# Overlapping Pfaffians

Donald E. Knuth, Computer Science Department, Stanford University

To Dominique Cyprien Foata on his 60th birthday

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**Abstract.** A combinatorial construction proves an identity for the product of the Pfaffian of a skew-symmetric matrix by the Pfaffian of one of its submatrices. Several applications of this identity are followed by a brief history of Pfaffians.

**0. Definitions.** Let $X$ be a possibly infinite index set. We consider quantities $f[xy]$ defined on ordered pairs of elements of $X$, satisfying the law of skew symmetry

$$f[xy] = -f[yx]\,, \qquad \text{for} \quad x, y \in X\,. \tag{0.0}$$

This notation is extended to $f[\alpha]$ for arbitrary words $\alpha = x_1 \ldots x_{2n}$ of even length over $X$ by defining the *Pfaffian*

$$f[x_1 \ldots x_{2n}] = \sum s(x_1 \ldots x_{2n}, y_1 \ldots y_{2n})\, f[y_1 y_2] \ldots f[y_{2n-1} y_{2n}]\,, \tag{0.1}$$

where the sum is over all $(2n-1)(2n-3)\ldots(1)$ ways to write $\{x_1, \ldots x_{2n}\}$ as a union of pairs $\{y_1, y_2\} \cup \cdots \cup \{y_{2n-1}, y_{2n}\}$, and where $s(x_1 \ldots x_{2n}, y_1 \ldots y_{2n})$ is the sign of the permutation that takes $x_1 \ldots x_{2n}$ into $y_1 \ldots y_{2n}$.

The Pfaffian is well defined, even though there are $2^n n!$ different permutations $y_1 \ldots y_{2n}$ that yield the same partition $\{y_1, y_2\} \cup \ldots \cup \{y_{2n-1}, y_{2n}\}$ into pairs. For if we interchange $y_{2j-1}$ with $y_{2j}$, we change the sign of both $s(x_1 \ldots x_{2n}, y_1 \ldots y_{2n})$ and $f[y_1, y_2] \ldots f[y_{2n-1} y_n]$, by (0.0); if we interchange $y_{2i-1}$ with $y_{2j-1}$ and $y_{2i}$ with $y_{2j}$, both factors stay the same. Thus, for example,

$$\begin{aligned} f[wxyz] &= f[wx]f[yz] - f[wy]f[xz] + f[wz]f[xy] \\ &= f[wx]f[yz] + f[wy]f[zx] + f[wz]f[xy]\,. \end{aligned} \tag{0.2}$$

A partition into pairs is commonly called a *perfect matching*. Therefore it is convenient to abbreviate (0.1) in the form

$$f[\alpha] = \sum_{\mu \in M(\alpha)} s(\alpha, \mu)\, \Pi f[\mu] \tag{0.3}$$

where $M(\alpha)$ is the set of perfect matchings of $\alpha$ represented as words $y_1 \ldots y_{2n}$ in some canonical way, and $\Pi f[y_1 \ldots y_{2n}] = f[y_1 y_2] \ldots f[y_{2n-1} y_{2n}]$.

Notice that we have

$$f[wxyz] = -f[xyzw]\,. \tag{0.4}$$

In general, an odd permutation of $\alpha$ will reverse the sign of $f[\alpha]$, because every term in (0.3) changes sign.

Pfaffians can also be defined recursively, starting with the null word $\epsilon$ and proceeding to words of greater length:

$$
\begin{aligned}
f[\epsilon] &= 1\,; \\
f[x_1 \ldots x_{2n}] &= \sum_{j=2}^{2n} f[x_1 x_j] f[x_{j+1} \ldots x_{2n} x_2 \ldots x_{j-1}]\,, \qquad n > 0\,.
\end{aligned}
\tag{0.5}
$$

This recurrence [9] corresponds to a procedure that constructs all perfect matchings by starting with $\{x_1, x_2\} \cup \cdots \cup \{x_{2n-1}, x_{2n}\}$ and making cyclic permutations of the indices in positions $\{2, \ldots, 2n\}$, $\{4, \ldots, 2n\}$, $\ldots$; each of these permutations is even.

It will be convenient in the sequel to extend the sign function $s$ to $s(\alpha, \beta)$ for arbitrary words $\alpha, \beta \in X^*$. We define $s(\alpha, \beta) = 0$ if either $\alpha$ or $\beta$ has a repeated letter, or if $\beta$ contains a letter not in $\alpha$. Otherwise $s(\alpha, \beta)$ is the sign of the permutation that takes $\alpha$ into the word

$$\beta\,(\alpha \backslash \beta)\,,$$

where $\alpha \backslash \beta$ is the word that remains when the elements of $\beta$ are removed from $\alpha$. Thus, for example,

$$
s(\alpha\beta\gamma, \beta) = \begin{cases} 0\,, & \text{if } \alpha\beta\gamma \text{ contains a repeated letter;} \\ (-1)^{|\alpha|\,|\beta|}\,, & \text{otherwise.} \end{cases}
\tag{0.6}
$$

We also have

$$s(\alpha, \beta\gamma) = s(\alpha, \beta) s(\alpha \backslash \beta, \gamma)\,, \tag{0.7}$$

since both sides vanish unless the letters of $\beta\gamma$ are distinct and contained in the distinct letters of $\alpha$, and in the latter case $s(\alpha, \beta\gamma)$ is the parity of the number of transpositions needed to bring $\beta$ to the left of $\alpha$ and $\gamma$ to the left of the remaining word $\alpha \backslash \beta$.

If $\alpha$ has repeated letters, the Pfaffian $f[\alpha]$ is zero, because $f[\alpha] = -f[\alpha]$ when we transpose two identical letters. Therefore our convention that $s(\alpha, \beta) = 0$ when $\alpha$ or $\beta$ has repeated letters does not invalidate definition (0.1), which used a different convention for $s(x_1 \ldots x_{2n}, y_1 \ldots y_{2n})$. One consequence of the new convention is the identity

$$f[\alpha] = \sum_{x_1 < \cdots < x_n} \sum_{y_1 > x_1} \cdots \sum_{y_n > x_n} s(\alpha, x_1 y_1 \ldots x_n y_n)\, f[x_1 y_1] \ldots f[x_n y_n] \tag{0.8}$$

for any word $\alpha$ of length $2n$, assuming that $X$ is an ordered set; the sum is over all conceivable perfect matchings $\mu = x_1 y_1 \ldots x_n y_n$, but $s(\alpha, \mu)$ is zero unless $\mu$ is a perfect matching of $\alpha$.

**1. The basic identity.** The following identity due to H. W. L. Tanner [24] can now be proved:

$$f[\alpha]\, f[\alpha\beta] = \sum_y s(\beta, xy)\, f[\alpha x y]\, f[\alpha\beta \backslash xy]\,, \qquad \text{for all } x \in \beta. \tag{1.0}$$

This formula is vacuous when $|\beta| = 0$ and trivial when $|\beta| = 2$, but when $|\beta| = 4$ it says in particular that

$$
\begin{aligned}
f[\alpha]\, f[\alpha w x y z] &= f[\alpha w x]\, f[\alpha y z] - f[\alpha w y]\, f[\alpha x z] + f[\alpha w z]\, f[\alpha x y] \\
&= f[\alpha w x]\, f[\alpha y z] + f[\alpha w y]\, f[\alpha z x] + f[\alpha w z]\, f[\alpha x y]\,.
\end{aligned}
\tag{1.1}
$$

We will demonstrate (1.0) by giving a combinatorial interpretation to each term on the left and right sides of the equation, when the Pfaffians are expanded as sums over perfect matchings.

A typical term on the right of (1.0) is

$$s(\beta, xy)\, s(\alpha xy, \mu)\, s(\alpha\beta\backslash xy, \nu)\, \Pi f[\mu]\, \Pi f[\nu]\,, \tag{1.2}$$

where $x$ and $y$ are distinct elements of $\beta$, $\mu$ is a perfect matching of $\alpha xy$, and $\nu$ is a perfect matching of $\alpha\beta\backslash xy$. Ignoring the sign for the moment, we can construct a graph by superimposing the matchings $\mu$ and $\nu$. In this graph all vertices of $\alpha$ have degree 2 because they are matched in both $\mu$ and $\nu$; all vertices of $\beta$ have degree 1.

There is a unique maximal path that starts at $y$ and uses edges from $\mu$ and $\nu$ alternately. This path ends at some element of $\beta$, call it $z$. Let $\mu_1$ and $\nu_1$ be the edges of $\mu$ and $\nu$ on this path; let $\mu_0$ and $\nu_0$ be the other edges. Then we define corresponding matchings

$$\mu' = \mu_0 \cup \nu_1\,, \qquad \nu' = \nu_0 \cup \mu_1\,, \tag{1.3}$$

which will be the key to establishing (1.0).

*Case 1*, $z \neq x$. In this case $|\mu_1| = |\nu_1|$, since the path from $y$ starts with an element of $\mu$ and ends with an element of $\nu$. Thus the matchings $\mu'$ and $\nu'$ correspond to another term on the right side of (1.0); we will prove that this other term cancels with (1.2). Since $\mu'' = \mu$ and $\nu'' = \nu$, this will set up an involution between cancelling terms.

We have

$$\Pi f[\mu]\, \Pi f[\nu] = \Pi f[\mu_0]\, \Pi f[\mu_1]\, \Pi f[\nu_0]\, \Pi f[\nu_1] = \Pi f[\mu']\, \Pi f[\nu']\,, \tag{1.4}$$

so (1.2) will cancel with its counterpart if the signs differ. The sign of (1.2) is

$$s(\alpha xyz, \mu_0\mu_1 z)\, s(\alpha\beta, xy\nu_0\nu_1)\,, \tag{1.5}$$

because $s(\beta, xy) = s(\alpha\beta, xy)$ and $s(\alpha\beta, xy)\, s(\alpha\beta\backslash xy, \nu) = s(\alpha\beta, xy\nu)$ by (0.7). The sign of the permutation that takes $\mu_1 z$ into $\nu_1 y$ is the same as the sign of the permutation that takes $y\nu_0\nu_1$ into $z\nu_0\mu_1$, hence (1.5) equals

$$s(\alpha xyz, \mu_0\nu_1 y)\, s(\alpha\beta, xz\nu_0\mu_1)\,.$$

But this is the negative of $s(\alpha xzy, \mu_0\nu_1 y)\, s(\alpha\beta, xz\nu_0\mu_1)$, the sign of the term that corresponds to $\mu'$ and $\nu'$.

*Case 2*, $z = x$. In this case we have $|\mu_1| = |\nu_1| + 2$, since $\mu_1$ includes both $x$ and $y$ while $\nu_1$ is contained in $\alpha$. It follows that $\mu'$ and $\nu'$ are perfect matchings of $\alpha$ and $\alpha\beta$, respectively, so they define a typical term

$$s(\alpha, \mu')\, s(\alpha\beta, \nu')\, \Pi f[\mu']\, \Pi f[\nu'] \tag{1.6}$$

from the left side of (1.0). Conversely, every such term corresponds to matchings $\mu$ and $\nu$ for a uniquely defined term (1.2) on the right. The sign of this term,

$$s(\alpha xy, \mu_0\mu_1)\, s(\alpha\beta, xy\nu_0\nu_1)\,,$$

agrees with $s(\alpha,\mu')\,s(\alpha\beta,\nu') = s(\alpha xy,\mu_0\nu_1 xy)\,s(\alpha\beta,\nu_0\mu_1)$, because the permutation that takes $\mu_1$ into $\nu_1 xy$ has the same sign as the permutation that takes $xy\nu_0\nu_1$ into $\nu_0\mu_1$.

**2. Basic applications.** The special case $\alpha=\epsilon$ of (1.0) reads

$$f[\beta] = \sum_y s(\beta,xy)\,f[xy]\,f[\beta\backslash xy]\,,\qquad \text{for all } x\in\beta. \tag{2.0}$$

This is a mild generalization of the recurrence (0.5); it tells us how to expand $f[\beta]$ with respect to any element of $\beta$. We can get rid of the constraint $x\in\beta$ by summing over all $x$:

$$f[\beta] = \frac{1}{|\beta|}\sum_x\sum_y s(\beta,xy)\,f[xy]\,f[\beta\backslash xy]\,. \tag{2.1}$$

Applying this rule to $f[\beta\backslash xy]$ and repeating until words of length 2 are reached yields a $|\beta|$-fold sum,

$$f[\beta] = \frac{1}{(2n)(2n-2)\ldots 2}\sum_{x_1}\cdots\sum_{x_{2n}} s(\beta,x_1\ldots x_{2n})\,f[x_1x_2]\ldots f[x_{2n-1}x_{2n}]\,, \tag{2.2}$$

when $|\beta|=2n$; this is, of course, the same as (0.8) when we collect equal terms.

Now let $\alpha$ be a fixed word such that $f[\alpha]\neq 0$, and consider the function

$$g(\beta) = f[\alpha\beta]/f[\alpha] \tag{2.3}$$

on the words of $X$. Tanner's identity (1.0) tells us that

$$g(\beta) = \sum_y s(\beta,xy)\,g(xy)\,g(\beta\backslash xy)\,,\qquad \text{for all } x\in\beta. \tag{2.4}$$

But this is the same relation as (2.0); so $g$ satisfies the Pfaffian recurrence (0.5). Therefore any identity for Pfaffians leads *a fortiori* to an identity for $g$. In particular, (0.3) tells us that

$$g(\beta) = \sum_{\mu\in M(\beta)} s(\beta,\mu)\,\Pi g(\mu)\,,$$

which is equivalent to

$$f[\alpha]^{n-1}\,f[\alpha\beta] = \sum_{M(\beta)} s(\beta,x_1y_1\ldots x_ny_n)\,f[\alpha x_1y_1]\ldots f[\alpha x_ny_n] \tag{2.5}$$

when $|\beta|=2n$, where the sum is over all perfect matchings $x_1y_1\ldots x_ny_n$ of $\beta$. The special case $n=2$ appears in (1.1).

We can also construct a dual formula by starting with a fixed $\alpha\beta$ such that $f[\alpha\beta]\neq 0$ and defining

$$h(\gamma) = s(\alpha\beta,\gamma)\,f[\alpha\beta\backslash\gamma]/f[\alpha\beta] \tag{2.6}$$

on the words $\gamma$ contained in $\alpha\beta$. Then (1.0) yields

$$h(\beta) = \sum_y s(\beta,xy)\,h(\beta\backslash xy)\,h(xy)\,,\qquad \text{for all } x\in\beta; \tag{2.7}$$

so we can derive a companion to (2.5) in a similar fashion:

$$f[\alpha]\,f[\alpha\beta]^{n-1}=\sum_{M(\beta)} s(\beta,x_1y_1\ldots x_ny_n)\,f[\alpha\beta\backslash x_1y_1]\ldots f[\alpha\beta\backslash x_ny_n]\,. \tag{2.8}$$

Identities (2.4) and (2.7) are the Pfaffian analogs of theorems about determinants that Muir called the Law of Extensible Minors and the Law of Complementaries. (See [15], §179 and §98 in the original edition; §187 and §179 in Metzler's revision.)

**3. Applications to determinants.** Determinants are the special case of Pfaffians in which the index set is bipartite with respect to $f$, in the sense that $f[xy]=0$ when $x$ and $y$ belong to the same part. It is convenient to imagine that the set of indices consists of two disjoint parts $X$ and $\bar X$, so that $x$ belongs to $X$ if and only if $\bar x$ belongs to $\bar X$, and $f[xy]=f[\bar x\,\bar y]=0$ for all $x,y\in X$. The independent quantities are now $f[x\bar y]=-f[\bar y x]$; we can regard $X$ as a set of "rows" and $\bar X$ as a set of "columns," so that $f[x\bar y]$ is essentially an element of the matrix $f$. We use $f[x,y]$ as an alternative notation for $f[x\bar y]$. In fact, when $\alpha$ and $\beta$ are arbitrary words of $X$ we write

$$f[\alpha,\beta]=f[\alpha\bar\beta^R] \tag{3.0}$$

for the determinant formed from rows $\alpha$ and columns $\beta$. Here $\bar\beta^R$ stands for the reverse complement of $\beta$:

$$\overline{y_1y_2\ldots y_n}^R=\bar y_n\ldots\bar y_2\bar y_1\,. \tag{3.1}$$

Definition (3.0) agrees with the usual definition of determinants, when $|\alpha|=|\beta|=n$, since the perfect matchings of $\alpha\bar\beta^R$ that do not have vanishing products correspond to the products

$$f[x_1\bar y_1]\ldots f[x_n\bar y_n]=f[x_1,y_1]\ldots f[x_n,y_n]\,, \tag{3.2}$$

where $\alpha=x_1\ldots x_n$ and $y_1\ldots y_n$ is a permutation of $\beta$; the corresponding sign $s(\alpha\bar\beta^R,x_1\bar y_1\ldots x_n\bar y_n)$ is just $s(\beta,y_1\ldots y_n)$, because the permutation that takes $x_1\ldots x_n\bar y_n\ldots\bar y_1$ to $x_1\bar y_1\ldots x_n\bar y_n$ is even. For example, we have

$$\begin{aligned}f[wx,yz]&=f[wx\bar z\bar y]\\&=f[wx]\,f[\bar z\bar y]-f[w\bar z]\,f[x\bar y]+f[w\bar y]\,f[x\bar z]\\&=0-f[w,z]\,f[x,y]+f[w,y]\,f[x,z]\,,\end{aligned}$$

the usual $2\times 2$ determinant

$$\begin{vmatrix}f[w,y]&f[w,z]\\f[x,y]&f[x,z]\end{vmatrix}\,.$$

Theorem (1.0) immediately yields a corresponding identity for determinants, when we apply these definitions:

$$f[\alpha,\beta]\,f[\alpha\gamma,\beta\delta]=\sum_y s(\gamma,x)\,s(\delta,y)\,f[\alpha x,\beta y]\,f[\alpha\gamma\backslash x,\beta\delta\backslash y]\,, \tag{3.3}$$

for all $x \in \gamma$. When $|\gamma| = |\delta|$ is 2 or 3, this identity reads

$$f[\alpha,\beta]\, f[\alpha wx,\beta yz] = f[\alpha w,\beta y]\, f[\alpha x,\beta z] - f[\alpha w,\beta z]\, f[\alpha x,\beta y]\,; \tag{3.4}$$

$$\begin{aligned} f[\alpha,\beta]\, f[\alpha uvw,\beta xyz] &= f[\alpha u,\beta x]\, f[\alpha vw,\beta yz] \\ &\quad - f[\alpha u,\beta y]\, f[\alpha vw,\beta xz] \\ &\quad + f[\alpha u,\beta z]\, f[\alpha vw,\beta xy]\,. \end{aligned} \tag{3.5}$$

Here are some small examples written in more conventional notation:

$$a_{11}\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} - \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}\,; \tag{3.6}$$

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \begin{vmatrix} a_{11} & a_{12} & a_{14} \\ a_{21} & a_{22} & a_{24} \\ a_{41} & a_{42} & a_{44} \end{vmatrix} - \begin{vmatrix} a_{11} & a_{12} & a_{14} \\ a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & a_{34} \end{vmatrix} \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{41} & a_{42} & a_{43} \end{vmatrix}\,; \tag{3.7}$$

$$\begin{aligned} a_{11}\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} &= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \begin{vmatrix} a_{11} & a_{13} & a_{14} \\ a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \end{vmatrix} - \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \begin{vmatrix} a_{11} & a_{12} & a_{14} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{vmatrix} \\ &\quad + \begin{vmatrix} a_{11} & a_{14} \\ a_{21} & a_{24} \end{vmatrix} \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{vmatrix}\,. \end{aligned} \tag{3.8}$$

Of course determinants have been investigated rather thoroughly for nearly 250 years, so it would be surprising indeed if these identities were new. Equation (3.6) was, for instance, noted by Lagrange in 1773 [16, page 39]; (3.7) and higher examples of (3.4) were discussed by Desnanot in 1819 [16, page 142].

One particularly interesting case in which (3.4) played a crucial role is C. L. Dodgson's elegant "condensation method" for determinant evaluation [7], discovered between the times when he wrote *Alice in Wonderland* and *Through the Looking Glass*: Suppose the index set $X$ is the integers, and let $f_0[x,y] = 1$ for all $x$ and $y$, while $f_1[x,y]$ is the entry in row $x$ and column $y$ of a given matrix. Then for $k \ge 1$ let

$$f_{k+1}[x,y] = \begin{vmatrix} f_k[x,y] & f_k[x,y+1] \\ f_k[x+1,y] & f_k[x+1,y+1] \end{vmatrix} \Big/ f_{k-1}[x+1,y+1]\,. \tag{3.9}$$

It follows that

$$f_k[x,y] = f_1[x(x+1)\ldots(x+k-1),\, y(y+1)\ldots(y+k-1)] \qquad \text{for } k \ge 0\,, \tag{3.10}$$

by induction on $k$ using (3.4). To evaluate the $n \times n$ determinant $f[1\,2\,\ldots\,n,\, 1\,2\,\ldots\,n]$, we may therefore simply compute $f_k[x,y]$ for $1 \le x,y \le n+1-k$ and $k = 2,\ldots,n$, hoping that it will not be necessary to divide by zero. Dodgson's condensation method provided the original motivation for Robbins and Rumsey's recent work on alternating sign matrices [19].

The earliest known identity involving products of determinants is

$$f[ab,12]\,f[ab,34]-f[ab,13]\,f[ab,24]+f[ab,14]\,f[ab,23]=0\,, \tag{3.11}$$

which Alexis Fontaine des Bertins proudly wrote out 126 times for different choices of the indices and then said "et cetera." He submitted this and other memoirs to the French academy in 1748, but the works remained unpublished until 1764 [16, pp. 10–11]. From (1.0) we can now recognize that the right-hand side of (3.8) is actually a Pfaffian product

$$f[ab]\,f[a\,b\,\bar{1}\,\bar{2}\,\bar{3}\,\bar{4}\,]\,,$$

which is indeed zero in the bipartite case. Bezout, in 1779, gave the similar formula

$$\begin{aligned} f[abc,123]\,f[abc,456]&-f[abc,124]\,f[abc,356]\\ &+f[abc,125]\,f[abc,346]-f[abc,126]\,f[abc,345]=0\,, \end{aligned} \tag{3.12}$$

and said "on voit qu'il y a une infinité d'autres combinaisons à faire" [16, page 51]; the right-hand side in this case is

$$f[a\,b\,c\,\bar{1}\,\bar{2}\,]\,f[a\,b\,c\,\bar{1}\,\bar{2}\,\bar{3}\,\bar{4}\,\bar{5}\,\bar{6}\,]$$

when we replace determinants by Pfaffians.

Another instance of (1.0) yields

$$\begin{aligned} f[ab]\,f[a\,b\,c\,\bar{1}\,\bar{2}\,\bar{3}\,\bar{4}\,\bar{5}\,]&=f[a\,b\,\bar{1}\,\bar{2}\,]\,f[a\,b\,c\,\bar{3}\,\bar{4}\,\bar{5}\,]-f[a\,b\,\bar{1}\,\bar{3}\,]\,f[a\,b\,c\,\bar{2}\,\bar{4}\,\bar{5}\,]\\ &\quad+f[a\,b\,\bar{1}\,\bar{4}\,]\,f[a\,b\,c\,\bar{2}\,\bar{3}\,\bar{5}\,]-f[a\,b\,\bar{1}\,\bar{5}\,]\,f[a\,b\,c\,\bar{2}\,\bar{3}\,\bar{4}\,]\\ &\quad-f[a\,b\,\bar{1}\,c]\,f[a\,b\,\bar{2}\,\bar{3}\,\bar{4}\,\bar{5}\,]\,. \end{aligned} \tag{3.13}$$

Under bipartite restrictions this becomes an identity in determinants,

$$f[ab,12]\,f[abc,345]-f[ab,13]\,f[abc,245]+f[ab,14]\,f[abc,235]-f[ab,15]\,f[abc,234]=0\,, \tag{3.14}$$

which Desnanot [6] seems to have known only in the special case

$$f[ab,12]\,f[abc,134]-f[ab,13]\,f[abc,124]+f[ab,14]\,f[abc,123]=0 \tag{3.15}$$

where column 1 = column 5, although he knew the general result (3.3) [16, page 145].

Thus we see that the single Pfaffian identity (1.0) unifies a variety of different-appearing determinant identities that arise when the indices are given bipartite structure in different ways.

When identity (2.8) is specialized to determinants, it gives a formula for minors of the adjugate of a matrix (i.e., determinants of cofactors):

$$\begin{aligned} &f[\alpha,\beta]\,f[\alpha x_1\ldots x_n,\beta u_1\ldots y_n]^{n-1}\\ &\quad=\begin{vmatrix} f[\alpha x_2\ldots x_n,\beta y_2\ldots y_n] & \ldots & f[\alpha x_2\ldots x_n,\beta y_1\ldots y_{n-1}]\\ \vdots & & \vdots\\ f[\alpha x_1\ldots x_{n-1},\beta y_2\ldots y_n] & \ldots & f[\alpha x_1\ldots x_{n-1},\beta y_1\ldots y_{n-1}] \end{vmatrix}\,. \end{aligned} \tag{3.16}$$

This general formula was first published by Jacobi in 1834, although special cases had been found by Lagrange in 1773 and Minding in 1829 [16, pp. 39, 197, 208–209]. The formula that corresponds to (2.5),

$$f[\alpha,\beta]^{n-1}\, f[\alpha x_1,\ldots x_n,\beta y_1\ldots y_n] = \begin{vmatrix} f[\alpha x_1,\beta y_1] & \ldots & f[\alpha x_1,\beta y_n] \\ \vdots & & \vdots \\ f[\alpha x_n,\beta y_1] & \ldots & f[\alpha x_n,\beta y_n] \end{vmatrix}, \tag{3.17}$$

is simpler but was not discovered until Sylvester introduced a new viewpoint in 1851 [17, pp. 60–61].

**4. Applications to closed forms.** Let $g$ be the skew-symmetric Blaschke operator

$$g[xy] = \frac{x-y}{1-xy}\,. \tag{4.0}$$

Laksov, Lascoux, and Thorup [10, (A.12.3)] and John R. Stembridge [23, Proposition 2.3(e)] independently discovered the remarkable identity

$$g[x_1x_2\ldots x_n] = \prod_{1\le i<j\le n} \frac{x_i-x_j}{1-x_ix_j}\,, \qquad n \text{ even}, \tag{4.1}$$

for which they gave an ingenious but quite special proofs.

We can, however, prove (4.1) as a special case of more general theorem that follows from a special case of (1.0):

**Theorem.** *The identity*

$$f[x_1\ldots x_n] = \prod_{1\le i<j\le n} f[x_ix_j] \tag{4.2}$$

*holds for all even $n$ if and only if it holds for $n=4$.*

**Proof.** If $n>4$ and the identity holds for smaller even values of $n$, let $\alpha$ be any word of length $n-4$. Then

$$\begin{aligned} f[\alpha]\, f[\alpha wxyz] &= f[\alpha wx]\, f[\alpha yz] - f[\alpha wy]\, f[\alpha xz] + f[\alpha wz]\, f[\alpha xy] \\ &= R(f[wx]\, f[yz] - f[wy]\, f[xz] + f[wz]\, f[xy]) \\ &= R\, f[wxyz] \\ &= R\, f[wx]\, f[wy]\, f[wz]\, f[xy]\, f[xz]\, f[yz]\,, \end{aligned}$$

where if $\alpha = x_1\ldots x_{n-4}$ the common factor $R$ is

$$\Bigl(\prod_{1\le i<j\le n-4} f[x_ix_j]^2\Bigr)\Bigl(\prod_{1\le i\le n-4} f[x_iw]\, f[x_ix]\, f[x_iy]\, f[x_iz]\Bigr).$$

Therefore

$$f[x_1\ldots x_{n-4}]\, f[x_1\ldots x_n] = f[x_1\ldots x_{n-4}] \prod_{1\le i<j\le n} f[x_ix_j]\,.$$

Equation (4.2) follows unless $f[x_1\ldots x_{n-4}]=0$.

If $f[y_1 \ldots y_{n-4}] = 0$ for all subwords $y_1 \ldots y_{n-4}$ of $x_1 \ldots x_n$, then $f[x_1 \ldots x_n] = 0$ and again (4.2) holds. Finally, if $y_1 \ldots y_{n-4}$ is a subword such that $f[y_1 \ldots y_{n-4}] \neq 0$, there is a permutation $y_1 \ldots y_n$ of $x_1 \ldots x_n$ for which our argument proves $f[y_1 \ldots y_n] = \prod_{1 \le i<j \le n} f[y_i y_j]$. This establishes (4.2), because permutations of the indices change the signs of both sides in the same manner. $\square$

The theorem is of interest because it applies not only to (4.0) but also to the simpler function

$$f[x_i x_j] = \frac{x_i - x_j}{c + x_i + x_j} \tag{4.3}$$

when $c$ is any complex constant. Thus we obtain a more-or-less "closed form" (4.2) for the Pfaffian of a new kind of matrix. (The special case $c = 0$ was previously noted by Schur [22, §36].)

In fact, the general function

$$f[x_i x_j] = \frac{x_i - x_j}{c + b(x_i + x_j) + a\, x_i x_j}\,, \qquad b^2 = ac \pm 1\,, \tag{4.4}$$

also satisfies the necessary conditions; this expression includes both (4.0) and (4.3).

Are there other skew-symmetric rational functions of two variables that satisfy

$$\begin{aligned} f[wx]\, f[yz] &+ f[wy]\, f[zx] + f[wz]\, f[xy] \\ &= f[wx]\, f[wy]\, f[wz]\, f[xy]\, f[xz]\, f[yz]\ ? \end{aligned} \tag{4.5}$$

One can, of course, replace $f[xy]$ by $f[r(x)r(y)]$ for any rational function $r$, so any solution of (4.5) implies an infinite class of equivalent solutions. Alain Lascoux [11] has recently found strong reasons for believing that there are no other solutions, up to changes of variables.

When $f[xy]$ is a polynomial, an amusing closed form of a similar type was noticed by G. Torelli [25]: Let $f_k[xy] = (x - y)^k$; then

$$f_{n-1}[x_1 \ldots x_n] = (-1)^{\binom{n/2}{2}} \left( \prod_{k=0}^{n/2-1} \binom{n-1}{k} \right) \prod_{1 \le i<j \le n} (x_i - x_j) \tag{4.6}$$

when $n$ is even. It is easy to prove this identity, as well as the fact that $f_{2m-1}[x_1 \ldots x_n] = 0$ for $2m < n$, by observing that the Pfaffian must vanish when $x_i = x_j$.

**5. Generalization of the basic identity.** Equation (1.0), which gives an expression for $f[\alpha]\, f[\alpha\beta]$ when $\alpha$ is a proper subword of $\alpha\beta$, leads to a similar identity that is useful when two words have an odd number of letters in common. Suppose $\alpha\beta\gamma$ has no repeated letters, and let $x \in \beta$. Then

$$\begin{aligned} f[\alpha\beta]\, f[\alpha\gamma] = \sum_y & s(\beta, xy)\, f[\alpha\beta \backslash xy]\, f[\alpha\gamma xy] \\ &+ \sum_y s(\beta, x)\, s(\gamma, y)\, f[\alpha y \beta \backslash x]\, f[\alpha x \gamma \backslash y]\,. \end{aligned} \tag{5.0}$$

For example, when $|\alpha|$ is odd we have

$$\begin{aligned} f[\alpha xyz]\, f[\alpha uvw] = {} & f[\alpha z]\, f[\alpha uvwxy] - f[\alpha y]\, f[\alpha uvwxz] \\ &+ f[\alpha uyz]\, f[\alpha xvw] - f[\alpha vyz]\, f[\alpha xuw] + f[\alpha wyz]\, f[\alpha xuv]\,. \end{aligned} \tag{5.1}$$

To prove (5.0), let $\gamma = x_1 \ldots x_k$. We will construct a "cancelling" word $\gamma' = x'_k \ldots x'_1$ on new indices, by defining

$$f[yx'_j] = 0 \quad \text{if} \quad y \neq x_j\,; \qquad f[x_j x'_j] = 1\,. \tag{5.2}$$

Then $f[\alpha\beta] = f[\alpha\gamma\gamma'\beta]$, and we can use (1.0) to conclude that

$$f[\alpha\beta]\, f[\alpha\gamma] = \sum_y s(\gamma'\beta, xy)\, f[\alpha\gamma\gamma'\beta\backslash xy]\, f[\alpha\gamma xy]\,. \tag{5.3}$$

Now if $y \in \beta$ we have $s(\gamma'\beta, xy) = s(\beta, xy)$, and $f[\alpha\gamma\gamma'\beta\backslash xy] = f[\alpha\beta\backslash xy]$. But if $y = x'_j$ we have $s(\gamma'\beta, xy) = (-1)^j s(\beta, x)$, $f[\alpha\gamma\gamma'\beta\backslash xy] = (-1)^{j-1} f[\alpha y\beta\backslash x]$, $f[\alpha\gamma xy] = (-1)^j f[\alpha x\gamma\backslash y]$, and $s(\gamma, y) = (-1)^{j-1}$.

**6. A brief history of Pfaffians.** Johann Friedrich Pfaff introduced the functions that now bear his name in 1815 [18] [16, pp. 396–401], while studying a general way to solve systems of first-order partial differential equations. He gave two procedures for listing all perfect matchings, and observed that when the matchings are ordered lexicographically the corresponding signs are strictly alternating $+, -, +, \ldots, +$.

Jacobi developed Pfaff's method further in 1827 [9], and discovered an analog of "Cramer's rule" for the solution of general systems of skew-symmetric linear equations

$$\sum_{j=1}^{2n} f[ij]\, z_j = f[i0]\,, \qquad n \text{ even}\,; \tag{6.0}$$

namely,

$$z_j = \frac{f[1 \ldots (j-1)\, 0\, (j+1) \ldots n]}{f[1 \ldots n]}\,. \tag{6.1}$$

This implicitly proves that the Pfaffian $f[1 \ldots n]$ is a factor of the general skew-symmetric determinant

$$\begin{vmatrix} f[11] & \ldots & f[1n] \\ \vdots & & \vdots \\ f[n1] & \ldots & f[nn] \end{vmatrix}\,, \qquad n \text{ even.} \tag{6.2}$$

Cayley proved in 1849 [3] that this determinant is in fact equal to the *square* of $f[1 \ldots n]$.

An elegant graph-theoretic proof of Cayley's theorem, somewhat analogous to the derivation of (1.0) above, was found by Veltmann in 1871 [26] and independently by Mertens in 1877 [14]. Their proof anticipated 20th-century studies on the superposition of two matchings, and the ideas have frequently been rediscovered. Cayley himself had claimed that such a proof would be possible, after doing the calculations for $n = 4$ on the final page of a paper he wrote in 1861 [5]. But we should note that his original method was simpler. In fact, Cayley originally [3] gave a short inductive proof of the more general formula

$$\begin{vmatrix} f[xy] & f[x2] & f[x3] & \ldots & f[xn] \\ f[2y] & f[22] & f[23] & \ldots & f[2n] \\ f[3y] & f[32] & f[33] & \ldots & f[3n] \\ \vdots & \vdots & \vdots & & \vdots \\ f[ny] & f[n2] & f[n3] & \ldots & f[nn] \end{vmatrix} = f[x23 \ldots n]\, f[y23 \ldots n]\,, \tag{6.3}$$

for arbitrary $x$ and $y$ when $n$ is even. And he proved several years later [17, pp. 269, 278] that the determinant on the left of (6.3) is $f[xy23\ldots n]\,f[23\ldots n]$ when $n$ is odd. (This determinant is incidentally *not* the same as $f[x2\ldots n, y2\ldots n]$; the elements of the latter are $f[x,y], f[x,2], \ldots = f[x\bar{y}], f[x\bar{2}], \ldots,$ not $f[xy], f[x2], \ldots,$ according to our conventions. Moreover, we generally use the notation $f[x,y]$ only when we assume that $f[xy] = 0$.)

It was Cayley who introduced the name *Pfaffian*, because of its "connexion with the researches of Pfaff on differential equations" [4]. Another name *semideterminant* (German *Halbdeterminant*) was proposed by Wilhelm Scheibner [21], but it did not gain many adherents.

Theorem (1.0) was discovered by Henry William Lloyd Tanner in 1878 [24], who gave inductive proofs for the cases $|\beta| = 4$ and $|\beta| = 6$ from which proof schemata for higher cases could be inferred. Władysław Zajaczkowski found another proof shortly afterward [28] [29] based on Jacobi's determinant theorem (3.16). The theorem was independently rediscovered in 1901 by J. Brill [1], who found a still better proof. He first established the identity

$$\binom{n-1}{k} f[x_1\ldots x_{2n}] = \sum_{1\le j_1<\cdots<j_{2k}\le 2n} s(x_1\ldots x_{2n}, x_1\ldots x_{2k}) f[x_1\ldots x_{2k}] f[x_1\ldots x_{2n}\backslash x_1\ldots x_{2k}] \tag{6.4}$$

by induction on $k$; then he made the left side zero by setting $x_{2n} = x_1$. A series of further steps led him to (1.0). But the combinatorial proof in section 1 above seems preferable to all three of these early approaches.

Identity (5.0) was recently discovered by Wenzel [27, Proposition 2.3], and demonstrated via exterior algebra by Dress and Wenzel [8].

The fact that Pfaffians are more fundamental than determinants, in the sense that determinants are merely the bipartite special case of a general sum over matchings, went unnoticed for a long time. The first person to observe that every $n \times n$ determinant is a Pfaffian was apparently Louis Saalschütz in 1908 [20], but the implicitly bipartite nature of his construction was not stated in his paper; a modern reader sees it only with hindsight. Brioschi had found a complicated way to express a $2n \times 2n$ determinant as a Pfaffian, in 1856 [2]: If $A$ is any $2n \times 2n$ matrix and if $Q = I_n \otimes \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, the determinant of $A$ is the Pfaffian of $A^T Q A$.

Pfaffians continue to find numerous applications, for example in matching theory [13] and in the enumeration of plane partitions [23]. It should prove interesting to extend Leclerc's combinatorics of relations for determinants [12] to the analogous rules for Pfaffians.

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## References

[1] J. Brill, "Note on the algebraic properties of Pfaffians," *Proceedings of the London Mathematical Society* **34** (1901), 143–151.

[2] F. Brioschi, "Sur l'analogie entre une classe de déterminants d'order pair; et sur les déterminants binaires," *Journal für die reine und angewandte Mathematik* **52** (1856), 133–141.

[3] A. Cayley, "Sur les déterminants gauches," *Journal für die reine und angewandte Mathematik* **38** (1849), 93–96. Reprinted in his *Collected Mathematical Papers* **1**, 410–413.

[4] A. Cayley, "On the theory of permutants," *Cambridge and Dublin Mathematical Journal* **7** (1852), 40–51. Reprinted in his *Collected Mathematical Papers* **2**, 16–26.

[5] A. Cayley, "Note on the theory of determinants," *Philosophical Magazine* **21** (1861), 180–185. Reprinted in his *Collected Mathematical Papers* **5**, 45–49.

[6] P. Desnanot, *Complément de la Théorie des Équations du Premier Degré* (Paris, 1819).

[7] C. L. Dodgson, "Condensation of determinants, being a new and brief method for computing their arithmetical values," *Proceedings of the Royal Society* **84** (1866), 150–155. Reprinted in *The Mathematical Pamphlets of Charles Lutwidge Dodgson and Related Pieces*, edited by Francine F. Abeles (Charlottesville, Virginia: The University Press of Virginia, 1994), 170–180.

[8] Andreas W. M. Dress and Walter Wenzel, "A simple proof of an identity concerning Pfaffians of skew symmetric matrices," *Advances in Mathematics* **112** (1995), 120–134.

[9] C. G. Jacobi, "Über die *Pfaff*sche Methode, eine gewöhnliche lineäre Differential-gleichung zwischen $2n$ Variabeln durch ein System von $n$ Gleichungen zu integriren," *Journal für die reine und angewandte Mathematik* **2** (1827), 347–357. Reprinted in *C. G. J. Jacobi's Gesammelte Werke* **4** (1886), 17–29.

[10] D. Laksov, A. Lascoux and A. Thorup, "On Giambelli's theorem on complete correlations," *Acta Mathematica* **162** (1989), 143–199.

[11] Alain Lascoux, personal communication, 10 April 1995.

[12] Bernard Leclerc, "On identities satisfied by minors of a matrix," *Advances in Mathematics* **100** (1993), 101–132.

[13] László Lovász and Michael D. Plummer, *Matching Theory* (Budapest: Akadémiai Kiadó, 1986); North-Holland Mathematics Studies **121**.

[14] F. Mertens, "Über die Determinanten, deren correspondirende Elemente $a_{pq}$ und $a_{qp}$ entgegengesetzt gleich sind," *Journal für die reine und angewandte Mathematik* **82** (1877), 207–211.

[15] Thomas Muir, *A Treatise on the Theory of Determinants* (London: Macmillan, 1882). Revised and enlarged by William H. Metzler (London: Longmans, Green, 1933; New York, Dover, 1960).

[16] Thomas Muir, *The Theory of Determinants in the Historical Order of Development* (London: MacMillan, 1906).

[17] Thomas Muir, *The Theory of Determinants in the Historical Order of Development*, volume 2 (London: MacMillan, 1911).

[18] J. F. Pfaff, "Methodus generalis, aequationes differentiarum partialium, nec non aequationes differentiales vulgares, utrasque primi ordinis, inter quotcunque variabiles, completi integrandi," *Abhandlungen der Königlich-Preußischen Akademie der Wissenschaften zu Berlin*, Mathematische Klasse (1814–1815), 76–136.

[19] David P. Robbins and Howard Rumsey, Jr., "Determinants and alternating sign matrices," *Advances in Mathematics* **62** (1986), 169–184.

[20] Louis Saalschütz, “Zur Determinanten-Lehre,” *Journal für die reine und angewandte Mathematik* **134** (1908), 187–197.

[21] W. Scheibner, “Über Halbdeterminanten,” *Berichte über die Verhandlungen der Königlich Sächsischen Gesellschaft der Wissenschaften zu Leipzig* **11** (1859), 151–159.

[22] J. Schur, “Über die Darstellung der symmetrischen und der alternierenden Gruppe durch gebrochene lineare Substitutionen,” *Journal für die reine und angewandte Mathematik* **139** (1911), 155–250. Reprinted in Issai Schur, *Gesammelte Abhandlungen* **1** (1973), 346–441.

[23] John R. Stembridge, “Nonintersecting paths, Pfaffians, and plane partitions,” *Advances in Mathematics* **83** (1990), 96–131.

[24] H. W. Lloyd Tanner, “A theorem relating to Pfaffians,” *Messenger of Mathematics* **8** (1878), 56–59.

[25] Gabriele Torelli, “Quistione 64,” *Giornale di Matematiche* **24** (1886), 377.

[26] W. Veltmann, “Beiträge zur Theorie der Determinanten,” *Zeitschrift für Mathematik und Physik* **16** (1871), 516–525.

[27] Walter Wenzel, “Pfaffian forms and $\Delta$-matroids,” *Discrete Mathematics* **115** (1993), 253–266.

[28] W. Zajaczkowski, “A theorem relating to Pfaffians,” *Messenger of Mathematics* **10** (1880), 36–37.

[29] W. Zajączkowski, “O pewnéj własności pfafianu,” *Rozprawy i Sprawozdania z Posiedzeń*, Wydziału Matematyczno-Przyrodniczego Akademii Umiejętności **7** (Krakow, 1880), 67–74.