Local equivalence of transversals in matroids

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Abstract

Given any system of n subsets in a matroid M, a *transversal* of this system is an n-tuple of elements of M, one from each set, which is independent. Two transversals differing by exactly one element are adjacent, and two transversals connected by a sequence of adjacencies are *locally equivalent*, the distance between them being the minimum number of adjacencies in such a sequence.

We give two sufficient conditions for all transversals of a set system to be locally equivalent. Also we propose a conjecture that the distance between any two locally equivalent transversals can be bounded by a function of n only, and provide an example showing that such function, if it exists, must grow at least exponentially.

Let M be a matroid, and $\mathcal{V} = (V_1, \ldots, V_n)$ a collection of subsets of M. By a *transversal* of $\mathcal{V}$ we mean a sequence $(v_1, \ldots, v_n)$ of elements of M such that $v_i \in V_i$ for $i = 1, \ldots, n$, and $v_1, \ldots, v_n$ are independent. By the well-known Rado's Theorem, transversals exist if and only if the following condition is satisfied:

$$\text{For every } \ X \subseteq \{1, \ldots, n\}, \ \operatorname{rank}\big(\bigcup_{i \in X} V_i\big) \geq |X|. \tag{1}$$

We say that a transversal $(v'_1, \ldots, v'_n)$ is a (result of a) *local replacement* of $(v_1, \ldots, v_n)$ at i if $v'_j = v_j$ for $j \neq i$; and call two transversals *locally equivalent* if one can be obtained from the other by a sequence of local replacements; the length of the shortest such sequence being *the distance* between the transversals.

In this note we address two questions: under what conditions are all transversals of a collection $\mathcal{V}$ locally equivalent; and how big (in terms of n) can be the distance between two locally equivalent transversals. Also, we shall consider in more detail the case when M is the free matroid (the matroid having no cycles).

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1 Sufficient conditions of local equivalence

Here we shall prove two sufficient conditions for all transversals of a set system to be locally equivalent.

THEOREM 1 *If a collection $\mathcal{V} = (V_1, \ldots, V_n)$ of subsets of a matroid M is such that*

$$\text{For every } \emptyset \neq X \subseteq \{1, \ldots, n\}, \ \operatorname{rank}\big(\bigcup_{i \in X} V_i\big) > |X| \tag{2}$$

then all transversals of $\mathcal{V}$ are locally equivalent.

The second theorem is a straightforward generalization of a result proved in [1].

Call a subset V of M *thick* if for every flat A of M either $V \subseteq A$, or $|V \cap A| < |V|/2$.

THEOREM 2 *If $V_1, \ldots, V_n$ are thick subsets of M and $\mathcal{V} = (V_1, \ldots, V_n)$ satisfies* (1) *then all transversals of $\mathcal{V}$ are locally equivalent, and the distance between any two of them does not exceed $2n - 1$.*

Some examples of thick subsets: one-element sets, cycles of size 3, subspaces of an n-dimensional vector space over GF(2).

A partial case of Theorem 2 for M a linear vector space over the field $GF(2)$, and V_i one- or two-dimensional subspaces was proved in [1] and independently in [4]. The proof from [1], almost unchanged, applies to the general situation.

We use the notation $\langle X \rangle$ for $X \subseteq M$ to mean the flat in M generated by X.

PROOF OF THEOREM 1.

Let $\overline{x} = (x_1, \ldots, x_n)$ and $\overline{y} = (y_1, \ldots, y_n)$ be two transversals of $\mathcal{V} = (V_1, \ldots, V_n)$. Let $D = D(\overline{x}, \overline{y}) = \{i \mid x_i \neq y_i\}$. We shall prove that $\overline{x}$ and $\overline{y}$ are locally equivalent by induction on $d = |D(\overline{x}, \overline{y})|$.

First we introduce some notation:

$I = \{1, \ldots, n\}$;

$X_J = \langle x_j \mid j \in J \rangle$ where $J \subseteq I$;

for $v \in \langle \overline{x} \rangle$, let $I_X(v)$ be the smallest set $J \subseteq I$ such that $v \in X_J$. (This set is uniquely determined.)

Suppose first that $\langle x_1, \ldots, x_n \rangle \neq \langle y_1, \ldots, y_n \rangle$. This means that $y_i \notin \langle x_1, \ldots, x_n \rangle$ for some i. Then the sequence $\overline{x}' = (x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n)$ is an independent transversal of $\mathcal{V}$. It is a local replacement of $\overline{x}$ at i; and $|D(\overline{x}', \overline{y})| < |D(\overline{x}, \overline{y})|$. By induction, we are done in this case.

So, let $X = \langle x_1, \ldots, x_n \rangle = \langle y_1, \ldots, y_n \rangle$. We construct inductively the sequence $\emptyset = I_0 \subseteq I_1 \subseteq \ldots \subseteq I_r \subseteq I$ as follows:

$I_{k+1} = I_k \cup \{j \in I \setminus I_k \mid V_j \not\subseteq X_{I \setminus I_k}\}$;

r is the first index for which $I_r \cap D \neq \emptyset$.

The property (2) implies that $I_k \subset I_{k+1}$ for all $0 \leq k < r$; in particular, the length of the sequence does not exceed $n - d + 1$, and the number r is well-defined.

Now we construct a sequence $(i_1, \ldots, i_r)$ of indices, and a sequence $(v_1, \ldots, v_r)$ of elements $v_k \in V_{i_k}$; starting with i_r and v_r. Choose any $i_r \in I_r \cap D$, and $v_r \in V_{i_r} \setminus X_{I \setminus I_r}$.

The choice of i_r and v_r implies that $i_r \in I_r \setminus I_{r-1}$, and that the set $I_X(v_r)$ contains an element $j \in I_{r-1}$ for which $V_j \not\subseteq X_{I \setminus I_{r-1}}$. Set $i_{r-1} = j$, and choose $v_{r-1} \in V_j \setminus X_{I \setminus I_{r-1}}$. Again we have $i_{r-1} \in I_{r-1} \setminus I_{r-2}$, and the set $I_X(v_{r-1})$ contains an element $j \in I_{r-2}$ for which $V_j \not\subseteq X_{I \setminus I_{r-2}}$. We continue in the same manner, and finally find $i_1 \in I_1$ and $v_1 \in V_{i_1}$ such that $v_1 \notin \langle X \rangle$.

To simplify the notation, let us assume that $i_1 = 1, \ldots, i_r = r$.

Now we shall perform certain local replacements of both $\overline{x}$ and $\overline{y}$. Consider the sequences $\overline{x}^{(i)} = (v_1, \ldots, v_i, x_{i+1}, \ldots, x_n)$ and $\overline{y}^{(i)} = (v_1, \ldots, v_i, y_{i+1}, \ldots, y_n)$ for $i = 1, \ldots, r$. Let also $\overline{x}^{(0)} = \overline{x}$, $\overline{y}^{(0)} = \overline{y}$.

By the choice of the elements v_i, we have $v_i \notin \langle x_i, \ldots, x_n \rangle$, and for $i \geq 2$, $v_i \in \langle x_{i-1}, \ldots, x_n \rangle$. Therefore all the sequences $\overline{x}^{(j)}$, $0 \leq j \leq r$, are transversals, and are locally equivalent to $\overline{x}$.

If we have $\langle \overline{x}^{(i)} \rangle = \langle \overline{y}^{(i)} \rangle$ for $i = 1, \ldots, j$ then all the sequences $\overline{y}^{(i)}$, $0 \leq i \leq j$, are also transversals, and are locally equivalent to $\overline{y}$. If this holds for $j = r$ then $|D(\overline{x}^{(r)}, \overline{y}^{(r)})| = |D(\overline{x}, \overline{y})| - 1$, and $\overline{x}$ and $\overline{y}$ are locally equivalent by induction. On the other side, if i is the first value for which $\langle \overline{x}^{(i)} \rangle \neq \langle \overline{y}^{(i)} \rangle$ then $\overline{x}^{(i)}$ is locally equivalent to $\overline{y}^{(i)}$ by the argument from the beginning of this proof. Thus, in either case $\overline{x}$ is locally equivalent to $\overline{y}$, and the theorem is proved. □

PROOF OF THEOREM 2.

The proof below exactly follows the proof of Proposition 5.1 in [1].

Take any two bases $\overline{x} = (x_1, \ldots, x_n)$ and $\overline{y} = (y_1, \ldots, y_n)$. Suppose that $x_i \neq y_i$ for some i. Let $X = \langle x_j \mid j \neq i \rangle$, and $Y = \langle y_j \mid j \neq i \rangle$. As $x_i \notin X$ and $y_i \notin Y$, and using the fact that V_i is thick, we have

$$|V_i \setminus (X \cup Y)| \geq |V_i| - |V_i \cap X| - |V_i \cap Y| > |V_i| - |V_i|/2 - |V_i|/2 = 0.$$

Therefore there exists an element z of V_i which belongs to neither X nor Y; and we can replace both x_i and y_i by z. Thus, using at most two local replacements, we can reduce by 1 the number of places in which $\overline{x}$ and $\overline{y}$ disagree.

This argument gives an upper bound of $2n-1$ on the maximum distance between transversals; because at the last stage, when $\overline{x}$ and $\overline{y}$ differ in only one place, one needs only one local replacement, and not two. □

The proof of Theorem 1 also gives an upper bound on the distance between transversals; the distance cannot exceed

$$2 + 4 + \ldots + 2(n-1) + 1 = n^2 - n + 1.$$

By all probability, this bound is far from sharp. It would be very interesting to find the exact bound on the distance between transversals under the assumptions of Theorem 1.

2 Free matroid

A free matroid M is a matroid with no cycles. In a free matroid all subsets are independent; and rank $(X) = |X|$ for all $X \subseteq M$. Let $\mathcal{V} = (V_1, \ldots, V_n)$ be a collection of subsets of M which has at least one transversal. Let $N = \{1, \ldots, n\}$. By the *kernel* of $\mathcal{V}$ we shall mean the largest subset $X \subseteq N$ for which

$$|\bigcup_{i \in X} V_i| = |X|.$$

The kernel exists since if X_1 and X_2 satisfy this property then so does $X_1 \cup X_2$.

THEOREM 3 *Two transversals of $\mathcal{V}$ are locally equivalent if and only if they agree on the kernel K of $\mathcal{V}$; the distance between them then does not exceed* $2(n - |K|) - 1$.

PROOF. We construct a bipartite graph $G = (N \cup M, E)$ with parts N and M; $i \in N$ is adjacent to $v \in M$ when $v \in V_i$. The transversals of $\mathcal{V}$ are in one-to-one correspondence with the matchings in G covering the part N; and a local replacement in this language corresponds to changing a single edge in the matching.

Let L be the set of vertices in M adjacent to vertices from K; by definition of K we have $|L| = |K|$. Thus, no edge incident to a vertex in K can ever be changed; and the "only if" part of the theorem is proved.

The set system corresponding to the graph induced on $(N \setminus K) \cup (M \setminus L)$ has an empty kernel, and at least one transversal. Thus, if $K \neq \emptyset$ then we can apply induction on $|N|$. So we assume that $K = \emptyset$, i.e. $\mathcal{V}$ satisfies the conditions of Theorem 1.

Let $A = (a_i \mid i \in N)$ and $B = (b_i \mid i \in N)$ be any two transversals. Colour the edges (i, a_i) blue, and (i, b_i) red. The multigraph formed by all coloured edges is a disjoint union of cycles (possibly of length 2 if $a_i = b_i$ for some i), paths, and isolated vertices. Let $\{C_1, \ldots, C_k\}$ be the set of all its cycles. We shall prove by induction on $n + k$ that the distance between A and B is at most $n + k$. Since $k \leq n$, and $k = n$ only if $A = B$, this will prove the theorem.

Let $C = \bigcup C_i$, $X = C \cap N$, $Y = C \cap M$. We have $|X| = |Y|$. Applying the inequality 2 to the set X we see that in G there is an edge between X and $M \setminus Y$; colour it green. This edge is incident to exactly one of the cycles $C_1, \ldots, C_k$; we delete it from C and apply the same argument to the set of remaining cycles; and continue in the same manner until we get k green edges. We shall consider the subgraph G' of G formed by all coloured edges (blue, red, and green). The system corresponding to G' also satisfies the property 2, and both A and B are its transversals.

Suppose first that there exists a green edge pq, $p \in N$, $q \in M$ such that q is not incident to any red edge. Then we can perform one local replacement on B replacing the red edge incident to p by pq, and reduce the number of cycles by 1. By induction, we are done in this case. Similarly we treat the case when q is not incident to any blue edge.

Thus, for every green edge pq, $q \in M$ the vertex q is incident to both red and blue edges. Let a be an end vertex of a red-blue path $abc\ldots$; we have $a \in M$, $b \in N$, $c \in M$. Say, the edge ba is red, and bc is blue. No green edge is incident to a. We perform one local replacement on A, replacing bc by ba, and then delete the vertices b and a. The system corresponding to the remaining graph still satisfies the property 2, and has $n-1$ sets. Thus, by induction, the theorem is proved. □

3 Lower bounds on the distance

We begin this section with a conjecture.

CONJECTURE. *For every natural n there exists $f(n)$ such that for every matroid M, if $\overline{x} = (x_1, \ldots, x_n)$ and $\overline{y} = (y_1, \ldots, y_n)$ are two locally equivalent transversals of a set system $(V_1, \ldots, V_n)$ in M then the distance between $\overline{x}$ and $\overline{y}$ does not exceed $f(n)$.*

I firmly believe this conjecture to be true. Trivially, $f(1) = 1$. It is easy to prove (and is left to the reader as an exercise) that $f(2) = 3$. The case $n = 3$ can possibly be dealt with by a long and tedious but not very difficult argument.

On the other hand, the function $f(n)$ if it exists must grow at least exponentially. Below we shall construct examples proving this, and an example showing that $f(3) \geq 7$.

EXAMPLE 1. Let M be a 3-dimensional space over any field; a, b, x three linearly independent vectors. Set

$$V_1 = \{a, b\},\ V_2 = \{a, b, x\},\ V_3 = \{a + x, b + x, a + b\}.$$

It is easy to check that this set system has eight independent transversals, and that the transversals $(a, b, a + x)$ and $(b, a, b + x)$ are at distance 7.

EXAMPLE 2. For $i = 1, \ldots, n$ let $V_i = \{e_i^0, e_i^1\}$ be n disjoint sets of size 2; $M = \bigcup V_i$, the matroid structure on M to be specified later.

The set $H = V_1 \times \ldots \times V_n$ of n-tuples of elements of M forms the Hamming graph; two n-tuples being adjacent whenever they differ in only one coordinate. Every matroid structure on M determines a subgraph of H induced on the vertices corresponding to independent subsets of M; and we need to choose a matroid structure on M so that the diameter of some connected component of this graph be as big as possible. We shall use the following easy lemma.

LEMMA. *For every set V, and every collection $\mathcal{X}$ of k-subsets of V such that $|X_1 \setminus X_2| \geq 2$ for any two different $X_1, X_2 \in \mathcal{X}$ there exists a matroid on V in which a k-set is independent if and only if it doesn't belong to $\mathcal{X}$.*

PROOF. Let the bases of the matroid be the k-subsets of V not belonging to $\mathcal{X}$. We only need to check that they satisfy the exchange axiom:

For any two bases X, Y, and any $x \in X$ there exists $y \in Y$ such that $X \setminus \{x\} \cup \{y\}$ is also a base.

If $x \in Y$ then we can take $y = x$. If $X \setminus Y = \{x\}$ then $Y \setminus X = \{y\}$, and we replace x by y. So, let $x \in X \setminus Y$, and $|Y \setminus X| \geq 2$, say, $\{y, z\} \subseteq Y \setminus X$. By our assumption on the collection $\mathcal{X}$ at least one of the sets $X \setminus \{x\} \cup \{y\}$, $X \setminus \{x\} \cup \{z\}$ does not belong to $\mathcal{X}$ and therefore is a base — the exchange property is proved. □

We shall denote vertices of H by $(0,1)$-vectors of length n; the vector $(\epsilon_1, \ldots, \epsilon_n)$ corresponding to the transversal $(e_1^{\epsilon_1}, \ldots, e_n^{\epsilon_n})$. The condition on the collection $\mathcal{X}$ from the Lemma now means simply that $\mathcal{X}$ corresponds to an independent set of vertices of H.

Let n be even, $n = 2m$. Denote by H_i the set of vectors of weight i: those having exactly i coordinates equal to 1.

We shall construct the set $\mathcal{X} = H_{m-2} \cup (H_m \setminus \mathcal{Y}) \cup H_{m+2}$ for some $\mathcal{Y} \subseteq H_m$ such that in the graph $H \setminus \mathcal{X}$ the set $\mathcal{Y}$ is contained in a connected compomnent of large diameter.

We define a graph on the set H_m; two vectors are adjacent if and only if they differ in exactly two coordinates. This is the Johnson graph $J(2m, m)$. The vertices of any induced path of length l in this graph form such set $\mathcal{Y}$ with the diameter of the connected component equal to $2l$. So, we need to find long induced paths in the graphs $J(2m, m)$. By a recent result of A.Evdokimov [2], one can find such paths of length $> (2-\epsilon)^n$ for any $\epsilon > 0$ and large enough n. This proves that $f(n)$ grows faster than any exponent $(2-\epsilon)^n$.

The mentioned theorem of A.Evdokimov is new and yet unpublished but it is easy to prove an exponential (though worse) lower bound using a well-known result by the same A.Evdokimov [3] that in the binary Hamming graph of dimension m one can find an induced path of length $c \cdot 2^m$. If $(v_1, \ldots, v_l)$ is such path then the sequence of vectors $(w_1, \ldots, w_l)$ where $w_i = v_i \overline{v}_i$ ($\overline{v}_i$ is the complement of v_i) gives an induced path in the graph $J(2m, m)$ of the same length. Thus, $f(n) \geq c \cdot 2^{n/2}$.

Finally, I conjecture that $f(n) \leq 2^n - 1$. This conjecture is not supported by any evidence, and is much more dubious than the first one.

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