New symmetric designs from regular Hadamard matrices

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Abstract

For every positive integer m, we construct a symmetric (v,k,λ)-design with parameters $v=\frac{h((2h-1)^{2m}-1)}{h-1}$, $k=h(2h-1)^{2m-1}$, and $\lambda=h(h-1)(2h-1)^{2m-2}$, where $h=\pm 3\cdot 2^d$ and $|2h-1|$ is a prime power. For $m\geq 2$ and $d\geq 1$, these parameter values were previously undecided. The tools used in the construction are balanced generalized weighing matrices and regular Hadamard matrices of order $9\cdot 4^d$.

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1 Introduction

Let $v>k>\lambda\geq 0$ be integers. A *symmetric* (v,k,λ)-*design* is an incidence structure $(P,\mathcal{B})$, where P is a set of cardinality v (the point-set) and $\mathcal{B}$ is a family of v k-subsets (blocks) of P such that any two distinct points are contained in exactly λ blocks. If $P=\{p_1,...,p_v\}$ and $\mathcal{B}=\{B_1,...,B_v\}$, then the $(0,1)$-matrix $M=[m_{ij}]$ of order v, where $m_{ij}=1$ if and only if $p_j\in B_i$, is the *incidence matrix* of the design. A $(0,1)$-matrix X of order v is the incidence matrix of a symmetric (v,k,λ)-design if and only if it satisfies the equation $XX^T=(k-\lambda)I+\lambda J$, where I is the identity matrix and J is the all-one matrix of order v. For references, see [1] or [3, Chapter 5].

A *Hadamard matrix* of order n is an n by n matrix H with entries equal to ± 1 satisfying $HH^T=nI$. A Hadamard matrix is *regular* if its row and column sums are constant. This sum is always even and if we denote it $2h$, then the order of the matrix is equal to $4h^2$. Replacing -1s in a regular Hadamard matrix of order $4h^2$ by 0s yields the incidence matrix of a symmetric $(4h^2,2h^2-h,h^2-h)$-design usually

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called a *Menon design*. Conversely, replacing 0s by -1s in the incidence matrix of a symmetric $(4h^2, 2h^2 - h, h^2 - h)$-design yields a regular Hadamard matrix of order $4h^2$. For references, see [9]. In this paper, we will be interested in regular Hadamard matrices of order $9 \cdot 4^d$, where d is a positive integer. If H is such a matrix, then the Kronecker product of a regular Hadamard matrix of order 4 and H is a regular Hadamard matrix of order $9 \cdot 4^{d+1}$. Therefore, one can obtain a family of regular Hadamard matrices of order $9 \cdot 4^d$, starting with a regular Hadamard matrix of order 36.

A *balanced generalized weighing matrix* $\mathrm{BGW}(v, k, \lambda)$ over a (multiplicatively written) group G is a matrix $W = [\omega_{ij}]$ of order v with entries from the set $G \cup \{0\}$ such that (i) each row and each column of W contain exactly k non-zero entries and (ii) for any distinct rows i and h, the multiset

$$\{\omega_{hj}^{-1}\omega_{ij} \colon 1 \le j \le v, \omega_{ij} \ne 0, \omega_{hj} \ne 0\}$$

contains exactly $\lambda/|G|$ copies of every element of G.

In this paper, we will use a balanced generalized weighing matrix $\mathrm{BGW}(q^m + q^{m-1} + \cdots + q + 1, q^m, q^m - q^{m-1})$ over a cyclic group G of order t, where q is a prime power, m is a positive integer, and t is a divisor of $q - 1$. Such matrices are known to exist [3, IV.4.22] and have been applied to constructing symmetric designs by Rajkundlia [8], Brouwer [2], Fanning [4], and the author [5, 6]. If $\mathcal{M}$ is a set of m by n matrices, G is a group of bijections $\mathcal{M} \to \mathcal{M}$, and W is a balanced generalized weighing matrix over G, then, for any $P \in \mathcal{M}$, $W \otimes P$ denotes the matrix obtained by replacing every entry σ in W by the matrix σP. In Section 2 (Lemma 2.1), we will prove the following modification of a result from [6]:

Let $\mathcal{M}$ be a set of matrices of order v containing the incidence matrix M of a symmetric (v, k, λ)-design with $q = \frac{k^2}{k-\lambda}$ a prime power. Let G be a finite cyclic group of bijections $\mathcal{M} \to \mathcal{M}$ such that (i) $(\sigma P)(\sigma Q)^T = PQ^T$ for any $P, Q \in \mathcal{M}$ and $\sigma \in G$, (ii) $\sum_{\sigma \in G} \sigma M = \frac{k|G|}{v} J$, and (iii) $|G|$ divides $q - 1$. If W is a balanced generalized weighing matrix $\mathrm{BGW}(q^m + \cdots + q + 1, q^m, q^m - q^{m-1})$ over G, then $W \otimes M$ is the incidence matrix of a symmetric $(v(q^m + q^{m-1} + \cdots + q + 1), kq^m, \lambda q^m)$-design.

In order to apply this lemma, we need a symmetric (v, k, λ)-design to start with. In the paper [6], we have shown that the designs corresponding to certain McFarland and Spence difference sets (or their complements) serve as such starters. In Section 3 of this paper, we show that for $h = \pm 3 \cdot 2^d$, if $|2h - 1|$ is a prime power, then there is a symmetric $(4h^2, 2h^2 - h, h^2 - h)$-design, which can also serve as a starter. As a result, we show that for any positive integers m and d, if $h = \pm 3 \cdot 2^d$ and $|2h - 1|$ is a prime power, then there exists a symmetric (v, k, λ)-design with

$$v = \frac{h((2h-1)^{2m} - 1)}{h - 1}, k = h(2h-1)^{2m-1}, \lambda = h(h-1)(2h-1)^{2m-2}.$$

These parameters are new, except $m = 1$ (Menon designs) and $d = 0$ (constructed by the author in [6]).

2 Preliminaries

Throughout this paper, we will denote identity, zero, and all-one matrices of suitable orders by I, O, and J, respectively.

If W is a balanced generalized weighing matrix of order w over a group G of bijections on a set $\mathcal{M}$ of matrices of order n, then, for any $P \in \mathcal{M}$, we will denote by $W \otimes P$ the matrix of order nw obtained by replacing every nonzero entry σ in W by the matrix σP and every zero entry in W by the zero matrix of order n.

The following lemma represents a slight modification of a result proven in [6]. Since it is crucial for this paper and the proof is short, we will repeat it here.

Lemma 2.1 *Let $v > k > \lambda \geq 0$ be integers. Let $\mathcal{M}$ be a set of matrices of order v and G a finite group of bijections $\mathcal{M} \to \mathcal{M}$ satisfying the following conditions:*
(i) $\mathcal{M}$ contains the incidence matrix M of a symmetric (v, k, λ)-design;
(ii) for any $P, Q \in \mathcal{M}$ and $\sigma \in G$,

$$(\sigma P)(\sigma Q)^T = PQ^T;$$

(iii) $\sum_{\sigma \in G} \sigma M = \frac{k|G|}{v} J$;
(iv) $q = \frac{k^2}{k-\lambda}$ is a prime power;
(v) G is cyclic and $|G|$ divides $q - 1$.
Then, for any positive integer m, there exists a symmetric $(vw, kq^m, \lambda q^m)$-design, where $w = \frac{q^{m+1}-1}{q-1}$.

Proof. Let $W = [\omega_{ij}]$, $i, j = 1, 2, \ldots, w$ be a balanced generalized weighing matrix $\mathrm{BGW}(w, q^m, q^m - q^{m-1})$ over G. We claim that $W \otimes M$ is the incidence matrix of a symmetric $(vw, kq^m, \lambda q^m)$-design. It suffices to show that, for $i, h = 1, 2, \ldots, w$,

$$\sum_{j=1}^{w} (\omega_{ij} M)(\omega_{hj} M)^T = \begin{cases} (k-\lambda)q^m I + \lambda q^m J \text{ if } i = h, \\ \lambda q^m J \text{ if } i \neq h. \end{cases}$$

If $i = h$, we have for some $\sigma_j \in G$,

$$\sum_{j=1}^{w} (\omega_{ij} M)(\omega_{hj} M)^T = \sum_{j=1}^{q^m} (\sigma_j M)(\sigma_j M)^T = \sum_{j=1}^{q^m} MM^T = (k-\lambda)q^m I + \lambda q^m J.$$

If $i \neq h$, we have for some $\sigma_j, \tau_j \in G$,

$$\sum_{j=1}^{w} (\omega_{ij} M)(\omega_{hj} M)^T = \sum_{j=1}^{q^m - q^{m-1}} (\sigma_j M)(\tau_j M)^T = \sum_{j=1}^{q^m - q^{m-1}} (\tau_j^{-1} \sigma_j M) M^T$$

$$= \frac{q^m - q^{m-1}}{|G|} \Big(\sum_{\sigma \in G} \sigma M \Big) M^T = \frac{k(q^m - q^{m-1})}{v} JM^T = \frac{k^2(q^m - q^{m-1})}{v} J = \lambda q^m J.$$

□

Definition 2.2 *Let $v > k > \lambda > 0$ be integers. A (v, k, λ)-difference set is a k-subset of an (additively written) group Γ of order v such that the multiset $\{x - y\colon x, y \in \Gamma\}$ contains exactly λ copies of each nonzero element of Γ.*

Several infinite families of difference sets are known (see [3] or [7] for references). We will mention the McFarland family having parameters $(p^{d+1}(r+1), p^d r, p^{d-1}(r-1))$, where p is a prime power, d is a positive integer, and $r = \frac{p^{d+1}-1}{p-1}$, and the Spence family having parameters $(3^{d+1}(3^{d+1}-1)/2, 3^d(3^{d+1}+1)/2, 3^d(3^d+1)/2)$, where d is a positive integer.

If Δ is a (v, k, λ)-difference set in a group Γ and $\mathcal{B} = \{\Delta + x\colon x \in \Gamma\}$, then $\mathrm{dev}(\Delta) = (\Gamma, \mathcal{B})$ is a symmetric (v, k, λ)-design.

In order to apply Lemma 2.1, we need a symmetric (v, k, λ)-design with $q = \frac{k^2}{k-\lambda}$ a prime power, a set $\mathcal{M}$ of matrices of order v containing the incidence matrix this design, and a cyclic group G satisfying conditions (ii), (iii), and (v) of Lemma 2.1. In the paper [6], we have shown that (v, k, λ) can be the parameters of any McFarland or Spence difference set or their complement with $q = \frac{k^2}{k-\lambda}$ a prime power. In this paper, we will use the Spence $(36, 15, 6)$-difference set in $\Gamma = \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_4$ and the complementary $(36, 21, 12)$-difference set. In the next section, we will reproduce the construction of the corresponding $\mathcal{M}$ and G given in [6]

3 $(36, 15, 6)$- and $(36, 21, 12)$-difference sets

We start with a brief description of the Spence $(36, 15, 6)$-difference set in $\Gamma = \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_4$.

We consider Γ as the set of triples (x_1, x_2, x_3), where $x_1, x_2 \in \{0, 1, 2\}$ and $x_3 \in \{0, 1, 2, 3\}$ with the mod 3 and the mod 4 addition, respectively. Consider $\mathbb{Z}_3 \oplus \mathbb{Z}_3$ as a 2-dimensional vector space over the field GF(3). Let L_1, L_2, L_3, L_4 be its 1-dimensional subspaces. Put $D_1 = \{(x_1, x_2, 0) \in \Gamma\colon (x_1, x_2) \notin L_1\}$ and, for $i = 2, 3, 4$, $D_i = \{(x_1, x_2, i-1) \in \Gamma\colon (x_1, x_2) \in L_i\}$. Then $D = D_1 \cup D_2 \cup D_3 \cup D_4$ is a $(36, 15, 6)$-difference set in Γ [7, Theorem 11.2].

In order to obtain the incidence matrix of the corresponding symmetric design, we have to select an order on Γ. We will assume that (x_1, x_2, x_3) precedes (y_1, y_2, y_3) in Γ if and only if there is i such that $x_i < y_i$ and $x_j = y_j$ whenever $j > i$. Let M be the $(0, 1)$-matrix of order 36 whose rows and columns are indexed by elements of Γ in this order and (x, y)-entry is equal to 1 if and only if $y - x \in D$. Then M is the incidence matrix of a symmetric $(36, 15, 6)$-design. In order to describe the structural properties of M which will be important in the sequel, we introduce the following operation ρ on the set of 3 by 3 block-matrices.

Definition 3.1 *Let $P = [P_{ij}]$ be a 3 by 3 block-matrix with square blocks (in particular, P can be a 3 by 3 matrix). Denote by ρP the matrix obtained by applying the* *cyclic permutation* $\rho = (123)$ *of degree* 3 *to the set of columns of* P*, i.e.,*

$$\rho \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix} = \begin{bmatrix} P_{13} & P_{11} & P_{12} \\ P_{23} & P_{21} & P_{22} \\ P_{33} & P_{31} & P_{32} \end{bmatrix}.$$

The above incidence matrix M of a symmetric $(36, 15, 6)$-design can be represented as a 4 by 4 block-matrix

$$M = \begin{bmatrix} M_1 & M_2 & M_3 & M_4 \\ M_4 & M_1 & M_2 & M_3 \\ M_3 & M_4 & M_1 & M_2 \\ M_2 & M_3 & M_4 & M_1 \end{bmatrix},$$

where each M_i is a 9 by 9 matrix. Further, each M_i can be represented as a 3 by 3 block-matrix

$$M_i = \begin{bmatrix} M_{i1} & M_{i2} & M_{i3} \\ M_{i3} & M_{i1} & M_{i2} \\ M_{i2} & M_{i3} & M_{i1} \end{bmatrix},$$

where each M_{ij} is a matrix of order 3, $M_{11} = O$, $M_{12} = M_{13} = J$, $M_{21} = M_{22} = M_{23} = M_{31} = M_{41} = I$, $M_{32} = M_{43} = \rho I$, and $M_{33} = M_{42} = \rho^2 I$.

Let $\mathcal{M}$ be the set of block-matrices $P = [P_{ij}]$, $i, j = 1, 2, 3, 4$, where each P_{ij} is a block-matrix $P_{ij} = [P_{ijkl}]$, $k, l = 1, 2, 3$, satisfying the following conditions:

(i) each P_{ijkl} is a $(0, 1)$-matrix of order 3;

(ii) for $i = 1, 2, 3, 4$, there is a unique $h_i = h_i(P) \in \{1, 2, 3, 4\}$ such that

$$(P_{ijk1}, P_{ijk2}, P_{ijk3}) \text{ is a permutation of } (O, J, J) \text{ for } j = h_i \text{ and all } k$$

and

$$P_{ijkl} \in \{I, \rho I, \rho^2 I\} \text{ for } j \neq h_i \text{ and all } k, l.$$

Clearly, the above matrix M is an element of $\mathcal{M}$.

Define a bijection $\sigma\colon \mathcal{M} \to \mathcal{M}$ by $\sigma P = P'$, where

(i) for $i = 1, 2, 3, 4$ and $j = 2, 3, 4$, $P'_{ij} = P_{i,j-1}$;

(ii) for $i = 1, 2, 3, 4$, if $h_i = 4$, then $P'_{i1} = \rho P_{i4}$;

(iii) for $i = 1, 2, 3, 4$, if $h_i \neq 4$, then $P'_{i1kl} = \rho P_{i4kl}$ for all k, l.

Let G be the cyclic group generated by σ. Then $|G| = 12$.

Claim. For any $P, Q \in \mathcal{M}$, $(\sigma P)(\sigma Q)^T = PQ^T$.

Proof. Let $P, Q \in \mathcal{M}$ and let $P' = \sigma P$ and $Q' = \sigma Q$. It suffices to show that, for $i = 1, 2, 3, 4$,

$$P'_{i1} Q'^T_{i1} = P_{i4} Q^T_{i4}. \tag{1}$$

If $h_i(P) = h_i(Q) = 4$ or $h_i(P) \neq 4$ and $h_i(Q) \neq 4$, then P'_{i1} is obtained from P_{i4} by the same permutation of columns as Q'_{i1} from Q_{i4}, so (1) is clear. Suppose $h_i(P) = 4$

and $h_i(Q) \neq 4$. Then $(P_{i4k1}, P_{i4k2}, P_{i4k3})$ is a permutation of (O, J, J) and matrices $Q_{i4k1}, Q_{i4k2}, Q_{i4k3}$ have the same row sum (equal to 1). Therefore

$$\sum_{l=1}^{3} P'_{i1kl} Q'^{T}_{i1kl} = \sum_{l=1}^{3} P_{i4kl} Q^{T}_{i4kl} = 2J,$$

and (1) follows. □

It is readily verified that

$$\sum_{n=0}^{11} \sigma^n M = 5J. \tag{2}$$

Thus, the set $\mathcal{M}$, the matrix M, and the group G satisfy Lemma 2.1 for $(v, k, \lambda) = (36, 15, 6)$ with $|G| = 12$. Note that the sum of the entries of any row of any matrix $P \in \mathcal{M}$ is equal to 15.

Let $\overline{M} = J - M$ and $\overline{\mathcal{M}} = \{J - P \colon P \in \mathcal{M}\}$. Without changing G, we obtain that $\overline{\mathcal{M}}$, $\overline{M}$, and G satisfy Lemma 2.1 for $(v, k, \lambda) = (36, 21, 12)$. The sum of the entries of any row of any matrix $P \in \overline{\mathcal{M}}$ is equal to 21.

Note that the described $(36, 15, 6)$-design and $(36, 21, 12)$-design are symmetric $(4h^2, 2h^2 - h, h^2 - h)$-designs with $h = 3$ and $h = -3$, respectively.

4 Using the Kronecker product

The next lemma will allow us to double the parameter h in a family of symmetric $(4h^2, 2h^2 - h, h^2 - h)$-designs satisfying Lemma 2.1.

Lemma 4.1 *Let an integer $h \neq 0$, a set $\mathcal{M}$ of matrices of order $4h^2$, and a finite cyclic group $G = \langle \sigma \rangle$ of bijections $\mathcal{M} \to \mathcal{M}$ satisfy the following conditions:*

(i) $\mathcal{M}$ contains the incidence matrix M of a symmetric $(4h^2, 2h^2 - h, h^2 - h)$-design;

(ii) for any $P, Q \in \mathcal{M}$, $(\sigma P)(\sigma Q)^T = PQ^T$;

(iii) $\sum_{n=0}^{|G|-1} \sigma^n M = \frac{(2h-1)|G|}{4h} J$.

(iv) the sum of the entries of any row of any matrix $P \in \mathcal{M}$ is equal to $2h^2 - h$.

Then there exists a set $\mathcal{M}_1$ of matrices of order $16h^2$ and a cyclic group $G_1 = \langle \tau \rangle$ of bijections $\mathcal{M}_1 \to \mathcal{M}_1$ satisfying the following conditions:

(a) $\mathcal{M}_1$ contains the incidence matrix M_1 of a symmetric $(16h^2, 8h^2 - 2h, 4h^2 - 2h)$-design;

(b) for any $R, S \in \mathcal{M}_1$, $(\tau R)(\tau S)^T = RS^T$;

(c) $\sum_{n=0}^{|G_1|-1} \tau^n M_1 = \frac{(4h-1)|G_1|}{8h} J$;

(d) the sum of the entries of any row of any matrix $R \in \mathcal{M}_1$ is equal to $8h^2 - 2h$;

(e) $|G_1| = 2|G|$.

Proof. For any $P \in \mathcal{M}$, define

$$R_P = \begin{bmatrix} J - P & P & P & P \\ P & J - P & P & P \\ P & P & J - P & P \\ P & P & P & J - P \end{bmatrix}.$$

It is well known and readily verified that $M_1 = R_M$ is the incidence matrix of a symmetric $(16h^2, 8h^2-2h, 4h^2-2h)$-design.

Let $\mathcal{M}_1 = \{R_P\colon P \in \mathcal{M}\}$. Then $M_1 \in \mathcal{M}_1$, so $\mathcal{M}_1$ satisfies (a). Condition (d) is implied by (iv). Any matrix $R \in \mathcal{M}_1$ can be divided into eight $4h^2$ by $8h^2$ cells R_{ij}, $1 \le i \le 4$, $1 \le j \le 2$. Observe that each R_{ij} is of one of the two following types:

(type 1) $R_{ij} = [P \quad J-P]$ or $R_{ij} = [J-P \quad P]$, $P \in \mathcal{M}$;

(type 2) $R_{ij} = [P \quad P]$, $P \in \mathcal{M}$.

Observe also that R_{i1} and R_{i2} are not of the same type.

For any $R \in \mathcal{M}_1$, denote by τR a $(0,1)$-matrix of order $16h^2$ divided into eight $4h^2$ by $8h^2$ cells τR_{ij}, $1 \le i \le 4$, $1 \le j \le 2$, where

$$\tau R_{i2} = R_{i1}$$

and

$$\tau R_{i1} = \begin{cases} J - R_{i2} & \text{if } R_{i2} \text{ is of type 1,} \\ [\sigma P \quad \sigma P] & \text{if } R_{i2} = [P \quad P]\,. \end{cases}$$

In order to verify (b), it suffices to show that, for $i = 1,2,3,4$, $(\tau R_{i1})(\tau S_{i1})^T = R_{i2}S_{i2}^T$.

If R_{i2} and S_{i2} are of type (1), then $(\tau R_{i1})(\tau S_{i1})^T = (J - R_{i2})(J - S_{i2})^T = 8h^2 J - R_{i2}J^T - JS_{i2}^T + R_{i2}S_{i2}^T = R_{i2}S_{i2}^T$ for the row sum of any matrix of type 1 is equal to $4h^2$. If $R_{i2} = [P \quad P]$ and $S_{i2} = [Q \quad Q]$, where $P, Q \in \mathcal{M}$, then $(\tau R_{i1})(\tau S_{i1})^T = 2(\sigma P)(\sigma Q)^T = 2PQ^T = R_{i2}S_{i2}^T$. If $R_{i2} = [P \quad P]$ and S_{i2} is of type 1, then $(\tau R_{i1})(\tau S_{i1})^T = (\sigma P)J = (2h^2 - h)J = R_{i2}S_{i2}^T$.

Let G_1 be the group of bijections $\mathcal{M}_1 \to \mathcal{M}_1$ generated by τ. Then (e) is satisfied, and we have to verify (c). For $n = 1, 2, \ldots, 2|G| - 1$, let A_n be the (i,j)-block of the 4 by 4 block-matrix $\tau^n M_1$. Then there is $P \in \mathcal{M}$ such that the multiset $\{A_n\colon 0 \le n \le 2|G|-1\}$ is the union of $\{\sigma^n P\colon 0 \le n \le |G|-1\}$ and the multiset consisting of $\frac{|G|}{2}$ copies of P and $\frac{|G|}{2}$ copies of $J - P$. Therefore,

$$\sum_{n=0}^{2|G|-1} A_n = \sum_{n=0}^{|G|-1} \sigma^n P + \frac{|G|}{2}J = \frac{(2h-1)|G|}{4h}J + \frac{|G|}{2}J = \frac{(4h-1)|G_1|}{8h}J.$$

□

The following theorem is now immediate by induction.

Theorem 4.2 *Let an integer $h \neq 0$, a set $\mathcal{M}$ of matrices of order $4h^2$, and a finite cyclic group G of bijections $\mathcal{M} \to \mathcal{M}$ satisfy conditions (i)–(iv) of Lemma 4.1. Then, for any positive integer d, there exists a non-empty set $\mathcal{M}_d$ of matrices of order $4^{d+1}h^2$ and a cyclic group G_d of bijections $\mathcal{M}_d \to \mathcal{M}_d$ satisfying the following conditions:*

(a) $\mathcal{M}_d$ contains the incidence matrix M_d of a symmetric design with parameters

$$(4^{d+1}h^2, 2^{2d+1}h^2 - 2^d h, 2^{2d}h^2 - 2^d h);$$

(b) for any $P, Q \in \mathcal{M}_d$ and $\tau \in G_d$, $(\tau P)(\tau Q)^T = PQ^T$;

(c) $\sum_{\tau \in G_d} \tau M_d = \frac{(2^{d+1}h-1)|G_d|}{2^{d+2}h} J$;

(d) the sum of the entries of any row of any matrix $R \in \mathcal{M}_d$ *is equal to* $2^{2d+1}h^2 - 2^d h$;

(e) $|G_d| = 2^d |G|$.

We combine Theorem 4.2 and Lemma 2.1 and obtain the main result of this paper.

Theorem 4.3 *If* $h = \pm 3 \cdot 2^d$*, where* d *is a positive integer and* $|2h-1|$ *is a prime power, then, for any positive integer* m*, there exists a symmetric* $(\frac{h((2h-1)^{2m}-1)}{h-1}, h(2h-1)^{2m-1}, h(h-1)(2h-1)^{2m-2})$*-design.*

Proof. We start with the set $\mathcal{M}$ or $\overline{\mathcal{M}}$ described in Section 3 and apply Theorem 4.2 to this set to obtain the set of matrices $\mathcal{M}_d$ or $\overline{\mathcal{M}}_d$ and the group G_d. Then we apply Lemma 2.1. Properties (ii) and (iii) required in Lemma 2.1 are implied by (b) and (c) of Theorem 4.2. The parameter q of Lemma 2.1 is equal to $(2h_d - 1)^2$, where $h_d = \pm 3 \cdot 2^d$, so q is a prime power. Since $|G| = 12$, we have $|G_d| = 3 \cdot 2^{d+2} = 4|h_d|$, so $|G_d|$ divides $q-1$. □

Remark 4.4 *These parameters are new, except* $m = 1$ *(Menon designs).*

References

[1] T. Beth, D. Jungnickel, and H. Lenz, *Design Theory*, B.I. Wissenschaftverlag, Mannheim, 1985, Cambridge Univ. Press, Cambridge, UK, 1986.

[2] A. E. Brouwer, An infinite series of symmetric designs, *Math. Centrum Amsterdam Report*, ZW 136/80 (1983).

[3] *The CRC Handbook of Combinatorial Designs*, eds. C. J. Colbourn and J. H. Dinitz, CRC Press, 1996.

[4] J. D. Fanning, A family of symmetric designs, *Discrete Mathematics* **146** (1995), pp. 307–312.

[5] Y.J. Ionin, Symmetric subdesigns of symmetric designs, *Journal of Combinatorial Mathematics and Combinatorial Computing* (to appear).

[6] Y.J. Ionin, A technique for constructing symmetric designs, *Designs, Codes and Cryptography* (to appear).

[7] D. Jungnickel, Difference sets, in: *Contemporary Design Theory: A Collection of Surveys* (J.H. Dinitz, D.R. Stinson; eds.), John Wiley & Sons, New York, 1992, 241–324.

[8] D.P. Rajkundlia, Some techniques for constructing infinite families of BIBDs, *Discrete Math.* **44** (1983), 61–96.

[9] J. Seberry and M. Yamada, Hadamard matrices, sequences, and block designs, in *Contemporary Design Theory*, eds. J.H. Dinitz and D.R. Stinson, John Wiley & Sons, 1992, 431–560.