A Short Proof of the Rook Reciprocity Theorem

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Abstract. Rook numbers of complementary boards are related by a reciprocity law. A complicated formula for this law has been known for about fifty years, but recently Gessel and the present author independently obtained a much more elegant formula, as a corollary of more general reciprocity theorems. Here, following a suggestion of Goldman, we provide a direct combinatorial proof of this new formula.

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A *board* B is a subset of $[d] \times [d]$ (where $[d]$ is defined to be $\{1, 2, \ldots, d\}$) and the *rook numbers* r_k^B of a board are the number of subsets of B of size k such that no two elements have the same first coordinate or the same second coordinate (i.e., the number of ways of "placing k non-taking rooks on B"). It has long been known [5] that the rook numbers of a board B determine the rook numbers of the complementary board $\overline{B}$ (defined to be $([d] \times [d]) \backslash B$) according to the polynomial identity

$$\sum_{k=0}^{d} r_k^B (d-k)!\, x^k = \sum_{k=0}^{d} (-1)^k r_k^{\overline{B}} (d-k)!\, x^k (x+1)^{d-k}.$$

Recently, a simpler formulation of this identity was found independently by Gessel [2] and Chow [1]. To state it, we follow [4] in defining

$$R(B; x) \stackrel{\text{def}}{=} \sum_{k=0}^{d} r_k^B x^{\underline{d-k}},$$

where $x^{\underline{n}} \stackrel{\text{def}}{=} x(x-1)(x-2)\cdots(x-n+1)$. Then we have the following reciprocity theorem.

Theorem. *For any board $B \subset [d] \times [d]$,*

$$R(\overline{B}; x) = (-1)^d R(B; -x-1).$$

The existing proofs derive this as a corollary of other reciprocity theorems, but Goldman [3] has suggested that a direct combinatorial proof ought to be possible. Indeed, it is, and the purpose of this note is to provide such a proof. The knowledgeable reader will recognize that the main idea is borrowed from [4].

Proof. Observe that

$$\begin{aligned}(-1)^d R(B;\,-x-1) &= (-1)^d \sum_{k=0}^{d} r_k^B(\,-x-1)^{\underline{d-k}} \\ &= \sum_{k=0}^{d} (-1)^k r_k^B (x+d-k)^{\underline{d-k}}.\end{aligned}$$

First assume x is a positive integer. Add x extra rows to $[d]\times[d]$. Then $r_k^B(x+d-k)^{\underline{d-k}}$ is the number of ways of first placing k rooks on B and then placing $d-k$ more rooks anywhere (i.e., on B, $\overline{B}$ or on the extra rows) such that no two rooks can take each other in the final configuration. By inclusion-exclusion, we see that the resulting configurations in which the set S of rooks on B is nonempty cancel out of the above sum, because they are counted once for each subset of S, with alternating signs. Thus what survives is the set of placements of d non-taking rooks on the extended board such that no rook lies on B. But it is clear that this is precisely what $R(\overline{B};x)$ enumerates. Therefore the theorem holds for all positive integers x and since it is a polynomial equation it holds for all x. □

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