Appendix to the article “On the number of Descendants and Ascendants in Random Search Trees” by C. Martínez, A. Panholzer, and H. Prodinger

Appendix A. Polynomials

We collect here various polynomials in two and three variables that appear as factors or terms in the explicit form of some the generating functions studied in this paper.

A.1. The polynomials $A_i(z,u,v)$.

▷ $$A_0(z,u,v)=\frac{1}{70}\Big(3410v^7u^6z^6+7356u^4v^4z+900v^8u^8z^7-1316u^9v^7z^5-1148u^9v^8z^6-880v^7u^8z^7-
210u^9v^8z^4-5874v^7u^5z^5+4886v^7u^8z^4+4080v^4u^6z-2564v^8u^8z^6-4648v^5u^3z+140u^{10}v^8z^7+3822u^5v^8z^5-
2378u^3v^3z-680u^9v^7z^7+7356v^4u^5z-120vu^2z-880v^7u^7z^7+1388v^7u^9z^6-490v^9u^5z+1042v^4u^2z+
3484u^7v^7z^6+120u^5z^2+60zu^3-120z^2u^4-140u^6v^7-140u^4v^8+70u^5v^8+140u^6v^6+70v^2u^3-140u^5v^4-
140v^6u^4+140u^5v^3-140u^4v^4-140v^4u^3-140v^2u^4+60uv^2z+10780v^7u^6z^2-130v^8u^8z^8-3920u^6v^8z^2+
4080u^3v^4z-5874u^8v^7z^5-1378v^5u^2z-4648v^5u^6z-9220v^5u^5z-9220v^5u^4z+12362v^7u^5z^4+60v^7u^8z^8-
80v^8u^9z^8+130u^9v^7z^8-45u^{10}v^8z^8+3410u^8v^7z^6+3822u^8v^8z^5-8606v^7u^7z^5-70v^9u^9z^4-84v^8u^{10}z^6+
1512v^9u^7z^6+868v^9u^8z^6+6986v^8z^5u^6+1106u^9v^8z^5+868v^9u^6z^6-400v^9u^7z^7+900v^8u^7z^7-2564v^8u^6z^6-
2590v^9u^7z^5+70v^9u^8z^8-3560v^8u^7z^6-826v^9u^8z^5-2590v^9u^6z^5-826v^9u^5z^5-2800v^9u^5z^3-400v^9u^8z^7-
5v^9u^{10}z^8-100v^9u^9z^7+1806v^5u^6z^6-2370v^5u^5z^5-520u^3v^2z^2+210uv^6z^4-14104v^4u^5z^2+56u^{10}v^7z^5-
220v^5z^7u^6-1612u^4v^2z^2+110v^3u^2z^2-1612u^6v^2z^2-1246v^6u^9z^6+8400u^6v^5z^4+140u^5v^7+140v^3u^3-
140v^6u^3+140v^4u^2+70v^4u-140v^3u^2-140u^5v^5-140v^5u-140v^5u^3-140u^5v^6+140u^6v^4+140v^7u^3+
560v^5u^4-140v^3u^6+70v^2u^5+60u^6z+140v^6u^2-140v^7u^2+70uv^6+70v^8u^3-140v^5u^7+70u^7v^4+
42v^9u^9z^5-126v^8u^{10}z^5+14v^9u^{10}z^5+84v^9u^9z^6+30v^9u^9z^8-28v^9u^{10}z^6-4062v^3u^4z-4062v^3u^5z+
7732u^5v^6z+770u^6v^8z+7732v^6u^4z+11760v^7u^5z^2-4620v^7u^5z-140v^9u^7z^2-4620v^7u^4z+6986v^8u^7z^5+
1890v^8u^5z+1534v^2u^4z-8606v^7u^6z^5+1960v^9u^5z^2-2800v^9u^6z^3+9240v^8u^6z^3-1260v^8u^7z^2-
15232u^6v^7z^3+12362v^7u^7z^4-3038v^8u^8z^4-7000v^8u^5z^2+2170v^9u^5z^4+3780v^9u^6z^4-7490v^8u^5z^4+
70v^9u^6z-580u^2v^3z+360u^2v^2z-15232v^7u^5z^3-700v^9u^7z^3-180vzu^3+746v^2zu^3+122v^4zu-180v^3uz+
9240v^8u^5z^3+980v^8u^8z^3+210v^9u^8z^4+2170v^9u^7z^4+13888v^7u^6z^4-7490v^8u^7z^4-10724v^8u^6z^4+
840v^9u^6z^2-9464v^7u^7z^3-2520v^7u^6z+4252v^6u^6z-1148u^5v^8z^6+2800u^3v^7z^2+126v^7u^2z^5-3038u^4v^8z^4+
1106u^4v^8z^5+826u^2v^6z-2520v^7u^3z+10v^8u^5z^8+500u^6v^8z^7-680v^7u^6z^7+140v^8u^5z^7+20v^9u^5z^7-
28v^9u^4z^6-210uv^6z-1316v^7u^4z^5-100v^7u^5z^7+25v^7u^6z^8+112v^7u^4z^6-126v^8u^3z^5+56v^7u^3z^5+
42v^9u^4z^5+14v^9u^3z^5-5v^9u^6z^8-84v^8u^4z^6-84v^7u^3z^6-210v^7u^2z^4-210v^8u^3z^4+56v^8u^3z^6-28v^8u^2z^5-
45v^8u^6z^8+1388v^7u^5z^6-70v^9u^3z^4-180u^6vz-140u^2v^8z+70uv^7z+280u^2v^8z^2-70u^2v^7z+140z^4u^2v^8+
126v^5uz+140v^7u^2z^3-2378u^6v^3z+448v^7u^3z^4+746u^6v^2z-300u^5vz+4886v^7u^4z^4-100v^9u^6z^7+
84v^9u^5z^6+140v^9u^3z^3+210v^9u^4z^4+980v^8u^3z^3-1624v^7u^3z^3+1534u^5v^2z-300u^4vz-700v^9u^4z^3-
1260u^3v^8z^2-140u^3v^9z^2+4252v^6u^3z+20v^7u^4z^7+4340v^8u^4z^3-9464v^7u^4z^3+872v^5u^8z^4+176v^4u^8z^4+
7824v^5u^7z^4-100v^7u^{10}z^7+6554u^7v^6z^5+3998u^8v^6z^5+14940u^4v^5z^2+1520v^3u^6z^4+2246v^4u^7z^5-
9220u^4v^4z^2-46u^8v^4z^5+320v^4z^7u^7+1280v^6u^9z^5+380v^6u^9z^7-2804u^5v^3z^3+1806v^5u^8z^6+540u^5vz^2+
120v^6u^{10}z^7-526v^5u^9z^6-286v^4u^8z^6-140v^7uz^2+840u^4v^9z^2+770u^3v^8z-13520u^6v^5z^3+14940v^5u^6z^2+
252u^2v^6z^2+420v^6uz^2+7880v^4u^5z^3+9428v^6u^7z^3-1344u^9v^6z^3+412u^8v^5z^3+2940v^6u^8z^3-5014v^6u^8z^4-
2416v^4u^7z^4+1302v^6u^{10}z^4+2246v^4z^5u^6-1834v^6z^6u^6+460v^3u^5z^4-5200v^4u^6z^4-70v^7u^7z+2800v^7u^7z^2+
1890v^8u^4z+448v^7u^9z^4-210v^7u^{10}z^4-2416v^4u^5z^4-3920v^8u^4z^2+336v^6u^9z^4-1876v^5u^9z^4+1560v^5u^7z^6-
490v^9u^4z-8008v^5u^7z^3+2948u^7v^4z^3+540u^5z^3v^2+150u^4z^2v+70v^9u^3z-5964v^6u^3z^2-2804u^6v^3z^3+
7880u^6v^4z^3+4856v^3u^4z^2+4336v^5u^3z^2+6552u^5v^3z^2+252v^6u^8z^2+4336v^5u^7z^2-1378v^5u^7z-
12914v^6u^4z^2-12914v^6u^6z^2+17336v^5u^5z^2+1042u^7v^4z-3634u^7v^4z^2-580u^7v^3z+1658u^7v^3z^2+
360u^7v^2z+196u^8v^5z^2-19028u^5v^6z^2-9220u^6v^4z^2+4856u^6v^3z^2-2096u^5v^2z^2-4366v^5z^5u^6+
432v^6u^{10}z^6+3998v^6z^5u^5+1812v^5z^5u^4+500v^8u^9z^7+112u^{10}v^7z^6-220u^9v^5z^7+308v^4z^6u^5-100v^4z^7u^6+
872v^5u^4z^4-1246v^6z^6u^5+6554v^6z^5u^6+17116v^6u^6z^3+1280v^6z^5u^4+7824v^5u^5z^4-700v^4z^5u^4-
46v^4z^5u^5-286v^4z^6u^6-1344v^6u^{10}z^5-9772u^7v^6z^4-1834u^8v^6z^6-546v^6z^5u^2+460u^5v^5z^7-70v^7uz^4-
952v^5u^4z^6-4366u^7v^5z^5+120v^6u^5z^7-2370u^8v^5z^5-1344u^3v^6z^5-300u^4v^6z^7+1302u^2v^6z^4+588u^3v^6z^6+
1148u^3v^5z^5+432u^4v^6z^6+10780v^7u^4z^2-13724u^6v^6z^4-2648u^7v^6z^6-526v^5z^6u^5-5964v^6u^7z^2+
1812u^9v^5z^5-34v^4z^2u^2-1396v^4u^7z^6+320v^4u^8z^7+4340v^8u^7z^3-3634v^4u^3z^2+196v^5u^2z^2+2940u^3v^6z^3+
412v^5u^3z^3-1344v^6u^2z^3-8008u^4v^5z^3-13520u^5v^5z^3-88v^4u^3z^3-856v^3u^4z^3+9428v^6u^4z^3-1624v^7u^8z^3+
17116v^6u^5z^3+1658u^3v^3z^2-500v^5u^8z^7+140v^7u^9z^3-420v^5z^2u+140v^7uz^3-280u^2v^8z^3-420v^6uz^3+$$

$336u^3v^6z^4-1876v^5u^3z^4-5014v^6u^4z^4-910v^5z^4u^2+896v^5z^3u^2+2948v^4u^4z^3-9772v^6u^5z^4+826v^6u^7z+$
$980v^4u^3z^4+176v^4u^4z^4+126v^5u^8z+122u^8v^4z-180u^8v^3z+60u^8v^2z-420u^9v^5z^2+420v^6u^9z^2+$
$210v^6u^{11}z^4+70v^7u^8z-70v^7u^{11}z^4-420u^{10}v^6z^3-120u^7vz-280v^8u^9z^3+140v^9u^8z^3-28v^8u^{11}z^5+$
$20v^9u^{10}z^7+56v^8u^{11}z^6+25u^{10}v^7z^8+10u^{11}v^8z^8-40u^{11}v^8z^7+140u^{10}v^8z^4+70u^7v^6-5v^7u^4z^8+$
$20u^6z^7v^2+14v^7u^{12}z^5+20v^7u^{12}z^7+15v^6u^{12}z^8-5v^7u^{12}z^8-20v^4u^{12}z^7+28v^4u^{12}z^6+60v^5u^{12}z^7+$
$5v^4u^{12}z^8-15v^5u^{12}z^8-42v^6u^{12}z^5+84v^6u^{12}z^6-28v^7u^{12}z^6-60v^6u^{12}z^7+42v^5u^{12}z^5+140v^4u^9z^2+$
$420v^5u^{10}z^3-210v^5u^{11}z^4-14v^4u^{12}z^5-84v^5u^{12}z^6-140v^4u^{10}z^3+70v^4u^{11}z^4+40u^{11}v^3z^7-10v^3z^8u^{11}+$
$5v^2z^8u^{10}-56u^{11}v^3z^6+280u^9v^3z^3-140u^{10}v^3z^4+70z^4u^9v^2+28u^{10}z^6v^2-20u^{10}z^7v^2-14u^{10}z^5v^2+$
$28u^{11}v^3z^5-210v^6u^8z+240u^7vz^2-120u^8z^2v^2-140z^3u^8v^2+140v^7u^{10}z^3-140v^7u^9z^2-60u^5vz^3+$
$150u^6vz^2+240u^3vz^2-520u^7z^2v^2+98z^5u^8v^2+260z^3u^7v^2+70z^4u^8v^2+14z^5u^7v^2-84z^6u^8v^2-$
$210z^4u^7v^2+28u^9z^6v^2+20u^9z^7v^2-98u^9z^5v^2+238u^{10}v^3z^5-70u^8v^3z^4+154u^9v^3z^5-140u^{10}v^3z^6+$
$80u^8v^3z^3-350u^9v^3z^4-856u^7v^3z^3+540u^6v^2z^3+460u^7v^3z^4+20u^6v^2z^4+70u^8v^3z^5-28u^9v^3z^6-$
$550v^3z^5u^7+14v^2z^5u^6+28v^3z^6u^8+56v^2z^6u^7+20u^{10}v^3z^7+20v^3z^7u^9+5v^3z^8u^{10}-10v^2z^8u^9+$
$98v^2z^5u^5-84v^2z^6u^6-210v^2z^4u^5+10v^2z^8u^8-550v^3u^6z^5+392v^3u^7z^6+70v^3z^5u^5+28v^3z^6u^6-$
$80v^3u^8z^7-10v^3u^9z^8-80v^3z^7u^7+30v^3z^8u^8-88v^4u^8z^3+308v^4u^9z^6+392v^4u^{10}z^6-100v^4u^9z^7-$
$700v^4u^9z^5-644v^4u^{10}z^5-100v^4u^{10}z^7-700v^4u^9z^3+490v^4u^{10}z^4+980v^4u^9z^4+5v^4z^8u^{10}-34v^4u^8z^2+$
$1148v^5u^{10}z^5+896v^5u^9z^3-910v^5u^{10}z^4-952v^5u^{10}z^6-182v^4u^{11}z^5-364v^5u^{11}z^6-100v^4u^{11}z^7+$
$434v^5u^{11}z^5+196v^4u^{11}z^6+140v^5u^{11}z^7+20v^4u^{11}z^8+460v^5u^{10}z^7-20v^5u^{11}z^8-95v^5z^8u^{10}+20v^4z^8u^9+$
$520v^6u^8z^7-100v^4z^8u^8+120v^5u^9z^8-500v^5z^7u^7+20v^5z^8u^8-75v^6z^8u^{10}-546v^6u^{11}z^5+588v^6u^{11}z^6-$
$84v^7u^{11}z^6-300v^6u^{11}z^7+126v^7u^{11}z^5+20v^7u^{11}z^7+60v^6u^{11}z^8-20v^6z^8u^9+520v^6z^7u^7-80v^6z^8u^8+$
$14v^7uz^5-182v^4u^2z^5-644u^3v^4z^5-364v^5u^3z^6+196v^4u^3z^6+420uv^5z^3+490v^4u^2z^4-14v^4z^5u+$
$70v^4uz^4-84v^5u^2z^6-700u^2v^4z^3-140uv^4z^3+140uv^4z^2+42v^5z^5u-40v^8u^4z^7+140u^4v^5z^7-42v^6z^5u+$
$434u^2v^5z^5+84u^2v^6z^6+28v^4u^2z^6-20v^4u^3z^7+60v^5u^3z^7-120u^2z^2v^2-140u^3z^3v^2-14z^5u^3v^2+$
$70z^4u^3v^2+28u^4z^6v^2+28u^5z^6v^2-98u^4z^5v^2-20u^5z^7v^2+260u^4z^3v^2+70u^4z^4v^2+20v^7u^3z^7-28u^2v^7z^6-$
$210uv^5z^4-350u^3v^3z^4+154u^4v^3z^5-28u^5v^3z^6+80u^3v^3z^3-70u^4v^3z^4+280u^2v^3z^3-140u^2v^3z^4+$
$238u^3v^3z^5-140u^4v^3z^6+28v^3z^5u^2-56v^3z^6u^3+20u^6v^3z^7-10u^7v^3z^8+5u^6v^2z^8+20u^5v^3z^7+5u^6v^3z^8+$
$40v^3z^7u^4-10v^3z^8u^5-10u^7z^8v^2+392v^4u^4z^6-100v^4u^4z^7-100v^4u^5z^7+20v^4z^8u^5+120v^5u^7z^8+$
$5v^4u^6z^8-95v^5u^6z^8-20v^5z^8u^5+5v^4z^8u^4+20v^4u^7z^8-60v^6u^3z^7-15v^5u^4z^8+60v^6z^8u^5+380v^6u^6z^7+$
$130v^7u^7z^8-75v^6u^6z^8-20v^6u^7z^8+15v^6z^8u^4+30v^9u^7z^8-80v^8u^7z^8-120u^6z^2+110u^8z^2v^3-60z^3u^6v+$
$280v^8u^8z^2-140v^8u^7z\Big)$

▷ $A_1(z,u,v)=\frac{6}{35}(1-z)^3\big(5z^3u^3+5z^2u^3-20u^2z^2+3zu^3-16u^2z+28uz+u^3-14-6u^2+14u\big)(2uv-u-1)u^2v^4$

▷ $A_2(z,u,v)=-\frac{6}{35}v^4(u-2v+1)(1-uz)^3\big(-5z^3u^3+20z^2u^3-5u^2z^2-28zu^3+16u^2z-3uz+14u^3-14u^2+6u-1\big)$

▷ $A_3(z,u,v)=\frac{1}{35}\Big(-672v^7u^6z^6-3444u^4v^4z+720v^8u^8z^7-600v^7u^8z^7+1848v^7u^5z^5-2520v^4u^6z-$
$1008v^8u^8z^6+1260v^5u^3z+504u^5v^8z^5-6888v^4u^5z-240v^7u^7z^7+2016u^7v^7z^6+630u^4v^8+4704u^5v^4+$
$5040v^6u^4-3564u^5v^3+5166u^4v^4-2394u^4v^3+1848v^4u^3+468v^2u^4-180v^8u^8z^8-252u^3v^4z-$
$420u^8v^7z^5+1512v^5u^6z+6552v^5u^5z+6048v^5u^4z-1680v^7u^5z^4+150v^7u^8z^8+840u^8v^7z^6+504u^8v^8z^5-$
$3528v^7u^7z^5+84v^{10}z^5u^7+1680v^9u^7z^6+336v^9u^8z^6+6552v^8z^5u^6-168v^{10}u^7z^6+2520v^9u^6z^6-$
$720v^9u^7z^7+1080v^8u^7z^7-2016v^8u^6z^6+840v^{10}u^6z^5-1848v^9u^7z^5+60v^9u^8z^8-3528v^8u^7z^6-168v^9u^8z^5-$
$4620v^9u^6z^5-3864v^9u^5z^5-5040v^9u^5z^3-240v^9u^8z^7+60u^6-420u^5v^7-378v^3u^3+3780v^6u^3+126v^4u^2-$
$4032u^5v^5-3654v^5u^3-3150u^4v^7+2520u^5v^6+1050u^6v^4-630u^6v^5-2310v^7u^3-630v^5u^2-6300v^5u^4+$
$1062v^2u^6-1332v^3u^6+1512v^2u^5+630u^2v^8+1260v^6u^2-1260v^7u^2-270u^5v-420u^6v+1260v^8u^3+$
$126v^5u^7-252u^7v^4-162u^7v^2+756v^3u^4z+4032v^3u^5z-5040u^5v^6z-5040v^6u^4z+4200v^7u^5z^2+$
$840v^7u^5z+3360v^7u^4z+4788v^8u^7z^5-1512v^7u^6z^5+1680v^9u^5z^2-1680v^9u^6z^3+5040v^8u^6z^3-$
$4200u^6v^7z^3+2100v^7u^7z^4-5040v^8u^5z^2+6300v^9u^5z^4+4200v^9u^6z^4-5040v^8u^5z^4-1680v^7u^5z^3+$
$7560v^8u^5z^3+840v^9u^7z^4+5040v^7u^6z^4-2520v^8u^7z^4-420v^{10}u^6z^4-8820v^8u^6z^4+1008u^5v^8z^6+$
$2520u^4v^8z^4-2016u^4v^8z^5+2520v^7u^3z+360v^7u^6z^7-360v^8u^5z^7+840v^{10}u^4z^5-168v^9u^3z^6-336v^9u^4z^6+$
$252v^7u^4z^5-252v^8u^3z^5-168v^{10}u^4z^6-30v^9u^5z^8+84v^{10}u^3z^5-336v^9u^4z^5+672v^9u^3z^5+120v^9u^4z^7+$

$84v^9u^2z^5 + 30v^9u^6z^8 + 504v^8u^4z^6 + 1260v^8u^3z^4 + 90v^8u^6z^8 - 504v^7u^5z^6 - 420v^9u^2z^4 - 840v^9u^3z^4 + 540u^6vz+3096u^6v^3z-2088u^6v^2z-1260v^7u^4z^4-600v^9u^6z^7+1008v^9u^5z^6-420v^{10}z^4u^3-1680v^{10}u^4z^4- 672v^{10}z^6u^5 + 840v^{10}u^3z^3 + 2520v^9u^4z^4 + 1512v^{10}u^5z^5 - 2520v^8u^3z^3 + 840v^9u^2z^3 - 840v^9u^2z^2 + 420v^9u^2z+120v^{10}z^7u^5-936u^5v^2z-4200v^9u^4z^3+2520u^3v^8z^2+840u^3v^9z^2-2520v^6u^3z+2520v^7u^4z^3+ 3360u^4v^9z^2 - 2520u^3v^8z + 1260v^8u^4z - 2520v^8u^4z^2 - 840v^9u^4z - 420v^9u^3z - 252v^5u^7z + 420u^7v^4z - 432u^7v^3z - 36u^7v^2z - 2520v^7u^4z^2 + 342u^7v^3 - 210v^9u^3 - 30u^8 + 30u^7 + 120u^7v^{10}z^7 - 840u^4v^{10}z^2 + 420u^3v^{10}z+840u^5v^{10}z^3+1680u^4v^{10}z^3-840u^3v^{10}z^2+360u^8v^2z+300u^7vz-1680u^5v^{10}z^4-252u^8v^3z- 240u^8vz - 210u^2v^9 - 672u^6v^{10}z^6 - 30u^6v^{10}z^8 - 30u^7v^{10}z^8 + 240u^6v^{10}z^7 + 84u^8v^4z - 30u^7v + 120u^8v - 90v^7u^7z^8 + 120v^9u^7z^8 - 90v^8u^7z^8 - 120u^7z + 60u^8z + 126u^8v^3 - 180u^8v^2 - 42u^8v^4\Big)$

▷ $A_4(z,u,v) = \frac{1}{35}\Big(30 - 840v^7u^6z^6 + 6888u^4v^4z - 1080v^8u^8z^7 - 252u^9v^7z^5 - 1008u^9v^8z^6 + 240v^7u^8z^7-1260u^9v^8z^4+420v^7u^5z^5+1260v^7u^8z^4+252v^4u^6z+2016v^8u^8z^6-1512v^5u^3z+360u^{10}v^8z^7- 504u^5v^8z^5 - 3096u^3v^3z - 360u^9v^7z^7 + 3444v^4u^5z - 300vu^2z + 600v^7u^7z^7 + 504v^7u^9z^6 + 840v^9u^5z - 420v^4u^2z - 2016u^7v^7z^6 + 420u^2v + 30uv - 60uz + 1260u^6v^7 - 630u^4v^8 - 630u^6v^8 - 1260u^5v^8 - 1260u^6v^6+210v^9u^5-1512v^2u^3-1848u^5v^4-5040v^6u^4+378u^5v^3-5166u^4v^4+2394u^4v^3-4704v^4u^3- 468v^2u^4 - 360uv^2z + 240uvz + 2520v^7u^6z^2 + 180v^8u^8z^8 + 2520u^6v^8z^2 + 2520u^3v^4z - 1848u^8v^7z^5 + 252v^5u^2z - 1260v^5u^6z - 6048v^5u^5z - 6552v^5u^4z - 420v^9u^7z - 2100v^7u^5z^4 - 150v^7u^8z^8 + 90v^8u^9z^8 + 90u^9v^7z^8-90u^{10}v^8z^8+672u^8v^7z^6-504u^8v^8z^5+1512v^7u^7z^5+1680v^{10}u^8z^4+840v^9u^9z^4-840v^{10}z^5u^7- 504v^8u^{10}z^6 - 1680v^9u^7z^6 - 2520v^9u^8z^6 - 4788v^8z^5u^6 + 2016u^9v^8z^5 - 240v^{10}u^9z^7 + 672v^{10}u^8z^6 + 168v^{10}u^7z^6+420v^{10}u^9z^4-840v^{10}u^9z^5-336v^9u^6z^6+240v^9u^7z^7+420v^9u^{10}z^4-720v^8u^7z^7+1008v^8u^6z^6- 84v^{10}u^6z^5+4620v^9u^7z^5-60v^9u^8z^8+3528v^8u^7z^6-120v^{10}u^8z^7+3864v^9u^8z^5-840v^{10}u^8z^3+840v^{10}u^7z^2- 420v^{10}u^6z+1848v^9u^6z^5+168v^9u^5z^5+30v^{10}u^9z^8-1512v^{10}u^8z^5+840v^9u^8z^2+1680v^9u^5z^3+720v^9u^8z^7- 840v^9u^9z^3+30v^9u^{11}z^8-30v^9u^{10}z^8+30v^{10}u^{10}z^8+600v^9u^9z^7+180v^2-120v+42v^4-126v^3+120u^2z- 60u^2 - 30u + 2310u^5v^7 - 342v^3u + 3564v^3u^3 + 162v^2u - 1062u^2v^2 - 2520v^6u^3 + 210v^9u^6 + 270u^3v - 1050v^4u^2 + 252v^4u + 1332v^3u^2 + 3654u^5v^5 - 126v^5u + 4032v^5u^3 + 3150u^4v^7 - 3780u^5v^6 - 126u^6v^4 + 630u^6v^5 + 420v^7u^3 + 630v^5u^2 + 6300v^5u^4 + 672v^{10}u^9z^6 + 336v^9u^9z^5 - 84v^{10}u^{10}z^5 - 120v^9u^{11}z^7 + 252v^8u^{10}z^5 - 672v^9u^{10}z^5 + 168v^9u^{11}z^6 - 84v^9u^{11}z^5 - 1008v^9u^9z^6 - 120v^{10}u^{10}z^7 - 120v^9u^9z^8 + 168v^{10}u^{10}z^6 + 336v^9u^{10}z^6 - 4032v^3u^4z - 756v^3u^5z - 1680v^{10}u^7z^3 + 840v^{10}u^6z^2 + 5040u^5v^6z + 2520u^6v^8z+5040v^6u^4z-4200v^7u^5z^2-3360v^7u^5z-840v^9u^7z^2-840v^7u^4z-6552v^8u^7z^5-1260v^8u^5z+ 936v^2u^4z + 3528v^7u^6z^5 - 1680v^9u^5z^2 - 840v^{10}u^6z^3 + 5040v^9u^6z^3 - 7560v^8u^6z^3 - 2520v^8u^7z^2 + 1680u^6v^7z^3 + 1680v^7u^7z^4 - 2520v^8u^8z^4 + 5040v^8u^5z^2 - 840v^9u^5z^4 - 4200v^9u^6z^4 + 2520v^8u^5z^4 + 420v^9u^6z + 432u^2v^3z + 36u^2v^2z + 4200v^7u^5z^3 + 4200v^9u^7z^3 - 540vzu^3 + 2088v^2zu^3 - 84v^4zu + 252v^3uz - 5040v^8u^5z^3 + 2520v^8u^8z^3 - 2520v^9u^8z^4 - 6300v^9u^7z^4 + 1680v^{10}u^7z^4 - 5040v^7u^6z^4 + 5040v^8u^7z^4 + 420v^{10}u^6z^4 + 8820v^8u^6z^4 - 3360v^9u^6z^2 - 2520v^7u^7z^3 - 2520v^7u^6z + 2520v^6u^6z\Big)$

A.2. **The polynomials $B_i(z,u)$.**

▷ $B_0(z,u) = 10uz-120u^2z+1140u^2z^2-8190u^5z^2-4236z^5u^6-1980z^3u^5-680z^4u^6-980u^4z- 605z^4u^4-4826z^5u^5+4354z^6u^6-5020z^3u^4+8585z^4u^5+500zu^3+2980z^3u^3-4140z^2u^3+8790z^2u^4- 1470u^6 - 490u^2 + 1470u^3 - 2450u^4 + 70u + 980u^6z + 2450u^5 + 8190u^6z^2 + 1040u^9z^7 + 960u^{10}z^7 - 1040u^7z^7+518u^7z^6+490u^7-70u^8+4236u^8z^5+605u^9z^4-518u^8z^6-4354u^9z^6-55u^8z^8-8585u^8z^4+ 4826u^9z^5+112u^{10}z^6-500u^7z+55u^9z^8-390u^{10}z^8+490u^{10}z^4+5020u^8z^3-8790u^7z^2+4140u^8z^2- 2980u^9z^3+150u^{11}z^8+680u^7z^4-1694u^{10}z^5-440u^{11}z^7+196u^{11}z^6+462u^{11}z^5+1980u^7z^3-112u^5z^6- 140uz^2+14u^2z^5+1694u^4z^5+40u^5z^8-960u^6z^7+440u^5z^7-5u^4z^8-28u^2z^6+140u^3z^6+20u^3z^7- 70uz^4+14z^5u-140u^4z^7-196u^4z^6-150u^6z^8-462u^3z^5+350u^2z^4-490u^3z^4+140uz^3-980u^2z^3- 20u^{13}z^7 + 140u^{10}z^2 - 140u^{11}z^3 + 70u^{12}z^4 + 5u^{13}z^8 + 28u^{13}z^6 - 14u^{13}z^5 - 40u^{12}z^8 + 140u^{12}z^7 - 140u^{12}z^6 - 350u^{11}z^4 - 14u^{12}z^5 - 10u^9z + 120u^8z + 390u^7z^8 - 1140u^9z^2 + 980u^{10}z^3$

▷ $B_1(z,u) = -6(1+u)u^5(1-z)^3\Big(5z^3u^3 + 5z^2u^3 - 20u^2z^2 + 3u^3z - 16u^2z + 28uz + u^3 - 6u^2 + 14u - 14\Big)$

▷ $B_2(z,u) = 6(1+u)\Big(5z^3u^3 - 20z^2u^3 + 5u^2z^2 - 16u^2z + 3uz + 28u^3z - 6u - 14u^3 + 14u^2 + 1\Big)(1-uz)^3$

A.3. The polynomials $C_i(z,u)$.

▷ $C_0(z,u) = 950uz - 6360u^2z + 5120u^2z^2 + 10850u^5z^2 - 58440z^5u^6 - 39420z^3u^5 + 32210z^4u^6 - 22540u^4z + 275z^4u^4 - 38218z^5u^5 + 38066z^6u^6 - 32540z^3u^4 + 69865z^4u^5 + 17260zu^3 + 17660z^3u^3 - 16280z^2u^3 + 29070z^2u^4 + 22540u^6z - 10850u^6z^2 + 13960u^9z^7 + 7320u^{10}z^7 - 13960u^7z^7 + 21574u^7z^6 + 58440u^8z^5 - 275u^9z^4 - 21574u^8z^6 - 38066u^9z^6 + 1600u^8z^8 - 69865u^8z^4 + 38218u^9z^5 - 1540u^{10}z^6 - 17260u^7z - 1600u^9z^8 - 2715u^{10}z^8 + 3710u^{10}z^4 + 32540u^8z^3 - 29070u^7z^2 + 16280u^8z^2 - 17660u^9z^3 + 885u^{11}z^8 - 32210u^7z^4 - 9352u^{10}z^5 - 2680u^{11}z^7 + 1484u^{11}z^6 + 2310u^{11}z^5 + 39420u^7z^3 + 1540u^5z^6 - 700uz^2 + 28u^2z^5 + 9352u^4z^5 + 215u^5z^8 - 7320u^6z^7 + 2680u^5z^7 - 25u^4z^8 - 140u^2z^6 + 784u^3z^6 + 100u^3z^7 - 350uz^4 + 70z^5u - 760u^4z^7 - 1484u^4z^6 - 885u^6z^8 - 2310u^3z^5 + 1960u^2z^4 - 3710u^3z^4 + 700uz^3 - 5320u^2z^3 - 100u^{13}z^7 + 700u^{10}z^2 - 700u^{11}z^3 + 350u^{12}z^4 + 25u^{13}z^8 + 140u^{13}z^6 - 70u^{13}z^5 - 215u^{12}z^8 + 760u^{12}z^7 - 784u^{12}z^6 - 1960u^{11}z^4 - 28u^{12}z^5 - 950u^9z + 6360u^8z + 2715u^7z^8 - 5120u^9z^2 + 5320u^{10}z^3$

▷ $C_1(z,u) = 18480u^5z^2 - 36960u^6z^3 - 18480z^5u^6 - 18480z^3u^5 + 36960z^4u^6 + 3024u^4z - 1848z^5u^5 + 3696z^6u^6 + 9240z^4u^5 - 504zu^3 - 1260u^6 + 252u^2 - 1260u^3 + 2268u^4 + 6384u^6z - 16800u^5z + 588u^5 + 18480u^6z^2 - 2640u^9z^7 - 2640u^7z^7 + 14784u^7z^6 + 1212u^7 - 600u^8 + 120u^9 - 18480u^8z^5 + 14784u^8z^6 + 3696u^9z^6 + 660u^8z^8 + 9240u^8z^4 - 1848u^9z^5 - 3864u^7z + 660u^9z^8 + 36960u^7z^4 - 5280u^8z^7 - 18480u^7z^3 - 33264u^7z^5 - 240u^9z + 1440u^8z$

▷ $C_2(z,u) = -120 + 240uz - 1440u^2z - 18480u^5z^2 + 36960u^6z^3 + 18480z^5u^6 + 18480z^3u^5 - 36960z^4u^6 - 6384u^4z + 1848z^5u^5 - 3696z^6u^6 - 9240z^4u^5 + 3864zu^3 + 1260u^6 - 1212u^2 + 1260u^3 - 588u^4 + 600u - 3024u^6z + 16800u^5z - 2268u^5 - 18480u^6z^2 + 2640u^9z^7 + 2640u^7z^7 - 14784u^7z^6 - 252u^7 + 18480u^8z^5 - 14784u^8z^6 - 3696u^9z^6 - 660u^8z^8 - 9240u^8z^4 + 1848u^9z^5 + 504u^7z - 660u^9z^8 - 36960u^7z^4 + 5280u^8z^7 + 18480u^7z^3 + 33264u^7z^5$

A.4. The polynomials $D_i(z,u)$.

▷ $D_0(z,u) = -\frac{u}{350}\Big(-1960uz^3 - 19000u^3z^3 + 19050u^3z^2 - 84uz^5 + 370z + 840uz^4 + 4844u^3z^5 + 400u^3z^4 + 2520u^9z^4 + 1960u^9z^3 + 140u^9z^2 - 8900u^8z^3 - 1650u^8z^2 + 336u^2z^6 - 644u^2z^5 - 2520u^2z^4 + 8900u^2z^3 - 7650u^2z^2 - 28uz^6 + 2450u - 1008u^3z^6 + 7650u^7z^2 + 1650uz^2 + 20u^2z^7 - 350 - 7350u^2 + 7350u^5 - 2450u^6 - 140z^2 - 70z^4 + 140z^3 + 14z^5 - 2340uz + 5900u^2z - 6860u^3z + 6860u^5z - 5900u^6z + 2340u^7z - 370u^8z - 20790u^4z^2 + 5840u^4z^3 + 22630u^4z^4 - 15546u^4z^5 + 20790u^5z^2 - 17040u^5z^4 - 1528u^5z^5 - 19050u^6z^2 - 5840u^6z^3 + 17040u^6z^4 - 400u^8z^4 + 15546u^8z^5 - 4844u^9z^5 + 12250u^3 - 12250u^4 + 350u^7 + 19000u^7z^3 - 22630u^7z^4 + 1\Big)$

▷ $D_1(z,u) = \frac{-6u}{35}\Big(-7 + 42u - 98u^2 - 98u^5 + 82u^6 + 14uz - 98u^2z + 294u^3z - 630u^4z + 434u^5z - 238u^6z + 74u^7z - 10u^8z + 280u^4z^2 - 280u^4z^3 + 140u^4z^4 - 28u^4z^5 + 280u^5z^2 - 560u^5z^3 + 560u^5z^4 - 280u^5z^5 - 280u^6z^3 + 560u^6z^4 - 504u^6z^5 - 28u^8z^5 + 98u^3 + 28u^4 - 32u^7 + 5u^8 + 140u^7z^4 - 280u^7z^5 + 224u^7z^6 + 224u^6z^6 + 56u^8z^6 - 80u^7z^7 - 40u^8z^7 + 10u^8z^8 + 56u^5z^6 - 40u^6z^7 + 10u^7z^8\Big)$

▷ $D_2(z,u) = -\frac{1}{35}\Big(30 - 192u + 492u^2 + 588u^5 - 588u^6 - 60uz + 444u^2z - 1428u^3z + 2604u^4z - 3780u^5z + 1764u^6z - 588u^7z + 84u^8z + 1680u^5z^2 - 1680u^5z^3 + 840u^5z^4 - 168u^5z^5 + 1680u^6z^2 - 3360u^6z^3 + 3360u^6z^4 - 1680u^6z^5 + 840u^8z^4 - 1680u^8z^5 - 168u^9z^5 - 588u^3 + 168u^4 + 252u^7 - 42u^8 - 1680u^7z^3 + 3360u^7z^4 - 3024u^7z^5 + 1344u^7z^6 + 336u^6z^6 + 1344u^8z^6 + 336u^9z^6 - 240u^7z^7 - 240u^9z^7 - 480u^8z^7 + 60u^8z^8 + 60u^9z^8\Big)$

Appendix B. Tables

The next Tables collect the full statements of the theorems, considering the general cases as they appear in the main text of the paper as well as the special cases not listed there.

j	$d_{n,j}$
1	$\frac{6}{5}$, for $n \geq 5$
	$d_{1,1} = 1,\ d_{2,1} = \frac{3}{2},\ d_{3,1} = 1,\ d_{4,1} = \frac{5}{4}$
2	$\frac{18}{5} - \frac{3}{n}$, for $n \geq 6$
	$d_{2,2} = \frac{3}{2},\ d_{3,2} = 3,\ d_{4,2} = \frac{11}{4},\ d_{5,2} = 3$
3	$\frac{136}{35} - \frac{6}{n}$, for $n \geq 7$
	$d_{3,3} = 1,\ d_{4,3} = \frac{11}{4},\ d_{5,3} = \frac{13}{5},\ d_{6,3} = \frac{29}{10}$
4	$\frac{9}{2} - \frac{9}{n}$, for $n \geq 8$
	$d_{4,4} = \frac{5}{4},\ d_{5,4} = 3,\ d_{6,4} = \frac{29}{10},\ d_{7,4} = \frac{113}{35}$
$5 \leq j \leq n-4$	$-\frac{12}{7}H_n + \frac{12}{7}H_j + \frac{12}{7}H_{n+1-j} - \frac{6}{7j} - \frac{6}{7(n+1-j)} + \frac{79}{70} - \frac{3(3j-5)}{7n} + \frac{6(j-1)^2}{7n^2} + \frac{2(2j-3)(j-1)^2}{7n^3} + \frac{3(j-2)(j-1)^3}{7n^4} - \frac{3(2j-5)(j-1)^4}{7n^5} + \frac{2(j-3)(j-1)^5}{7n^6}$

TABLE 1. Average number of descendants of the j^{th} node in a random LBST of size n.

j	$d_{n,j}^{(2)}$
1	$\frac{2}{5}$ for $n \geq 5$
	$d_{1,1}^{(2)} = 0,\ d_{2,1}^{(2)} = 1,\ d_{3,1}^{(2)} = 0,\ d_{4,1}^{(2)} = \frac{1}{2}$
2	$\frac{36}{5} H_n - \frac{206}{25}$ for $n \geq 6$
	$d_{2,2}^{(2)} = 1,\ d_{3,2}^{(2)} = 6,\ d_{4,2}^{(2)} = \frac{13}{2},\ d_{5,2}^{(2)} = \frac{41}{5}$
3	$\frac{72}{5} H_n - \frac{4534}{175} + \frac{6}{n}$ for $n \geq 7$
	$d_{3,3}^{(2)} = 0,\ d_{4,3}^{(2)} = \frac{13}{2},\ d_{5,3}^{(2)} = 8,\ d_{6,3}^{(2)} = \frac{52}{5}$
4	$\frac{108}{5} H_n - \frac{7886}{175} - \frac{36}{5(n-1)} + \frac{126}{5n}$ for $n \geq 8$
	$d_{4,4}^{(2)} = \frac{1}{2},\ d_{5,4}^{(2)} = \frac{41}{5},\ d_{6,4}^{(2)} = \frac{52}{5},\ d_{7,4}^{(2)} = \frac{468}{35}$
$5 \leq j \leq n-4$	$\left(\frac{36n}{5} - \frac{12}{35}\right) H_n + \left(\frac{36j}{5} - \frac{36n}{5} - \frac{48}{7}\right) H_{n+1-j} + \left(\frac{12}{35} - \frac{36j}{5}\right) H_j$ $- \frac{132}{35j} - \frac{132}{35(n+1-j)} + \frac{3489}{175} - \frac{33j}{5} + \left(\frac{66}{7} - \frac{429j}{35} + \frac{33j^2}{5}\right) \frac{1}{n}$ $+ \frac{132(j-1)^2}{35n^{\underline{2}}} + \frac{44(2j-3)(j-1)^{\underline{2}}}{35n^{\underline{3}}} + \frac{66(j-2)(j-1)^{\underline{3}}}{35n^{\underline{4}}}$ $- \frac{66(2j-5)(j-1)^{\underline{4}}}{35n^{\underline{5}}} + \frac{44(j-3)(j-1)^{\underline{5}}}{35n^{\underline{6}}}$

TABLE 2. Second factorial moment of the number of descendants of the j^{th} node in a random LBST of size n.

m	$p_{n,m} = \mathbb{P}\left[D_n = m\right]$
1	$\frac{3}{7}\left(1+\frac{1}{n}\right)$ for $n \geq 6$
	$p_{1,1} = 1,\ p_{2,1} = \frac{1}{4},\ p_{3,1} = \frac{2}{9},\ p_{4,1} = \frac{1}{8},\ p_{5,1} = \frac{13}{125}$
2	$\frac{1}{7}\left(1+\frac{1}{n}\right)$, for $n \geq 6$
	$p_{2,2} = \frac{1}{4},\ p_{3,2} = 0,\ p_{4,2} = \frac{1}{16},\ p_{5,2} = \frac{4}{125}$
3	$\frac{3}{35}\left(1+\frac{1}{n}\right)$, for $n \geq 6$
	$p_{3,3} = \frac{1}{9},\ p_{4,3} = 0,\ p_{5,3} = \frac{3}{5}$
4	$\frac{2}{35}\left(1+\frac{1}{n}\right)$ for $n \geq 6$
	$p_{4,4} = \frac{1}{16},\ p_{5,4} = 0$
$5 \leq m < n$	$\frac{12}{7}\frac{(n+2+m)(n-1-m)}{n^2(m+2)(m+1)} - \frac{12}{7}\frac{m^{\underline{5}}}{n^2 n^{\underline{5}}} + \frac{12}{7n^2}$
n	$\frac{1}{n}$

TABLE 3. Probability that a random node in a random LBST of size n has m descendants.

j	$a_{n,j}$
1	$\frac{6}{5}H_n - \frac{6}{25}$, for $n \geq 5$
	$a_{1,1} = 1,\ a_{2,1} = \frac{3}{2},\ a_{3,1} = 2,\ a_{4,1} = \frac{9}{4}$
2	$\frac{6}{5}H_n - \frac{21}{25}$, for $n \geq 6$
	$a_{2,2} = \frac{3}{2},\ a_{3,2} = 1,\ a_{4,2} = \frac{7}{4},\ a_{5,2} = \frac{19}{10}$
3	$\frac{6}{5}H_n - \frac{87}{175} - \frac{6}{5}\frac{1}{n^{\underline{2}}}$, for $n \geq 7$
	$a_{3,3} = 2,\ a_{4,3} = \frac{7}{4},\ a_{5,3} = \frac{11}{5},\ a_{6,3} = \frac{12}{5}$
4	$\frac{6}{5}H_n - \frac{11}{25} - \frac{18}{5}\frac{1}{n^{\underline{2}}} - \frac{12}{5}\frac{1}{n^{\underline{3}}}$, for $n \geq 8$
	$a_{4,4} = \frac{9}{4},\ a_{5,4} = \frac{19}{10},\ a_{6,4} = \frac{12}{5},\ a_{7,4} = \frac{18}{7}$
$5 \leq j \leq n-4$	$\frac{24}{35}H_n + \frac{18}{35}H_j + \frac{18}{35}H_{n+1-j} + \frac{12}{35j} + \frac{12}{35(n+1-j)} - \frac{279}{175} - \frac{6}{7n} + \frac{18j}{35n} - \frac{12(j-1)^2}{35n^{\underline{2}}} - \frac{4(2j-3)(j-1)^{\underline{2}}}{35n^{\underline{3}}} - \frac{6(j-2)(j-1)^{\underline{3}}}{35n^{\underline{4}}} + \frac{6(2j-5)(j-1)^{\underline{4}}}{35n^{\underline{5}}} - \frac{4(j-3)(j-1)^{\underline{5}}}{35n^{\underline{6}}}$

TABLE 4. Average number of ascendants of the j^{th} node in a random LBST of size n.

j	$a_{n,j}^{(2)}$
1	$\frac{36}{25}H_n^2 - \frac{144}{125}H_n - \frac{36}{25}H_n^{(2)} + \frac{766}{625} + \frac{3}{125}\frac{1}{n} - \frac{12}{125}\frac{1}{n-1} + \frac{18}{125}\frac{1}{n-2} - \frac{12}{125}\frac{1}{n-3} + \frac{3}{125}\frac{1}{n-4}$ for $n \geq 5$
	$a_{1,1}^{(2)} = 0$, $a_{2,1}^{(2)} = 1$, $a_{3,1}^{(2)} = 2$, $a_{4,1}^{(2)} = 3$
2	$\frac{36}{25}H_n^2 - \frac{324}{125}H_n - \frac{36}{25}H_n^{(2)} + \frac{1726}{625} - \frac{27}{125}\frac{1}{n} + \frac{108}{125}\frac{1}{n-1} - \frac{162}{125}\frac{1}{n-2} + \frac{108}{125}\frac{1}{n-3} - \frac{27}{125}\frac{1}{n-4}$ for $n \geq 6$
	$a_{2,2}^{(2)} = 1$, $a_{3,2}^{(2)} = 0$, $a_{4,2}^{(2)} = 2$, $a_{5,2}^{(2)} = \frac{11}{5}$
3	$\frac{36}{25}H_n^2 - \left(\frac{1548}{875} + \frac{72}{25}\frac{1}{n(n-1)}\right)H_n - \frac{36}{25}H_n^{(2)} + \frac{77094}{30625} + \frac{2169}{875}\frac{1}{n} - \frac{2892}{875}\frac{1}{n-1} + \frac{1476}{875}\frac{1}{n-2} - \frac{1044}{875}\frac{1}{n-3} + \frac{321}{875}\frac{1}{n-4} - \frac{36}{875}\frac{1}{n-5} + \frac{6}{875}\frac{1}{n-6} - \frac{72}{25}\frac{1}{n^2} + \frac{72}{25}\frac{1}{(n-1)^2}$ for $n \geq 7$
	$a_{3,3}^{(2)} = 2$, $a_{4,3}^{(2)} = 2$, $a_{5,3}^{(2)} = \frac{18}{5}$, $a_{6,3}^{(2)} = \frac{22}{5}$
4	$\frac{36}{25}H_n^2 + \left(-\frac{204}{125} + \frac{144}{25}\frac{1}{n} - \frac{72}{25}\frac{1}{n-1} - \frac{72}{25}\frac{1}{n-2}\right)H_n - \frac{36}{25}H_n^{(2)} + \frac{11077}{4375} + \frac{5517}{875}\frac{1}{n} - \frac{786}{125}\frac{1}{n-1} - \frac{246}{125}\frac{1}{n-2} + \frac{378}{125}\frac{1}{n-3} - \frac{177}{125}\frac{1}{n-4} + \frac{54}{125}\frac{1}{n-5} - \frac{12}{125}\frac{1}{n-6} + \frac{6}{875}\frac{1}{n-7} - \frac{144}{25}\frac{1}{n^2} + \frac{72}{25}\frac{1}{(n-1)^2} + \frac{72}{25}\frac{1}{(n-2)^2}$ for $n \geq 8$,
	$a_{4,4}^{(2)} = 3$, $a_{5,4}^{(2)} = \frac{11}{5}$, $a_{6,4}^{(2)} = \frac{22}{5}$, $a_{7,4}^{(2)} = \frac{36}{7}$

j	$a^{(2)}_{n,j}$
$5 \le j \le n-4$	$\frac{576}{245}H_nH_{n-j} + \frac{576}{245}H_nH_j - \frac{288}{245}H_n^2 + \frac{324}{1225}H_{n-j}^2 - \frac{1368}{1225}H_{n-j}H_j + \frac{324}{1225}H_j^2$ $+ \Big(\frac{72}{42875}\frac{227j+525}{j} + \frac{792}{245}\frac{1}{n+1-j} - \frac{2}{1225}\frac{2j^6-6j^5-55j^4+120j^3+953j^2-1014j+540}{n}$ $+ \frac{4}{245}\frac{(j-1)^2(j^4-4j^3-11j^2+30j+72)}{n-1} - \frac{4}{245}\frac{(j-1)(j-2)(2j^4-12j^3+13j^2+15j-36)}{n-2}$ $+ \frac{8}{245}\frac{(j-1)^2(j-2)^2(j-3)^2}{n-3} - \frac{2}{245}\frac{(j-1)(j-2)(j-3)(j-4)(2j^2-10j+15)}{n-4}$ $+ \frac{4}{1225}\frac{(j-1)(j-2)(j-3)^2(j-4)(j-5)}{n-5}\Big)H_n$ $+ \Big(-\frac{36}{42875}\frac{5697j+70}{j} - \frac{288}{245}\frac{1}{n+1-j} + \frac{1}{6125}\frac{20j^6-102j^5-445j^4+2040j^3+8165j^2+402j+360}{n}$ $- \frac{2}{1225}\frac{(j-1)(10j^5-71j^4+14j^3+557j^2-42j-1224)}{n-1} + \frac{2}{1225}\frac{(j-1)(j-2)(20j^4-162j^3+319j^2+129j-612)}{n-2}$ $- \frac{4}{1225}\frac{(j-1)(j-2)(j-3)(10j^3-81j^2+194j-102)}{n-3} + \frac{1}{1225}\frac{(j-1)(j-2)(j-3)(j-4)(20j^2-142j+255)}{n-4}$ $- \frac{2}{6125}\frac{(j-1)(j-2)(j-3)(j-4)(j-5)(10j-51)}{n-5}\Big)H_{n-j}$ $+ \Big(-\frac{36}{42875}\frac{2267j+2030}{j} - \frac{288}{245}\frac{1}{n+1-j} + \frac{1}{6125}\frac{20j^6-18j^5-655j^4+360j^3+10895j^2-20682j+10440}{n}$ $- \frac{2}{1225}\frac{(j-1)(10j^5-29j^4-154j^3+263j^2+882j-216)}{n-1} + \frac{2}{1225}\frac{(j-1)(j-2)(20j^4-78j^3-59j^2+171j-108)}{n-2}$ $- \frac{4}{1225}\frac{(j-1)(j-2)(j-3)(10j^3-39j^2+26j-18)}{n-3} + \frac{1}{1225}\frac{(j-1)(j-2)(j-3)(j-4)(20j^2-58j+45)}{n-4}$ $- \frac{2}{6125}\frac{(j-1)(j-2)(j-3)(j-4)(j-5)(10j-9)}{n-5}\Big)H_j$ $+ \frac{288}{245}H_n^{(2)} - \frac{3204}{1225}H_{n-j}^{(2)} - \frac{3204}{1225}H_j^{(2)} + \frac{12}{1225}\frac{1}{j+4} - \frac{24}{175}\frac{1}{j+3} + \frac{792}{1225}\frac{1}{j+2} - \frac{2088}{1225}\frac{1}{j+1}$ $- \frac{53148}{42875}\frac{1}{j} - \frac{1044}{875}\frac{1}{j-1} + \frac{396}{875}\frac{1}{j-2} - \frac{12}{125}\frac{1}{j-3} + \frac{6}{875}\frac{1}{j-4} + \frac{144}{175}\frac{1}{j^2} + \frac{6}{875}\frac{1}{n-3-j} - \frac{12}{125}\frac{1}{n-2-j}$ $+ \frac{396}{875}\frac{1}{n-1-j} - \frac{1044}{875}\frac{1}{n-j} - \frac{24}{8575}\frac{1123j-420}{(n+1-j)j} - \frac{2088}{1225}\frac{1}{n+2-j} + \frac{792}{1225}\frac{1}{n+3-j} - \frac{24}{175}\frac{1}{n+4-j}$ $+ \frac{12}{1225}\frac{1}{n+5-j} - \frac{792}{245}\frac{1}{(n+1-j)^2} + \frac{2297696}{214375}$ $- \frac{5522j^7-8166j^6-167395j^5+142845j^4+3249683j^3-1044579j^2-2445930j+1096200}{643125\,nj}$ $+ \frac{2(2761j^7-12366j^6-25787j^5+116883j^4+176746j^3-361269j^2+120672j+22680)}{128625\,(n-1)j}$ $- \frac{2(5522j^7-41298j^6+82913j^5+11427j^4-226090j^3+256698j^2-49482j-22680)}{128625\,(n-2)j}$ $+ \frac{4(j-1)(j-3)(2761j^5-17888j^4+38198j^3-28798j^2-5718j+3780)}{128625\,(n-3)j}$ $- \frac{5522j^7-74430j^6+407885j^5-1158375j^4+1755053j^3-1233465j^2+169920j+113400}{128625\,(n-4)j}$ $+ \frac{2(2761j^7-45498j^6+299185j^5-999255j^4+1757614j^3-1456887j^2+313560j+113400)}{643125\,(n-5)j}$ $+ \frac{2(2j^6-6j^5-55j^4+120j^3+953j^2-1014j+540)}{1225\,n^2} - \frac{4(j-1)^2(j^4-4j^3-11j^2+30j+72)}{245\,(n-1)^2}$ $+ \frac{4(j-1)(j-2)(2j^4-12j^3+13j^2+15j-36)}{245\,(n-2)^2} - \frac{8(j-1)^2(j-2)^2(j-3)^2)}{245\,(n-3)^2}$ $+ \frac{2(j-1)(j-2)(j-3)(j-4)(2j^2-10j+15)}{245\,(n-4)^2} - \frac{4(j-1)(j-2)(j-3)^2(j-4)(j-5)}{1225\,(n-5)^2}$

TABLE 5. Second factorial moment of the number of ascendants of the j^{th} node in a random LBST of size n.