New Bounds for Codes Identifying Vertices in Graphs

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Abstract

Let $G = (V, E)$ be an undirected graph. Let C be a subset of vertices that we shall call a code. For any vertex $v \in V$, the neighbouring set $N(v, C)$ is the set of vertices of C at distance at most one from v. We say that the code C *identifies* the vertices of G if the neighbouring sets $N(v, C), v \in V$, are all nonempty and different. What is the smallest size of an identifying code C ? We focus on the case when G is the two-dimensional square lattice and improve previous upper and lower bounds on the minimum size of such a code.

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1 Introduction

In this paper, we investigate a problem initiated in [3]: given an undirected graph $G = (V, E)$, we define $B(v)$, the *ball* of radius one centered at a vertex $v \in V$, by

$$B(v) = \{x \in V : d(x, v) \leq 1\},$$

where $d(x, v)$ represents the number of edges in a shortest path between v and x. The vertex v is then said to *cover* all the elements of $B(v)$. We often refer to a distinguished subset C of V as a *code*, and to its elements as *codewords*.

A code C is called a *covering* if the sets $B(v) \cap C$, $v \in V$, are all nonempty; if furthermore they are all different, C is called an *identifying code*. The set of codewords covering a vertex v is called the *identifying set* (I-set) of v.

Now, what is the minimum cardinality of an identifying code ? This problem originates in [3] and is also taken up in [1].

Let us mention an application. A processor network can be modeled by an undirected graph $G = (V, E)$, where V is the set of processors and E the set of their links. A selected subset C of the processors constitutes the code. Its codewords report to a central controler the state of their neighbourhoods (typically, balls of radius one) by sending one bit of information (e.g., 1 if it does not contain a faulty processor, 0 otherwise). Based on these $|C|$ bits, the controler must locate the faulty processor. Common network architectures are the n-cube or the two-dimensional mesh or grid.

In this paper we focus on the case when G is a square grid drawn on a torus, that is G is the graph $\mathbb{T}_{nm}$ with vertex set $V = \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$ and edge set $E = \{\{u, v\} : u - v = (\pm 1, 0) \text{ or } u - v = (0, \pm 1)\}$. We shall also consider the limiting infinite case, i.e. when G is the graph $\mathbb{T}$ with vertex set $\mathbb{Z} \times \mathbb{Z}$. The *density* $D(C)$ of $C \subseteq V$ is defined as $|C|/|V|$ for $\mathbb{T}_{nm}$ and for the infinite graph $\mathbb{T}$ as

$$D(C) = \limsup_{n \to \infty} \frac{|C \cap Q_n|}{|Q_n|}$$

where Q_n is the set of vertices $(x, y) \in V$ such that $|x| \leq n$ and $|y| \leq n$.

An example of an identifying code of $\mathbb{T}$ is given in figure 1. It is taken from [3] and its density is 3/8. Our purpose is to determine the minimum density D of an identifying code of $\mathbb{T}$. It is proved in [3] that $1/3 \leq D \leq 3/8$. We shall improve this to

$$\frac{23}{66} \leq D \leq \frac{5}{14}.$$

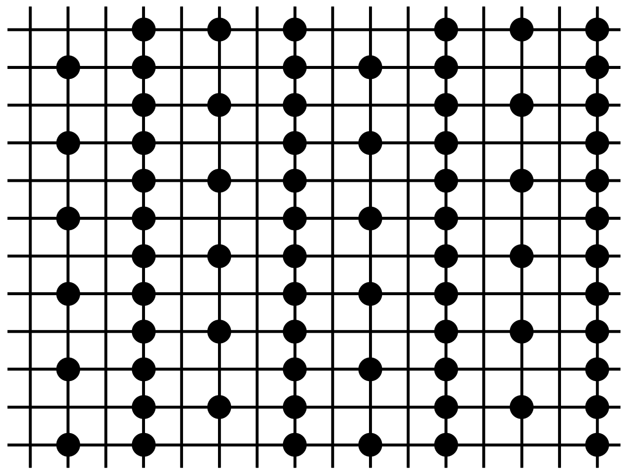

Figure 1: The pattern is periodic and extends to $\mathbb{Z}^2$ with density 3/8.

2 Lower bounds

For a given finite regular graph $G = (V, E)$, let $B = |B(v)|$ denote the size (independent of its centre) of a ball of radius one; let C be an identifying code. Since C is a covering of V, the *sphere-covering bound* holds:

$$|C| \cdot B \geq |V|.$$

But the identifying property implies a strictly better bound : let L_1 denote the set of vertices identified by singletons; now $|V| - |L_1|$ vertices have I-sets of size at least two. In other words, C is a double covering (see [2, Ch. 14]) of these vertices; thus, using the fact that $|L_1| \leq |C|$, we have:

$$|C| \cdot B \geq 2(|V| - |L_1|) + |L_1| = 2|V| - |L_1| \geq 2|V| - |C|.$$

We obtain, [3]

$$|C| \cdot \frac{B+1}{2} \geq |V|. \tag{2.1}$$

Bound (2.1) can be tight in some graphs, for example the triangular lattice, see [3].

2.1 The graphs $\mathbb{T}_{nm}$

Until the end of this section G will be a finite torus $\mathbb{T}_{nm}$ with $n, m \geq 30$, say. All balls of radius one have cardinality five. For $i = 1, 2, 3, 4, 5$, let L_i be the set of vertices identified by a set of exactly i codewords. Set $\ell_i = |L_i|$, $L_{\geq 3} = L_3 \cup L_4 \cup L_5$ and

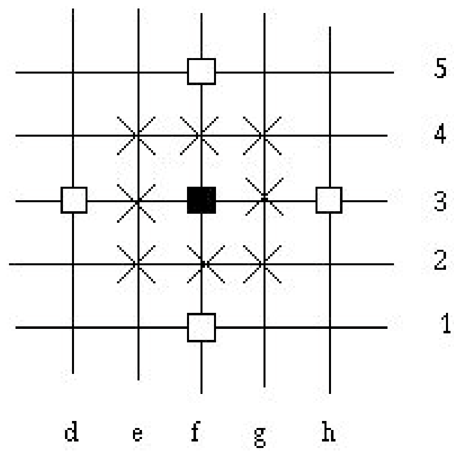

Figure 2: An element of C'.

$\ell_{\geq 3} = |L_{\geq 3}|$. Counting in two ways the number of couples (c, x) such that $c \in C$, $x \in V$ and $d(c, x) \leq 1$, we get:

$$5|C| = \sum_{1 \leq i \leq 5} i\ell_i. \tag{2.2}$$

From (2.2), we infer that $5|C| = \ell_1 + 2(|V| - \ell_1 - \ell_{\geq 3}) + 3\ell_{\geq 3} + \ell_4 + 2\ell_5$. Since $\ell_1 \leq |C|$, we obtain:

$$6|C| \geq 2|V| + \ell_{\geq 3} + \ell_4 + 2\ell_5. \tag{2.3}$$

If it were possible that $\ell_{\geq 3} = 0$ then the bound (2.3) would collapse to (2.1). But this is not the case for the square grids and for the rest of this section we shall bound $\ell_{\geq 3}$ from below as tightly as we can.

2.2 Partitioning C

We partition the code C into two subcodes C' and C'', with C'' consisting of all codewords belonging to at least one I-set of cardinality at least three. Thus, C' is the set of all codewords belonging only to I-sets of size one or two. Our strategy will be to bound $\ell_{\geq 3}$ from below by a function of $|C'|$. First, some facts about C' and C''.

In G, any vertex $c' \in C'$ has the neighbouring configuration of figure 2, where the black square represents c', a white square represents an element of C, and a cross represents a vertex not in C.

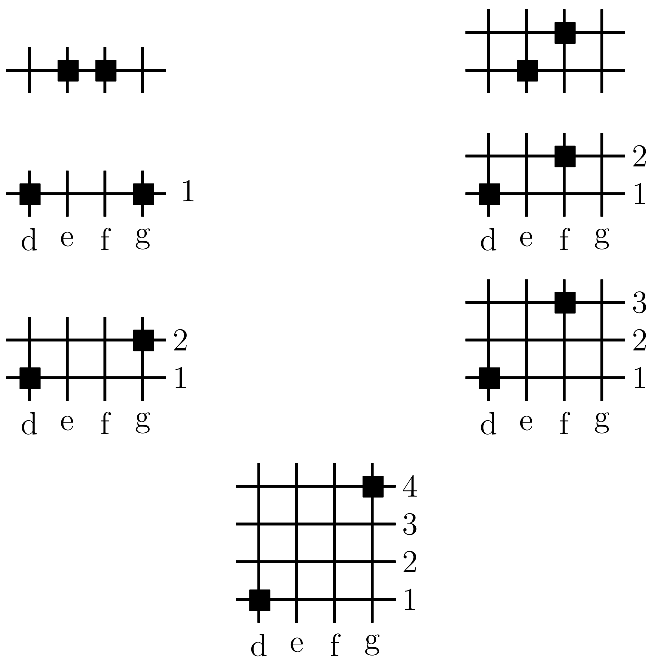

Figure 3: Forbidden configurations of two elements of C'.

Indeed, suppose that a codeword $c \in C$ is on $e3$; then, in order to give c' and c distinct I-sets, c' should belong to an I-set of size at least three. If $c \in C$ is on $e2$, then, in order to give $e3$ and $f2$ distinct I-sets, again c' must belong to an I-set of size at least three. This contradicts the definition of C'. Finally, $d3$, $f1$, $f5$ and $h3$ belong to C because $e3$, $f2$, $f4$ and $g3$ must have an I-set which is not reduced to $\{c'\}$. Actually, using similar arguments, it is easy to check (see figure 3) that two elements of C' cannot be at Euclidean distance 3 (e.g., on $d1$ and $g1$), $\sqrt{5}$ (on $d1$ and $f2$), $\sqrt{10}$ (on $d1$ and $g2$), $2\sqrt{2}$ (on $d1$ and $f3$), and even $3\sqrt{2}$ (on $d1$ and $g4$) from one another.

Obviously, we have $3\ell_3 + 4\ell_4 + 5\ell_5 \geq |C''|$, i.e.,

$$3\ell_{\geq 3} + \ell_4 + 2\ell_5 \geq |C''|. \tag{2.4}$$

Let $\ell_4 = \alpha\ell_{\geq 3}$, $\ell_5 = \beta\ell_{\geq 3}$ (with $\alpha, \beta, \alpha + \beta \in [0, 1]$). Then

$$\ell_{\geq 3} \geq \frac{|C''|}{3 + \alpha + 2\beta}.$$

Combining with (2.3), this leads to

$$6|C| \geq 2|V| + |C''|(1 - \frac{2}{3 + \alpha + 2\beta}).$$

The right hand side is smallest when $\alpha = \beta = 0$, hence

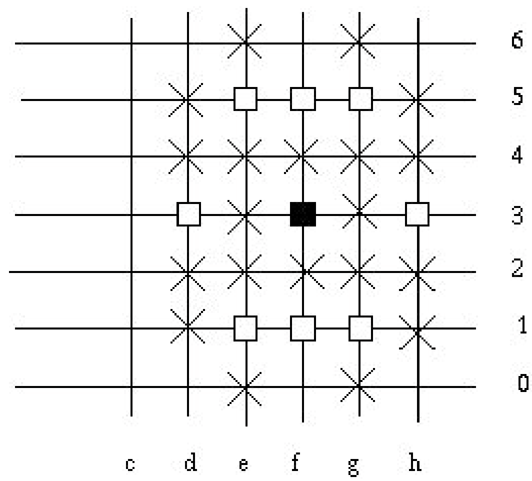

Figure 4: An element of C' with degree two in Γ.

Lemma 2.1 $6|C| \geq 2|V| + |C''|/3$. $\triangle$

2.3 An incidence relation between C' and $L_{\geq 3}$

For any vertex v, let $R(v)$ be the set of points at Euclidean distance either 2 or $\sqrt{5}$ from v. Now let us consider the bipartite graph Γ whose set of vertices is $C' \cup L_{\geq 3}$, and whose set of edges is included in $C' \times L_{\geq 3}$, with an edge between $c' \in C'$ and $x \in L_{\geq 3}$ if and only if $x \in C \cap R(c')$. We now study possible degrees in Γ.

Lemma 2.2 *Any element of C' has degree at least two in Γ.*

Proof. Consider again figure 2. To identify $e4$, we can assume, without loss of generality, that there is a codeword in $e5$. Since $e5$ and $f5$ must have distinct I-sets, at least one of them must have at least a third element in its I-set. The same is true for $f1$ and $g1$, or $h2$ and $h3$, according to which place you choose for covering $g2$. Actually, the only way for $c' \in C'$ to have degree exactly two is given by figure 4 (or its rotation). $\triangle$

Lemma 2.3 *Any element of $L_{\geq 3}$ has degree at most three in Γ.*

Proof. Assume that a codeword x in $L_{\geq 3}$ has degree four: four distinct codewords c'_1, c'_2, c'_3, and c'_4 of C' are adjacent to x in Γ. For each i, $c'_i \in R(x)$, because $x \in R(c'_i)$, and figure 5 shows, with black squares, the twelve possible locations for the four c'_i's around x; figure 5 also gives the two possible ways of identifying the vertex x on $f3$

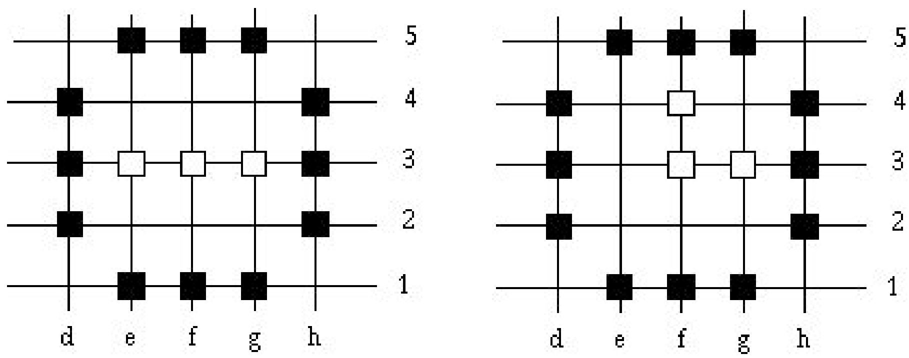

Figure 5: $R(x)$, the set of possible locations for elements of C'.

with three codewords, represented as white squares (more elements in the I-set of x would only mean more restrictions on the c_i''s). Now, keeping in mind figure 2 and the forbidden configurations of figure 3 it is not difficult to check that choosing four c_i''s among these twelve positions is impossible, and furthermore that figure 6 gives the only possible configurations with three elements of C' in $R(x)$ (this will help in proving our following lemma). △

Lemma 2.4 *If an element of $L_{\geq 3}$ has degree three in Γ, then at least two of its neighbours in Γ have degree at least four.*

Proof. Let us consider Configuration (b) of figure 6. There is necessarily a codeword on $f7$, in order to identify $f6$. The points $f2$ and $f4$ have different I-sets, so there is a codeword on $e2$. So in Γ we have the edges $(e5, f7)$, $(e5, f3)$, $(e5, e3)$; $(g5, f7)$, $(g5, f3)$; $(g1, f3)$, $(g1, e2)$. Now in order to cover $d6$ and $d4$, we must increase the degree of $e5$, and this will do nothing for the covering of $h6$, $h4$, $h2$, $f0$ and $h0$. For $h4$ and $h6$ we have two possibilities. Either we do not take $h3$ as a codeword: this allows the degree of $g5$ to increase by one only (if we take $i4$ and $i6$ as codewords). But then the covering of $h2$, $f0$ and $h0$ requires an increase of the degree of $g1$ of at least two, and in the best case we end up with degrees four, three and four for $e5, g5$ and $g1$, respectively. Or we take $h3$ in C: now $g3$ is in $L_{\geq 3} \cap C$ and the degrees of $g5$ and $g1$ both increase. The covering of $h6$, $f0$ and $h0$ will necessarily lead to another increase, and we end up with degrees at least four in Γ.

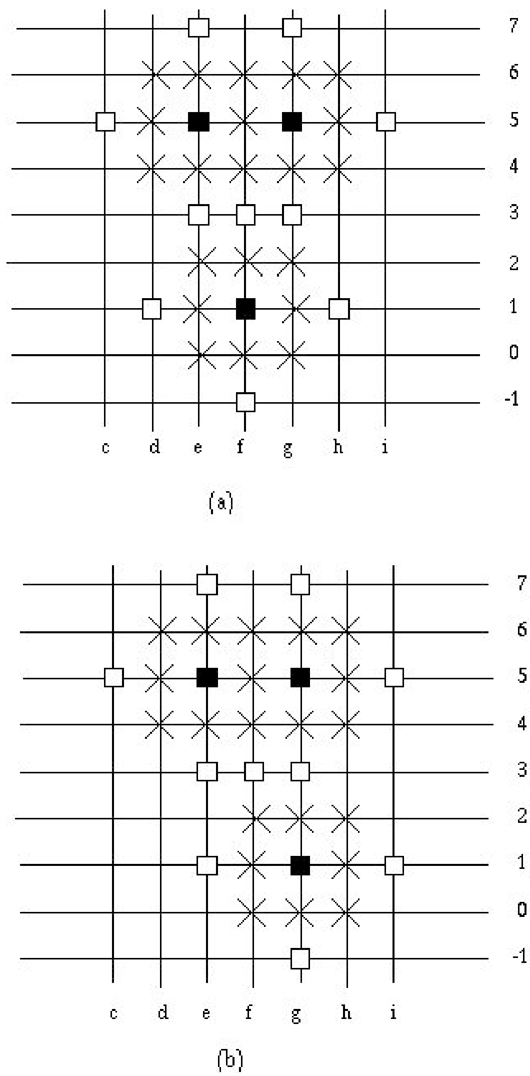

Figure 6: Possible locations for three elements of C' in $R(x)$.

In Configuration (a) of figure 6, there must also be a codeword on $f7$, so the two elements of C', $e5$ and $g5$, have $f7$ and $f3$ as neighbours in Γ. We now prove that $g5$ has at least two more edges in Γ; by symmetry, the same will be true for $e5$, proving our lemma.

Because $h6$ must be covered, $h7$ or $i6$ are in C. If $h7 \in C$, then the fact that $h4$ has to be covered gives the claim. Assume that $i6 \in C$. Since $h4$ must be covered, $h3$ or $i4$ belong to C. If $h3 \in C$, we are done. If $i4 \in C$ and $h3 \notin C$, then $i3 \in C$, because $h3$ and $g2$ must have distinct I-sets.

In all cases, $g5$ has degree at least four in Γ. △

Corollary 2.5 $\ell_{\geq 3} \geq |C'|$.

Proof. We partition $L_{\geq 3}$ into two sets, A and B: A is the set of vertices with degree exactly three in Γ and B is the set of vertices with degree at most two in Γ. We partition C' into two sets, X and Y: X contains the vertices having degree two or three in Γ and Y contains the vertices having degree at least four in Γ. Let a, b, c and d be the number of edges between X and A, X and B, Y and A, Y and B, respectively. Counting in different ways the edges of Γ, we obtain:

$$c + d \geq 4|Y|,\ a + b \geq 2|X|,\ a + c = 3|A|,\ b + d \leq 2|B|,$$

or

$$4|Y| - d \leq c = 3|A| - a \tag{2.5}$$

and

$$2|X| - a \leq b \leq 2|B| - d. \tag{2.6}$$

This leads to $4|C'| \leq 3|A| + 4|B| + a - d$. But Lemma 2.4 implies

$$a \leq |A|. \tag{2.7}$$

Therefore, $4|C'| \leq 4\ell_{\geq 3} - d \leq 4\ell_{\geq 3}$. △

We will now improve on this last result by showing that X and B cannot be both made up only of vertices of degree two in Γ.

2.4 A refined analysis of the degrees in Γ

Let us further partition the sets X and B: let C'_2 and C'_3 be the subsets of X with vertices of degree two and three in Γ, respectively; let B_0, B_1, and B_2 be the subsets of B containing vertices of degree zero, one, and two in Γ, respectively.

We study the elements of C'_2 and start from figure 4. Because $d2, d3$ and $d4$ must have distinct I-sets, we see that at least one of $c2$ and $c4$ must belong to C: we can assume, by symmetry, that $c4 \in C$. Then $c3$ or $c2$ are in C, and $c3 \in L_{\geq 3}$.

Case A: $c3 \notin C$. It implies that $c2 \in C$ and $c3$ has degree zero in Γ.
Case B: $c3 \in C$. What degree can $c3$ have in Γ? There are only four possible places for elements of C' around $c3$: $a2$, $a3$, $a4$ and $c1$. Keeping in mind the forbidden distances between two elements of C', it is easy to check that there are three possibilities: 1) $c3$ has degree zero in Γ; 2) $c3$ has degree one in Γ, and any of these four places is possible; 3) $c3$ has degree two in Γ and necessarily $a4 \in C'$ (the other neighbour of $c3$ in Γ being $a2$ or $c1$).
Case B1: $c3$ has degree zero in Γ.
Case B2: $c3$ has degree one in Γ.
Case B3: $c3$ has degree two in Γ. This implies that $a4 \in C'$ (and $c1$ or $a2$ is in C').
Case B3a: $c5 \in C$. This implies that $c4 \in L_{\geq 3} \cap C$; moreover, $c4$ has degree one in Γ, $a4$ being its only neighbour.
Case B3b: $c5 \notin C$. This implies that $b6 \in C$ (to cover $b5$) and $d6 \in C$ (because $e4$ and $d5$ have distinct I-sets). The vertex $e6$ is not a codeword, and, since its I-set is different from that of $d5$, $e6 \in L_{\geq 3}$, with degree zero in Γ.

In these five cases, we have exhibited a vertex with degree zero or one in Γ. Of course, each time, a second one exists in a symmetric position, on column g or i.

Now we gather Cases A and B3b, which generated elements of $L_{\geq 3} \setminus C$ (of degree zero in Γ); and Cases B2 and B3a, which generated codewords of degree one in Γ. Case B1 has produced a codeword with degree zero in Γ. The point is to see how many elements of C'_2 could produce the **same** vertex. Then we can have an estimate on the number of elements which have degree zero or one in Γ, thus improving the inequality linking $|C'|$ and $\ell_{\geq 3}$.

We give a sketch only for Cases A and B3b. The other cases are very similar. The following remark will be useful: two elements of C'_2 cannot be at distance two from each other.

In Case A (resp., B3b), we produced an element of $L_{\geq 3} \setminus C$, $c3$ (resp., $e6$), at Euclidean distance 3 (resp., $\sqrt{10}$) from our starting point $f3 \in C'_2$. In Case A, apart from $f3$, the only possible location for an element of C'_2 at Euclidean distance 3 from $c3$ is $z3$. In Case B3b, apart from $f3$, the only possible locations for an element of C'_2 at Euclidean distance $\sqrt{10}$ from $e6$ are $d9$ and $f9$, but, using our preliminary remark, at most one is possible. One "crossing" between Case A and Case B3b can occur only when there is an element of C'_2 on $e9$, which excludes $d9$ and $f9$. So in this case, one vertex with degree zero in Γ is shared by at most two elements of C'_2.

In Cases B2 and B3a, one vertex with degree one is shared by at most two elements of C'_2. In case B1, at most two elements of C'_2 generate the same vertex of degree zero.

Since, by symmetry, one element in C'_2 produces two vertices with degree zero or one in Γ, we have shown:

Lemma 2.6 $|B_0| + |B_1| \geq |C'_2|$. △

Now, following (2.6), we have $2|C'_2|+3|C'_3|-a = b \leq 2|B_2|+|B_1|-d$, or $3|X|-|C'_2| \leq 2|B_2| + |B_1| + a - d$. By the previous lemma, this implies that

$$3|X| \leq 2|B_2| + 2|B_1| + |B_0| + a - d = 2|B| - |B_0| + a - d.$$

Thus

$$3|X| \leq 2|B| + a - d, \tag{2.8}$$

which improves on (2.6) and, together with (2.5) and (2.7), leads to

$$4|C'| \leq 3|A| + \frac{8}{3}|B| + \frac{1}{3}a - \frac{1}{3}d \leq \frac{10}{3}|A| + \frac{8}{3}|B| - \frac{1}{3}d \leq \frac{10}{3}\ell_{\geq 3},$$

and we have just proved:

Lemma 2.7 $\ell_{\geq 3} \geq 6|C'|/5$. △

Corollary 2.8 $6|C| \geq 2|V| + 6|C'|/5$.

Proof. By (2.3), $6|C| \geq 2|V| + \ell_{\geq 3} \geq 2|V| + 6|C'|/5$. △
Since $|C'| + |C''| = |C|$, Lemma 2.1 and the above corollary yield:

$$66|C| \geq 23|V|. \tag{2.9}$$

By letting the two dimensions of $\mathbb{T}_{mn}$ grow to infinity, we obtain

Theorem 2.9 *The minimum density of an identifying code of the infinite square lattice* $\mathbb{T}$ *satisfies* $D \geq 23/66$. △

Remark : more detailed study of the possible degrees in Γ can lead to small improvements in the lower bound. For example, further refining the above argument can lead to the condition $d \geq a$ which gives $\ell_{\geq 3} \geq 4|C'|/3$ and $D \geq 15/43 \approx 23/66 + 0.00035$ (see [4]). But analysis of the above type tends to become more and more intricate and the improvements to the lower bound less and less significant.

3 A new construction

Consider the pattern of figure 7. This is an alternative construction to figure 1. One readily checks that it makes up an identifying code of density 3/8. Notice that it can be modified to yield the construction of figure 8 with the same density. But this identifying code is not optimal. Codewords can be deleted without losing the identifying property. We obtain the code of figure 9. Hence :

Theorem 3.1 *The minimum density of an identifying code of the infinite square lattice* $\mathbb{T}$ *satisfies* $D \leq 5/14$*.* $\triangle$

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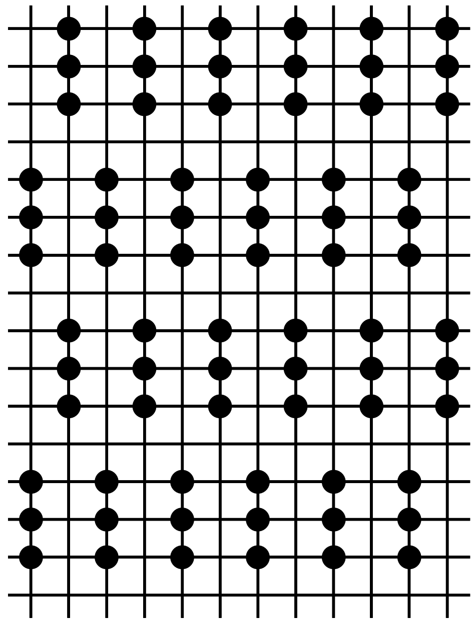


Figure 7: An alternative periodic identifying code of density 3/8.

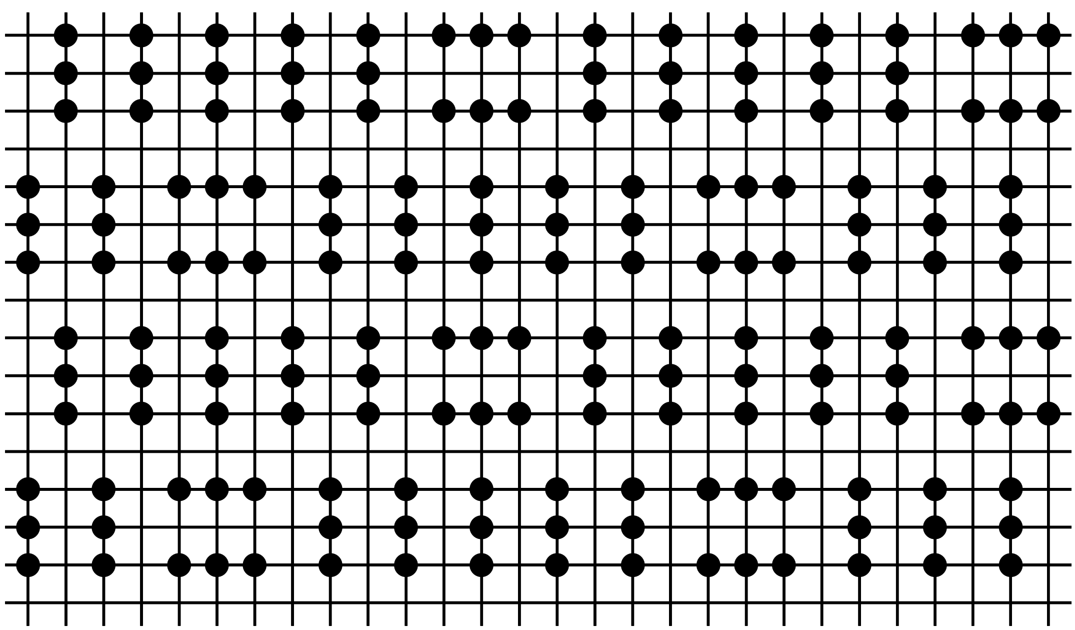
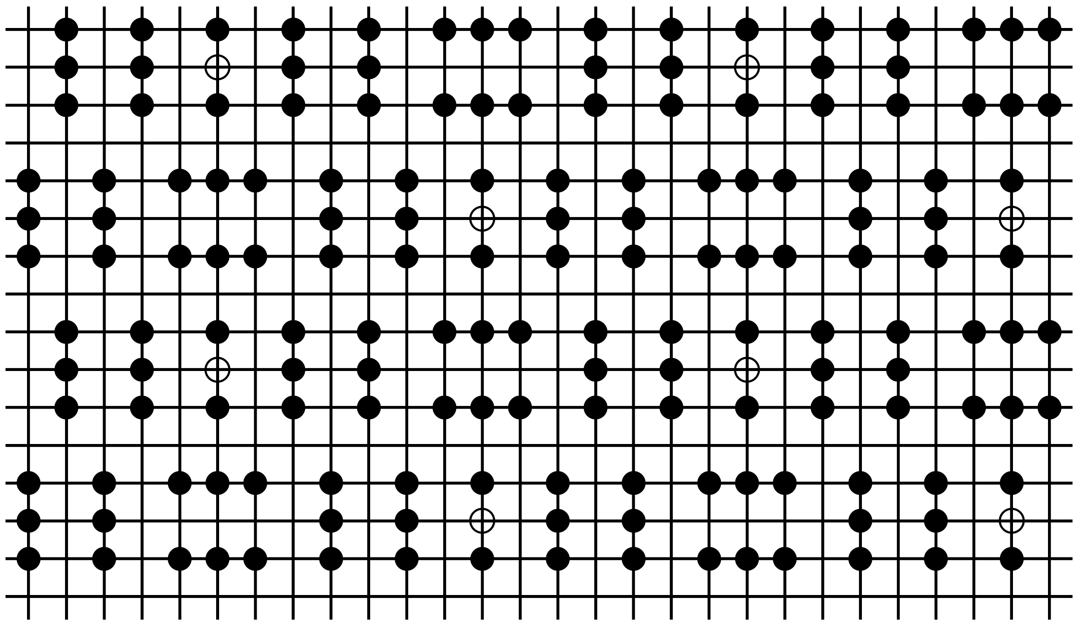


Figure 8: Another periodic identifying code of density 3/8.



The eight white codewords in the picture can be deleted without losing the identifying property. We obtain a periodic tiling of $\mathbb{Z}^2$ by the tile below.

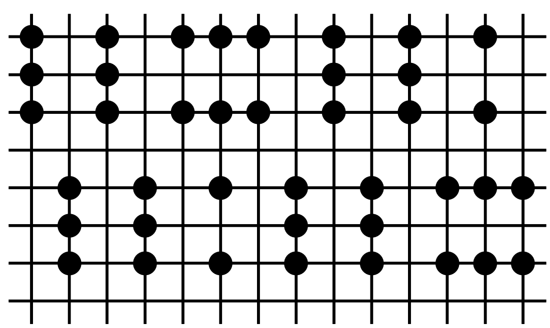


Figure 9: The improved identifying code : the tile is of size 112 and contains 40 codewords. Hence the density $40/112 = 5/14$.