The Borwein conjecture and partitions with prescribed hook differences

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Dedicated to Dominique Foata: teacher, mentor, and friend

Abstract

Peter Borwein has conjectured that certain polynomials have non-negative coefficients. In this paper we look at some generalizations of this conjecture and observe how they relate to the study of generating functions for partitions with prescribed hook differences. A combinatorial proof of the generating function for partitions with prescribed hook differences is given.

1 Introduction

In a personal communication to George Andrews in 1990, Peter Borwein made the following three conjectures. We use the notation

$$\begin{aligned} (a;q)_n &= \prod_{j=0}^{n-1}(1-a\,q^j), \\ \begin{bmatrix} N \\ M \end{bmatrix} &= \frac{(q^{N-M+1};q)_M}{(q;q)_M}. \end{aligned}$$

Conjecture 1 *The polynomials $A_n(q)$, $B_n(q)$, and $C_n(q)$ defined by*

$$(q;q^3)_n(q^2;q^3)_n = A_n(q^3) - qB_n(q^3) - q^2C_n(q^3) \tag{1}$$

have non-negative coefficients.

Conjecture 2 *The polynomials $A^*_n(q)$, $B^*_n(q)$, and $C^*_n(q)$ defined by*

$$(q;q^3)^2_n(q^2;q^3)^2_n = A^*_n(q^3) - qB^*_n(q^3) - q^2C^*_n(q^3) \tag{2}$$

have non-negative coefficients.

Conjecture 3 *The polynomials $A_n^\star(q)$, $B_n^\star(q)$, $C_n^\star(q), D_n^\star(q)$ and $E_n^\star(q)$ defined by*

$$\begin{aligned}(q;q^5)_n(q^2;q^5)_n(q^3;q^5)_n(q^4;q^5)_n \\ = \quad A_n^\star(q^5) - qB_n^\star(q^5) - q^2C_n^\star(q^5) - q^3D_n^\star(q^5) - q^4E_n^\star(q^5) \qquad (3)\end{aligned}$$

have non-negative coefficients.

George Andrews [1] has generalized the first two conjectures:

Conjecture 4 *For $m \geq 1$, the polynomials $A^\dagger(m,n,t,q)$, $B^\dagger(m,n,t,q)$, and $C^\dagger(m,n,t,q)$ defined by*

$$\begin{aligned}(q;q^3)_m(q^2;q^3)_m(zq;q^3)_n(zq^2;q^3)_n \\ = \sum_{t=0}^{2n} z^t \left[A^\dagger(m,n,t,q^3) - q\, B^\dagger(m,n,t,q^3) - q^2\, C^\dagger(m,n,t,q^3)\right] \quad (4)\end{aligned}$$

have non-negative coefficients.

Dennis Stanton has discovered a generalization of the first conjecture. We can use the q-binomial theorem to expand (k odd, $1 \leq a < k/2$)

$$(q^a;q^k)_m(q^{k-a};q^k)_n = \sum_{\nu=(1-k)/2}^{(k-1)/2} (-1)^\nu q^{k(\nu^2+\nu)/2-a\nu} F_\nu(q^k), \qquad (5)$$

where

$$F_\nu(q) = \sum_{j=-\infty}^{\infty} (-1)^j q^{j(k^2j+2k\nu+k-2a)/2} \begin{bmatrix} m+n \\ m+\nu+kj \end{bmatrix}. \qquad (6)$$

Each monomial in $q^{k(\nu^2+\nu)/2-a\nu}F_\nu(q^k)$ involves a power of q for which the exponent is congruent to $-a\nu$ modulo k.

Conjecture 5 *If a is relatively prime to k and $m = n$, then the coefficients of $F_\nu(q)$ are non-negative.*

The polynomial $F_\nu(q)$ appears to be a special case of the generating function for partitions "with prescribed hook differences," [2]. In particular, it is shown in that paper that if $\alpha + \beta < 2K$ and $-K + \beta \leq n - m \leq K - \alpha$, then

$$G(\alpha,\beta,K;q) = \sum_j (-1)^j q^{j[K(\alpha+\beta)j + K(\alpha-\beta)]/2} \begin{bmatrix} m+n \\ m+Kj \end{bmatrix} \qquad (7)$$

is the generating function for partitions inside an $m \times n$ rectangle with "hook difference conditions" specificed by α, β, and K. The polynomial $F_\nu(q)$ is simply the special case

$$K = k, \quad \alpha = \nu + \frac{k+1}{2} - \frac{a}{k}, \quad \beta = -\nu + \frac{k-1}{2} + \frac{a}{k}.$$

Since we know that this is a generating function, it follows that the coefficients are non-negative.

The only problem with this analysis is that the hook difference conditions defined in [2] only make sense for integer values of α, β, and K. In section 2, we will examine these hook difference conditions, and in section 3 we will consider what is involved in extending the definition to non-integer values. We are not able to show that $F_\nu(q)$ is a generating function. However, it is possible to construct a family of partition generating functions, $\mathcal{A}_{m,n}(q)$, that are remarkably close to $A_n(q)$ when $m = n$. Furthermore, it appears that conjecture 5 can be strengthened to the following.

Conjecture 6 *Let α and β be positive rational numbers and K an integer greater than 1 such that αK and βK are integers. If $1 \leq \alpha + \beta \leq 2K - 1$ (with strict inequalities when $K = 2$) and $-K + \beta \leq n - m \leq K - \alpha$, then $G(\alpha, \beta, K; q)$ has non-negative coefficients.*

This conjecture is justified heuristically by the arguments of section 3. Several special cases have been verified. For $K = 2$ and $m = n$, this author has proven identities that imply the conjecture for $\alpha = 1$, $\beta = 1/2$ and $\alpha = 3/2$, $\beta = 1$ [4]:

$$G(1, 1/2, 2; q) = \sum_{j=0}^{m} q^{jm} \begin{bmatrix} m \\ j \end{bmatrix}, \tag{8}$$

$$G(3/2, 1, 2; q) = \sum_{j=0}^{m} q^{j^2} \begin{bmatrix} m \\ j \end{bmatrix}. \tag{9}$$

Mourad Ismail and Dennis Stanton [5] have proven that the conjecture holds if

$$\alpha + \beta = K \qquad \text{and} \qquad \alpha - \beta = m - n + 1.$$

Experimentally, it appears that the bounds on $n - m$ are sharp. For example, $A_n(q) = G(5/3, 4/3, 3; q)$ with $m = n$. The conjecture states that $G(5/3, 4/3, 3; q)$ has non-negative coefficients when $|n - m| \leq 1$:

m	n	$G(5/3,4/3,3;q)$
2	1	$1+q+q^2$
	2	$1+q+2q^2+q^3+q^4$
	3	$1+q+2q^2+2q^3+2q^4+q^6$
	4	$1+q+2q^2+2q^3+3q^4+q^5+q^6-q^9-q^{10}$
3	1	$1+q+q^2+q^3-q^4$
	2	$1+q+2q^2+2q^3+q^4+q^5+q^6$
	3	$1+q+2q^2+3q^3+2q^4+2q^5+3q^6+2q^7+q^8+q^9$
	4	$1+q+2q^2+3q^3+3q^4+3q^5+4q^6+3q^7+3q^8+2q^9+q^{10}+q^{12}$
	5	$1+q+2q^2+3q^3+3q^4+4q^5+5q^6+4q^7+4q^8+3q^9+2q^{10}-q^{13}$ $-q^{14}-q^{15}-q^{16}-q^{17}$
4	2	$1+q+2q^2+2q^3+2q^4+q^5+q^6-q^9$
	3	$1+q+2q^2+3q^3+3q^4+2q^5+4q^6+3q^7+3q^8+2q^9+q^{10}+q^{11}$ $+q^{12}$
	4	$1+q+2q^2+3q^3+4q^4+3q^5+5q^6+5q^7+6q^8+5q^9+5q^{10}+3q^{11}$ $+4q^{12}+3q^{13}+2q^{14}+q^{15}+q^{16}$
	5	$1+q+2q^2+3q^3+4q^4+4q^5+6q^6+6q^7+8q^8+7q^9+8q^{10}+6q^{11}$ $+6q^{12}+5q^{13}+5q^{14}+3q^{15}+3q^{16}+q^{17}+q^{18}+q^{20}$
	6	$1+q+2q^2+3q^3+4q^4+4q^5+7q^6+7q^7+9q^8+9q^9+10q^{10}+8q^{11}$ $+9q^{12}+6q^{13}+6q^{14}+4q^{15}+3q^{16}-q^{19}-2q^{20}-2q^{21}-2q^{22}-2q^{23}$ $-q^{24}-q^{25}-q^{26}$

I wish to acknowledge Dennis Stanton's contribution to this paper in the form of many fruitful discussions.

2 Partitions with prescribed hook differences

Given a partition λ whose ith largest part is λ_i, we define λ'_i to be the ith largest part in the conjugate partition (λ'_i is the number of parts that are greater than or equal to i). We say that λ fits inside an $m \times n$ rectangle if $m \geq \lambda'_1$ and $n \geq \lambda_1$. If $(i,j) \in \lambda$ (equivalently, if $\lambda_i \geq j$), then we define the **hook difference** at position (i,j) to be $\lambda_i - \lambda'_j$. The **diagonal** δ is the set of all positions $(i,j) \in \lambda$ for which $i - j = \delta$. The following proposition is a special case of theorem 1 in [2].

Proposition 1 *If $-K+\beta \leq n-m \leq K-\alpha$ where α, β, and K are positive integers, $\alpha+\beta < 2K$, then $G(\alpha,\beta,K;q)$ as defined in equation (7) is the generating function for partitions inside an $m \times n$ rectangle for which the hook differences on diagonal $\alpha - 1$ are less than or equal to $K-\alpha-1$ and the hook differences on diagonal $1-\beta$ are greater than or equal to $\beta+1-K$.*

The proof of this proposition given in [2] relies on recurrences and does not lend itself to non-integer values of α or β. However, as we shall demonstrate, there is a combinatorial proof of this proposition that uses the approach of [3]. It is this proof that appears to be amenable to generalization.

Proof: We shall use the Frobenius representation of a partition,

$$\lambda = \begin{pmatrix} a_1, & a_2, & \dots, & a_t \\ b_1, & b_2, & \dots, & b_t \end{pmatrix},$$

where $a_i = \lambda_i - i$, $b_i = \lambda'_i - i$, and t is the largest integer for which $\lambda_t \geq t$. We note that $a_1 > a_2 > \cdots > a_t \geq 0$, $b_1 > b_2 > \cdots > b_t \geq 0$, and the number being partitioned is $t + \sum(a_i + b_i)$. We want to show that $G(\alpha, \beta, K; q)$ is the generating function for partitions whose Frobenius representation satisfies

$$\begin{aligned} a_1 &< n, & b_1 &< m \\ a_i - b_{i-\alpha+1} &\leq K - 2\alpha, & b_i - a_{i-\beta+1} &\leq K - 2\beta, \quad \text{for all } i. \end{aligned} \tag{10}$$

We shall say that a partition has an (α, β, K) **positive oscillation of length** j, $j \geq 1$, if we can find a sequence $i_1 < i_2 < \cdots < i_j$ for which

$$\begin{aligned} a_{i_1} - b_{i_1-\alpha+1} &> K - 2\alpha, \\ b_{i_2} - a_{i_2-\beta+1} &> K - 2\beta, \\ &\vdots \\ &\begin{cases} a_{i_j} - b_{i_j-\alpha+1} > K - 2\alpha, & j \text{ odd}, \\ b_{i_j} - a_{i_j-\beta+1} > K - 2\beta, & j \text{ even}. \end{cases} \end{aligned} \tag{11}$$

A partition has an (α, β, K) **negative oscillation of length** j, $j \geq 1$, if we can find a sequence $i_1 < i_2 < \cdots < i_j$ for which

$$\begin{aligned} b_{i_1} - a_{i_1-\beta+1} &> K - 2\beta, \\ a_{i_2} - b_{i_2-\alpha+1} &> K - 2\alpha, \\ &\vdots \\ &\begin{cases} a_{i_j} - b_{i_j-\alpha+1} > K - 2\alpha, & j \text{ even}, \\ b_{i_j} - a_{i_j-\beta+1} > K - 2\beta, & j \text{ odd}. \end{cases} \end{aligned} \tag{12}$$

Lemma 1 *If α, β, and K are positive integers with $\alpha+\beta < 2K$ and if $-K+\beta \leq n - m \leq K - \alpha$, then the generating function for partitions inside an $m \times n$ rectangle with an (α, β, K) positive oscillation of length j is*

$$f^{+}_{\alpha,\beta,K}(j; q) = q^{j[K(\alpha+\beta)j + K(\alpha-\beta)]/2} \begin{bmatrix} m+n \\ m+Kj \end{bmatrix}. \tag{13}$$

By conjugating the partition (interchanging the as and bs in the Frobenius representation), this lemma implies that the generating function for partitions inside an $m \times n$ rectangle with an (α, β, K) negative oscillation of length j is

$$f^{-}_{\alpha,\beta,K}(j;q) = q^{j[K(\alpha+\beta)j - K(\alpha-\beta)]/2} \begin{bmatrix} m+n \\ m-Kj \end{bmatrix}. \tag{14}$$

Lemma 1 thus implies that

$$G(\alpha,\beta,K;q) = \begin{bmatrix} m+n \\ m \end{bmatrix} + \sum_{j=1}^{\infty} (-1)^j \left[f^{+}_{\alpha,\beta,K}(j;q) + f^{-}_{\alpha,\beta,K}(j;q) \right]. \tag{15}$$

Proposition 1 is an immediate consequence of equation (15): $\left[\begin{smallmatrix} m+n \\ m \end{smallmatrix}\right]$ is the generating function for all partitions that sit inside an $m \times n$ box, and if such a partition has a positive or negative oscillation and if j is the length of the longest such oscillation, then the alternating sum will count it with a total weight of

$$-2 \left\lfloor \frac{j}{2} \right\rfloor + 2 \left\lfloor \frac{j-1}{2} \right\rfloor + (-1)^j = -1.$$

Proof of lemma 1: Let λ be a partition into at most $m + Kj$ parts, each part less than or equal to $n - Kj$. If i is greater than the number of parts in λ, then we define $\lambda_i = 0$. We let t be the largest integer (≥ 0) such that

$$\lambda_{2\lceil j/2 \rceil \alpha + 2 \lfloor j/2 \rfloor \beta + t} \geq t, \tag{16}$$

and then define sequences $a_1, \ldots, a_\mu$ and $b_1, \ldots, b_\mu$, $\mu = \lceil j/2 \rceil \alpha + \lfloor j/2 \rfloor \beta + t$, as follows. If j is even, then

$$a_i = \begin{cases} \lambda_i + jK - i, & 1 \leq i \leq \alpha, \\ \lambda_{i+\alpha+\beta} + (j-2)K + \alpha + \beta - i, & \alpha + 1 \leq i \leq 2\alpha + \beta, \\ \lambda_{i+2(\alpha+\beta)} + (j-4)K + 2(\alpha+\beta) - i, & 2\alpha + \beta + 1 \leq i \leq 3\alpha + 2\beta, \\ \vdots & \\ \lambda_{i+j(\alpha+\beta)/2} + j(\alpha+\beta)/2 - i, & \\ & j(\alpha+\beta)/2 - \beta + 1 \leq i \leq j(\alpha+\beta)/2 + t, \end{cases} \tag{17}$$

$$b_i = \begin{cases} \lambda_{\alpha+i} + (j-1)K + \alpha - i, & 1 \leq i \leq \alpha + \beta, \\ \lambda_{2\alpha+\beta+i} + (j-3)K + 2\alpha + \beta - i, & \alpha + \beta + 1 \leq i \leq 2(\alpha+\beta), \\ \lambda_{3\alpha+2\beta+i} + (j-5)K + 3\alpha + 2\beta - i, & 2(\alpha+\beta) + 1 \leq i \leq 3(\alpha+\beta), \\ \vdots & \\ \lambda_{j(\alpha+\beta)/2-\beta+i} + K + j(\alpha+\beta)/2 - \beta - i, & \\ & (\frac{j}{2} - 1)(\alpha+\beta) + 1 \leq i \leq j(\alpha+\beta)/2, \\ \lambda'_{i-j(\alpha+\beta)/2} - j(\alpha+\beta)/2 - i, & \\ & j(\alpha+\beta)/2 + 1 \leq i \leq j(\alpha+\beta)/2 + t. \end{cases} \tag{18}$$

We note that

$$\begin{aligned}\frac{j}{2}(\alpha+\beta)+t+\sum_i(a_i+b_i) &= \sum_{i=1}^{j(\alpha+\beta)+t}\lambda_i \;+\; \sum_{i=1}^{t}\lambda'_i \;-\; t[j(\alpha+\beta)+t] \\ &\quad +\frac{j}{2}[K(\alpha+\beta)j+K(\alpha-\beta)] \\ &= \sum_{i\geq 1}\lambda_i \;+\; \frac{j}{2}[K(\alpha+\beta)j+K(\alpha-\beta)].\end{aligned} \tag{19}$$

If j is odd then,

$$a_i = \begin{cases} \lambda_i + jK - i, & 1 \leq i \leq \alpha, \\ \lambda_{i+\alpha+\beta} + (j-2)K + \alpha + \beta - i, & \alpha+1 \leq i \leq 2\alpha+\beta, \\ \quad\vdots & \\ \lambda_{i+(j-1)(\alpha+\beta)/2} + K + (j-1)(\alpha+\beta)/2 - i, & \\ \qquad (j-1)(\alpha+\beta)/2-\beta+1 \leq i \leq \alpha+(j-1)(\alpha+\beta)/2, & \\ \lambda'_{i-\alpha-(j-1)(\alpha+\beta)/2} - \alpha - (j-1)(\alpha+\beta)/2 - i, & \\ \qquad \alpha+(j-1)(\alpha+\beta)/2+1 \leq i \leq \alpha+(j-1)(\alpha+\beta)/2+t, & \end{cases} \tag{20}$$

$$b_i = \begin{cases} \lambda_{\alpha+i} + (j-1)K + \alpha - i, & 1 \leq i \leq \alpha+\beta, \\ \lambda_{2\alpha+\beta+i} + (j-3)K + 2\alpha + \beta - i, & \alpha+\beta+1 \leq i \leq 2(\alpha+\beta), \\ \quad\vdots & \\ \lambda_{\alpha+(j-1)(\alpha+\beta)/2+i} + \alpha + (j-1)(\alpha+\beta)/2 - i, & \\ \qquad (j-1)(\alpha+\beta)/2+1 \leq i \leq (j-1)(\alpha+\beta)/2+\alpha+t. & \end{cases} \tag{21}$$

Here we have that

$$\begin{aligned}\alpha+\frac{j-1}{2}(\alpha+\beta)+t+\sum_i(a_i+b_i) &= \sum_{i=1}^{(j+1)\alpha+(j-1)\beta+t}\lambda_i \;+\; \sum_{i=1}^{t}\lambda'_i \\ &\quad -\; t[(j+1)\alpha+(j-1)\beta+t] \\ &\quad +\; \frac{j}{2}[K(\alpha+\beta)j+K(\alpha-\beta)] \\ &= \sum_{i\geq 1}\lambda_i \;+\; \frac{j}{2}[K(\alpha+\beta)j+K(\alpha-\beta)].\end{aligned} \tag{22}$$

The α_i and β_i are non-negative integers because of the choice of t. Since $\lambda_1 \leq n - jK$ and $n - m \leq K - \alpha$, we have that

$$a_1 \;=\; \lambda_1 + jK - 1 \;<\; n, \tag{23}$$

$$b_1 \quad = \quad \lambda_{\alpha+1} + (j-1)K + \alpha - 1 < m. \tag{24}$$

Furthermore, if we take $i_1 = \alpha$, $i_2 = \alpha + \beta$, $i_3 = 2\alpha + \beta$, $i_4 = 2\alpha + 2\beta$, $\ldots$, $i_j = \lceil j/2 \rceil \alpha + \lfloor j/2 \rfloor \beta$, then these sequences satisfy the inequalities in (11) that characterize a partition with an (α, β, K) positive oscillation of length j. The only reason why these sequences might not represent a partition with an (α, β, K) positive oscillation of length j is that we might have

$$b_{j(\alpha+\beta)/2} \;\leq\; b_{j(\alpha+\beta)/2+1}$$

when j is even or

$$a_{\alpha+(j-1)(\alpha+\beta)/2} \;\leq\; a_{\alpha+(j-1)(\alpha+\beta)/2+1}$$

when j is odd.

2.1 Combinatorial proof of equivalence: first direction

We now perform a shifting operation that is done, successively, for each integer value of r from j down through 1. Initially, we take i_{j+1} to be ∞, $i_r = \lceil r/2 \rceil \alpha + \lfloor r/2 \rfloor \beta$ for $j \geq r \geq 1$. Our objective is to define a bijection between the pairs of sequences given above and the pairs of sequences that give the Frobenius representation for partitions inside an $m \times n$ rectangle with an (α, β, K) positive oscillation of length j.

If r is even:

$$\tau \quad = \quad b_{i_r} - a_{i_r - \beta + 1}, \tag{25}$$
$$\kappa \quad = \quad \max\{\nu \mid i_r < \nu \leq i_{r+1} - \alpha,\; b_\nu - a_{\nu-\beta+1} > \tau\}, \tag{26}$$
$$\gamma(\nu) \quad = \quad \max\{b_i - a_{i-\beta+1} - \tau \mid \nu \leq i \leq \kappa\}. \tag{27}$$

If the set that defines κ is empty, then we do no shifting for this value of r. Otherwise, for $i_r < \nu \leq \kappa$, we set

$$b_\nu \quad \longleftarrow \quad b_\nu - \gamma(\nu), \tag{28}$$
$$a_{\nu-\beta} \quad \longleftarrow \quad a_{\nu-\beta} + \gamma(\nu), \tag{29}$$

and then reset the value of i_r to κ.

If r is odd:

$$\tau \quad = \quad a_{i_r} - b_{i_r - \alpha + 1}, \tag{30}$$
$$\kappa \quad = \quad \max\{\nu \mid i_r < \nu \leq i_{r+1} - \beta,\; a_\nu - b_{\nu-\alpha+1} > \tau\}, \tag{31}$$
$$\gamma(\nu) \quad = \quad \max\{a_i - b_{i-\alpha+1} - \tau \mid \nu \leq i \leq \kappa\}. \tag{32}$$

If the set that defines κ is empty, then we do no shifting for this value of r. Otherwise, for $i_r < \nu \leq \kappa$, we set

$$a_\nu \longleftarrow a_\nu - \gamma(\nu), \tag{33}$$
$$b_{\nu-\alpha} \longleftarrow b_{\nu-\alpha} + \gamma(\nu), \tag{34}$$

and then reset the value of i_r to κ.

We claim that this shifting procedure yields a pair of sequences that give the Frobenius representation for a partition inside an $m \times n$ rectangle with an (α, β, K) positive oscillation of length j.

If j is even, then after doing the shift for $r = j$, the new value of $b_{j(\alpha+\beta)/2}$ is strictly larger than the new value of $b_{j(\alpha+\beta)/2+1}$. To see this, we observe that the value of $b_{j(\alpha+\beta)/2}$ does not change, and if the old value of $b_{j(\alpha+\beta)/2+1}$ is greater than or equal to the old value of $b_{j(\alpha+\beta)/2}$, then

$$\gamma(j(\alpha+\beta)/2+1) \geq b_{j(\alpha+\beta)/2+1} - a_{j(\alpha+\beta)/2-\beta+2} - \left(b_{j(\alpha+\beta)/2} - a_{j(\alpha+\beta)/2-\beta+1}\right), \tag{35}$$

so that the new value of $b_{j(\alpha+\beta)/2+1}$ equals

$$\begin{aligned} b_{j(\alpha+\beta)/2+1} - \gamma(j(\alpha+\beta)/2+1) &\leq b_{j(\alpha+\beta)/2} + a_{j(\alpha+\beta)/2-\beta+2} - a_{j(\alpha+\beta)/2-\beta+1} \\ &< b_{j(\alpha+\beta)/2}. \end{aligned} \tag{36}$$

The function γ is weakly decreasing, and so the new values of $a_{\nu-\beta}$ are still strictly decreasing. We do need to verify that

$$b_\nu - \gamma(\nu) > b_{\nu+1} - \gamma(\nu+1).$$

This will be true if $\gamma(\nu) = \gamma(\nu+1)$. If these values of γ are not equal, then by definition:

$$\gamma(\nu) = b_\nu - a_{\nu-\beta+1} - \tau, \qquad \gamma(\nu+1) \geq b_{\nu+1} - a_{\nu-\beta+2} - \tau. \tag{37}$$

It follows that

$$b_\nu - \gamma(\nu) = a_{\nu-\beta+1} + \tau > a_{\nu-\beta+2} + \tau \geq b_{\nu+1} - \gamma(\nu+1). \tag{38}$$

We note that after the shift for $r = j$, the value of $a_{j(\alpha+\beta)/2-\beta}$ might be less than or equal to the new value of $a_{j(\alpha+\beta)/2-\beta+1}$ which will be bounded by

$$a_{j(\alpha+\beta)/2-\beta+1} < \lambda'_1 - j\alpha - (j-2)\beta - K. \tag{39}$$

After the next shift, for $r = j-1$, the new value of $a_{j(\alpha+\beta)/2-\beta}$ will be strictly greater than the new value of $a_{j(\alpha+\beta)/2-\beta+1}$.

The same argument holds *mutatis mutandis* if j is odd and for each successive value of r. If r is even, then the new value of $a_{\lceil r/2\rceil\alpha+\lfloor r/2\rfloor\beta-\beta+1}$ is bounded by

$$a_{\lceil r/2\rceil\alpha+\lfloor r/2\rfloor\beta-\beta+1} < \lambda'_1 - r\alpha - (r-2)\beta - (j-r+1)K. \tag{40}$$

If r is odd, then the new value of $b_{\lceil r/2\rceil\alpha+\lfloor r/2\rfloor\beta-\alpha+1}$ is bounded by

$$b_{\lceil r/2\rceil\alpha+\lfloor r/2\rfloor\beta-\alpha+1} < \lambda_1' - (r-1)\alpha - (r-1)\beta - (j-r+1)K. \tag{41}$$

We observe that the value of a_1 is left unchanged and so is strictly less than n, and that the final value of b_1 (after the shift that corresponds to $r=1$) is strictly less than $\lambda_1' - jK \le m$.

2.2 Combinatorial proof of equivalence: other direction

To see that we do, in fact, have a bijection we note that we can uniquely reconstruct the values of τ, κ, and $\gamma(\nu)$ for each shift as r runs from 1 back up to j. We first choose the sequence $i_1 < i_2 < \cdots < i_j$ maximally. That is to say, we find the largest integer i_j for which $b_{i_j} - a_{i_j-\beta+1} > K - 2\beta$ (if j is even) or $a_{i_j} - b_{i_j-\alpha+1} > K - 2\alpha$ (if j is odd), and then after each i_r is chosen, we choose the largest possible value for i_{r-1}. To reverse the shifting process, we perform the following operation for each r from 1 through j.

If r is even:

$$\tau^* = \max\{b_\nu - a_{\nu-\beta+1} \mid r(\alpha+\beta)/2 \le \nu \le i_r\}, \tag{42}$$
$$\kappa^* = \min\{\nu \mid r(\alpha+\beta)/2 \le \nu \le i_r,\ b_\nu - a_{\nu-\beta+1} = \tau^*\}, \tag{43}$$
$$\gamma^*(\nu) = \min\{\tau^* - (b_i - a_{i-\beta+1}) \mid r(\alpha+\beta)/2 \le i < \nu\}. \tag{44}$$

For $r(\alpha+\beta)/2 < \nu \le \kappa^*$, we set

$$b_\nu \longleftarrow b_\nu + \gamma^*(\nu), \tag{45}$$
$$a_{\nu-\beta} \longleftarrow a_{\nu-\beta} - \gamma^*(\nu). \tag{46}$$

If r is odd:

$$\tau^* = \max\{a_\nu - b_{\nu-\alpha+1} \mid \lceil r/2\rceil\alpha + \lfloor r/2\rfloor\beta \le \nu \le i_r\}, \tag{47}$$
$$\kappa^* = \min\{\nu \mid \lceil r/2\rceil\alpha + \lfloor r/2\rfloor\beta \le \nu \le i_r,\ a_\nu - b_{\nu-\alpha+1} = \tau^*\}, \tag{48}$$
$$\gamma^*(\nu) = \min\{\tau^* - (a_i - b_{i-\alpha+1}) \mid \lceil r/2\rceil\alpha + \lfloor r/2\rfloor\beta \le i < \nu\}. \tag{49}$$

For $\lceil r/2\rceil\alpha + \lfloor r/2\rfloor\beta < \nu \le \kappa^*$, we set

$$a_\nu \longleftarrow a_\nu + \gamma^*(\nu), \tag{50}$$
$$b_{\nu-\alpha} \longleftarrow b_{\nu-\alpha} - \gamma^*(\nu). \tag{51}$$

It is left to the reader to verify that this does uniquely reverse the shifting done in section 2.1. To prove that we have a bijection between pairs of sequences generated by $f^+_{\alpha,\beta,K}(j;q)$ and partitions inside an $m\times n$ rectangle with an (α,β,K) positive oscillation of lenth j, we need to verify that if we start with an arbitrary partition, we get a pair of sequences generated by $f^+_{\alpha,\beta,K}(j;q)$. The only condition on these sequences that is not straightforward to verify is that they are strictly decreasing with the possible exception that if j is even then we might have $b_{j(\alpha+\beta)/2+1} \geq b_{j(\alpha+\beta)/2}$ and if j is odd then we might have $a_{\lceil j/2\rceil\alpha+\lfloor j/2\rfloor\beta+1} \geq a_{\lceil j/2\rceil\alpha+\lfloor j/2\rfloor\beta}$.

We observe that in applying the shift given above for $r = 1$, the new value for a_α might be less than or equal to the new value for $a_{\alpha+1}$. If $j = 1$, there is no problem. If j is larger than 1, then on the $r = 2$ shift we replace $a_{\alpha+1}$ with

$$a_{\alpha+1} \longleftarrow a_{\alpha+1} - (\tau^* - b_{\alpha+\beta} + a_{\alpha+1}) \;=\; b_{\alpha+\beta} - \tau^*.$$

We note that if the new value of $a_{\alpha+1}$ is greater than or equal to the new value of a_α, then the value of b_1 after the $r = 1$ shift is strictly less than its original value. This implies that after the $r = 1$ shift we have $a_\alpha - b_1 > K - 2\alpha$. We combine this observation with the following inequalities:

$$\begin{aligned} b_1 - b_{\alpha+\beta} &\geq \alpha + \beta - 1, &(52)\\ \tau^* &\geq K - 2\beta, &(53)\\ 2K &> \alpha + \beta, &(54) \end{aligned}$$

to see that

$$b_{\alpha+\beta} - \tau^* \;<\; a_\alpha. \tag{55}$$

The new value of $a_{\alpha+1}$ after the $r = 2$ shift is strictly less than a_α. This argument continues to hold for each $r < j$ so that after all of the shifting we have at most one pair of consecutive elements in the sequences for which we do not have strict decrease.

Q.E.D.

3 Prescribed Hook Differences with non-integer parameters

We want to define a prescribed hook difference condition when α and β are not integers. While there does not seem to be hope for doing this in general, the particular case

$$\alpha = \nu + \frac{K+1}{2} - \frac{a}{K}, \qquad \beta = -\nu + \frac{K-1}{2} + \frac{a}{K} \tag{56}$$

does hold promise. In particular, let

$$\alpha = \overline{\alpha} - a/K, \qquad \beta = \overline{\beta} + a/K, \tag{57}$$

where $\overline{\alpha}$ and $\overline{\beta}$ are positive integers and a is a positive integer less than or equal to $\min\{\overline{\alpha},\overline{\beta}\}$. Let $\{\overline{a}_i\}$ and $\{\overline{b}_i\}$ be the pair of sequences generated by

$$f^{+}_{\overline{\alpha},\overline{\beta},K}(j;q) = q^{j[K(\overline{\alpha}+\overline{\beta})j+K(\overline{\alpha}-\overline{\beta})]/2} \begin{bmatrix} m+n \\ m+Kj \end{bmatrix}$$

as given in equations (17–18) and (20–21). We now define

$$a_i = \overline{a}_i - \begin{cases} 1, & \text{if } \overline{\alpha}-a+1 \le (i \mod \overline{\alpha}+\overline{\beta}) \le \overline{\alpha} \\ & \quad \text{and } i \le \lceil j/2 \rceil(\overline{\alpha}+\overline{\beta})-\overline{\beta}, \\ 0, & \text{otherwise,} \end{cases} \tag{58}$$

$$b_i = \overline{b}_i - \begin{cases} 1, & \text{if } \overline{\alpha}+\overline{\beta}-a+1 \le (i \mod \overline{\alpha}+\overline{\beta}) \le \overline{\alpha}+\overline{\beta}, \\ & \quad \text{and } i \le \lfloor j/2 \rfloor(\overline{\alpha}+\overline{\beta}), \\ 0, & \text{otherwise,} \end{cases} \tag{59}$$

where $(i \mod \overline{\alpha}+\overline{\beta})$ is the least positive residue of i modulo $(\overline{\alpha}+\overline{\beta}=\alpha+\beta)$. We have subtracted a total of aj from the pair of sequences. We are left with a pair of sequences generated by

$$f^{+}_{\alpha,\beta,K}(j;q) = q^{j[K(\alpha+\beta)j+K(\alpha-\beta)]/2} \begin{bmatrix} m+n \\ m+Kj \end{bmatrix}.$$

To get a pair of sequences generated by $f^{-}_{\alpha,\beta,K}(j;q)$, we find the sequences generated by $f^{+}_{\beta,\alpha,K}(j;q)$ and then interchange the as and bs. This means that we start with the pair of sequences generated by $f^{+}_{\overline{\beta},\overline{\alpha},K}(j;q)$ and then add aj to them by defining

$$a_i = \overline{a}_i + \begin{cases} 1, & \text{if } \overline{\beta}+1 \le (i \mod \overline{\beta}+\overline{\alpha}) \le \overline{\beta}+a \\ & \quad \text{and } i \le \lfloor j/2 \rfloor(\overline{\beta}+\overline{\alpha}), \\ 0, & \text{otherwise,} \end{cases} \tag{60}$$

$$b_i = \overline{b}_i + \begin{cases} 1, & \text{if } 1 \le (i \mod \overline{\beta}+\overline{\alpha}) \le a, \\ & \quad \text{and } i \le \lceil j/2 \rceil\overline{\beta}+\lfloor j/2 \rfloor\overline{\alpha}, \\ 0, & \text{otherwise.} \end{cases} \tag{61}$$

It is not clear to what partitions these pairs of sequences correspond. The sequences generated by $f^{+}_{\alpha,\beta,K}(j;q)$ satisfy a weakened form of the oscillating condition. If we set $i_r = \lceil r/2 \rceil\overline{\alpha}+\lfloor r/2 \rfloor\overline{\beta}$, then

$$a_{i_r} - b_{i_r-\overline{\alpha}+1} \ge K-2\overline{\alpha}, \qquad j \text{ odd}, \tag{62}$$

$$b_{i_r} - a_{i_r-\overline{\beta}+1} \ge K-2\overline{\beta}, \qquad j \text{ even}. \tag{63}$$

We also introduce additional inequalities:

$$a_{i_r-a} > a_{i_r-a+1}, \qquad j \text{ odd}, \tag{64}$$

$$b_{i_r-a} > b_{i_r-a+1}, \qquad j \text{ even}. \tag{65}$$

3.1 $\mathcal{A}_{m,n}(q)$: a related partition generating function

If we restrict our attention to the polynomial $A_n(q) = G(5/3, 4/3, 3; q)$ given in conjecture 1—see equations (2) and (7)—then we have

$$K = 3, \quad a = 1, \quad ,\overline{\alpha} = 2, \quad \overline{\beta} = 1.$$

A family of partitions whose generating function appears to be very closely related to $A_n(q)$ consists of those that fit inside an $m \times n$ rectangle and satisfy the following conditions for all i such that $\lambda_i \geq i$:

$$\text{either} \quad \lambda_i - \lambda_i' \ \geq \ 0 \quad \text{or} \quad \lambda_i \ = \ \lambda_{i+1}, \tag{66}$$

$$\text{either} \quad \lambda_{i+1} - \lambda_i' \ \leq \ -1 \quad \text{or} \quad \lambda_i \ = \ \lambda_{i+1}. \tag{67}$$

These two conditions can be combined into

$$\text{either} \quad \lambda_i \ = \ \lambda_{i+1} \quad \text{or} \quad \lambda_i \ \geq \ \lambda_i' \ > \ \lambda_{i+1}. \tag{68}$$

If we designate the generating function for these partitions by $\mathcal{A}_{m,n}(q)$, then we have a simple recursion from which they can be computed:

$$\mathcal{A}_{m,n} \ = \ 0, \quad \text{if } m < 0 \quad \text{or} \quad n < 0, \tag{69}$$

$$\mathcal{A}_{m,n} \ = \ 1, \quad \text{if } mn = 0, \quad m \geq 0, \ n \geq 0 \tag{70}$$

$$\mathcal{A}_{1,n} \ = \ \begin{bmatrix} n+1 \\ 1 \end{bmatrix}, \quad \text{if } n \geq 0, \tag{71}$$

$$\begin{aligned} \mathcal{A}_{m,n} \ = \ & \mathcal{A}_{m-1,n} + \mathcal{A}_{m,n-1} - \mathcal{A}_{m-1,n-1} \\ & + q^{m+n-1} \left[\mathcal{A}_{m-1,n-1} - \mathcal{A}_{m-1,n-2} + \chi(m \leq n) \mathcal{A}_{m-1,m-2} \right], \\ & \qquad \text{if } m \geq 2, \quad n \geq 1. \end{aligned} \tag{72}$$

We compare $\mathcal{A}_{n,n}$ with A_n:

$$A_n(q) - \mathcal{A}_{n,n}(q) \ = \ 0, \quad n \leq 4, \tag{73}$$

$$A_5(q) - \mathcal{A}_{5,5}(q) \ = \ q^{11} - q^{12} - q^{13} + q^{14}, \tag{74}$$

$$A_6(q) - \mathcal{A}_{6,6}(q) \ = \ q^{11} - q^{13} - q^{15} + q^{17} + q^{19} - q^{21} - q^{23} + q^{25}, \tag{75}$$

$$\begin{aligned} A_7(q) - \mathcal{A}_{7,7}(q) \ = \ & q^{11} - q^{15} - q^{16} + q^{19} + q^{20} + 2q^{22} - q^{23} - 2q^{24} \\ & - 2q^{25} - q^{26} + 2q^{27} + q^{29} + q^{30} - q^{33} - q^{34} + q^{38}. \end{aligned} \tag{76}$$

It is easily verified by induction that

$$\mathcal{A}_{m,n}(1) \ = \ \begin{cases} 2 \times 3^{n-1}, & m = n, \\ 3^{\min(m,n)}, & |m-n| = 1. \end{cases} \tag{77}$$

so that in fact

$$A_n(1) \ = \ \mathcal{A}_{n,n}(1). \tag{78}$$

As n increases, the coefficients of $A_n(q) - \mathcal{A}_{n,n}(q)$ do increase, but remain substantially less than the coefficients of either $A_n(q)$ or $\mathcal{A}_{n,n}(q)$. Plots of these coefficients for $n = 8, 10, 12, 15$, and 18 are included in the file **borwein.ps**. While we do get coefficients on the order of 4000 when $n = 18$, this is less than 1/500 of the corresponding coefficient in either $A_n(q)$ or $\mathcal{A}_{n,n}(q)$.

One interesting pattern does emerge. For $5 \leq n \leq 17$,

$$P_n(q) \;=\; \frac{A_n(q) - \mathcal{A}_{n,n}(q)}{q^{11}(1-q)(1-q^2)}$$

is a symmetric, unimodal, monic polynomial of degree $n^2 - 25$ with strictly positive coefficients. The fact that it is symmetric and monic of degree $n^2 - 25$ follows from the fact that both $\mathcal{A}_{n,n}(q)$ and $A_n(q)$ are symmetric polynomials. There is no apparent reason why it should be unimodal with strictly positive coefficients.

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