}{2}\right)^{m-1} + \frac{2}{k^2-4}} \end{aligned}$$

which clearly is maximum only at $m = 3$. Since $k \geq 3$ it is easy to see that $R(m)$ is decreasing on $[3, \infty)$ and that $\lim_{m\to\infty} R(m) = R(2) = 0$. We have proved the following result.

Theorem 1 *Let m be a positive integer, k a positive integer greater than or equal to 3, a_1 and c real numbers such that $0 < c < \frac{k}{2}a_1, y = \frac{c}{a_1}$, and let $S_k(m,y) = \cup_{i=1}^m (e_i, f_i)$ where $\{e_i\}$ and $\{f_i\}$ are defined by (2.2) and (2.3). Then $\mu(S_k(m,y))$ is a maximum only when $m = 3$ and $y = 1$ and*

$$\mu\left(S_k(3,1)\right) = \frac{k(k-2)}{k^2-2} + \frac{8(k-2)}{k(k^2-2)(k^4-2k^2-4)}$$

Furthermore, if m is greater than 2, then $\mu(S_k(m,1)) > \mu(S_k(m+1,1))$ *and* $\mu(S_k(m,1)) > \mu(S_k(2,1)) = \mu(S_k(\infty)) = \dfrac{k(k-2)}{k^2-2}$.

We remark that while the construction of $S_k(m,y)$ above makes sense for any real number k greater than 2, the maximum of $\mu(S_k(m,y))$ is at $m=3$ only if $k \le \sqrt{2+2\sqrt{2}} \approx 2.20$. In fact, it can be shown that for each integer t greater than or equal to 3, there exists a real number $k(t) \in (2, 2.2)$ such that the maximum value of $\mu(S_k(m,y))$ is at $m=t$ for $k=k(t)$ (though $\mu(S_k(\infty)) = \dfrac{k(k-2)}{k^2-2}$ for any value of k greater than 2).

3 Maximum k-sum-free sets are in the family.

In Section 2 we constructed a family $\mathcal{S} = \{S_k(m,y)\}$ of k-sum-free sets and showed that if $k \ge 3$ then $\mu(S_k(m,y))$ is a maximum over $\mathcal{S}$ only when $m=3$ and $y=1$. In this section we will show that if $k \ge 4$ and S is a maximum k-sum-free subset of $(0,1]$ (so both S and $\mu(S)$ are maximal) then S can be obtained by adding an end-point to each of the three disjoint open interval components of $S_k(3,1)$.

The proof is quite long, so we have broken it up into several lemmas. The over-all procedure is basically to assume that S is a maximum k-sum-free set and then to construct it from right to left. There are two techniques that we use frequently in proving the lemmas. The first is that if every element of a k-sum-free set T is multiplied by a positive real number y, then the new set Ty is also k-sum-free (while the translated set $T+y$ may not be k-sum-free). The second is that if $x \in S$ then not both y and $kx-y$ can be in S. We refer to this as "forbidden pairs with respect to x". We can use this idea to show that $\mu(S \cap T) \le \dfrac{1}{2}\mu(T)$ for certain subsets T of $(0,1]$. Since $\mu(S) > \dfrac{k(k-2)}{k^2-2} \ge \dfrac{1}{2}$ for $k > 2+\sqrt{2}$, forbidden pairing can be used to learn about the structure of S. For example, we know immediately that $\dfrac{1}{k}$ is not in S, since if it were in S then not both y and $1-y$ could be in S for any $y \in (0,1]$, so $\mu(S) \le \dfrac{1}{2}$, a contradiction.

Finding the value of $u_2 = \sup\{x \in S \mid x < \dfrac{2}{k}\}$ is the key point in determining the structure of S; u_2 will turn out to be the right end-point of the second component from the right in S. Lemmas 2,3,and 4 deal primarily with the value of u_2. In Lemma 5 it is shown that $[\ (\frac{2}{k}u_2, u_2) \cup (\frac{2}{k}, 1)\ ] \subseteq S$ and that $u_3 = \sup\{x \in S \mid x < \frac{2}{k}u_2\}$ can be determined in much the same way as u_2. In Lemma 6 it is shown that if $u_i = \sup\ \{x \in S \mid x < \frac{2}{k}u_{i-1}\ \}$ for $i = 2, 3, \cdots$, then there exists a positive integer $m \geq 3$ such that u_m exists but u_{m+1} does not. The sequence $1, u_2, u_3, \cdots, u_m$ then gives the right-hand end points of the components of S, and S turns out to be $S_k(m, y)$ for some m and y, *i.e.* $S \in \mathcal{S}$.

Lemma 2 *If S is a maximum k-sum-free subset of $(0,1]$ where k is an integer greater than or equal to 4 and if $u_2 = \sup\left\{x \in S | x < \frac{2}{k}\right\}$ then $\frac{2}{k^2} < u_2 < \frac{2}{k^2-2}$.*

Proof: If $\frac{1}{k} < u_2 \leq \frac{2}{k}$ then there exists a real number x in $S \cap \left(\frac{1}{k}, \frac{2}{k}\right)$. Then $0 < kx - 1 < 1$, and for each $y \in [kx-1, 1]$, not both y and $kx - y$ are in S. Because of these "forbidden pairs with respect to x,"

$$\mu\left(S \cap [kx-1, 1]\right) \leq \frac{1}{2}[1 - (kx-1)]. \tag{3.1}$$

If we now let

$$S' = \left\{\frac{1}{kx-1}w | w \in S \cap (0, kx-1]\right\}$$

then S' is k-sum-free and

$$\begin{aligned}
\mu(S') &= \frac{1}{kx-1}\mu(S \cap (0, kx-1]\Big) \\
&= \frac{1}{kx-1}\left(\mu(S) - \mu(S \cap [kx-1, 1])\right) \\
&\geq \frac{1}{kx-1}\Big((kx-1)\mu(S) + [1-(kx-1)]\mu(S) - \frac{1}{2}[1-(kx-1)]\Big) \\
&> \mu(S)
\end{aligned}$$

where the first inequality follows from (3.1) and the second follows because $\mu(S) > \frac{k^2-2k}{k^2-2} \geq \frac{1}{2}$ for $k \geq 2+\sqrt{2}$. We have contradicted the assumption that S is a maximum set, so $u_2 \leq \frac{1}{k}$.

Now suppose $u_2 \in \left[\frac{2}{k^2-2}, \frac{1}{k}\right]$. For each $\epsilon > 0$ there exists a real number x in S such that $0 \leq u_2 - x < \epsilon$. If $u_2 < ku_2 - \frac{2}{k}$ then ϵ can be chosen such that $x < kx - \frac{2}{k}$, and for each $y \in (0, x]$, not both y and $kx - y$ can be in S. Because of this "forbidden pairing with respect to x" of $(0, x]$ with $[kx - x, kx)$, and since $\frac{2}{k} < kx - x < kx < 1$,

$$\begin{aligned}\mu(S) &= \mu\left(S \cap \left[x, \frac{2}{k}\right]\right) + \mu\left(S \cap \left((0, x] \cup \left(\frac{2}{k}, 1\right]\right)\right) \\ &\leq (u_2 - x) + 1 - \frac{2}{k} \\ &< 1 - \frac{2}{k} + \epsilon\end{aligned}$$

and $\mu(S)$ is not a maximum since $1 - \frac{2}{k} < \frac{k^2-2k}{k^2-2}$.

If $u_2 \geq ku_2 - \frac{2}{k}$ then since $x \geq kx - \frac{2}{k}$ and due to the forbidden pairing with respect to x of $\left(0, kx - \frac{2}{k}\right]$ with $\left[\frac{2}{k}, kx\right)$,

$$\begin{aligned}\mu(S) &\leq \left(1 - \frac{2}{k}\right) + u_2 - \left(kx - \frac{2}{k}\right) \\ &= 1 - (k-1)u_2 + k(u_2 - x) \\ &\leq 1 - (k-1)\frac{2}{k^2-2} + k(u_2 - x) \\ &< \frac{k^2-2k}{k^2-2} + k\epsilon.\end{aligned}$$

But $\frac{k^2-2k}{k^2-2}$ is the size of $S_k(2,1)$, so $\mu(S)$ is not a maximum. Hence $u_2 < \frac{2}{k^2-2}$.

If $u_2 \leq \frac{2}{k^2}$ then the set

$$S' = \left\{\frac{k^2}{2}x | x \in S \cap (0, u_2]\right\}$$

has size

$$\begin{aligned}
\mu(S') &= \frac{k^2}{2}\mu(S \cap (0, u_2]) \\
&\geq \frac{k^2}{2}\left(\frac{2}{k^2}\mu(S) + \frac{k^2-2}{k^2}\mu(S) - \left(1 - \frac{2}{k}\right)\right) \\
&= \mu(S) + \frac{k^2-2}{2}\left(\mu(S) - \frac{k^2-2k}{k^2-2}\right) \\
&> \mu(S)
\end{aligned}$$

which again is a contradiction, so $\mu_2 > \dfrac{2}{k^2}$ completing the proof.

We remark that the bounds for u_2 in Lemma 2, $\dfrac{2}{k^2}$ and $\dfrac{2}{k^2-2}$, are the right end-points of the second component from the right in $S_k(\infty)$ and $S_k(2,1)$ respectively.

Lemma 3 *If S is a maximum k-sum-free subset of $(0,1]$ where k is an integer greater than or equal to 4 and if $u_2 = \sup\left\{x \in S | x < \dfrac{2}{k}\right\}$, then $(ku_2, 1) \subseteq S$ and $\mu\left(S \cap \left(0, ku_2 - \dfrac{2}{k}\right]\right) + \mu\left(S \cap \left[\dfrac{2}{k}, 1\right]\right) = \dfrac{k-2}{k}$.*

Proof: First we will show that if S is a maximum k-sum-free subset of $(0,1]$ then $S \cup (ku_2, 1)$ is also k-sum-free. If x and y are in S and $z \in (ku_2, 1)$ then $x+y < kz$, since $ku_2 > \dfrac{2}{k}$ by Lemma 2. If $x \in (ku_2, 1)$ and $z \geq \dfrac{2}{k}$ then $x + y > kz$, while if $x \in (ku_2, 1)$ and $z < \dfrac{2}{k}$ then $x + y > kz$, since $z \leq u_2$. Thus $S \cup (ku_2, 1)$ is k-sum-free and hence $(ku_2, 1) \subseteq S$.

As in the proof of Lemma 2, for each x in $S \cap \left(\dfrac{2}{k^2}, u_2\right]$ there is a forbidden pairing with respect to x of $\left(0, kx - \dfrac{2}{k}\right]$ and $\left[\dfrac{2}{k}, kx\right)$, so

$$\mu\left(S \cap \left(\left(0, ku_2 - \frac{2}{k}\right] \cup \left[\frac{2}{k}, ku_2\right]\right)\right) \leq ku_2 - \frac{2}{k}. \tag{3.2}$$

If the inequality in (3.2) is strict, then the set $S' = \left(S \cap \left(ku_2 - \dfrac{2}{k}, u_2\right]\right) \cup \left(\dfrac{2}{k}, 1\right]$ is also k-sum-free and $\mu(S) < \mu(S')$. Thus equality holds in (3.2).

Lemma 4 *If S is a maximum k-sum-free subset of $(0,1]$ where k is an integer greater than or equal to 4 and if $u_2 = \sup\left\{x \in S | x < \frac{2}{k}\right\}$ then $\mu(S) > \frac{1}{2}u_2 + \left(1 - \frac{2}{k}\right)$.*

Proof: If not then since $u_2 < \frac{2}{k^2-2}$ (Lemma 2) we have

$$\begin{aligned}\mu(S) &< \frac{1}{k^2-2} + \left(1 - \frac{2}{k}\right)\\ &\leq \frac{1}{k^2-2} \cdot \frac{2(k-2)}{k} + \frac{k-2}{k}\\ &= \frac{k^2-2k}{k^2-2}\end{aligned}$$

which is a contradiction since this is the size of $S_k(2,1)$. The second inequality above is because $\frac{2(k-2)}{k} \leq 1$ when $k \geq 4$. We remark that this is the only place where the proof does not work for all real $k \geq 2+\sqrt{2}$, the bound imposed by the necessity of having $\frac{k(k-2)}{k^2-2} \geq \frac{1}{2}$ to make forbidden pairing arguments work.

Lemma 5 *If S is a maximum k-sum-free subset of $(0,1]$ where k is an integer greater than or equal to 4 and if $u_2 = \sup\left\{x \in S | x < \frac{2}{k}\right\}$ then*

(a) $\mu\left(S \cap \left(0, \frac{2}{k}u_2\right]\right) > 0$.

(b) If $u_3 = \sup\left\{x \in S | x < \frac{2}{k}u_2\right\}$ then $u_3 \leq \frac{1}{k}u_2$.

(c) There exists a positive number c such that $ku_3 - \frac{2}{k}u_2 = ku_2 - \frac{2}{k} = c$.

(d) $\left[\left(\frac{2}{k}u_2, u_2\right) \cup \left(\frac{2}{k}, 1\right)\right] \subseteq S$ and $S \cap (0,c) = \emptyset$.

Proof:

(a) If $\mu\left(S \cap \left(0, \frac{2}{k}u_2\right]\right) = 0$ then, since $u_2 < \frac{2}{k^2-2}$ (Lemma 2), $\mu(S) \leq \left(1 - \frac{2}{k}\right) + u_2\left(1 - \frac{2}{k}\right) < \frac{k^2-2k}{k^2-2}$

(b) if $x \in S \cap \left(\frac{1}{k}u_2, \frac{2}{k}u_2\right)$ then there are forbidden pairs with respect to x in $[kx - u_2, u_2]$: If $y \in S \cap [kx - u_2, u_2]$ then $kx - y \notin S$. If we let $|S \cap [kx - u_2, u_2]| = r$ and $\mu\left(S \cap \left[ku_2 - \frac{2}{k}, u_2\right]\right) = p$ then

$$r < \frac{1}{2}[u_2 - (kx - u_2)] \tag{3.3}$$

because of the forbidden pairing and

$$p > \frac{1}{2}u_2 \tag{3.4}$$

by Lemma 3 and Lemma 4. From equations (3.3) and (3.4) we get

$$p - r > \frac{1}{2}(kx - u_2). \tag{3.5}$$

Now let $S' = \left\{\frac{u_2}{kx - u_2}w | w \in S \cap \left[ku_2 - \frac{2}{k}, kx - u_2\right]\right\} \cup \left(\frac{2}{k}, 1\right]$. It is easy to check that S' is k-sum-free and

$$\begin{aligned}
\mu(S') &= \frac{u_2}{kx - u_2}(p - r) + \left(1 - \frac{2}{k}\right) \\
&= (p - r) + \frac{u_2 - (kx - u_2)}{kx - u_2}(p - r) + \left(1 - \frac{2}{k}\right) \\
&> (p - r) + \frac{2r}{2(p - r)}(p - r) + \left(1 - \frac{2}{k}\right) \\
&= \mu(S)
\end{aligned}$$

where the inequality follows from equations (3.3) and (3.5) and the last equality follows from Lemma 3. Hence $S \cap \left(\frac{1}{k}u_2, \frac{2}{k}u_2\right) = \emptyset$ and $u_3 \leq \frac{1}{k}u_2$.

(c) By Lemma 2, $ku_2 - \frac{2}{k}$ is equal to some positive number c. Let $ku_3 - \frac{2}{k}u_2 = b$ and assume $b < c$. Let

$S' = A \cup B \cup C$ where
$A = \left\{ \dfrac{c + \frac{2}{k}u_2}{ku_3} x | x \in S \cap (c, u_3) \right\}, B = \left(\dfrac{2}{k}u_2, u_2\right]$, and $C = \left(\dfrac{2}{k}, 1\right]$.
If $z \in B \cup C$ or if $\{x, y, z\} \subseteq A$ it is clear that S' has no solution to $x + y = kz$. If $z \in A$ and $y \notin A$ then $x + y > c + \frac{2}{k}u_2 > kz$, so S' is k-sum-free. And

$$\begin{aligned}
\mu(S') &= \frac{c + \frac{2}{k}u_2}{ku_3} \mu\left(S \cap (c, u_3)\right) + \left(1 - \frac{2}{k}\right) u_2 + \left(1 - \frac{2}{k}\right) \\
&> \mu\left(S \cap (c, u_3)\right) + \left(1 - \frac{2}{k}\right) u_2 + \left(1 - \frac{2}{k}\right) \\
&> \mu(S)
\end{aligned}$$

where the first inequality is because $\mu\left(S \cap (c, u_3)\right) > 0$ (otherwise $\mu(S) \leq \mu\left(S_k(2, 1)\right)$ and $ku_3 = b + \frac{2}{k}u_2 < c + \frac{2}{k}u_2$, while the second inequality follows from Lemma 3.
On the other hand, if $b > c$ then b is positive and for each $x \in S \cap \left[\dfrac{2}{k^2}u_2, u_3\right]$ there is a forbidden pairing with respect to x of $\left(0, kx - \dfrac{2}{k}u_2\right]$ and $\left[\dfrac{2}{k}u_2, kx\right)$. Since x can be arbitrarily close to u_3,

$$\mu\left(S \cap \left((0, b] \cup \left[\frac{2}{k}u_2, ku_3\right]\right)\right) \leq b. \tag{3.6}$$

It is not hard to check that the set

$$S_0 = \left(S \cap (b, u_3]\right) \cup \left(\frac{2}{k}u_2, u_2\right) \cup \left(\frac{2}{k}, 1\right] \tag{3.7}$$

is k-sum-free and that $\mu(S) \leq \mu(S_0)$ (by (3.6) and Lemma 5(b)). We define the set S' by

$$S' = \left(S \cap (b, u_3]\right) \cup \left(\frac{2}{k}{u'}_2, {u'}_2\right] \cup \left(\frac{2}{k}, 1\right]$$

where u'_2 is chosen so that

$$ku_3 - \frac{2}{k}u'_2 = ku'_2 - \frac{2}{k}.$$

Since $b > c$ we have $k^2u_3 + 2 > k^2u_2 + 2u_2$, so

$$u'_2 = \frac{k^2u_3 + 2}{k^2 + 2} > \frac{k^2u_2 + 2u_2}{k^2 + 2} = u_2.$$

Hence $\mu(S') > \mu(S_0) \geq \mu(S)$. It remains to show S' is k-sum-free. If $z \in (\frac{2}{k}, 1]$, or if $z \in (\frac{2}{k}u'_2, u'_2]$ and neither x nor y is in $(\frac{2}{k}, 1]$, or if $x, y, z \in S \cap (b, u_3]$ then clearly $x + y < kz$. If $x \in S, y \in S \backslash (b, u_3]$ and $z \in S \cap (b, u_3]$ then

$$\begin{aligned} kz &< ku_3 + \frac{2}{k}(u'_2 - u_2) \\ &= b + \frac{2}{k}u' \\ &< x + y \end{aligned}$$

while if $x \in S, y \in \left(\frac{2}{k}, 1\right]$, and $z \in \left(\frac{2}{k}u'_2, u'_2\right]$, then

$$\begin{aligned} kz &\leq \left(ku'_2 - \frac{2}{k}\right) + \frac{2}{k} \\ &= \left(ku_3 - \frac{2}{k}u'_2\right) + \frac{2}{k} \\ &< \left(ku_3 - \frac{2}{k}u_2\right) + \frac{2}{k} \\ &< x + y \end{aligned}$$

Hence $b = c$.

(d) If $\mu(S \cap (0, c]) = \delta > 0$ then, because of the forbidden pairing of $(0, c]$ with each of $\left[\frac{2}{k}u_2, ku_3\right)$ and $\left[\frac{2}{k}, ku_2\right)$ we have

$$\mu(S) \leq \mu(S \cap (0, u_3]) + \left[\left(1 - \frac{2}{k}\right)u_2 - \delta\right] + \left(1 - \frac{2}{k}\right) - \delta < \mu(S_0)$$

where S_0 is given by (3.7) with $b = c$. And if $\delta = 0$ but

$$\mu\left(S \cap \left(\left(\frac{2}{k}u_2, u_2\right] \cup \left(\frac{2}{k}, 1\right]\right)\right) \quad < \quad u_2\left(1 - \frac{2}{k}\right) + \left(1 - \frac{2}{k}\right)$$

we would still have $\mu(S) < \mu(S_0)$. So $S \cap (0, c)$ and

$$\left[\left(\frac{2}{k}u_2, u_2\right) \cup \left(\frac{2}{k}, 1\right)\right] \backslash S$$

are both sets of measure 0; we will now show each is the empty set. If $y \in S \cap (0, c)$ we choose r and t in S such that

$$\frac{1}{k}y + \frac{2}{k^2} < r < t \leq u_2$$

(such r and t exist because S is missing at most a set of measure zero in $\left[\frac{2}{k^2}, u_2\right]$). Then

$$\frac{2}{k} < kr - y < kt - y \leq ku_2$$

and for each $q \in S \cap [r, t], kq - y \notin S$. Hence $\mu(S \cap [kr - y, kt - y]) = 0$, so $\mu\left(S \cap \left(\frac{2}{k}, ku_2\right)\right) < ku_2 - \frac{2}{k}$ and $\mu(S) < \mu(S_0)$. Thus we have shown $S \cap (0, c) = \emptyset$. It is each to see that any such maximum S contains $\left(\frac{2}{k}u_2, u_2\right)$ and $\left(\frac{2}{k}, 1\right)$. □

Lemma 6 *Let S be a maximum k-sum-free subset of $(0, 1]$ where k is an integer greater than or equal to 4 with the sequence $\{u_i\}$ defined by $u_1 = 1$, $u_i = \sup\left\{x \in S | x < \frac{2}{k}u_{i-1}\right\}$ for $i = 2, 3, \ldots$. Then*

(a) There exists a positive integer $m \geq 3$ such that u_m exists but u_{m+1} does not.

(b) There exists a positive number $c \in \left[\frac{2}{k(k-1)}u_m, u_m\right]$ such that $[0, c) \cap S = \emptyset$ and $ku_{i+1} - \frac{2}{k}u_i = c \;\; i = 1, 2, \ldots, m - 1$.

(c) $\left(\frac{2}{k}u_i, u_i\right) \subseteq S \quad i = 1, 2, \ldots, m - 1$ and $(t, u_m) \subseteq S$ where $t = \max\left\{\frac{2}{k}u_m, c\right\}$.

Proof: By Lemma 5 there exists a positive number c such that

$$ku_{i+1} - \frac{2}{k}u_i = c \tag{3.8}$$

and

$$\left(\frac{2}{k}u_i, u_i\right) \subseteq S \tag{3.9}$$

for $i = 1$ and 2 and where $S \cap [0, c) = \emptyset$. Since c is a fixed positive number it is clear that the statement in (a) is true. We will show by induction that equations (3.8) and (3.9) hold for all positive integers integers less than m. Assume (3.8) and (3.9) hold for all i less than j where $j \in \{3, 4, \ldots, m-1\}$; we will show they hold for $i = j$ as well.

First we will show that

$$\mu\left(S \cap (0, u_j)\right) > \frac{1}{2}u_j. \tag{3.10}$$

Since u_{j+1} exists and $S \cap (0, c) = \emptyset$ we must have $c < \frac{2}{k}u_j$. Hence the set $S' = \cup_{i=1}^{j}\left(\frac{2}{k}u_i, u_i\right)$ is k-sum-free and

$$\begin{aligned} \mu\left(S \cap (0, u_j)\right) &= \mu(S) - \mu(S') + \left(1 - \frac{2}{k}\right)u_j \\ &\geq \frac{1}{2}u_j + \frac{k-4}{2k}u_j \end{aligned} \tag{3.11}$$

since $\mu(S) \geq \mu(S')$. This verifies (3.10) for any k greater than 4. But for $k \geq 4$ the value of $\mu(S')$ is given by Lemma 1 with $c < a_1 = \frac{2}{k}u_j$. Hence $y < 1$ and we use formula (2.6). As remarked earlier, this is an increasing function of y, so $\mu(S')$ is not a maximum and $\mu(S) - \mu(S') > 0$. This makes the inequality in (3.11) strict and verifies (3.10).

Next we will show

$$u_{j+1} \leq \frac{1}{k}u_j. \tag{3.12}$$

If there exists $x \in S \cap \left(\frac{1}{k}u_j, \frac{2}{k}u_j\right)$ then

$$\mu\left(S \cap [kx - u_j, u_j]\right) \leq \frac{1}{2}[u_j - (kx - u_j)] \tag{3.13}$$

by a forbidden pair argument. If we now let

$$S' = \left\{\frac{u_j}{kx - u_j} w | w \in S \cap (c, kx - u_j)\right\} \cup \left(S \cap \left[\frac{2}{k}u_{j-1}, 1\right]\right)$$

then it is easy to show S' is k-sum-free and

$$\begin{aligned}
\mu(S') - \mu(S) &= \frac{u_j}{kx - u_j}\left(\mu\left(S \cap (0, u_j]\right) - \mu\left(S \cap [kx - u_j, u_j]\right)\right) \\
&\quad -\mu\left(S \cap (0, u_j]\right) \\
&\geq \frac{u_j - (kx - u_j)}{kx - u_j}\left(\mu\left(S \cap (0, u_j]\right)\right) \\
&\quad -\frac{u_j}{kx - u_j} \cdot \frac{1}{2}[u_j - (kx - u_j)] \\
&= \frac{u_j - (kx - u_j)}{kx - u_j}\left(\mu\left(S \cap (0, u_j]\right) - \frac{1}{2}u_j\right) \\
&> 0
\end{aligned}$$

where the first inequality follows by (3.13) and the second by (3.10). Thus we have verified equation (3.12).

Now let $b = ku_{j+1} - \frac{2}{k}u_j$. We wish to show $b = c$. If $b < c$ then we let

$$S' = \left\{\frac{\frac{1}{k}\left(\frac{2}{k}u_j + c\right)}{u_{j+1}} w | w \in S \cap (0, u_{j+1})\right\} \cup \left(S \cap \left[\frac{2}{k}u_j, 1\right]\right).$$

It is easy to check that S' is k-sum-free and we have

$$\mu(S') - \mu(S) = \left(\frac{\frac{1}{k}\left(\frac{2}{k}u_j + c\right)}{u_{j+1}} - 1\right)\mu\left(S \cap (0, u_{j+1})\right). \tag{3.14}$$

If $\mu(S \cap (0, u_{j+1})) = 0$ then (as in the discussion following inequality (3.11) $\mu(S)$ is given by formula (2.6) with $y < 1$, so it cannot be a maximum. Since $u_{j+1} = \frac{1}{k}\left(\frac{2}{k}u_j + b\right)$, each factor in (3.14) is positive, which is a contradiction.

If $b > c$ then $\frac{2}{k}u_j < ku_{j+1} \leq u_j$ and (as in the proof of Lemma 5(c)) from a forbidden pairing we get

$$\mu\left(S \cap \left((0, b] \cup \left[\frac{2}{k}u_j, ku_{j+1}\right)\right)\right) \leq b. \tag{3.15}$$

Now we let $S' = (S \cap (b, u_{j+1})) \cup \cup_{i=3}^{j}\left(\frac{2}{k}u_i, u_i\right) \cup \left(\frac{2}{k}u'_2, u'_2\right) \cup \left(\frac{2}{k}, 1\right)$ where

$$u'_2 = \frac{1}{k}\left(b + \frac{2}{k}\right) > \frac{1}{k}\left(c + \frac{2}{k}\right) = u_2. \tag{3.16}$$

So S' is obtained from S by replacing $S \cap \left((0, b] \cup \left[\frac{2}{k}u_j, ku_{j+1}\right]\right)$ by $\left(\frac{2}{k}u_j, ku_{j+1}\right)$, replacing $S \cap \left[\frac{2}{k}u_2, u_2\right]$ by $\left(\frac{2}{k}u'_2, u'_2\right)$ and possibly omitting finitely many points (certain end-points). It is easy to check that S' is k-sum-free and

$$\mu(S') - \mu(S) \geq \left(1 - \frac{2}{k}\right)u'_2 - \left(1 - \frac{2}{k}\right)u_2 > 0$$

by (3.15) and (3.16), so again $\mu(S)$ is not a maximum. Therefore $b = c$ which verifies (3.8) for $i = j$. And clearly $S \cup \left(\frac{2}{k}u_j, u_j\right)$ is k-sum-free, which verifies (3.9) for $i = j$. Thus we have shown by induction that (3.8) and (3.9) hold for $i = 1, 2, \ldots, m - 1$.

Since u_{m+1} does not exist, if we let $t = \max\left\{\frac{2}{k}u_m, c\right\}$ then $S \cup (t, u_m)$ is k-sum-free, so $(t, u_m) \subseteq S$ verifying (c). Finally, if

$$kc < c + \frac{2}{k}u_m \tag{3.17}$$

then we can choose a real number y greater than c such that $ky < y + \frac{2}{k}u_m$. But then $S \cup \{y\}$ is k-sum-free which violates the maximality of S. Hence (3.17) must be false which shows $c \in \left[\frac{2}{k(k-1)}u_m, u_m\right]$. $\square$

Theorem 2 *If k is an integer greater than or equal to 4 and S is a maximum k-sum-free subset of $(0, 1]$ then S is the union of the set $S_k(3, 1) = \cup_{i=1}^{3}(e_i, f_i)$ (of Lemma 1) and three points, one end-point of each interval. Any of the eight possible ways of choosing the end-points is all right except $\{e_1, f_2, e_3\}$.*

Proof: By Lemma 6, if we ignore end-points, S has the form of $S_k(m, y)$ for some $m \geq 3$. By Theorem 1, $S_k(3, 1)$ is the largest of these. To get a maximal set we need to put in one end-point of each interval, but since $e_1 + e_3 = kf_2$ we cannot choose $\{e_1, f_2, e_3\}$. $\square$

The end-points turn out to be

$$f_1 = \frac{4}{k^4 - 2k^2 - 4}, \quad f_2 = \frac{2(k^2-2)}{k^4 - 2k^2 - 4}, \quad f_3 = 1,$$

with $e_i = \frac{2}{k}f_i \;\; i = 1, 2, 3$. For $k = 4$ one gets

$$S_4(3, 1) = \left(\frac{1}{110}, \frac{2}{110}\right) \cup \left(\frac{7}{110}, \frac{14}{110}\right) \cup \left(\frac{1}{2}, 1\right).$$

Corollary 1 *Let $f(n, k)$ denote the maximum size of a k-sum-free subset of $\{1, 2, \ldots n\}$. If $k \geq 4$ then $\lim_{n\to\infty} \frac{f(n,k)}{n} \geq \mu\left(S_k(3, 1)\right)$.*

4 Remarks

Moving right to left the greedy procedure does not produce a maximum k-sum-free set. One gets $\left(\frac{2}{k}, 1\right]$ for the first interval and then $\frac{2}{k(k-1)}$ is the largest number that can be added. However the set is now maximal if $k \geq 3$. In fact $\frac{2}{k(k-1)}$ is the only real number x such that $\{x\} \cup \left(\frac{2}{k}, 1\right]$ is a maximal k-sum-free subset of $(0, 1]$. If you first put $\frac{8}{k(k^4 - 2k^2 - 4)}$ in S and then work right to left from 1 following the greedy procedure, you do get a maximum k-sum-free set.

We have already noted that we assumed $k \geq 4$ for the proof of Lemma 4 (so Theorem 2 holds for all real $k \geq 4$) and $k \geq 2+\sqrt{2}$ for forbidden pair arguments. But if $k \geq 2.2$ then $\mu(S_k(m,y))$ is still maximized when $m=3$ and $y=1$. Namely, $\mu(S_3(3,1)) = 77/177$.

Conjecture 3 *Theorem 2 holds for $k=3$ as well.*

As mentioned in the Introduction, maximum k-sum-free subsets of $(0,1]$ and of $\{1,2,\ldots,n\}$ have very different structures for $k=3$. (There is no maximum 3-sum-free analogue on $(0,1]$ of the all odd number maximum 3-sum-free subset of $\{1,2,\ldots,n\}$). However, we think they have the same structures for $k \geq 4$.

Conjecture 4 *Equality holds in Corollary 1.*

We believe that if n is sufficiently large, to get a maximum k-sum-free subset of $\{1,2,\ldots,n\}$ one takes the integers within the three intervals obtained by multiplying each real number in $S_3(3,1)$ by n (with slight modification of the end-points due to integer round-off). To prove this integral version one can probably use the general outline of the above proof for $(0,1]$. There are some technical difficulties due to the fact that if one multiplies each member of a set of integers by a real number greater than 1, the result may not be a set of integers, and even if it is, the size of the set is the same as the size of the original set (as opposed to what happens with the measure of a set of real numbers).

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