Combinatorial Approaches and Conjectures for 2-Divisibility Problems Concerning Domino Tilings of Polyominoes

Lior Pachter

Department of Mathematics
MIT, Cambridge, MA 02139
lpachter@math.mit.edu

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Abstract

We give the first complete combinatorial proof of the fact that the number of domino tilings of the $2n \times 2n$ square grid is of the form $2^n(2k+1)^2$, thus settling a question raised in [4] . The proof lends itself naturally to some interesting generalizations, and leads to a number of new conjectures.

Mathematical Subject Classification. Primary 05C70.

1 Introduction

The number of domino tilings of the $n \times m$ square grid was first calculated in a seminal paper by Kasteleyn [6] . He showed that, for n, m even, the number of tilings $N(n,m)$ is given by

$$N(n,m) = \prod_{j=1}^{\frac{n}{2}} \prod_{k=1}^{\frac{m}{2}} (4\cos^2 \frac{\pi j}{n+1} + 4\cos^2 \frac{\pi k}{m+1}). \tag{1}$$

This result, while interesting in its own right, does not reveal all of the properties of $N(n,m)$ at first glance. For example, $N(2n,2n)$ is either a perfect square or twice a perfect square (this was first proved by Montroll [7] using linear algebra and later proved by Jokusch [5] and others). Another interesting observation is that

$$N(2n,2n) = 2^n(2k+1)^2. \tag{2}$$

A derivation of this fact from (**1**) has been obtained independently by a number of authors; we refer the reader to [4] . A combinatorial proof of (**2**) has proved more elusive, although partial results have been established [2] . As we shall show, a direct combinatorial proof of (**2**) illuminates the combinatorics behind $N(2n, 2n)$ and leads directly to generalizations.

Interestingly, perhaps because of the closed form of equation (**1**), observations other than the ones mentioned above have been scarce. Propp has remarked [9] that "Aztec diamonds and their kin have (so far) been much more fertile ground for exact combinatorics than the seemingly more natural rectangles".

We hope to show that there is a rich source of problems to be found in the enumeration of perfect matchings of rectangular grids. In fact, it seems that the tools needed to resolve many of the problems have yet to be discovered.

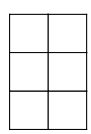
2 The square grid

Theorem 1 *Let $N(2n, 2n)$ be the number of domino tilings of the $2n \times 2n$ square grid.*

$$N(2n, 2n) = 2^n(2k + 1)^2. \tag{3}$$

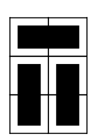
Our proof is broken down into two parts. The first part is not new, in fact it appears as a very special case in a theorem in [2] . Since we are interested in this special case only, we provide a simplified version of the proof in [2] that sacrifices much of the generality but illustrates the elegant combinatorial nature of the argument.

We begin by introducing the notation we will use. Rather than discussing perfect matchings of graphs, we will use the dual graph and think of edges in the perfect matching as dominoes covering two adjacent squares. We will, on occasion, use the two descriptions interchangeably. For an arbitrary region R, we will use the notation $\#\ R$ for the number of domino tilings of R. For example,

\#  = 3.

We will use the notation $\#_2 R$ for the parity of the number of domino tilings of R.

The *direction* of a domino from a fixed square is either up, down, left or right. We shall say that a domino is *oriented* in the *positive* (resp. *negative*) direction from a given square if its direction is up or to the right (resp. down or to the left). For example, in the tiling below, the top left square has a domino that is *positively oriented* and whose direction is *right.*



Lemma 1 *Label the diagonal squares on the $2n \times 2n$ square grid from the bottom left to the top right with the labels $a_1, b_1, a_2, b_2, \ldots, a_n, b_n$. The number of domino*

tilings of the square grid with dominoes placed at $a_1, a_2, \ldots, a_n$ *is dependent only on the orientation of the dominoes and not their direction.*

Figure 1 illustrates the labeling of the diagonal for the 8×8 square grid:

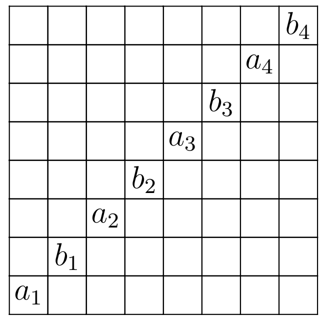


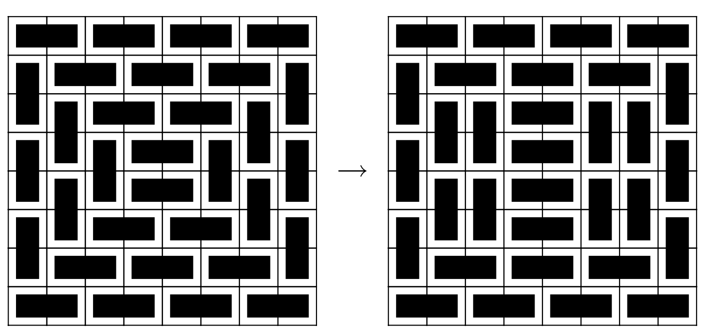
Figure 1

Proof of lemma: Let M be any domino tiling of the $2n \times 2n$ square grid. Let M' be the tiling obtained by reflecting M across the diagonal and define $D = M \cup M'$ (D is allowed to consist of multiple dominoes). Notice that in the dual graph of the $2n \times 2n$ square grid, D is a 2-factor and is therefore a disjoint union of even-length cycles. Furthermore, since D is symmetric across across the diagonal, any cycle maps to another cycle under the reflection.

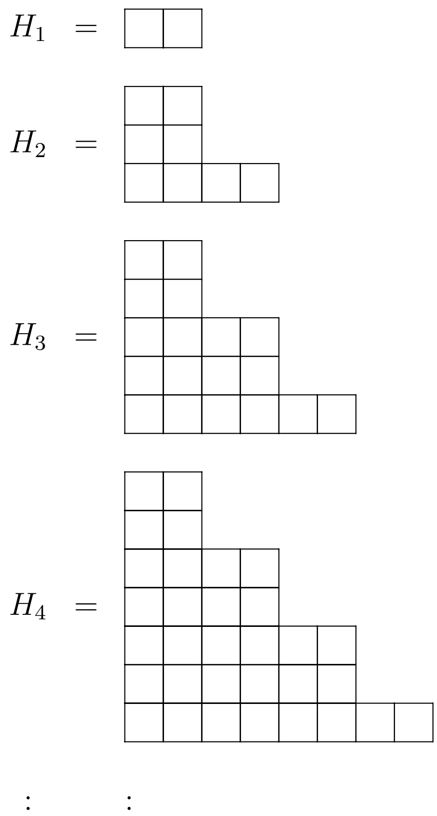
Now define C_i' to be the cycle containing a_i. C_i' can have at most one other vertex on the diagonal because every vertex in C_i' has degree 2. Furthermore, such a vertex must be of the type b_j, for otherwise the number of vertices enclosed by C is odd (contradicting the fact that D is a disjoint union of even length cycles). It follows that all the cycles C_i' are distinct.

Finally, let $C_i = C_i' \cap M$ be the alternating cycles (cycles in the dual graph alternating between edges in the tiling and edges not in the tiling) in M obtained from C_i'. By the above arguments, the alternating cycles C_i are disjoint. Thus, there is a bijection between any two sets of tilings with fixed dominoes of the same orientation on the a_i's. We simply select all the dominoes on the a_i's that have switched direction and rotate the appropriate alternating cycles.

Example 1 *Changing the direction of the domino at* a_2 *we have*



We now define a class of grids, H_n (first introduced by Ciucu [2]), as follows:



H_n is defined from H_{n-1} by adding a grid of size $2 \times (2n-1)$ to the left of H_{n-1}.

Lemma 2 *The number of domino tilings of the square grid is given by*

$$N(2n, 2n) = 2^n(\#H_n)^2. \tag{4}$$

Proof of lemma: Consider a fixed orientation for the dominoes covering the a_i's. We can assume (using Lemma 1) that the directions of the dominoes are all either down or to the right (call such a configuration *reduced*). Notice that the square grid decomposes naturally into two halves. Figure 2 illustrates an example of a reduced configuration.

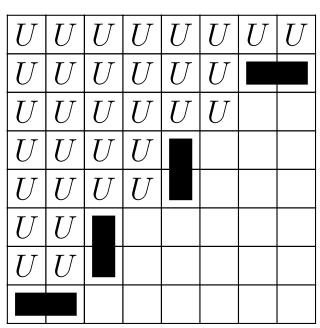


Figure 2

Notice that the region filled with U is equivalent to H_n, as is its complement. Now consider the standard checkerboard 2-coloring of the square grid. All the U's which are adjacent to empty squares have the same color. It follows that in any reduced configuration, every domino covers either two U's or none at all. We have from Lemma 1 that

$$N(2n, 2n) = 2^n \sum_C \#C \tag{5}$$

where C ranges over all reduced configurations. From the remarks above it follows that

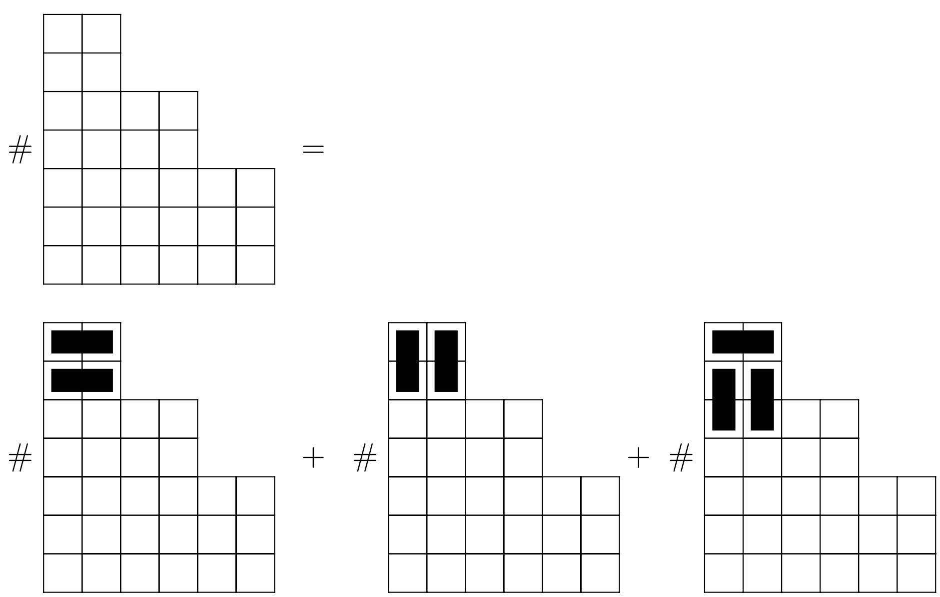
$$\sum_C \#C = (\#H_n)^2, \tag{6}$$

which completes the proof of the lemma.

Lemma 3 *$\#H_n$ is odd.*

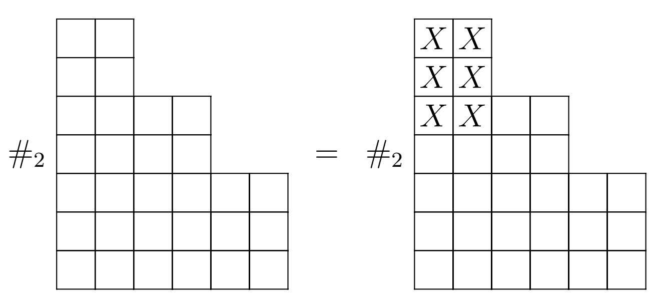
Proof of lemma: Our proof is by induction. The case when $n = 1, 2$ is trivial. We illustrate the general case by showing the step $n = 3 \Rightarrow n = 4$.

Begin by observing that



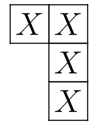
(7)

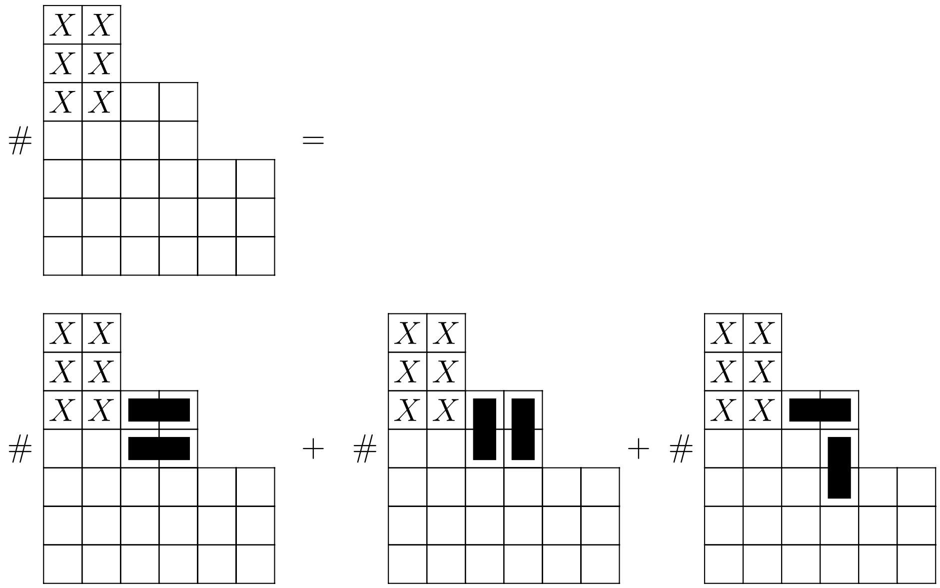
The first two terms in (7) are equal, so we have



(8)

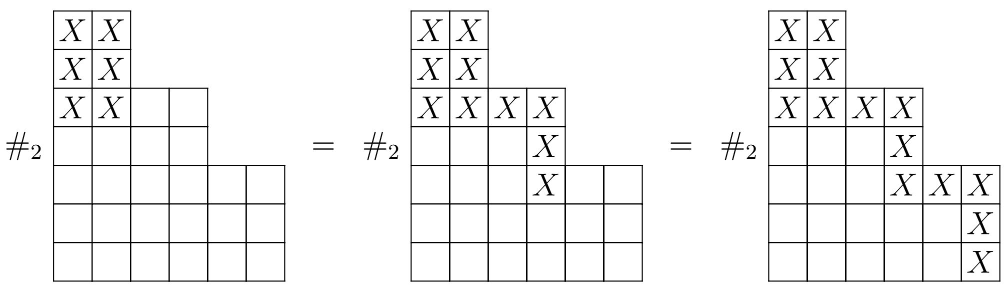
where the X's denote squares that cannot be used.

We now begin removing shapes of the form  from the diagonal, using a similar idea:



(9)

Hence, we can conclude that



(10)

Our last shape is H_{n-1} (minus the forced domino on the bottom right), flipped and rotated by 90°! It follows that

$$\#_2 H_n = \#_2 H_{n-1}. \tag{11}$$

Proof of theorem: The theorem follows immediately by applying Lemmas 2 and 3.

3 Rectangular Grids

The exact formula for the largest power of 2 appearing in $N(2n, 2n)$ suggests an investigation of the same question for $n \times m$ rectangular grids.

We use the notation (a, b) to denote the greatest common divisor of a and b.

Problem 1 *Let $N(n,m)$ be the number of domino tilings of the $n \times m$ rectangular grid. Prove combinatorially that*

$$N(2n,2m) = 2^{\frac{(2n+1,2m+1)-1}{2}}(2r_1+1) \tag{12}$$

$$N(2n+1,2m) = 2^{\frac{(n+1,2m+1)-1}{2}(3+j)}(2r_2+1) \tag{13}$$

where j is defined by $n+1 = 2^j(2t+1)$. (In the above r_1, r_2, t are natural numbers that may vary for different n, m.)

Equation (**12**) [4] . (This has been observed by Saldanha [10]). Indeed, the other case should follow by similar methods. A combinatorial proof is not known for either case. Combinatorial proofs are important in this context because other methods fail for regions that are more complicated. Section 4 contains numerous examples where an analogous formula to (**1**) is lacking, and therefore there is no closed form formula from which to work.

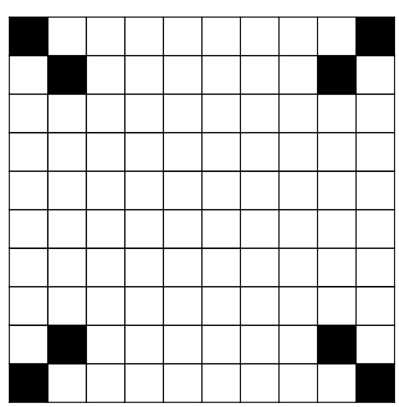
Stanley [11] has conjectured that for fixed m (and n varying), $N(n,m)$ satisfies a linear recurrence with constant coefficients that is of order $2^{\frac{m+1}{2}}$ (he established this when $m+1$ is an odd prime). Such recurrences have been obtained for small m and can be used to provide proofs of special cases of Problem **1**. Indeed, Bao [1] has used such recurrences together with the reduction techniques we use above to establish combinatorial proofs for the formulas in Problem **1**for $n \leq 2$. Unfortunately, the difficulty in establishing recurrences for $N(n,m)$ combinatorially probably precludes the general applicability of the above method for finding combinatorial proofs for (12) and (13).

Equation (13), which remains to be verified using algebraic methods, was checked extensively for various values of n with $m \leq 10$.

4 Conjectures

4.1 Deleting From Diagonals

We begin with an intriguing "power of 2" conjecture for a new type of region we call the *spider*.



The $(5,2)$ spider

Define the (n, k) spider to be the region obtained by deleting k consecutive squares (from the corner) along each diagonal of the $2n \times 2n$ square grid.

Conjecture 1 *Let $S(n, k)$ be the number of domino tilings of the (n, k) spider.*

$$S(n, k) = 2^{n+k(2k-1)}(2r + 1), \quad k \leq \lfloor \frac{n}{2} \rfloor. \tag{14}$$

When $k > \lfloor \frac{n}{2} \rfloor$ the region reduces to an Aztec diamond after the removal of forced dominoes (for a definition and extensive discussion of Aztec diamonds see [3]). If n is even we see that (**14**) reduces to the formula for the number of domino tilings of the Aztec diamond when $k = \frac{n}{2}$. Conjecture 1 has been checked numerically for $n \leq 10$.

Values of $S(n, k)$ for $n = \{2, \ldots, 6\}$, $k \leq \lfloor \frac{n}{2} \rfloor$

n/k	2	3	4	5	6
0	$2^2 3^2$	$2^3 29^2$	$2^4 17^2 53^2$	$2^5 241^2 373^2$	$2^6 5^4 31^2 53^2 701^2$
1	2^3	$2^4 7^2$	$2^5 13^4$	$2^6 3^4 11^2 139^2$	$2^7 5^2 744397^2$
2	$-$	$-$	2^{10}	$2^{11} 31^2$	$2^{12} 3617^2$
3 $-$	$-$	$-$	$-$	$-$	2^{21}

4.2 Deleting From Step Diagonals

The acute reader will have noticed that the arguments in Lemma 1 establish that any domino tiling of the $2n \times 2n$ square grid contains at least n disjoint alternating cycles. The tiling in Example 1 illustrates that this is the best result possible (for other results along these lines see [8]). Figure 3 shows how to place n dominoes so as to ensure the remaining figure has only one tiling (the n dominoes "block" the n cycles).

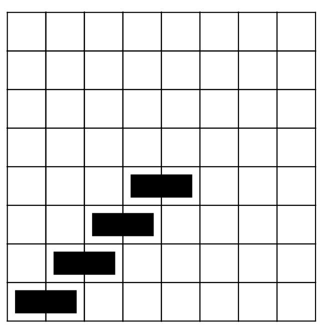


Figure 3

We shall call the set of the first n stepwise horizontal edges in the $2n \times 2n$ square grid the *step-diagonal*.

The above observation has led Propp [9] to ask whether removal of only half the dominoes from the bottom of the step diagonal results in a graph whose number of tilings is interesting. Indeed, drawing on his idea, we have formulated the following remarkable conjecture:

Conjecture 2 *Let G be the grid obtained after the removal of* **any** *k edges from the step-diagonal of the $2n \times 2n$ square grid. Then the number of domino tilings of G is of the form*

$$\#G = 2^{n-k}(2r+1). \tag{15}$$

In addition, if the k edges removed are consecutive from the lower left corner then $2r+1$ is a perfect square.

Also related to the step-diagonal is the following conjecture:

Conjecture 3 *Let G be the grid obtained after the removal of one edge from the step-diagonal of the $2n \times 2n$ square grid. Using the notation that $N(2n, 2n) = 2^n(2k+1)^2$, the number of domino tilings of G satisfies:*

$$\#G = c(2k+1) \tag{16}$$

where c is a constant which depends upon which edge was removed.

Conjecture 2 was checked extensively for $n \leq 10$ (the exponential growth of the number of configurations to be tested precluded exhaustive checking of this conjecture). Conjecture 3 was checked for all $n \leq 10$.

Edward Early has considered the number of tilings of *holey squares*. The holey square $H(n, m)$ is a $2n \times 2n$ square with a hole of size $2m \times 2m$ removed from the center. He has conjectured

Conjecture 4

$$\#H(n, m) = 2^{n-m}(2k+1)^2. \tag{17}$$

The fact that $2^{n-m}|H(n, m)$ is easily obtained using Lemma 1 (the fact that $H(n, m)$ is either a perfect square or twice a perfect square also follows). The fact that $n - m$ is the highest power of 2 dividing $H(n, m)$ does not follow inductively in this case. Bao [1] has established that the conjecture is true for $m = 1, 2$ by showing that a region similar to H_n has an odd number of domino tilings. Unfortunately, algebraic methods using (**1**) fail in this case since no analogous formulas from which to work are known.

Finally, based on numerical evidence, we present our grand conjecture:

Conjecture 5 *Conjecture 2 is true for all holey squares (with n replaced by $n - m$ in (**15**)). Conjecture 3 is true for all holey squares (with $(2k+1)$ replaced by the square root of the odd part of $\#H(n, m)$).*

5 Conclusion

The results and conjectures of the previous sections point to an underlying combinatorial principle which is most likely the basis of the nice patterns of powers of 2. While such a result eludes us, the following old (somewhat forgotten) result which appears in [7] may hint at an algebraic approach to "power of 2" conjectures:

Proposition 1 *A graph G has an even number of perfect matchings iff there is a non-empty set $S \subseteq V(G)$ such that every point is adjacent to an even number of points of S.*

6 Acknowledgments

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