ON NONCROSSING AND NONNESTING PARTITIONS FOR CLASSICAL REFLECTION GROUPS

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Abstract. The number of noncrossing partitions of $\{1, 2, \ldots, n\}$ with fixed block sizes has a simple closed form, given by Kreweras, and coincides with the corresponding number for nonnesting partitions. We show that a similar statement is true for the analogues of such partitions for root systems B and C, defined recently by Reiner in the noncrossing case and Postnikov in the nonnesting case. Some of our tools come from the theory of hyperplane arrangements.

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1. Introduction

A *noncrossing* partition of the set $[n] = \{1, 2, \ldots, n\}$ is a set partition π of $[n]$ such that if $a < b < c < d$ and a, c are contained in a block B of π, while b, d are contained in a block B' of π, then $B = B'$. Noncrossing partitions are classical combinatorial objects with an extensive literature, see [7, 9, 11, 12, 13, 17, 18, 19, 22]. Natural analogues of noncrossing partitions for the classical reflection groups of type B, C and D were introduced by Reiner [16] and were shown to have similar enumerative and structural properties with those of the noncrossing partitions, which are associated to the reflection groups of type A.

Nonnesting partitions were recently defined by Postnikov (see [16, Remark 2]) in a uniform way for all irreducible root systems associated to Weyl groups. Let Φ be such a root system and Φ^+ be a choice of positive roots. Define the *root order* on Φ^+ by $\alpha \leq \beta$ if $\alpha, \beta \in \Phi^+$ and $\beta - \alpha$ is a linear combination of positive roots with nonnegative coefficients. A *nonnesting partition* on Φ is simply an antichain in the root order of Φ. Postnikov observed that the nonnesting partitions on Φ are in bijection with certain regions of an affine hyperplane arrangement related to the Coxeter arrangement associated to Φ. For $\Phi = A_{n-1}$, nonnesting partitions are naturally in bijection with set partitions π of $[n]$ such that if $a < b < c < d$ and a, d are *consecutive* elements of a block B of π, then b, c are not both contained in a block B' of π. This concept has reappeared in a geometric context in [3].

A number of striking similarities between noncrossing and nonnesting partitions were pointed out by Postnikov and recorded by Reiner [16, Remark 2]. For the case

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of the root system A_{n-1}, the number of both noncrossing and nonnesting partitions is the nth Catalan number and their distribution according to the number of blocks is the same. Moreover, it follows from Postnikov's observation and one of the results in [1] [2, Part II] that, for $\Phi = B_n, C_n$ or D_n, as well as A_{n-1}, the number of nonnesting partitions on Φ coincides with that of noncrossing partitions, as computed in [16].

In this paper we strengthen these observations by fixing the block sizes. Our motivation comes from a simple formula of Kreweras [11] for the number of noncrossing partitions of $[n]$ of a fixed type λ, the integer partition of n whose parts are the sizes of the blocks. It is not hard to prove (see e.g. [3, §4]) that the number of nonnesting partitions of $[n]$ of type λ is given by the same formula. We prove similar formulas for the root systems B_n and C_n which again coincide in the noncrossing and nonnesting case.

The paper is structured as follows. In Section 2 we give some more background and definitions and state our results, after we extend the notion of type λ to nonnesting partitions on B_n, C_n and D_n. In Section 3 we discuss the case of A_{n-1} and provide an explicit bijection between noncrossing and nonnesting partitions which preserves the type λ. In Section 4 we prove the analogue of the result of Kreweras for noncrossing partitions for the other classical reflection groups. In Section 5 we show that the number of nonnesting partitions on B_n and C_n of type λ is given by the same formula as the corresponding number of noncrossing partitions. Our arguments exploit the connections between nonnesting partitions and hyperplane arrangements and use the "finite field method" of [1] [2, Part II]. Section 6 contains some concluding remarks and related questions.

2. Background and results

Noncrossing partitions. We first recall the definition of noncrossing partitions for the classical reflection groups from [16]. In this section, Φ denotes a root system in one of the infinite families A_{n-1}, B_n, C_n and D_n.

Partitions of $[n]$ are naturally in bijection with intersections of the reflecting hyperplanes $x_i - x_j = 0$ in $\mathbb{R}^n$ of the Coxeter group of type A_{n-1} and are refered to as A_{n-1}-partitions. Φ-partitions are defined by analogy. The reflecting hyperplanes in the case of the Coxeter group of type B_n are

$$\begin{aligned} x_i &= 0 \quad \text{for} \quad 1 \le i \le n, \\ x_i - x_j &= 0 \quad \text{for} \quad 1 \le i < j \le n, \\ x_i + x_j &= 0 \quad \text{for} \quad 1 \le i < j \le n. \end{aligned} \tag{1}$$

The subspace of $\mathbb{R}^8$

$$\{x \in \mathbb{R}^8 : x_1 = -x_5 = -x_8, x_2 = x_3, x_6 = x_7, x_4 = 0\}$$

is a typical intersection of such hyperplanes when $n = 8$ which is encoded by the partition having blocks $\{1, -5, -8\}$, $\{-1, 5, 8\}$, $\{2, 3\}$, $\{-2, -3\}$, $\{6, 7\}$, $\{-6, -7\}$

and $\{4,-4\}$. A B_n-partition is a partition π of the set $\{1,2,\ldots,n,-1,-2,\ldots,-n\}$ which has at most one block (called the *zero block*, if present) containing both i and $-i$ for some i and is such that for any block B of π, the set $-B$, obtained by negating the elements of B, is also a block of π. It follows that the zero block, if present in π, is a union of pairs $\{i,-i\}$.

The same hyperplanes as in (1) are the reflecting hyperplanes in the case of C_n and those of the second and third kind are the ones in the case of D_n. Thus the notion of a C_n-partition coincides with that of a B_n-partition while a D_n-partition is defined as a B_n-partition in which the zero block does not consist of a single pair $\{i,-i\}$, if present. The partition with blocks $\{1,-3,5\},\{-1,3,-5\},\{4\},\{-4\}$ and $\{2,6,-2,-6\}$ is a D_6-partition which corresponds to the intersection of hyperplanes

$$\{x\in\mathbb{R}^6 : x_1=-x_3=x_5, x_2=x_6, x_2=-x_6\}$$

in $\mathbb{R}^6$.

A Φ-partition π can be represented pictorially by placing the integers $1,2,\ldots,n$, if $\Phi=A_{n-1}$, and $1,2,\ldots,n,-1,-2,\ldots,-n$ otherwise, in this order, along a line and drawing arcs above the line between i and j whenever i and j lie in the same block B of π and no other element between them does so. We call π *noncrossing* if no two of the arcs cross. This is equivalent to the definition given in the Introduction in the case of A_{n-1}. Note that the notions of B_n and C_n noncrossing partitions coincide. Figure 1 shows that the B_8-partition with blocks $\{1,-5,-8\}$, $\{-1,5,8\}$, $\{2,3\}$, $\{-2,-3\}$, $\{6,7\}$, $\{-6,-7\}$ and $\{4,-4\}$, discussed earlier, is noncrossing.

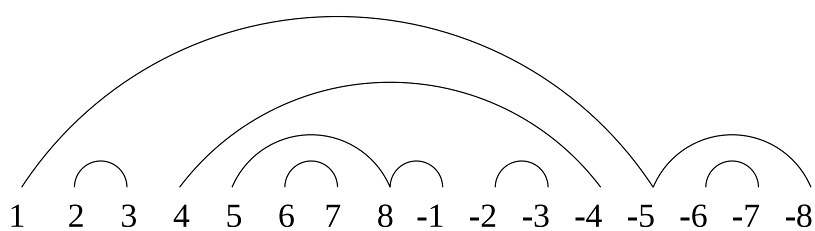

FIGURE 1. A B_8-noncrossing partition

The following theorem was proved by Kreweras [11] in the case $\Phi=A_{n-1}$ and by Reiner [16] in the remaining cases.

Theorem 2.1. ([11, 16]) *The number of noncrossing Φ-partitions is the nth Catalan number $\frac{1}{n+1}\binom{2n}{n}$ if $\Phi=A_{n-1}$, $\binom{2n}{n}$ if $\Phi=B_n$ or C_n and $\binom{2n}{n}-\binom{2(n-1)}{n-1}$ if $\Phi=D_n$.*

The *type* of a Φ-partition π is the integer partition λ whose parts are the sizes of the nonzero blocks of π, including one part for each pair of blocks $\{B,-B\}$ if $\Phi=B_n,C_n$ or D_n. Thus if λ is a partition of the nonnegative integer k, then $k=n$ if $\Phi=A_{n-1}$

and $k \leq n$ if $\Phi = B_n, C_n$ or D_n, with $k \neq n-1$ if $\Phi = D_n$. The type of the partition of Figure 1 is $(3,2,2)$. The number of noncrossing partitions of $[n]$ with fixed type was given by Kreweras [11]. For any integer partition λ we let $m_\lambda = r_1! r_2! \cdots$, where r_i denotes the number of parts of λ equal to i.

Theorem 2.2. (Kreweras [11, Theorem 4]) *The number of noncrossing partitions of $[n]$ of type λ is equal to*

$$\frac{n!}{m_\lambda\,(n-d+1)!},$$

where d is the number of parts of λ.

Let λ be a partition of $k \leq n$. Recall that there are no D_n-partitions of type λ if $k = n-1$. The following analogue of the previous theorem will be proved in Section 4.

Theorem 2.3. *The number of noncrossing B_n-partitions of type λ (equivalently C_n, or D_n if λ is not a partition of $n-1$) is equal to*

$$\frac{n!}{m_\lambda\,(n-d)!},$$

where d is the number of parts of λ.

Nonnesting partitions. From now and on we choose Φ and Φ^+ explicitly as in [10, 2.10], so that positive roots are of the form e_i, $2e_i$ and $e_i \pm e_j$ for $i < j$, where the e_i denote standard coordinate vectors. We rely on [10] for any undefined terminology on root systems. Recall from the introduction that a nonnesting partition π on Φ is an antichain in the root order on Φ^+. Such a partition π determines a Φ-partition in a way that we describe next.

For $\Phi = A_{n-1}$ we have $\Phi^+ = \{e_i - e_j\}_{1 \leq i < j \leq n}$. The A_{n-1}-partition which is associated to π is the one whose diagram contains an arc between i and j, with $i < j$, if and only if $e_i - e_j$ is in π. It follows that nonnesting partitions of A_{n-1} are in bijection with partitions of $[n]$ whose diagrams have no two arcs "nested" one within the other. Equivalently, if $a < b < c < d$, a, d are contained in a block B and no m with $a < m < d$ is in B, then b, c are not both contained in a block B'. This is the alternative description given in the introduction and becomes the definition of a *nonnesting permutation* of a multiset [3, §2] if the blocks are labeled. Figure 2 shows the diagram of the A_{10}-partition associated to $\pi = \{e_1 - e_4, e_2 - e_5, e_3 - e_6, e_5 - e_7, e_7 - e_9\}$.

If $\Phi = B_n$ we have the extra positive roots e_i, for $1 \leq i \leq n$ and $e_i + e_j$, for $1 \leq i < j \leq n$. A diagram representing π can be drawn by placing the integers $1, 2, \ldots, n, 0, -n, \ldots, -2, -1$, in this order, along a line and arcs between them. For $i, j \in [n]$, we include arcs between i and j and between $-i$ and $-j$ if π contains $e_i - e_j$, an arc between i and $-j$ if π contains $e_i + e_j$ and arcs between i and 0 and between

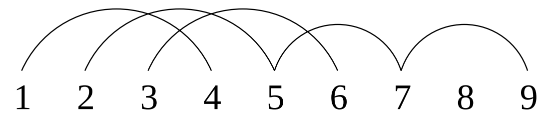

FIGURE 2. A nonnesting partition of [9]

0 and $-i$ if π contains e_i. The chains of successive arcs in the diagram become the blocks of a B_n-partition, after dropping 0, which is the partition we associate to π. This map defines a bijection between nonnesting partitions on B_n and B_n-partitions whose diagrams, in the above sense, contain no two arcs nested one within the other. We call this diagram the *nonnesting diagram* of π, to distinguish it from the diagram of the B_n-partition associated to π. Figure 3 shows the nonnesting diagram of the B_6-partition associated to $\pi = \{e_4, e_1 - e_3, e_2 - e_5, e_5 + e_6\}$. The blocks are $\{1, 3\}$, $\{-1, -3\}$, $\{2, 5, -6\}$, $\{6, -5, -2\}$ and $\{4, -4\}$.

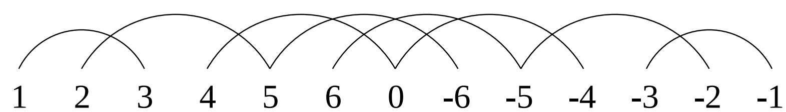

FIGURE 3. A B_6-nonnesting partition

The positive roots of C_n are obtained from those of B_n by replacing e_i by $2e_i$, for $1 \leq i \leq n$. The C_n-partition and nonnesting diagram associated to π in this case are determined as before, except that i and $-i$ are connected by an arc if π contains $2e_i$ and that 0 does not appear in the diagram. Again, the diagrams obtained in this way contain no two arcs nested one within the other. Figure 4 shows the nonnesting diagram of the C_6-partition associated to $\pi = \{2e_5, e_1 - e_4, e_3 - e_5, e_4 - e_6\}$ with blocks $\{1, 4, 6\}$, $\{-6, -4, -1\}$, $\{2\}$, $\{-2\}$ and $\{3, 5, -5, -3\}$.

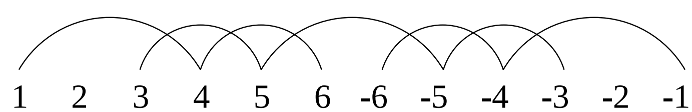

FIGURE 4. A C_6-nonnesting partition

The positive roots of D_n are those of B_n other than e_i, $1 \leq i \leq n$. The same rules as before determine the diagram of a nonnesting partition π on D_n. However, nestings can occur in the diagram, e.g. if $e_i - e_n$ and $e_i + e_n$ are both in π for some $i < n$ (see Figure 6). Note that these two elements are related in the root order of B_n and C_n but not of D_n. The chains in the diagram, which we still call the nonnesting diagram, determine the nonzero blocks of the D_n-partition associated to π and the zero block is formed by the connected component which contains n if a nesting $\{e_i - e_n, e_i + e_n\}$ appears in π.

We will usually not distinguish between a nonnesting partition π and its associated Φ-partition or nonnesting diagram. In particular, the *type* of π is the type λ of the associated Φ-partition. The partition of Figure 3 has type $(3, 2)$ and that of Figure 4 has type $(3, 1)$.

Recall that a *hyperplane arrangement* $\mathcal{A}$ is a finite set of affine hyperplanes in $\mathbb{R}^n$. The *regions* of $\mathcal{A}$ are the connected components of the space obtained from $\mathbb{R}^n$ by removing the hyperplanes of $\mathcal{A}$. The *Catalan arrangement* associated to Φ (see [1, §5] [2, Chapter 7] [8, §3] and [21, §2] [3, §1] [15, §7] for the A_{n-1} case) consists of the hyperplanes

$$\alpha \cdot x = k \quad \text{for} \quad \alpha \in \Phi^+ \quad \text{and} \quad k = -1,\ 0,\ 1.$$

It was observed by Postnikov (see Section 6 and [16, Remark 2]) that the nonnesting partitions on Φ are in bijection with the regions of the Φ-Catalan arrangement which lie inside the fundamental chamber of the underlying Coxeter arrangement. The next theorem follows from this observation and a special case of [1, Theorem 5.5] [2, Corollary 7.2.3] and is stated in [16, Remark 2].

Theorem 2.4. *For $\Phi = A_n, B_n, C_n$ or D_n, the number of nonnesting partitions on Φ is equal to*

$$\prod_{i=1}^{n} \frac{e_i + h + 1}{e_i + 1},$$

where $e_1, e_2, \ldots, e_n$ are the exponents of Φ and h is its Coxeter number.

This quantity coincides with the number of noncrossing partitions on Φ given in Theorem 2.1 and is denoted by $Catalan(\Phi)$. The similarity between the enumerative properties of noncrossing and nonnesting partitions is further demonstrated by the next theorem, which follows e.g. from [3, Corollary 4.3].

Theorem 2.5. ([3]) *The number of nonnesting partitions of $[n]$ of type λ is equal to*

$$\frac{n!}{m_\lambda\,(n-d+1)!},$$

where d is the number of parts of λ.

In Section 3 we give an explicit bijection between noncrossing and nonnesting partitions of $[n]$ which preserves type. The following analogue of Theorem 2.5 is proved in Section 5.

Theorem 2.6. *The number of nonnesting partitions either on B_n or on C_n of type λ is equal to*

$$\frac{n!}{m_\lambda\,(n-d)!},$$

where d is the number of parts of λ. If λ is a partition of an integer less than $n-1$ then the number of nonnesting partitions on D_n of type λ is equal to

$$\frac{(n-1)!}{m_\lambda\,(n-d-1)!}.$$

We do not know of a uniform formula for the number of nonnesting partitions on D_n of type λ if λ is a partition of n.

3. The case $\Phi = A_{n-1}$

In this section we discuss further the case $\Phi = A_{n-1}$. We give a simple bijection between noncrossing and nonnesting partitions of $[n]$ which preserves type and explains directly the fact that the two quantities of Theorems 2.2 and 2.5 are identical. We do not know of such a bijection for the case $\Phi = B_n$ or C_n. To be self-containt, we also include a proof of Theorems 2.2 and 2.5.

Given a partition π of $[n]$ of type λ, let $B_1, B_2, \dots, B_d$ be the blocks of π, numbered so that if a_i is the least element of B_i then $1 = a_1 < a_2 < \cdots < a_d$. We write $a = a(\pi) = (a_1, a_2, \dots, a_d)$ and $\mu = \mu(\pi) = (\mu_1, \mu_2, \dots, \mu_d)$, where μ_i is the cardinality of B_i, so that μ is a permutation of λ. For the partition of Figure 2 we have $a = (1, 2, 3, 8)$ and $\mu = (2, 4, 2, 1)$.

Theorem 3.1. *Given a nonnesting partition π of $[n]$, there is a unique noncrossing partition $\pi' := \sigma_n(\pi)$ such that $a(\pi') = a(\pi)$ and $\mu(\pi') = \mu(\pi)$. The map σ_n is a bijection between nonnesting and noncrossing partitions of $[n]$ which preserves type.*

Proof. Let π be nonnesting and a_i, μ_i, B_i for $1 \le i \le d$ be as before. Let C_i be a chain of $\mu_i - 1$ successive arcs for each i. We refer to the μ_i endpoints of these arcs as the elements of C_i. To construct the diagram of π', we place successively the chains C_i relative to each other as follows. Assume that we have already placed the chains C_i for $i < j$. Note that the total number $\mu_1 + \mu_2 + \cdots + \mu_{j-1}$ of elements of the chains already placed is at least $a_j - 1$. We insert the leftmost element of C_j in position a_j, counting from the left, relative to the elements of $C_1, \dots, C_{j-1}$. There is a unique way to place the other elements of the chain to the right without forming any pair of crossing arcs. The resulting diagram determines a noncrossing partition π' with the desired properties. The inverse of σ_n is defined in the same way except that, for each j, we place the elements of C_j to the right of the leftmost one in the unique way in which no pair of nesting arcs is formed. □

Figure 5 shows the diagram of the noncrossing partition which corresponds under the bijection σ_9 to the nonnesting partition of Figure 2. Its blocks are $\{1,9\}$, $\{2,5,6,7\}$, $\{3,4\}$ and $\{8\}$.

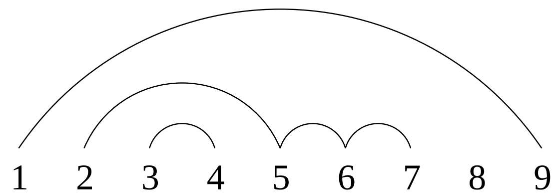

FIGURE 5. A noncrossing partition of [9]

The proof of Theorems 2.2 and 2.5 that follows was outlined in Remark 1 of [3, §5] for the nonnesting case. We will need the following version of the Cycle Lemma [6] (see also [5], the references cited there and Lemmas 3.6 and 3.7 in [23, Chapter 5]).

Lemma 3.2. ([6]) *Let $b_1, b_2, \ldots, b_m$ be integers which sum to -1 and set $b_{m+i} = b_i$ for $1 \le i \le m-1$. There is a unique $j \in [m]$ such that the cyclic permutation $b_j, b_{j+1}, \ldots, b_{j+m-1}$ has its partial sums $S_1, S_2, \ldots, S_{m-1}$ nonnegative, where $S_r = b_j + b_{j+1} + \cdots + b_{j+r-1}$.*

Proof of Theorems 2.2 and 2.5. A *labeled* partition π of $[n]$ of type $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_d)$ is a set partition of $[n]$ of type λ whose blocks are labeled with the integers $1, 2, \ldots, d$ so that the block labeled with i has cardinality λ_i. We show that the number of nonnesting, as well as noncrossing, labeled partitions of $[n]$ of type λ is equal to $\frac{n!}{(n-d+1)!}$. This implies the results since any partition of $[n]$ of type λ can be labeled in m_λ ways. For $1 \le i \le d$, let j_i be the least element of the block of π labeled with i. It follows from the proof of Theorem 3.1 that the map $\pi \mapsto (j_1, j_2, \ldots, j_d)$ induces a bijection between either nonnesting or noncrossing labeled partitions of type λ with sequences $(j_1, j_2, \ldots, j_d)$ of distinct elements of $[n]$ such that for all $1 \le k \le n$,

$$\sum_{j_r \le k} \lambda_r \ge k.$$

Lemma 3.2, applied with $m = n+1$, $b_{j_i} = \lambda_i - 1$ and $b_j = -1$ for the other values of j, implies that these sequences are in bijection with elements $(j_1, j_2, \ldots, j_d) + H$ of the quotient of the abelian group $\mathbb{Z}_{n+1}^d$ by the cyclic subgroup H generated by $(1, 1, \ldots, 1)$ for which all j_i are mutually distinct. Clearly, the number of such cosets is $n(n-1)\cdots(n-d+2)$. $\square$

4. NONCROSSING PARTITIONS OF FIXED TYPE

In this section we prove Theorem 2.3 bijectively.

Proof of Theorem 2.3. It suffices to prove the statement in the case of B_n. We describe a bijection between noncrossing B_n-partitions of type λ and pairs (S, f), where S is a subset of $[n]$ with d elements and f is a map which assigns to each element of S a part of λ so that each part is hit by f as many times as its multiplicity in λ. The number of such pairs is

$$\binom{n}{d} \frac{d!}{m_\lambda} = \frac{n!}{m_\lambda \, (n-d)!}.$$

Let π be a noncrossing B_n-partition of type $\lambda = (\lambda_1, \ldots, \lambda_d)$. To construct (S, f), choose for each pair $\{B, -B\}$ of blocks of π the leftmost element of the block which either lies entirely to the left or is nested within its negative in the diagram of π. The d elements thus chosen are the elements of S and for $s \in S$, $f(s)$ is defined to be the cardinality of the block of π which contains s. For example, for the partition whose diagram is shown in Figure 1 we have $S = \{2, 5, 6\}$, $f(2) = 2$, $f(5) = 3$ and $f(6) = 2$.

To show that this correspondence is a bijection we describe the inverse. We may assume that λ is not the empty partition, i.e. $d \geq 1$. We first place the integers $1, \ldots, n, -1, \ldots, -n$, in this order, along a line. Given (S, f) as above, we call an element s of S *admissible* if none of the $f(s) - 1$ integers on the line immediately to its right are in S or $-S$. We claim that admissible elements exist. Indeed, for $s \in S$ let $g(s) - 1$ be the number of integers strictly between s and the next element of S or $-S$ to its right. Since there are exactly n integers between the smallest integer i in S and its negative $-i$, including i, the numbers $g(s)$ sum to n. On the other hand, the sum of the parts $f(s)$ of λ is at most n. Hence we have $f(s) \leq g(s)$ for at least one s in S, which means that s is admissible.

For each admissible element s, let s and the $f(s) - 1$ integers immediately to its right form a block B and let $-B$ be another block. We now remove from the picture the blocks already constructed and continue similarly, until all elements of S are removed. The remaining elements, if any, form the zero block. This proceedure defines a noncrossing B_n-partition of type λ. If $n = 8$, $S = \{2, 5, 6\}$, $f(2) = 2$, $f(5) = 3$ and $f(6) = 2$ then the blocks $\{2, 3\}$ and $\{6, 7\}$ are constructed first, along with their negatives. The resulting partition is the one in Figure 1.

We leave it to the reader to check that the two maps are indeed inverses of each other. Note that the blocks constructed from the admissible elements of S by the second map, along with their negatives, correspond to the blocks B of π which have no other block nested within B, along with their negatives. □

The argument in the previous proof refines the one given by Reiner in the proof of the following result.

Corollary 4.1. ([16, Proposition 6]) *The number of noncrossing B_n-partitions whose type has d parts is equal to $\binom{n}{d}^2$. The total number of noncrossing B_n-partitions is $\binom{2n}{n}$.*

5. Nonnesting partitions of fixed type

To prove Theorem 2.6 we need some more background from the theory of hyperplane arrangements [14] (see also Section 2). The *characteristic polynomial* [14, §2.3] of a hyperplane arrangement $\mathcal{A}$ in $\mathbb{R}^d$ is defined as

$$\chi(\mathcal{A}, q) = \sum_{x \in L_{\mathcal{A}}} \mu(\hat{0}, x)\ q^{\dim x},$$

where $L_{\mathcal{A}}$ is the poset of all affine subspaces of $\mathbb{R}^d$ which can be written as intersections of some of the hyperplanes of $\mathcal{A}$, $\hat{0} = \mathbb{R}^d$ is the unique minimal element of $L_{\mathcal{A}}$ and μ stands for its Möbius function [20, §3.7]. The characteristic polynomial will be important for us because of the following theorem of Zaslavsky.

Theorem 5.1. (Zaslavsky [24]) *The number of regions into which the hyperplanes of $\mathcal{A}$ dissect $\mathbb{R}^d$ is given by*

$$r(\mathcal{A}) = (-1)^d\, \chi(\mathcal{A}, -1).$$

Our strategy towards Theorem 2.6 is to find hyperplane arrangements whose regions are in bijection with appropriately labeled nonnesting partitions of various types. We then use the finite field method of [1] [2, Part II] to compute the characteristic polynomials and Theorem 5.1 to derive the number of regions of the arrangements. A similar proof was given in [3, §4] for Theorem 2.5.

For the rest of this section let $\lambda = (\lambda_1, \ldots, \lambda_d)$ be a partition with $\lambda_1 + \cdots + \lambda_d = n - m$, for some nonnegative integer m.

The case of B_n. A *labeled* nonnesting partition π on B_n of type λ is a nonnesting partition on B_n of type λ whose pairs of nonzero blocks $\{B, -B\}$ are labeled with the integers $1, 2, \ldots, d$ so that if $\{B, -B\}$ is labeled with i then B has cardinality λ_i. We say that π is *signed* if a sign $+$ or $-$ is assigned to each nonzero block of π so that the sign of $-B$ is the negative of that of B.

We associate a region in $\mathbb{R}^d$ to a signed labeled nonnesting partition π on B_n of type λ as follows. If B is the nonzero block of π labeled with i, we write the variables $x_i, x_i + 1, \ldots, x_i + \lambda_i - 1$, if the sign of B is $+$, and $-x_i - \lambda_i + 1, \ldots, -x_i - 1, -x_i$, if the sign of B is $-$, in this order, from left to right in place of the elements of B in the nonnesting diagram of π. We also write the numbers $-m, \ldots, -1, 0, 1, \ldots, m$, in this order, from left to right in place of the elements of the zero block, so that a 0 is written again in place of 0 in the middle. If $\tau_1, \tau_2, \ldots, \tau_{2n+1}$ are the quantities that appear from left to right in the modified nonnesting diagram of π then the region of $\mathbb{R}^d$ which we associate to π is the one defined by the inequalities

$$\tau_1 < \tau_2 < \cdots < \tau_{2n+1}. \qquad (2)$$

For the partition of Figure 3, if the blocks $\{2,5,-6\}$, $\{1,3\}$, labeled with 1 and 2, respectively, have the sign $+$ then the associated region of $\mathbb{R}^2$ is defined by $x_2 < x_1 < x_2+1 < -1 < x_1+1 < -x_1-2 < 0 < x_1+2 < -x_1-1 < 1 < -x_2-1 < -x_1 < -x_2$.

Since (2) are linear in the x_i's, they define a region of a hyperplane arrangement. To be more precise, let $\mathcal{B}_n(\lambda)$ denote the arrangement which consists of the hyperplanes in $\mathbb{R}^d$ of the form $\alpha_i(x) = \alpha_j(x)$ for $i \neq j$, where α_i are the affine forms

$$\begin{array}{ll} x_i, x_i+1, \ldots, x_i+\lambda_i-1 & \text{for } 1 \leq i \leq d, \\ -x_i, -x_i-1, \ldots, -x_i-\lambda_i+1 & \text{for } 1 \leq i \leq d, \\ -m, \ldots, -1, 0, 1, \ldots, m. & \end{array} \tag{3}$$

Lemma 5.2. *The signed, labeled nonnesting partitions on B_n of type λ are in bijection with the regions of the arrangement $\mathcal{B}_n(\lambda)$.*

Proof. In view of the above discussion we only need to check that the region defined by (2) is nonempty. This is equivalent to the statement that we can draw the nonnesting diagram of π on the real line symmetrically around the origin so that all arcs have length 1. This follows by induction on the number of arcs after we assume that π has no singleton blocks, as we may, and remove the two extreme arcs to get the nonesting diagram of a partition π' with two arcs less. $\square$

Proposition 5.3. *We have*

$$\chi(\mathcal{B}_n(\lambda), q) = \prod_{j=n-d}^{n-1} (q-2j-1).$$

Proof. Let q be a large prime number and $\mathbb{F}_q$ denote the finite field of integers mod q. Theorem 2.2 in [1] (see also [14, Theorem 2.69] and the original formulation in [4, §16]) implies that $\chi(\mathcal{B}_n(\lambda), q)$ counts the number of d-tuples $(x_1, x_2, \ldots, x_d) \in \mathbb{F}_q^d$ for which (3) are distinct as classes mod q. To count these d-tuples we first note that for each i, exactly one of the strings

$$\begin{array}{ll} x_i, x_i+1, \ldots, x_i+\lambda_i-1, & 1 \leq i \leq d, \\ -x_i-\lambda_i+1, \ldots, -x_i-1, -x_i, & 1 \leq i \leq d \end{array}$$

will have its values within the classes $1, 2, \ldots, t$ mod q, where $t = \frac{q-1}{2}$. There are 2^d ways to choose these strings. We first permute the d strings in $d!$ ways and place them along a line. Then we place the class 0 mod q first from the left and m indistinguishable boxes immediately to its right and distribute $t-n$ more boxes in the $d+1$ spaces between successive strings as well as to the left of the leftmost string and to the right of the rightmost one in $\binom{t-n+d}{d}$ ways. Finally we naturally assign to the t objects to the right of 0 (which are either boxes or elements of the form $x_i + r$) the values $1, 2, \ldots, t \in \mathbb{F}_q$, in this order, to get appropriate tuples $(x_1, x_2, \ldots, x_d)$.

The product

$$2^d\, d! \binom{\frac{q-1}{2} - n + d}{d}$$

is the expression for $\chi(\mathcal{B}_n(\lambda), q)$ we have claimed. □

Corollary 5.4. *The number of nonnesting partitions on B_n of type λ is equal to*

$$\frac{n!}{m_\lambda\,(n-d)!}.$$

Proof. It follows from Theorem 5.1 and Proposition 5.3 that the number of regions of $\mathcal{B}_n(\lambda)$ is $2^d \frac{n!}{(n-d)!}$. By Lemma 5.2, this is also the number of signed, labeled nonnesting partitions on B_n of type λ, which is clearly $2^d\, m_\lambda$ times the number of nonnesting partitions on B_n of type λ. □

The case of C_n. Signed, labeled nonnesting partitions on C_n of type λ are defined exactly as in the case of B_n. Recall that 0 does not appear in the nonnesting diagram of such a partition π. We associate a region of $\mathbb{R}^d$ to π as in the case of B_n, except that we write the numbers $-m+1/2, \dots, -3/2, -1/2, 1/2, 3/2, \dots, m-1/2$ in place of the elements of the zero block, so the number of the τ_i in (2) is now $2n$.

For the partition of Figure 4, if the blocks $\{1,4,6\}$, $\{2\}$, labeled with 1 and 2, respectively, have the sign $+$ then the associated region of $\mathbb{R}^2$ is defined by $x_1 < x_2 < -\frac{3}{2} < x_1 + 1 < -\frac{1}{2} < x_1 + 2 < -x_1 - 2 < \frac{1}{2} < -x_1 - 1 < \frac{3}{2} < -x_2 < -x_1$.

Let $\mathcal{C}_n(\lambda)$ denote the arrangement which consists of the hyperplanes in $\mathbb{R}^d$ of the form $\beta_i(x) = \beta_j(x)$ for $i \neq j$, where β_i are the affine forms

$$\begin{array}{l} x_i, x_i+1, \dots, x_i+\lambda_i-1 \quad \text{for} \quad 1 \le i \le d, \\ -x_i, -x_i-1, \dots, -x_i-\lambda_i+1 \quad \text{for} \quad 1 \le i \le d, \\ -m+\frac{1}{2}, \dots, -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \dots, m-\frac{1}{2}. \end{array} \tag{4}$$

The following lemma can be proved as Lemma 5.2.

Lemma 5.5. *The signed, labeled nonnesting partitions on C_n of type λ are in bijection with the regions of the arrangement $\mathcal{C}_n(\lambda)$.*

Proposition 5.6. *We have*

$$\chi(\mathcal{C}_n(\lambda), q) = \prod_{j=n-d}^{n-1} (q - 2j - 1).$$

Proof. Let y_i denote any of the forms $x_i + r$ or $-x_i - r$. The equations $y_i = k - 1/2$ for $k \in \mathbb{Z}$ can be written as $2y_i = 2k-1$, so $\mathcal{C}_n(\lambda)$ can be defined over the integers. Hence [1, Theorem 2.2] applies again and $\chi(\mathcal{C}_n(\lambda), q)$ counts the number of d-tuples $(x_1, x_2, \dots, x_d) \in \mathbb{F}_q^d$ for which (4) are distinct as classes mod q. Note that the conditions $y_i \neq -y_i \bmod q$ imply $y_i \neq 0 \bmod q$ and the ones of the form $y_i \neq k - 1/2$ can be written as $y_i \neq k + \frac{q-1}{2} \bmod q$. The rest of the proof follows the one of Proposition 5.3 except that we now place the m boxes to the right of the rightmost string, instead of to the right of 0 mod q, to guarantee that the forms y_i will be assigned values mod q different from $t, t-1, \ldots, t-m+1$, where $t = \frac{q-1}{2}$. □

The following corollary is obtained as in the case of B_n.

Corollary 5.7. *The number of nonnesting partitions on C_n of type λ is equal to*

$$\frac{n!}{m_\lambda\,(n-d)!}.$$

The following is the analogue of Corollary 4.1 for nonnesting partitions on B_n and C_n.

Corollary 5.8. *The number of nonnesting partitions on B_n whose type has d parts is equal to $\binom{n}{d}^2$. The total number of nonnesting partitions on B_n is $\binom{2n}{n}$. The same is true if B_n is replaced by C_n.*

The case of D_n. Recall that a nonnesting partition π on D_n has a zero block if $e_i - e_n$ and $e_i + e_n$ are both in π for some $i < n$. The zero block is formed by the connected component of the nonnesting diagram which contains n. Figure 6 shows a nonnesting partition on D_5 with zero block $\{1, 3, 5, -1, -3, -5\}$.

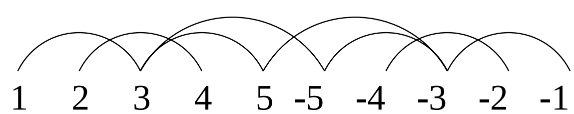

FIGURE 6. A D_5-nonnesting partition

We now complete the proof of Theorem 2.6.

Proposition 5.9. *If λ is a partition of an integer less than $n-1$ then the number of nonnesting partitions on D_n of type λ is equal to*

$$\frac{(n-1)!}{m_\lambda\,(n-d-1)!}.$$

Proof. Let π be a nonnesting partition on D_n of type λ. By the assumption on λ, π has a zero block which contains $\{n, -n\}$ and at least one more pair $\{i, -i\}$. Merge n and $-n$ in the nonnesting diagram of π to a single element, labeled 0, to get the diagram of a nonnesting partition on B_{n-1} of type λ. For example, the partition of Figure 6 becomes the B_4-nonnesting partition with nonzero blocks $\{2, 4\}$ and $\{-2, -4\}$. This correspondence is a bijection between nonnesting partitions on

D_n of type λ and nonnesting partitions on B_{n-1} of type λ, so the result follows from Corollary 5.4. $\square$

6. Remarks

1. Let Φ be an irreducible root system in $\mathbb{R}^n$ associated to a Weyl group W, as in the introduction. Let Φ^+ be a choice of positive roots and $\mathcal{C}_\Phi$ denote the corresponding *fundamental alcove* in $\mathbb{R}^n$, defined by the inequalities $\alpha \cdot x > 0$ for $\alpha \in \Phi^+$. The bijection that we refered to before Theorem 2.4 sends a nonnesting partition π on Φ to the region in $\mathcal{C}_\Phi$ defined by $\beta \cdot x > 1$, if $\beta \geq \alpha$ in the root order of Φ for some $\alpha \in \pi$ and $\beta \cdot x < 1$, if otherwise. It follows that the number of nonnesting partitions on Φ is equal to the number of regions of the Φ-Catalan arrangement, divided by the order of the group W.

2. One can naturally try to carry structural properties of noncrossing partitions over to nonnesting partitions. However, the set of nonnesting partitions of $[n]$ partially ordered by refinement will typically not be a lattice, as is the case for noncrossing partitions. Already for $n = 6$, the partitions $\{1,3,6\}\{2,5\}\{4\}$ and $\{1,4,6\}\{2,5\}\{3\}$ do not have a meet in this poset.

3. We do not know of a more direct proof of Theorem 2.6. In particular, we do not know of any bijections between noncrossing and nonnesting partitions in the cases of the root systems B, C or D, similar to the one in Theorem 3.1.

4. We do not know of a formula for the number of nonnesting partitions on D_n of type λ when λ is a partition of n. This number is equal to the number of noncrossing D_n-partitions (or B_n) of type λ if λ has no part greater than 2 but not otherwise. For example, there are 3 noncrossing D_3-partitions of type (3) but 4 nonnesting partitions of the same kind, namely the 3 nonnesting partitions on B_3 of type (3) and the one with blocks $\{1,-2,-3\}$ and $\{2,3,-1\}$. Table 1 shows the number of nonnesting partitions on D_n of type λ for the partitions λ of $n \leq 6$ with at least one part greater than 2.

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type	number of nonnesting partitions on D_3
(3)	4
type	number of nonnesting partitions on D_4
(4)	6
(3, 1)	15
type	number of nonnesting partitions on D_5
(5)	8
(4, 1)	28
(3, 2)	24
(3, 1, 1)	36
type	number of nonnesting partitions on D_6
(6)	10
(5, 1)	45
(4, 2)	40
(4, 1, 1)	80
(3, 3)	20
(3, 2, 1)	140
(3, 1, 1, 1)	70

TABLE 1. Nonnesting partitions on D_n by type

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