A 2-COLORING OF $[1, N]$ CAN HAVE $(1/22)N^2 + O(N)$ MONOCHROMATIC SCHUR TRIPLES, BUT NOT LESS!

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Submitted: March 3, 1998; Accepted: March 25, 1998

Abstract: We prove the statement of the title, thereby solving a \$100 problem of Ron Graham. This was solved independently by Tomasz Schoen.

Tianjin, June 29, 1996: In a fascinating invited talk at the SOCA 96 combinatorics conference organized by Bill Chen, Ron Graham proposed (see also [GRR], p. 390):

Problem (\$100): Find (asymptotically) the least number of monochromatic Schur triples $\{i, j, i+j\}$ that may occur in a 2-coloring of the integers $1, 2, \dots, n$.

By naming the two colors 0 and 1, the above is equivalent to the following

Discrete Calculus Problem: Find the minimal value of

$$F(x_1, \dots, x_n) := \sum_{\substack{1 \le i < j \le n \\ i+j \le n}} [\ x_i x_j x_{i+j} + (1 - x_i)(1 - x_j)(1 - x_{i+j})\ ] ,$$

over the n-dimensional (discrete) unit cube $\{(x_1, \dots, x_n) | x_i = 0, 1\}$. We will determine *all* local minima (with respect to the Hamming metric), then determine the global minimum.

Partial Derivatives: For any function $f(x_1, \dots, x_n)$ on $\{0, 1\}^n$ define the discrete *partial derivatives* $\partial_r f$ by $\partial_r f(x_1, \dots, x_r, \dots, x_n) := f(x_1, \dots, x_r, \dots, x_n) - f(x_1, \dots, 1 - x_r, \dots, x_n)$.

If $(z_1, \dots, z_n)$ is a local minimum of F, then we have the n inequalities:

$$\partial_r F(z_1, \dots, z_n) \le 0 \quad , \quad 1 \le r \le n.$$

A purely routine calculation shows that (below $\chi(S)$ is 1(0) if S is true(false))

$$\partial_r F(x_1, \dots, x_n) =$$

$$(2x_r - 1)\left\{ \sum_{i=1}^{n} x_i + \sum_{i=1}^{n-r} x_i - (n - \left\lfloor \frac{r}{2} \right\rfloor) - \chi(r > \frac{n}{2}) - (2x_r - 1) + x_r \chi(r > \frac{n}{2}) + 1 - (x_{\frac{r}{2}} + x_{2r}) \chi(r \le \frac{n}{2}) \right\} .$$

Since we are only interested in the *asymptotic* behavior, we can modify F by any amount that is $O(n)$. In particular, we can replace $F(x_1, \dots, x_n)$ by

$$G(x_1, \dots, x_n) = F(x_1, \dots, x_n) + \sum_{i=1}^{n/2} x_i(x_{2i} - 1) - \frac{1}{2} \sum_{i=1}^{n} x_i.$$

[1] Supported in part by the NSF.
Mathematics Classification Numbers: Primary: 05D10, 05A16; Secondary: 04A20

Noting that $(2x_r-1)^2 \equiv 1$ and $(2x_r-1)x_r \equiv x_r$ on $\{0,1\}^n$, we see that for $1 \le r \le n$,

$$\partial_r G(x_1,\dots,x_n) = (2x_r-1)\left\{\sum_{i=1}^{n} x_i + \sum_{i=1}^{n-r} x_i - (n - \left\lfloor \frac{r}{2} \right\rfloor) - \frac{1}{2}\chi(r \le n/2)\right\} - \frac{1}{2}\chi(r \le n/2) - 1/2.$$

Let $k = \sum_{i=1}^n x_i$. By symmetry we may assume that $k \ge n/2$. Since at a local minimum $(z_1,\dots,z_n)$ we have $\partial_r G(z_1,\dots,z_n) \le 0$, it follows that any local minimum $(z_1,\dots,z_n)$ satisfies the

Ping-Pong Recurrence: Let

$$H_c(y) := \begin{cases} 0, & \text{if } y \ge c; \\ 1, & \text{if } y < c. \end{cases}$$

For $r = n, n-1, \dots, n - \lfloor n/2 \rfloor + 1$

$$z_r = H_{1/2}\left(k - n + \left\lfloor \frac{r}{2} \right\rfloor + \sum_{j=1}^{n-r} z_j\right), \qquad (Right\,Volley)$$

$$z_{n-r+1} = H_1\left(2k - n - 1/2 + \left\lfloor \frac{n-r+1}{2} \right\rfloor - \sum_{j=r}^{n} z_j\right), \qquad (Left\,Volley)$$

and if n is odd then $z_{(n+1)/2} = H_{1/2}(k - n + \lfloor \frac{n+1}{4} \rfloor + \sum_{j=1}^{(n-1)/2} z_j)$.

These equations uniquely determine z (if it exists), in the order $z_n, z_1, z_{n-1}, z_2, \dots$. When we solve the Ping-Pong recurrence we forget the fact that $\sum_{i=1}^n z_i = k$. Most of the time, the unique solution will not satisfy this last condition, but when it does, we have a genuine local minimum. Note that *any* local minimum must show up in this way.

The Solution of the Ping-Pong Recurrence: By playing around with the Maple package `RON` (available from either author's website), we were able to find the following *explicit* solution, for n sufficiently large, to the Ping-Pong recurrence. As usual, for any word (or letter) W, W^m means 'W repeated m times'.

Let $w = 2k - n$, $k > n/2$ (the case $k = n/2$ is treated seperately). Then $0 < w \le n$. If $w \ge n/2$ then the (only) solution is 0^n. If $w < n/2$, then let s be the (unique) integer $0 \le s < \infty$, that satisfies $n/(12s+14) \le w < n/(12s+2)$.

Case I: If $n/(12s+8) \le w < n/(12s+2)$ then the unique solution is

$$\begin{cases} 0^{\lfloor \frac{n}{2} \rfloor} 1^{n - \lfloor \frac{n}{2} \rfloor - w - 1} 0^{w+1} & \text{for } s = 0; \\ 0^{4w} (1^{6w-1} 0^{6w-1})^{\frac{s-1}{2}} 1^{\lfloor \frac{n}{2} \rfloor - (6s-2)w + s - 1} 0^{n - \lfloor \frac{n}{2} \rfloor - (6s+1)w + s - 1} 1^{6w-1} (0^{6w-1} 1^{6w-1})^{\frac{s-1}{2}} 0^{w+1} & \text{for } s \text{ odd;} \\ 0^{4w} (1^{6w-1} 0^{6w-1})^{\frac{s-2}{2}} 1^{6w-1} 0^{\lfloor \frac{n}{2} \rfloor - (6s-2)w + s - 1} 1^{n - \lfloor \frac{n}{2} \rfloor - (6s+1)w + s - 1} (0^{6w-1} 1^{6w-1})^{\frac{s}{2}} 0^{w+1} & \text{otherwise.} \end{cases}$$

Case II: If $n/(12s+14) \le w < n/(12s+8)$ then the unique solution is

$$\begin{cases} 0^{4w} (1^{6w-1} 0^{6w-1})^{\frac{s-1}{2}} 1^{6w-1} 0^{n - (12s+5)w + 2s - 1} 1^{6w-1} (0^{6w-1} 1^{6w-1})^{\frac{s-1}{2}} 0^{w+1} & \text{for } s \text{ odd;} \\ 0^{4w} (1^{6w-1} 0^{6w-1})^{\frac{s}{2}} 1^{n - (12s+5)w + 2s - 1} (0^{6w-1} 1^{6w-1})^{\frac{s}{2}} 0^{w+1} & \text{for } s \text{ even.} \end{cases}$$

Case III: if $w = 0$ (i.e. $s = \infty$), the unique solution is

$$\begin{cases} 1(0^3 1^3)^{k/6} 1^2 (0^3 1^3)^{(k-6)/6} 0^3 & \text{if } k \equiv 0 \pmod 6; \\ 1(0^3 1^3)^{(k-1)/6} 0 1^3 (0^3 1^3)^{(k-7)/6} 0^3 & \text{if } k \equiv 1 \pmod 6; \\ 1(0^3 1^3)^{(k-2)/3} 0^3 & \text{if } k \equiv 2 \pmod 6; \\ 1(0^3 1^3)^{(k-3)/6} 0^2 (0^3 1^3)^{(k-3)/6} 0^3 & \text{if } k \equiv 3 \pmod 6; \\ 1(0^3 1^3)^{(k-4)/6} 0^3 1 (0^3 1^3)^{(k-4)/6} 0^3 & \text{if } k \equiv 4 \pmod 6; \\ 1(0^3 1^3)^{(k-2)/3} 0^3 & \text{if } k \equiv 5 \pmod 6. \end{cases}$$

Proof: Routine verification!

Now it is time to impose the extra condition that $\sum_{i=1}^n z_i = k$ $(= (w+n)/2)$. With Case I we get a contradiction of the applicable range of w, but Case II yields that $w = \frac{n+2(s+1)}{12s+11}$, which is a local minimum for n sufficiently large. Case III gives a local minimum when $k \equiv 0, 1 \pmod 6$. Hence

The Local Minima Are:

$$\begin{cases} Z_s := 0^{4w_s} (1^{6w_s-1} 0^{6w_s-1})^{\frac{s}{2}} 1^{6w_s-3} (0^{6w_s-1} 1^{6w_s-1})^{\frac{s}{2}} 0^{w_s+1} & \text{for } 0 \le s < \infty \text{ (where } w_s := \frac{n+2(s+1)}{12s+11}), \\ Z_\infty^0 = 1(0^3 1^3)^{k/6} 1^2 (0^3 1^3)^{(k-6)/6} 0^3 & \text{for } w = 0 \text{ and } k \equiv 0 \pmod 6\text{, and} \\ Z_\infty^1 = 1(0^3 1^3)^{(k-1)/6} 0 1^3 (0^3 1^3)^{(k-7)/6} 0^3 & \text{for } w = 0 \text{ and } k \equiv 1 \pmod 6. \end{cases}$$

A routine calculation [R] shows that for $0 \le s < \infty$

$$F(Z_s) = \frac{12s+8}{16(12s+11)} n^2 + O(n),$$

which is strictly increasing in s. An easy calculation shows $F(Z_\infty^0) = F(Z_\infty^1) = (1/16)n^2 + O(n)$.

...And The Winner Is: $Z_0 = 0^{4n/11} 1^{6n/11} 0^{n/11}$ setting the world-record of $(1/22)n^2 + O(n)$.

Note: Tomasz Schoen[S], a student of Tomasz Luczak, has independently solved this problem.

An Extension: Here we note that our result implies a good lower bound for the general r-coloring of the first n integers; if we r-color the integers (with colors $C_1 \ldots C_r$) from 1 to n then the minimum number of monochromatic Schur triples is bounded above by

$$\frac{n^2}{2^{2r-3} 11} + O(n).$$

This comes from the following coloring:

$$\begin{cases} Color(i) = C_j & \text{if } \frac{n}{2^j} < i \le \frac{n}{2^{j-1}} \quad \text{for } 1 \le j \le r-2, \\ Color(i) = C_{r-1} & \text{if } 1 \le i \le \frac{4n}{2^{r-2} 11} \text{ or } \frac{10n}{2^{r-2} 11} < i \le \frac{n}{2^{r-2}}, \\ Color(i) = C_r & \text{if } \frac{4n}{2^{r-2} 11} < i \le \frac{10n}{2^{r-2} 11}. \end{cases}$$

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