Multimatroids II. Orthogonality, minors and connectivity

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Abstract

A multimatroid is a combinatorial structure that encompasses matroids, delta-matroids and isotropic systems. This structure has been introduced to unify a theorem of Edmonds on the coverings of a matroid by independent sets and a theorem of Jackson on the existence of pairwise compatible Euler tours in a 4-regular graph. Here we investigate some basic concepts and properties related with multimatroids: matroid orthogonality, minor operations and connectivity. **Mathematical Reviews: 05B35**

1 Introduction

In a preceding paper [5] we unified a theorem of Jackson [15], on the existence of pairwise compatible Euler tours in a 4-regular graph, with a theorem of Edmonds [12], on the minimum number of independent sets to cover the ground-set of a matroid. For this purpose we introduced a new combinatorial structure, called a multimatroid, which unifies matroids, delta-matroids and isotropic systems. We complete in the present paper and subsequent ones [6, 7] the basic properties of multimatroids.

In Section 2 we review the results already proved in [5]. We also introduce the extended submodularity inequality, equivalent to a kind of supermodularity inequality used by Jackson [15], and we relate it with the bisubmodularity inequality introduced by Kabadi and Chandrasekaran [16]. In Section 3 we introduce an orthogonality relation between matroids, similar to the classical strong map relation, and we show that a multimatroid gives raise to orthogonal matroids. Conversely we derive in Section 4 a multimatroid from a sequence of orthogonal matroids and we retrieve as a particular case the generalized matroids of Tardos [17]. We introduce the minor operations and the separators in Sections 5 and 6. Finally we study some relations between multimatroids and Eulerian graphs in Section 7.

2 A survey

Consider a partition Ω of a finite set U. Each class of Ω is called a *skew class*. Each pair of distinct elements belonging to the same skew class is called a *skew pair*. A *subtransversal* (resp. *transversal*) of Ω is a subset A of U such that $|A \cap \omega| \leq 1$ (resp. $|A \cap \omega| = 1$) holds for every ω in Ω. Two subtransversals are *compatible* if their union is also a subtransversal. We denote by $\mathcal{S}(\Omega)$ (resp. $\mathcal{T}(\Omega)$) the set of subtransversals (resp. transversals) of Ω.

A *weak multimatroid* is a triple $Q = (U, \Omega, r)$ with a partition Ω of a finite set U and a *rank function* $r : \mathcal{S}(\Omega) \to \mathbf{N}$ satisfying the three following axioms:

2.1 $r(\emptyset) = 0$*;*

2.2 $r(A) \leq r(A + x) \leq r(A) + 1$ *is satisfied for every subtransversal A of Ω and every x in U provided that A is disjoint from the skew class containing x;*

2.3 Submodularity inequality: $r(A) + r(B) \geq r(A \cup B) + r(A \cap B)$ *is satisfied for every pair of compatible subtransversals A and B of Ω;*

The following axiom has also to be satisfied in order to derive interesting properties. Then Q is called a *multimatroid*.

2.4 $r(A + x) - r(A) + r(A + y) - r(A) \geq 1$ *is satisfied for every subtransversal A of Ω and every skew pair $\{x, y\}$ provided that A is disjoint from the skew class including $\{x, y\}$.*

If each skew class has cardinality equal to the positive integer q, then Q a *q-matroid*. An *independent set* is a subtransversal I of Ω such that $r(I) = |I|$, a *base* is a maximal independent set, and a *circuit* is a subtransversal C of Ω that is not independent and is minimal with this property. We denote by $\mathcal{I}(Q)$, $\mathcal{B}(Q)$ and $\mathcal{C}(Q)$ the collections of independent sets, bases and circuits, respectively.

If A is a subtransversal of Ω, then $r(P)$ is defined for every subset P of A. The axioms 2.1 to 2.3 imply that the restriction of r to the power-set of A is the rank function of a matroid on the set A, denoted by $Q[A]$ and called the *submatroid* induced on A. The independent sets (resp. circuits) of $Q[A]$ are the independent sets (resp. circuits) of Q included in A. If Q is a 1-matroid, then U is a transversal of Ω and we identify Q to the matroid $Q[U]$. The inverse construction that associates a 1-matroid to a matroid is obvious. The multimatroid Q may be thought as the aggregation of the submatroids $Q[A]$, when A ranges in the collection of subtransversals of Ω, which gives the name to the structure.

A multimatroid Q will often be given with a *projection* onto a set V: this is a surjective mapping $p : U \to V$ such that $p(x_1) = p(x_2)$ is satisfied if and only if the elements x_1 and x_2 belong to the same skew class. We set $\Omega_v = \{v : p(x) = v\}$ for every element v in V, so that $\Omega = \{\Omega_v : v \in V\}$. We also say that Q is *indexed* on V. For every transversal T of Ω, the restriction $p_{|T}$ is a bijection from T onto V. The isomorphic image of $Q[T]$ by $p_{|T}$ is called the projection of $Q[T]$ and is denoted by $p(Q[T])$.

Properties of the independent sets, circuits, and bases

Consider a, possibly weak, multimatroid $Q = (U, \Omega, r)$. For every subtransversal A of Ω, the relation

$$r(A) = \max_{I \subseteq A, I \in \mathcal{I}(Q)} |I|$$

is satisfied. Therefore Q is determined when either $\mathcal{I}(Q)$, $\mathcal{B}(Q)$ or $\mathcal{C}(Q)$ is known. In the two following characterizations the properties (a) to (c) correspond to the axioms 2.1 to 2.3 and the property (d) corresponds to Axiom 2.4. A pair (U, Ω) with a finite set U and a partition Ω is called a *partitioned set*.

Proposition 2.5 [5] *Let (U, Ω) be a partitioned set. A subset $\mathcal{I}$ of $\mathcal{S}(\Omega)$ is the collection of independent sets of a multimatroid on (U, Ω) if and only if the following properties are satisfied:*

(a) $\emptyset \in \mathcal{I}$;

(b) *If $I \in \mathcal{I}$ and $J \subseteq I$ then $J \in \mathcal{I}$;*

(c) **Augmentation:** *If $I, J \in \mathcal{I}$ are compatible and $|I| < |J|$ then $I + x \in I$ for some $x \in J \setminus I$;*

(d) *If $I \in \mathcal{I}$ and $\{x, y\}$ is a pair included in a class of Ω disjoint from I, then $I + x \in \mathcal{I}$ or $I + y \in \mathcal{I}$.*

Proposition 2.6 [5] *Let (U, Ω) be a partitioned set. A subset $\mathcal{C}$ of $\mathcal{S}(\Omega)$ is the collection of circuits of a multimatroid on (U, Ω) if and only if the following properties are satisfied:*

(a) $\emptyset \notin \mathcal{C}$;

(b) *If $C_1, C_2 \in \mathcal{C}$ and $C_1 \subseteq C_2$ then $C_1 = C_2$;*

(c) **Elimination:** *If $C_1, C_2 \in \mathcal{C}$ are distinct and compatible and $x \in C_1 \cap C_2$, then $C \subseteq (C_1 \cup C_2) - x$ for some $C \in \mathcal{C}$;*

(d) *If $C_1, C_2 \in \mathcal{C}$, then $C_1 \cup C_2$ cannot include precisely one skew pair.*

A multimatroid is said to be *nondegenerate* if each of its skew classes has at least cardinality 2.

Proposition 2.7 *The bases of a nondegenerate multimatroid are transversal.*

Proof. Suppose indirectly that a base B of a nondegenerate multimatroid is not transversal. Consider a skew class ω disjoint from B. Since Q is nondegenerate we can chose distinct elements x and y in ω. Proposition 2.5(d) implies that $B+x$ or $B+y$ is independent, and so B cannot be a base. □

Corollary 2.8 *The bases of a q-matroid are transversal if $q \geq 2$.*

Let U' be a subset of U. The *restriction* of Q to U' is $Q[U'] = (U', \Omega', r')$, where $\Omega' = \{\omega \cap U' : \omega \in \Omega, \omega \cap U' \neq \emptyset\}$ and r' is the restriction of r to $\mathcal{S}(\Omega')$. Clearly $Q[U']$ is a multimatroid. We say that $Q[U']$ is *spanning* if $U' \cap \omega$ is nonempty for every skew class ω of Q.

Proposition 2.9 *If $Q[U']$ is a nondegenerate spanning restriction of a (nondegenerate) multimatroid Q, then the bases of $Q[U']$ are the bases of Q contained in U'.*

Proof. Set $Q = (U, \Omega, r)$ and $Q' = (U', \Omega', r')$. Every base of Q contained in U' is obviously a base of Q'. Conversely let B' be a base of Q'. Then B' is an independent set of Q contained in U'. Since $Q[U']$ is nondegenerate, B' is a transversal of Ω' by Proposition 2.7. Since $Q[U']$ is spanning, B' is also a transversal of Ω. Hence B' is a transversal independent set of Q, which is a base of Q. □

Proposition 2.7 implies that the bases of a nondegenerate multimatroid are equicardinal. It is easy to construct a degenerate multimatroid where this property is false. Proposition 2.9 also is false when it is applied to a restriction that is degenerate or not spanning.

Relation with delta-matroids

The structure of delta-matroid has been independently introduced by Dress and Havel [11], Chandrasekaran and Kabadi [9], and the author [2]. A *delta-matroid* is a set-system $D = (V, \mathcal{F})$, where V is a finite set and $\mathcal{F}$ is a nonempty collection of subsets of V, called the *feasible sets* or *bases*, satisfying the following *symmetric exchange axiom*:

2.10 *For $F_1, F_2 \in \mathcal{F}$, for $v \in F_1 \Delta F_2$, there is $w \in F_1 \Delta F_2$ with $F_1 \Delta \{v, w\} \in \mathcal{F}$.*

Proposition 2.11 [2] *A nonempty collection $\mathcal{F}$ of subsets of a finite set V is the collection of bases of a matroid if and only if $\mathcal{F}$ satisfies the symmetric exchange axiom and the members of $\mathcal{F}$ are equicardinal.*

Accordingly one identifies a matroid to a delta-matroid with equicardinal bases. For a set system $D = (V, \mathcal{F})$ and a subset X of V, set $\mathcal{F} \Delta X = \{F \Delta X : F \in \mathcal{F}\}$ and $D \Delta X = (V, \mathcal{F} \Delta X)$. If $\mathcal{F}$ satisfies the symmetric exchange axiom then $\mathcal{F} \Delta X$ also clearly satisfies the same axiom. Hence $D \Delta X$ is a delta-matroid if D is a delta-matroid. The transformation $D \mapsto D \Delta X$ is called *twisting*. If D is a matroid and

$X = V$, then $D\Delta X$ is the matroid dual of D. A *paired set* is a pair (U, Ω) with a finite set U and a partition Ω of U into pairs.

Theorem 2.12 [5] *Let (U, Ω) be a paired set and let T be a transversal of Ω. A nonempty collection $\mathcal{B}$ of transversals of Ω is the set of bases of a 2-matroid Q defined on (U, Ω) if and only if $\{B \cap T : B \in \mathcal{B}\}$ is the collection of bases of a delta-matroid.*

The delta-matroid of Theorem 2.12 is called the *trace* of Q on T and is denoted by $Q \cap T$. Consider a projection p of Q onto a set V. The isomorphic image of $Q \cap T$ by $p_{|T}$ is a delta-matroid on the ground-set V, which we denote by $p(Q \cap T)$. For every transversal T' of Ω, we easily verify that

$$p(Q \cap T') = p(Q \cap T)\Delta p(T \Delta T').$$

The subset $X = p(T\Delta T')$ ranges in the power-set of V when T' ranges in the set of transversals of Ω. Hence, if we fix T and we set $D = p(Q \cap T)$, the delta-matroid $p(Q \cap T') = D\Delta X$ ranges in the twisting class of D. Conversely the following construction shows that every twisting class of delta-matroids can be derived from an indexed 2-matroid.

Construction 2.13 Let $D = (V, \mathcal{F})$ be a delta-matroid. Set

$$\begin{aligned}
V_i &= \{v_i : v \in V\},\ i = 1, 2,\\
U &= V_1 + V_2\\
\Omega_v &= \{v_1, v_2\},\ v \in V\\
\Omega &= \{\Omega_v : v \in V\}\\
F_i &= \{v_i : v \in F\},\ F \in \mathcal{F},\ i = 1, 2,\\
\mathcal{B} &= \{F_1 \cup (V_2 \setminus F_2) : F \in \mathcal{F}\}.
\end{aligned}$$

Theorem 2.12 implies that $\mathcal{B}$ is the collection of bases of a 2-matroid Q defined on (U, Ω). We have $D = p(Q \cap V_1)$, where p is the projection of Q onto V defined by the relation $p(v_1) = p(v_2) = v$ for every v in V. We call Q the *lift* of D.

Eulerian multimatroids

A graph (finite and undirected) G is said to be *Eulerian* if each vertex has even degree. The number of components of G is denoted by $k(G)$. We consider that each edge e of G is incident to two *half-edges* h_1 and h_2, each of them incident to one vertex, the ends of e being the vertices incident to h_1 and h_2. The set of half-edges incident to a vertex v is denoted by $h(v)$. A pair of half-edges incident to the same vertex (resp. edge) is called a *vertex-transition* (resp. *edge-transition*).

Assume G is Eulerian. A *local splitter* incident to v is a pair $S_v = \{S'_v, S''_v\}$, where S'_v and S''_v are complementary subsets of $h(v)$ having even cardinalities. If S'_v and S''_v are nonempty, then S_v is said to be *proper*. A *splitter* is a set $S = \{S_v : v \in W\}$, where W is a subset of vertices, and S_v is a proper local splitter incident to v. The splitter S is *complete* if W is equal to the set of vertices of G.

To *detach* the proper local splitter S_v is to replace the vertex v by two vertices v' and v'' such that $h(v') = S'_v$ and $h(v'') = S''_v$. The resulting graph, denoted by $G||S_v$, is still an Eulerian graph. To *detach* the splitter S is to replace G by $G||S = G||S_{v_1}||S_{v_2}||\cdots||S_{v_p}$, where $(v_1, v_2, \cdots, v_p)$ is an enumeration of W. (Obviously $G||S$ does not depend on the actual enumeration.) The *rank* of the splitter S is $|S| - k(G||S) + k(G)$.

Consider a subset U of proper local splitters of G. A splitter contained in U is said to be *allowed* and the pair $G_U = (G, U)$ is called a *restricted Eulerian graph*. Denote by $V(G_U) = V$ the subset of vertices of G that are incident to some local splitter in U and, for each v in V, denote by Ω_v the set of local splitters in U incident to v. The set $\Omega = \{\Omega_v : v \in V\}$ is a partition of U and $\mathcal{S}(\Omega)$ is the set of allowed splitters. Denote by r the restriction of the splitter rank function to $\mathcal{S}(\Omega)$ and set $Q(G_U) = (U, \Omega, r)$. It is proved in [5] that $Q(G_U)$ is a weak multimatroid. It is a multimatroid if the following *skewness condition* is satisfied:

2.14 *If $S_v = \{S'_v, S''_v\}$ and $T_v = \{T'_v, T''_v\}$ are distinct allowed local splitters incident to the same vertex v, then $|S'_v \cap T'_v|$ is odd.*

Note that $Q(G_U)$ is naturally indexed on V. We set $Q(G) = Q(G_U)$ when all splitters are allowed. The (weak) multimatroid $Q(G_U)$ is said to be *Eulerian*.

The 3-matroid of a 4-regular Graph

In the particular case where G is a 4-regular graph, every proper local splitter is made of two disjoint vertex-transitions. Accordingly it is also called a *bitransition*. The skewness condition is satisfied because, if $\{S'_v, S''_v\}$ and $\{T'_v, T''_v\}$ are two bitransitions incident to the same vertex, we have $|S'_v \cap T'_v| = 1$. Moreover there are three bitransitions incident to each vertex. Hence $Q(G)$ is a 3-matroid.

Assume G is connected. We describe an *Euler tour* T by an enumeration of the half-edges $h'_0 h''_0 h'_1 h''_1 \cdots h'_{m-1} h''_{m-1}$ such that $\{h'_i, h''_i\}$ is an edge-transition and $\{h''_i, h'_{i+1}\}$ is a vertex-transition, for $0 \le i < m$, with the convention $h'_{i+1} = h'_0$ when $i = m-1$ [1]. For each vertex v let T_v be the bitransition made of the two vertex-transitions incident to v and belonging to $\{\{h''_i, h'_{i+1}\} : 0 \le i < m\}$. Then $B(T) := \{T_v : v \in V\}$ is a complete splitter and $G||B(T)$ is a regular graph of degree 2 that admits T as a (unique) Euler tour. We have $k(G||B(T)) = k(G) = 1$, and so $B(T)$ is a base of the 3-matroid $Q(G)$. Conversely if B is a base of $Q(G)$, then the unique Euler tour T of $G||B$ is also an Euler tour of G such that $B = B(T)$. Hence there is a bijective correspondance between the Euler tours of G and the bases of $Q(G)$.

[1] An Euler tour is usually defined by means of an alternate sequence of edges and vertices. Note that the graph consisting of one vertex v incident to two loops e_1 and e_2, where e_i is incident to the half-edges h'_i and h''_i, for $i = 1, 2$, has two Euler tours described by $h'_1 h'_2 h''_1 h''_2$ and $h'_1 h'_2 h''_2 h''_1$, whereas the usual definition gives only one Euler tour described by $v e_1 v e_2$.

Theorems of Jackson and Edmonds

Let $Q = (Q^j : j \in J)$ be a finite family of multimatroids defined on the same partitioned set (U, Ω). Denote by $\mathcal{B}(Q)$ the set of families $B = (B^j : j \in J)$, where B^j is a base of Q^j. Set $Cov(B) = \bigcup_{j \in J} B^j$ for every B in $\mathcal{B}(Q)$. The *rank function* of Q is the mapping r, defined for S in $\mathcal{S}(\Omega)$ by the formula $r(S) = \sum_{j \in J} r^j(S)$, where r^j is the rank function of Q^j.

Theorem 2.15 [5] *A finite family $Q = (Q^j : j \in J)$ of multimatroids defined on the same partitioned set (U, Ω), with the rank function r, satisfies*

$$\max_{B \in \mathcal{B}(Q)} |Cov(B)| = \min_{S \in \mathcal{S}(\Omega)} (r(S) + |U \setminus S|),$$

provided that each skew class ω is such that $3 \leq |\omega| \leq |J|$. A base B of Q and a subtransversal S of Ω satisfying the equality can be efficiently computed.

The theorem still holds when every skew class ω satisfies $|\omega| = 1$: then each Q^j is a matroid and the statement is a theorem of Edmonds [12]. However the theorem is false when $|J| = 2$ and every skew class ω satisfies $|\omega| = 2$: it is shown in [5] that the parity problem for matroids can be transformed into the problem of searching for B in $\mathcal{B}(Q)$ maximizing $|Cov(B)|$ with these assumptions.

Consider now a connected 4-regular graph G. We say that a bitransition is *covered* by an Euler tour T if it belongs to $B(T)$. Set $J = \{1, 2, 3\}$ and apply Theorem 2.15 to $Q = (Q^1, Q^2, Q^3)$, where $Q^1 = Q^2 = Q^3 = Q(G)$. We find that the maximal number of bitransitions covered by three Euler tours of G is equal to

$$\min_{S \in \mathcal{S}(\Omega)} (3|V| + 2|S| - 3k(G||S) + 3).$$

In particular there are three Euler tours that cover all the bitransitions if and only if

$$2|S| \geq 3k(G||S) - 1$$

holds for every splitter S. This result has been originally proved by Jackson [15, 14], and a polynomial algorithm to find three Euler tours covering a maximal number of bitransitions is given in [4].

Extended submodularity inequality

Let $Q = (U, \Omega, r)$ be a multimatroid. If A_1 and A_2 are subtransversals of Ω then $sk(A_1, A_2)$ denotes the number of skew pairs included in $A_1 \cup A_2$, and $A_1 \uplus A_2$ denotes the union of A_1 and A_2 less the union of the skew pairs included in $A_1 \cup A_2$. A function $f : \mathcal{S}(\Omega) \to \mathbf{N}$ is said to satisfy the *extended submodularity inequality* if

$$f(A) + f(B) \geq f(A \cap B) + f(A \uplus B) + sk(A, B) \qquad (1)$$

holds for every pair of subtransversals A_1 and A_2.

Theorem 2.16 *A triple $Q = (U, \Omega, r)$ is a multimatroid if and only if r satisfies the axioms 2.1 and 2.2, and the extended submodularity inequality.*

We refer the reader to a paper of ALLYS [1] for a short proof of that theorem. A kind of extended submodularity inequality, obtained by inverting $\geq$ in the relation (1), was introduced by JACKSON [15].

Bisubmodularity inequality

Denote by 3^V the set of ordered pairs (P, Q), where P and Q are disjoint subsets of V. For $X_1 = (P_1, Q_1)$ and $X_2 = (P_2, Q_2)$ in 3^V, set

$$\begin{aligned} X_1 \wedge X_2 &= (P_1 \cap P_2, Q_1 \cap Q_2), \\ X_1 \vee X_2 &= ((P_1 \cup P_2) \setminus (Q_1 \cup Q_2), (Q_1 \cup Q_2) \setminus (P_1 \cup P_2)). \end{aligned}$$

A function $f : 3^V \to \mathbf{R}$ is said to be *bisubmodular* if

$$f(X_1) + f(X_2) \geq f(X_1 \wedge X_2) + f(X_1 \vee X_2) \tag{2}$$

always holds. This inequality has been introduced by CHANDRASEKARAN and KABADI [9, 16]. They proved that, for a delta-matroid $D = (V, \mathcal{F})$, the function $R : 3^V \to \mathbf{Z}$, defined by

$$R(P, Q) = \max_{F \in \mathcal{F}} (|P \cap F| - |Q \cap (V \setminus F)|)$$

is bisubmodular. Moreover the convex hull of the characteristic vectors of the bases of D is the set of vectors x in $\mathbf{R}^V$ satisfying

$$x(P) - x(Q) \leq R(P, Q), \ (P, Q) \in 3^V,$$

where the notation $x(W)$ stands for $\sum_{w \in W} x_w$. The integral bisubmodular functions, when they are allowed to take infinite values, have also been used by BOUCHET and CUNNINGHAM [8] to study the jump systems (a generalization of delta-matroids in $\mathbf{Z}^V$).

The fact that R is bisubmodular can be retrieved as follows. Use Construction 2.13 to lift D into a 2-matroid $Q = (U, \Omega, r)$.

It is easy to verify that R satisfies the bisubmodularity inequality (2) if and only if the function $r' : \mathcal{S}(\Omega) \to \mathbf{Z}$, defined by the relation

$$r'(P_1 \cup Q_2) = R(P, Q) + |Q|,$$

satisfies the extended submodularity inequality (1). Since the collection of bases of Q is equal to $\{F_1 \cup (V_2 \setminus F_2) : F \in \mathcal{F}\}$, the rank of the subtransversal $P_1 \cup Q_2$ is such that

$$\begin{aligned}
r(P_1 \cup Q_2) &= \max_{F \in \mathcal{F}} |(P_1 \cup Q_2) \cap (F_1 \cup (V_2 \setminus F_2)| \\
&= \max_{F \in \mathcal{F}} (|P_1 \cap F_1| + |Q_2 \cap (V_2 \setminus F_2)|) \\
&= \max_{F \in \mathcal{F}} (|P \cap F| + |Q \cap (V \setminus F)|) \\
&= R(P, Q) + |Q| \\
&= r'(P_1 \cup Q_2).
\end{aligned}$$

The rank function r satisfies the extended submodularity inequality by Theorem 2.16. So we retrieve that R is bisubmodular.

3 Orthogonality relation

Let M_1 and M_2 be two matroids on the same set E, with rank functions r_1 and r_2, respectively. The matroid M_1 is a *strong map* of the matroid M_2 if $r_1 - r_2$ is an increasing function, that is

$$r_1(X) - r_2(X) \leq r_1(X + x) - r_2(X + x) \tag{3}$$

holds whenever X is a subset of E and x is an element of $E \setminus X$. The matroids M_1 and M_2 are *orthogonal* if M_1 is a strong map of M_2^*. In this section we show that, if T_1 and T_2 are disjoint transversals of a multimatroid $Q = (U, \Omega, r)$ indexed on a set V, then the projections of the submatroids $Q[T_1]$ and $Q[T_2]$ are orthogonal.

The next proposition is known when it is expressed in terms of strong maps. We recall its proof for the reader's convenience. The properties (b) and (c) imply that the orthogonality relation is symmetric.

Proposition 3.1 *Let M_1 and M_2 be two matroids on the same set E, with rank functions r_1 and r_2, respectively. The following properties are equivalent:*

(a) *M_1 is orthogonal to M_2;*

(b) *$r_1(X_1 + x) - r_1(X_1) + r_2(X_2 + x) - r_2(X_2) \geq 1$ holds whenever X_1 and X_2 are disjoint subsets of E, and x belongs to $E \setminus (X_1 \cup X_2)$;*

(c) *$|C_1 \cap C_2| \neq 1$ holds for every circuit C_1 of M_1 and every circuit C_2 of M_2.*

Proof. (a) $\Longleftrightarrow$ (b). Let r_2^* be the rank function of M_2^*. The relation ([18] p. 35)

$$r_2^*(A) = r_2(E \setminus A) - r_2(E) + |A|$$

is satisfied for every subset A of E. The relation (3), applied to r_1 and r_2^*, implies that M_1 and M_2 are orthogonal if and only if

$$r_1(X+x)-r_1(X) \geq r_2(E-X-x)-r_2(E-X)+1 \tag{4}$$

holds for every subset X of E and every element x in $E \setminus X$. Set $Y = E - X - x$. The relation (4) can be written

$$r_1(X+x)-r_1(X)+r_2(Y+x)-r_2(Y) \geq 1. \tag{5}$$

Since r_1 and r_2 are submodular functions, the preceding inequality also holds when one replaces X by a subset X_1 of X and Y by a subset X_2 of Y. This proves (b). Conversely (b) $\Longrightarrow$ (5) $\Longrightarrow$ (4) $\Longrightarrow$ (3).

(b) $\Longrightarrow$ (c). Assume $|C_1 \cap C_2| = 1$ and consider the unique element x in $C_1 \cap C_2$. Set $X_1 = C_1 - x$ and $X_2 = C_2 - x$. One has $r_1(X_1+x) = r_1(X_1)$ and $r_2(X_2+x) = r_2(X_2)$, which contradict (b).

(c) $\Longrightarrow$ (b). Assume (b) is false. Since r_1 and r_2 are increasing functions we have $r_1(X_1+x) = r_1(X_1)$ and $r_2(X_2+x) = r_2(X_2)$. The element x belongs to the closure of X_1 in M_1. So there is a circuit C_1 of M_1 such that $x \in C_1 \subseteq X_1 + x$. Similarly there exists a circuit C_2 of M_2 such that $x \in C_2 \subseteq X_2 + x$. These circuits contradict (c). □

We informally represent a multimatroid Q indexed on a set V by drawing V and some transversals of interest as horizontal lines. An element v of V and the elements of Ω_v are placed on the same vertical line. We think of the projection associated to the indexing as an orthogonal projection onto V.

Theorem 3.2 *If T_1 and T_2 are disjoint transversals of a multimatroid Q indexed on a set V, then the projections of $Q[T_1]$ and $Q[T_2]$ are orthogonal matroids.*

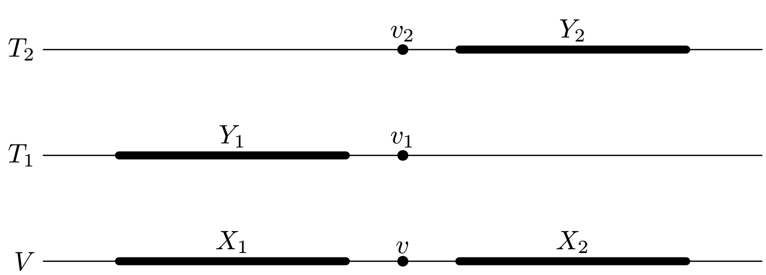

Figure 1: Illustration of the proof of Theorem 3.2.

Proof. (See Figure 1) Let r_i be the rank function of the projection of $Q[T_i]$, for $i = 1, 2$. According to Proposition 3.1 we have to verify that, for every pair of disjoint subsets X_1 and X_2 of V and every element v in $V \setminus (X_1 \cup X_2)$, we have

$$r_1(X_1+v)-r_1(X_1)+r_2(X_2+v)-r_2(X_2) \geq 1. \tag{6}$$

Let p be the projection of Q onto V and, for $i = 1, 2$, let Y_i be the subset of T_i such that $p(Y_i) = X_i$ and let v_i be the element of T_i such that $p(v_i) = v$. The inequality (6) is equivalent to

$$r(Y_1 + v_1) - r(Y_1) + r(Y_2 + v_2) - r(Y_2) \geq 1. \tag{7}$$

The set $Y = Y_1 + Y_2$ is a subtransversal of Ω and $\{v_1, v_2\}$ is a skew pair included in a skew class disjoint from Y. Axiom 2.4 implies

$$r(Y + v_1) - r(Y) + r(Y + v_2) - r(Y) \geq 1.$$

Since Y_1 and Y_2 are subsets of Y and the restriction of r to the powerset of Y is submodular by Axiom 2.3, the last inequality implies (7). □

Let $D = (V, \mathcal{F})$ be a delta-matroid. Denote by $\max(\mathcal{F})$ and $\min(\mathcal{F})$ the collections of (inclusionwise) maximal members and minimal members of $\mathcal{F}$, respectively. The set systems $M(D) = (V, \max(\mathcal{F}))$ and $m(D) = (V, \min(\mathcal{F}))$ are matroids [2, 3], called the *upper matroid* and *lower matroid* of D, respectively.

Theorem 3.3 *If D is a delta-matroid, then $M(D)$ is a strong map of $m(D)$.*

Proof. Consider the lift of D arising from Construction 2.13. Set $\mathcal{F}_i = \{F_i : F \in \mathcal{F}\}$, for $i = 1, 2$. The equality $\mathcal{B}(Q) = \{F_1 \cup (V_2 \setminus F_2) : F \in \mathcal{F}\}$ implies

$$\mathcal{I}(Q) = \{P_1 \cup Q_2 : P \subseteq F, Q \cap F = \emptyset, \text{ for some } F \in \mathcal{F}\}.$$

The independent sets of the submatroid $Q[V_1]$ are the independent sets of Q included in V_1. Hence

$$\mathcal{I}(Q[V_1]) = \{P_1 : P \subseteq F, \text{ for some } F \in \mathcal{F}\},$$

which implies

$$\mathcal{B}(Q[V_1]) = \max(\mathcal{F}_1).$$

We similarly find

$$\mathcal{B}(Q[V_2]) = \max(\mathcal{F}_2 \Delta V_2).$$

Therefore $Q[V_1]$ and $Q[V_2]$ are projected onto the set systems $M(D)$ and $m(D)\Delta V$, respectively. So $M(D)$ and $m(D)\Delta V$ are orthogonal matroids by Theorem 3.2. (We also retrieve that $M(D)$ and $m(D)$ are actually matroids.) □

4 Free sums of orthogonal matroids

If $Q = (U, \Omega, r)$ is a q-matroid indexed on a set V, and $(V_1, V_2, \cdots, V_q)$ is a partition of U into transversals of Ω, then $Q[V_1]$, $Q[V_2]$, $\cdots$, $Q[V_q]$ are projected onto pairwise orthogonal matroids, by Theorem 3.2. Conversely, given a sequence $(M_1, M_2, \cdots, M_q)$ of orthogonal matroids on the set V, we construct here a q-matroid $Q = Q(M_1, M_2, \cdots, M_q)$ and a partition $(V_1, V_2, \cdots, V_q)$ of the ground-set of Q into transversals such that $Q[V_1]$, $Q[V_2]$, $\cdots$, $Q[V_q]$ are projected onto M_1, M_2, $\cdots$, M_q, respectively.

A q-matroid $Q = (U, \Omega, r)$ is *free* if there exists a partition of U into a sequence of transversals $(V_1, V_2, \cdots, V_q)$ such that

$$r(S) = \sum_{1 \leq i \leq q} r(S \cap V_i)$$

holds for every subtransversal S of Ω.

Proposition 4.1 *Let $Q = (U, \Omega, r)$ be a q-matroid and let $(V_1, V_2, \cdots, V_q)$ be a partition of U into transversals of Ω. The following properties are equivalent:*

(a) *Q is free with respect to $(V_1, V_2, \cdots, V_q)$;*

(b) *a subtransversal I of Ω is an independent set of Q if and only if $I \cap V_i$ is an independent set of $Q[V_i]$ for every i, $1 \leq i \leq q$;*

(c) *a subtransversal C of Ω is a circuit of Q if and only if C is a circuit of $Q[V_i]$ for some i, $1 \leq i \leq q$.*

Proof. This readily follows from the definitions. □

Construction 4.2 Let M_1, M_2, $\cdots$, M_q be pairwise orthogonal matroids on the set V, with rank functions ρ_1, ρ_2, $\cdots$, ρ_q, respectively. Set

$$\begin{aligned}
&V_i = \{v_i : v \in V\},\ 1 \leq i \leq q,\\
&U = \bigcup_{1 \leq i \leq q} V_i,\\
&\Omega_v = \{v_i : 1 \leq i \leq q\},\ v \in V,\\
&\Omega = \{\Omega_v : v \in V\},\\
&P_i = \{v_i : v \in P\} \text{ and } r_i(P_i) = \rho_i(P),\ P \subseteq V,\ 1 \leq i \leq q,\\
&r(S) = \sum_{1 \leq i \leq q} r_i(S \cap V_i),\ S \in \mathcal{S}(\Omega).
\end{aligned}$$

Proposition 4.3 *The triple $Q = (U, \Omega, r)$ arising from Construction 4.2 is a free q-matroid and M_i is the projection of $Q[V_i]$, for $1 \leq i \leq q$.*

Proof. We have to verify that r satisfies the axioms 2.1 to 2.4. Axiom 2.1 is obvious. Let us verify Axiom 2.4. (The verifications of the two remaining axioms are similar.) We have to prove that

$$r(S+v_j)-r(S)+r(S+v_k)-r(S)\geq 1 \tag{8}$$

holds for every subtransversal S of Ω and every skew pair $\{v_j, v_k\}$ contained in a skew class Ω_v disjoint from S.

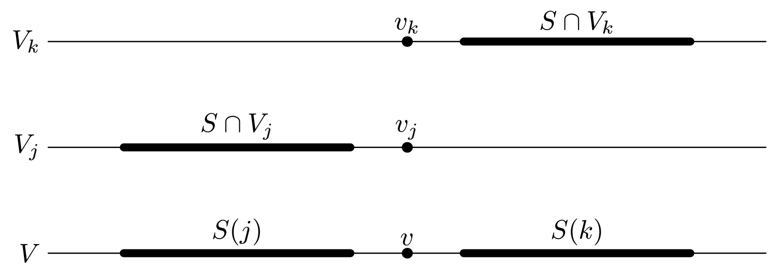

Figure 2: Verifying Axiom 1.1.4.

For $1 \leq i \leq q$, let $S(i)$ denote the subset of V that is equal to the projection of $S \cap V_i$ (see Fig. 2). By the construction of Q we have

$$r(S)=\sum_{1\leq i\leq q}\rho_i(S(i)), \tag{9}$$

The subsets $S(j)$ and $S(k)$ are disjoint and do not contain v, and M_j and M_k are orthogonal. Therefore

$$\rho_j(S(j)+v)-\rho_j(S(j))+\rho_k(S(k)+v)-\rho_k(S(k))\geq 1$$

is satisfied by Proposition 3.1(b). The equality (9) implies

$$\begin{aligned} r(S+v_j) &= r(S)-\rho_j(S(j))+\rho_j(S(j)+v), \\ r(S+v_k) &= r(S)-\rho_k(S(k))+\rho_k(S(k)+v). \end{aligned}$$

The three preceding relations imply the inequality (8). □

We denote by $Q(M_1, M_2, \cdots, M_q)$ the multimatroid arising from Construction 4.2 and we call it the *free sum* of M_1, M_2, $\cdots$, M_q. According to Proposition 4.1, a subtransversal I of Ω is an independent set of Q if and only if $I \cap V_i$ is an independent set of the submatroid $Q[V_i]$, for $1 \leq i \leq q$. If Q' is another q-matroid defined on the same partitioned set (U, Ω) and indexed on the same set V, and such that $Q'[V_i] = Q[V_i]$ for $1 \leq i \leq q$, then every independent set I' of Q' is such that $I' \cap V_i$ is an independent set of $Q'[V_i]$ for $1 \leq i \leq q$, and so I' is also an independent set of Q. Hence Q is the 'most free' q-matroid among all the q-matroids that admit (M_1, M_2, $\cdots$, M_q) as a sequence of projected submatroids.

If M_1 and M_2 are orthogonal matroids on the set V, then the set system $D(M_1, M_2) = (V, \mathcal{F})$, where

$$\begin{aligned}\mathcal{F} &= \{F : F \subseteq B_1, F \cap B_2 = \emptyset \text{ for some } B_1 \in \mathcal{B}(M_1) \text{ and } B_2 \in \mathcal{B}(M_2)\} \\ &= \{F : F \in \mathcal{I}(M_1), V \setminus F \in \mathcal{I}(M_2)\}.\end{aligned}$$

has been introduced by TARDOS [17] under the name of *generalized matroid*. Clearly $D(M_2, M_1) = D(M_1, M_2)\Delta V$. We also note that $D(M, M^*) = M$, for every matroid M.

Proposition 4.4 *If M_1 and M_2 are orthogonal matroids on the set V, then $D(M_1, M_2)$ is equal to the projection of $Q(M_1, M_2) \cap V_1$.*

Proof. Let $Q = Q(M_1, M_2) = (U, \Omega, r)$ and let D be the projection of $Q \cap V_1$. A subset F of V is a feasible set of D if and only if $F_1 \cup (V_2 \setminus F_2)$ is a base of Q. According to Proposition 4.1 (b), this is equivalent to F_1 and $V_2 \setminus F_2$ being independent in $Q[V_1]$ and $Q[V_2]$, respectively. Since $Q[V_1]$ and $Q[V_2]$ are projected onto M_1 and M_2, respectively, this is equivalent to F and $V \setminus F$ being independent sets of M_1 and M_2, respectively. □

Corollary 4.5 [2] *Generalized matroids are delta-matroids.*

Corollary 4.6 *Let $Q = Q(M, M^*) = (U, \Omega, r)$ be the free sum of two dual matroids on the set V. The submatroids $Q[V_1]$ and $Q[V_2]$ are projected onto M and M^*, respectively. A transversal B of Ω is a base of Q if and only if $B \cap V_i$ is a base of $Q[V_i]$, for $i = 1, 2$.*

5 Minors

Consider a multimatroid $Q = (U, \Omega, r)$ and a subtransversal X of Ω. Set

$$\begin{aligned}\Omega' &= \{\omega \in \Omega : \omega \cap X = \emptyset\}, \\ U' &= \bigcup_{\omega \in \Omega'} \omega, \\ r'(S) &= r(S + X) - r(X),\ S \in \mathcal{S}(\Omega').\end{aligned}$$

The triple (U', Ω', r') is a multimatroid, which we denote by $Q|X$ and call a *minor* of Q. If X and Y are disjoint compatible subtransversals, then $Q|X|Y = Q|Y|X = Q|(X \cup Y)$. The minor $Q|X$ is *proper* if $X \neq \emptyset$, *elementary* if $|X| = 1$. If $|X| > 1$, then $Q|X = Q|x_1|x_2|\cdots|x_p$, where $x_1, x_2, \cdots, x_p$ is any enumeration of the elements of X.

Proposition 5.1 *Let $Q(G_U)$ be an Eulerian multimatroid. For every allowed splitter X we have*

$$Q(G_U)|X = Q(G'_{U'}), \tag{10}$$

where $G' = G||X$ *and* U' *is the set of allowed local splitters incident to the vertices of* G'*.*

Proof. The multimatroids $Q_1 = Q(G_U)|X$ and $Q_2 = Q(G'_{U'})$ are defined on the same partitioned set (U', Ω'). Let r be the rank function of $Q(G_U)$ and let r_i be the rank function of Q_i, for $i = 1, 2$. For every subtransversal S of Ω' we have

$$\begin{aligned} r_1(S) &= r(S+X) - r(X) \\ &= |S+X| - k(G||(S+X)) + k(G) - |X| + k(G||X) - k(G) \\ &= |S| - k((G||X)||S) + k(G||X) \\ &= r_2(S). \end{aligned}$$

□

When G is a 4-regular graph, the graph $G' = G||X$ in the relation (10) has vertices of degree 2 that we may wish to erase in order to obtain another 4-regular graph. In general to *erase* a vertex w of degree 2 in a graph H is to delete w as well as the edges and half-edges incident to w then, if there remains two half-edges h_1 and h_2 that are no longer incident to an edge (which happens if w was not incident to a loop in H), to add a new edge and make it incident to h_1 and h_2. To *open* X in G is to construct the detachment $G||X$, then to successively erase the vertices of degree 2. The new graph, denoted by $G|X$, is a 4-regular graph if G is 4-regular. We have $k(G|X) = k(G||X) - k_2$, where k_2 is the number of components of $G||X$ regular of degree 2, and $Q(G||X) = Q(G|X)$. Set $G_U|X = (G|X)_{U'}$, where U' is the set of allowed local splitters incident to the vertices of $G|X$. Then the relation (10) can be written

$$Q(G_U)|X = Q(G_U|X). \tag{11}$$

For a matroid M on the set V and an element v of V, we denote by $M \setminus v$ and M/v the matroids obtained by deleting v and by contracting v, respectively.

Proposition 5.2 *Let* $Q = Q(M_1, M_2, \cdots, M_q)$ *be a free sum of orthogonal matroids on a set* V*. For every* v *in* V *and every* j *in* $\{1, 2, \cdots, q\}$*, we have* $Q|v_j = Q(M'_1, M'_2, \cdots, M'_q)$ *with*

$$\begin{aligned} M'_j &= M_j/v, \\ M'_k &= M_k \setminus v, \ k \in \{1, 2, \cdots, q\} - j. \end{aligned}$$

Proof. Set $(U', \Omega', r') = Q|v_j$. We may assume $j = 1$. For every subtransversal S of Ω' we have

$$\begin{aligned} r'(S) &= r(S+v_1) - r(v_1) \\ &= r_1(S_1+v_1) - r_1(v_1) + \sum_{2\le j\le q} r_j(S_j) \\ &= r'_1(S_1) + \sum_{2\le j\le q} r'_j(S_j), \end{aligned}$$

where S_j is the projection of $S \cap V_j$, r_j is the rank function of M_j, r'_j is the rank function of $M_j \setminus v$ if $j \neq 1$ and r'_1 is the rank function of M_1/v. □

Corollary 5.3 *Let M be a matroid on a set V and let $Q = Q(M, M^*)$. For every element v in V we have*

$$\begin{aligned} Q|v_1 &= Q(M/v, M^* \setminus v) = Q(M/v, (M/v)^*), \\ Q|v_2 &= Q(M \setminus v, M^*/v) = Q(M \setminus v, (M \setminus v)^*). \end{aligned}$$

An element x in U is *singular* if $r(x) = 0$. A skew class that contains a singular element is *singular*.

Proposition 5.4 *A singular skew class contains precisely one singular element.*

Proof. If there were distinct singular elements, x and y, in the same skew class of a multimatroid with rank function r, we should have

$$r(x) - r(\emptyset) + r(y) - r(\emptyset) = 0,$$

contradicting Axiom 2.4. □

Proposition 5.5 *If a skew class ω of a multimatroid Q is singular, then the elementary minor $Q|x$ does not depend from the choice of z in ω, namely $Q|z = Q[U \setminus \omega]$.*

Proof. Let r be the rank function of Q and let x be the singular element of ω. For every subtransversal S, disjoint from ω, the submodularity inequality 2.3 implies

$$r(S+x) - r(S) \le r(x) - r(\emptyset) = 0.$$

Since $r(x) = 0$, it follows

$$r(S+x) - r(x) = r(S). \tag{12}$$

For every element y in $\omega - x$, Axiom 2.4 implies

$$r(S+y) - r(S) + r(S+x) - r(S) \ge 1.$$

Since $r(y) = 1$, by Proposition 5.4, this implies

$$r(S + y) - r(y) = r(S). \tag{13}$$

The equations (12) and (13) imply $Q|x = Q|y = Q[U \setminus \omega]$. □

Theorem 5.6 *For every minor $Q|X$ of a nondegenerate multimatroid Q there exists an independent set Y such that $Q|Y = Q|X$.*

Proof. We use induction on $|X|$. The property is trivial if $|X| = 0$. Assume $|X| > 0$ and consider an element x in X. Set $Q' = Q|x$ and $X' = X - x$. By induction there exists an independent set Y' of Q' such that $Q'|Y' = Q'|X'$. This implies

$$Q|X = Q|x|X' = Q|x|Y' = Q|(Y' + x).$$

If $Y' + x$ is an independent set of Q the proof is done. Assume $Y' + x$ is dependent and denote by r' the rank function of Q'. We have

$$r(Y' + x) \leq |Y'|$$

and

$$|Y'| = r'(Y') = r(Y' + x) - r(x).$$

These relations imply $r(x) = 0$, and so the skew class ω that contains x is singular. Consider an element y in $\omega - x$, which exists because Q is nondegenerate. Proposition 5.4 implies $r(y) = 1$, and Proposition 5.5 implies $Q' = Q|x = Q|y$. Set $Y = Y' + y$. We have

$$Q|X = Q|x|X' = Q'|X' = Q'|Y' = Q|y|Y' = Q|Y$$

and

$$r(Y) = r(Y' + y) = r'(Y') + r(y) = |Y'| + 1 = |Y|,$$

which completes the proof. □

The following result is often called the *scum theorem* [10]

Corollary 5.7 *For every minor M' of a matroid M there exists an independent set I of M and an independent set J of M^*, such that $I \cap J = \emptyset$ and $M' = M/I \setminus J$.*

Proof. Consider the free sums $Q = Q(M, M^*)$ and $Q' = Q(M', M'^*)$. Corollary 5.3 implies that Q' is a minor of Q. Theorem 5.6 implies the existence of an independent set Y of Q such that $Q' = Q|Y$. For $i = 1, 2$, the set $Y \cap V_i$ is an independent set of $Q[V_i]$. The projection I of $Y \cap V_1$ is an independent set of M, and the projection J of $Y \cap V_2$ is an independent set of M^*. By using Corollary 5.3 again, we have

$$\begin{aligned} Q(M', M'^*) &= Q|Y \\ &= Q|(Y \cap V_1)|(Y \cap V_2) \\ &= Q(M/I \setminus J, M^* \setminus I/J), \end{aligned}$$

and so $M' = M/I \setminus J$. □

6 Separators

Let us recall that a *separator* of a matroid M on the set V, with rank function r, is a subset W of V such that

$$r(S) = r(S \cap W) + r(S \setminus W) \tag{14}$$

is satisfied for every subset S of V. We similarly define a *separator of the elements* of a multimatroid $Q = (U, \Omega, r)$ as a subset W of U that is a union of skew classes and satisfies the equality (14) for every subtransversal S of Ω. If Q is indexed on a set V, we define a *separator of the indices* as a subset W of V such that $\bigcup_{v \in W} \Omega_v$ is a separator of the elements. When using the term *separator*, without specifying the elements or the indices, we implicitly refer to a separator of the indices if Q is indexed, and to a separator of the elements if no indexing of Q is specified. The multimatroid Q is *connected* if it has no proper separator.

There is also a weaker notion of separator that has some interest: a subset $W \subseteq U$ is a *weak separator* if the equality (14) holds for every subtransversal S of V (but W is not necessarily a union of skew classes). For example in a free sum $Q(M_1, M_2, \cdots, M_q)$ of orthogonal matroids defined on a set V, the transversals V_1, V_2, $\cdots$, V_q are weak separators.

Proposition 6.1 *The set of (weak) separators is closed under union, intersection, and complementation.*

Proof. This readily follows from the definition and the submodularity of the rank function. □

We recall the following basic relation between the separators and the circuits of a matroid.

Theorem 6.2 *A subset W of the elements of a matroid M is a separator if and only if every circuit of M is either included in W or disjoint from W.*

Corollary 6.3 *Let $Q = (U, \Omega, r)$ be a multimatroid and let X be a subset of U. The following properties are equivalent:*

(a) *X is a weak separator of Q;*

(b) *$X \cap T$ is a separator of the submatroid $Q[T]$ for every transversal T of Ω;*

(c) *every circuit of Q is either included in X or disjoint from X.*

Proof. The equivalence of (a) and (b) readily follows from the definition. The equivalence of (b) and (c) is a simple consequence of Theorem 6.2. □

Although the class of connected matroids is clearly equal to the class of connected 1-matroids, many basic properties of connected matroids cannot be generalized to arbitrary multimatroids. For example a matroid is connected if and only if every pair of elements of that matroid belong to the same circuit. That property no longer holds for a 2-matroid $Q = (U, \Omega, r)$. Indeed let $U = \{a, a', b, b', c, c', d, d'\}$, let $\Omega = \{aa', bb', cc', dd'\}$, and let the set of circuits be equal to $\{a'b'cd, a'bcd', ab'c'd, abc'd'\}$ (we omit the braces around the elements of a powerset). There exists no proper subset W of U such that every circuit is either included in W or disjoint from W. Accordingly Q is connected. However there is no circuit including $\{a', c'\}$. In a subsequent paper [6] we show that some basic properties of connected matroids can be generalized to the subclass of tight multimatroids.

7 Cyclic splitters

The cyclic splitters are particular splitters associated to the cycles of an Eulerian graph G. We show that the circuits of $Q(G)$ are the minimal nonempty cyclic splitters. We define a transformation of G that preserves the cyclic splitters and we prove the existence of a connected Eulerian graph G' such that $Q(G') = Q(G)$.

Definitions and basic properties

Let Γ be a subset of edges of G. The set of half-edges incident to the edges in Γ is denoted by $h(\Gamma)$. We recall that the set of half-edges incident to a vertex v is denoted by $h(v)$. The set Γ is a *cycle* of G if $|h(\Gamma) \cap h(v)|$ is even for every vertex v. Set $V(\Gamma) = \{v \in V : \emptyset \neq h(\Gamma) \cap h(v) \neq h(v)\}$ and $S(\Gamma) = \{S(\Gamma)_v : v \in V(\Gamma)\}$, where $S(\Gamma)_v = \{h(\Gamma) \cap h(v), h(v) \setminus h(\Gamma)\}$. So Γ is a cycle if and only if $S(\Gamma)$ is a splitter. Then we call $S(\Gamma)$ a *cyclic splitter*. Figure 3 depicts a cycle Γ, drawn with thick edges, and the detachment $G||S(\Gamma)$. The following property is a direct consequence of the definitions.

Proposition 7.1 *If Γ is a cycle of an Eulerian graph G, then the edge-set of every component of $G||S(\Gamma)$ is contained in Γ or disjoint from Γ.*

A bicoloring of the half-edges, say in black and white, is *compatible* with a splitter $S = \{S_v : v \in W\}$, $S_v = \{S'_v, S''_v\}$, if the three following conditions are satisfied :

7.2 *two half-edges incident to a same edge have the same color;*

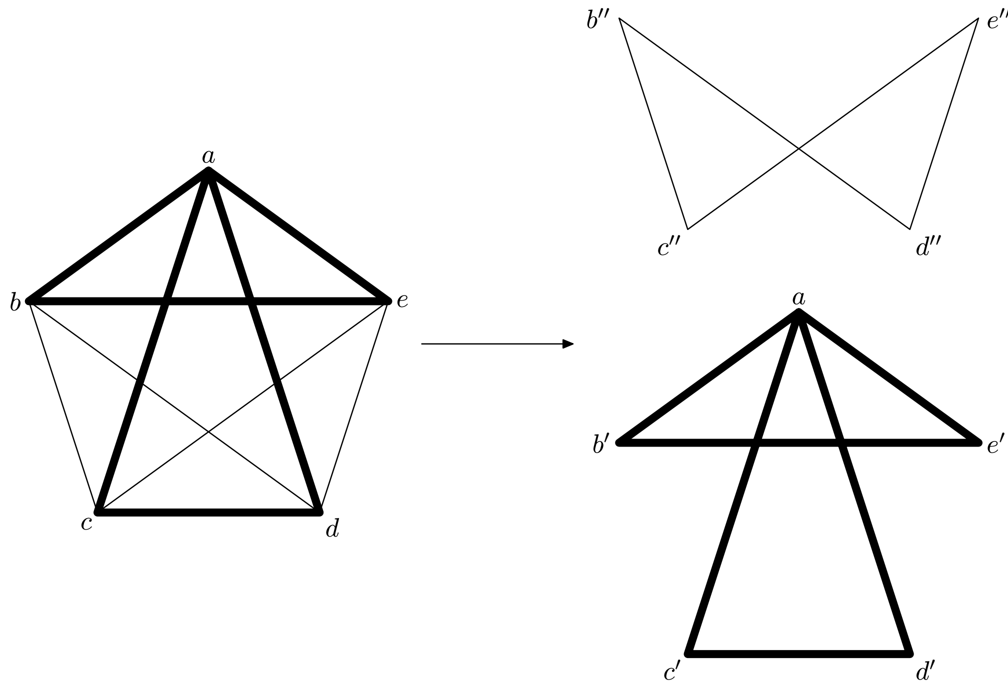

Figure 3: Detachment of a cyclic splitter

7.3 *two half-edges incident to a same vertex v in $V - W$ have the same color;*

7.4 *two half-edges incident to a same vertex v in W have the same color if and only if both belong to either S'_v or S''_v.*

Proposition 7.5 *A splitter $S = \{S_v : v \in W\}$ is cyclic if and only if there is a bicoloring of the half-edges compatible with S.*

Proof. If $S = S(\Gamma)$, for some cycle Γ, then we obtain a compatible bicoloring by letting the half-edges in $h(\Gamma)$ be black and the other half-edges be white. Conversely if there is a compatible bicoloring , then the subset Γ of the edges incident to the black half-edges is a cycle such that $S = S(\Gamma)$. □

If H is a component of $G||S$, where S is a cyclic splitter, and if we consider a coloring of the half-edges compatible with S, then Proposition 7.1 implies that the half-edges incident to H have the same color, which we call the *color* of H.

Circuits of $Q(G)$

In view of the following properties we point out that a circuit of $Q(G)$ is not to be confused with a circuit of G. The former is a splitter of G and the latter is a set of edges of G.

Proposition 7.6 *Let G be an Eulerian graph. A nonempty cyclic splitter S of G is* *dependent in $Q(G)$.*

Proof. Set $S = \{S_v : v \in W\}$, $S_v = \{S'_v, S''_v\}$. Consider a vertex v in W and the components X' and X'' of $G||S$ that contain S'_v and S''_v, respectively. In a bicoloring of the half-edges compatible with S, the half-edges in S'_v have not the same color as the half-edges in S''_v, by the condition 7.4. Hence X' and X'' have distinct colors, and so $X' \neq X''$. If we reconstruct G from $G||S$ by identifying each pair of vertices of $G||S$ corresponding to the same vertex of G, the components X' and X'' are merged into the same component. Accordingly $k(G||S) > k(G)$, and so S is dependent in $Q(G)$. □

Theorem 7.7 *Let G be an Eulerian graph. The minimal nonempty cyclic splitters of G are the circuits of $Q(G)$.*

Proof. By the preceding proposition we know that every minimal nonempty cyclic splitter of G includes a circuit of $Q(G)$. It remains to prove that every circuit C of $Q(G)$ is a cyclic splitter.

Set $C = \{C_v : v \in W\}$, $C_v = \{C'_v, C''_v\}$, and denote by v' and v'' the vertices of $G||C$ such that $h(v') = C'_v$ and $h(v'') = C''_v$. Since C is a circuit of $Q(G)$ we have $k(G||C) - k(G) = 1$ and $k(G||C) = k(G||(C - C_v)) + 1$ for every v in W.

Claim. *For every v in W, v' and v'' are in different components of $G||C$.*

Proof. By identifying v' and v'' in $G||C$ we obtain $G||(C - C_v)$. No component of $G||C$ is modified after this identification, except for the components X' and X'' containing v' and v'', respectively, which are merged together. If these components are equal, then the number of components is not modified by the identification, which contradicts the equality $k(G||C) = k(G||(C - C_v)) + 1$. □

Let $G^1, G^2, \cdots, G^k$ be the components of G and, for $1 \leq j \leq k$, let $C^j = \{C_v : v \in W \cap V(G^j)\}$. The detachment $G||C$ can be constructed by successively constructing each detachment $G^j||C^j$, for $j = 1, 2, \cdots, k$. Therefore

$$1 = k(G||C) - k(G) = \sum_{j=1}^{j=k} (k(G^j||C^j) - 1).$$

Accordingly we may assume $G^j||C^j$ is connected, for $2 \leq j \leq k$, and $G^1||C^1$ has two components, say X' and X''.

Each subset C^j, $2 \leq j \leq k$, is empty. Indeed if there was a local splitter C_v in C^j, then v' and v'' would be vertices of $G^j||C^j$, which is a component of $G||C$, contradicting the claim. So, for every v in W, the vertices v' and v'' belong to $G^1||C^1$. Moreover they do not belong to the same component of $G||C$ according to the claim. Hence we may assume C'_v is a subset of half-edges of X' and C''_v is a subset of half-edges of X''. Then, by coloring the half-edges of X' in black and all the other half-edges in white, we obtain a bicoloring compatible with C. □

Corollary 7.8 *If two Eulerian graphs G' and G'' have the same cyclic splitters, then $Q(G') = Q(G'')$.*

Breaking 2-cuts

Let $\{V^1, V^2\}$ be a bipartition of the vertex-set of G. The set C of edges of G that have one end in V^1 and one end in V^2 is called a *k-cut* if $|C| = k$. Since G is Eulerian, k is even. To *break* a 2-cut $\{e_1, e_2\}$ is to construct the graph depicted in Figure 4. Formally, denoting by h_i^j the half-edge incident to e_i and V^j, we replace e_1 and e_2 by two edges e^1 and e^2, where e^j is incident to h_1^j and h_2^j, for $j = 1, 2$. So we split the component of G that contains e_1 and e_2 into a component that contains e^1 and a component that contains e^2. The reverse operation, which consists in replacing $\{e^1, e^2\}$ by $\{e_1, e_2\}$, is called *glueing* along $\{e^1, e^2\}$.

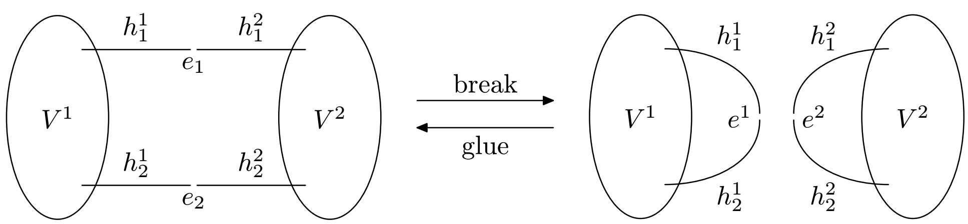

Figure 4: Breaking a 2-cut and glueing along a pair of non-connected edges

Proposition 7.9 *If an Eulerian graph G' is derived from another one G by either breaking a 2-cut or glueing a pair of nonconnected edges, then G and G' have the same cyclic splitters.*

Proof. Consider a cyclic splitter S of G and a bicoloring of the half-edges compatible with S. The same bicoloring, with respect to G', still satisfies the conditions 7.3 and 7.4. If it also satisfies Condition 7.2, with respect to G', then S is a cyclic splitter of G', and the property is proved. In the other case we have to modify the bicoloring in order to prove the property. Let Γ be the cycle of G incident to the black edges.

Case 1: Breaking a 2-cut $\{e_1, e_2\}$. Since the intersection of a cut with a cycle has an even cardinality, $\{e_1, e_2\}$ is either contained in Γ or disjoint from Γ. In both cases the four half-edges incident to e_1 and e_2 have the same color. These half-edges are also incident to e^1 and e^2 in G'. Therefore Condition 7.2 is satisfied.

Case 2: Glueing along a pair of nonconnected edges $\{e^1, e^2\}$. By Condition 7.2, with respect to G, the half edges incident to e^1 have the same color χ_1, and the half-edges incident to e^2 have the same color χ_2. If $\chi_1 = \chi_2$ Condition 7.2 still holds with respect to G'. If $\chi_1 \neq \chi_2$ we exchange the colors black and white on the half-edges of the component of G that contain the edge e^2. We have still a bicoloring compatible with S, and $\chi_1 = \chi_2$. □

Corollary 7.10 *For every Eulerian graph there exists a connected Eulerian graph that admits the same splitter rank function.*

Proof. Delete the vertices of null degree, which play no role in the cyclic splitters. Then make successive glueings until obtaining only one component. □

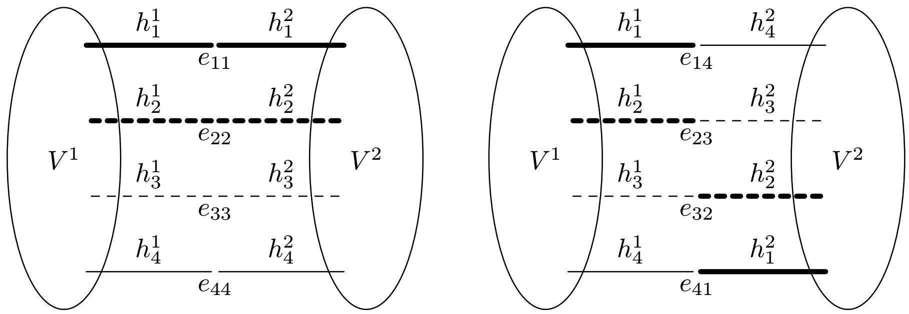

Figure 5: Switching a 4-cut, $\tau = (1\,4)\,(2\,3)$.

Switching a 4-cut

Let $C = \{e_{ii} : i \in I\}$, $I = \{1, 2, 3, 4\}$, be a 4-cut of G defined by the bipartition $\{V^1, V^2\}$ of the vertex-set and let τ be a fixed point free involution on I. (Thus τ is a permutation of I such that $\tau(i) \neq i$ and $\tau^2(i) = i$ hold for all i in I. We note that precisely three such involutions exist.) Denote by h_i^j the half-edge incident to e_{ii} and V^j, for all i in I and all j in $\{1, 2\}$. For each i in I, remove the edge e_{ii} and replace it by an edge $e_{i\tau(i)}$ incident to h_i^1 and $h_{\tau(i)}^2$. This transformation, called a *switching*, is illustrated in Figure 5. By performing again the same switching one regains the original graph.

Proposition 7.11 *If an Eulerian graph G' is derived from another one G by switching a 4-cut, then G and G' have the same cyclic splitters.*

Proof. The proof is similar to the preceding one and we use the same notation. The cut C contains an even number of edges of Γ, say p. If $p = 0$ or $p = 4$, the half-edges incident to C have the same color, and so the coloring is still compatible in G'. If $p = 2$ and $C \cap \Gamma = \{e_{ii}, e_{\tau(i)\tau(i)}\}$, for some i in I, again the coloring is compatible in G'. In the remaining case we may assume without loss of generality $\tau = (1\,4)\,(2\,3)$, the half-edges incident to e_{11} and e_{22} are colored in black, the half-edges incident to e_{33} and e_{44} are colored in white. The coloring of the half-edges, with respect to G', is illustrated in Figure 6. We obtain a coloring compatible in G' by exchanging the colors black and white on the half-edges incident to V^2. □

Question. Given two connected Eulerian graphs G and G' without 2-cut and such that $Q(G) = Q(G')$, is it true that G' can be derived from G by a succession of 4-cut switchings? This has been proved when G and G' are 4-regular by Ghier [13].

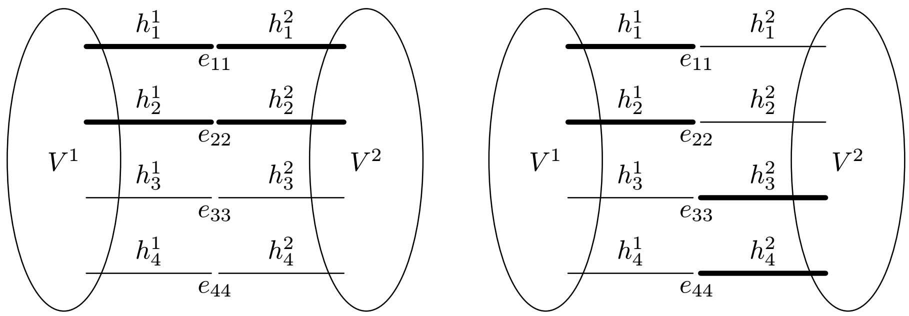

Figure 6: Exchanging the colors of the half-edges incident to V^2.

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