# Improved LP Lower Bounds for Difference Triangle Sets

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**Abstract.** In 1991 Lorentzen and Nilsen showed how to use linear programming to prove lower bounds on the size of difference triangle sets. In this note we show how to improve these bounds by including additional valid linear inequalities in the LP formulation. We also give some new optimal difference triangle sets found by computer search.

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Following Kløve [2] we define an $(I, J)$ difference triangle set, $\Delta$, as a set of integers $\{a_{ij} \mid 1 \leq i \leq I,\ 0 \leq j \leq J\}$ such that all the differences $a_{ij} - a_{ik}$, $1 \leq i \leq I$, $0 \leq k < j \leq J$ are positive and distinct. Let $m = m(\Delta)$ be the maximum difference. The difference triangle set problem is to minimize $m$ given $I$ and $J$. Kløve defined $M(I, J)$ as this minimum. Lorentzen and Nilsen [3] showed how to use linear programming (LP) methods to prove lower bounds on $M(I, J)$. This was a generalization and improvement of earlier lower bounds. Here we show how to improve the LP bound (for $I > 1$) by adding inequalities to the LP formulation. We also announce some new values of $M(I, J)$ found by computer search.

Given a difference triangle set $\{a_{ij} \mid 1 \leq i \leq I,\ 0 \leq j \leq J\}$ we have associated difference triangles $\{X_{ijk} \mid 1 \leq i \leq I,\ 1 \leq j \leq J,\ 1 \leq k \leq J + 1 - j\}$ where $X_{ijk} = a_{i,j+k-1} - a_{i,k-1}$. The ith difference triangle is $\{X_{ijk} \mid 1 \leq j \leq J,\ 1 \leq k \leq J + 1 - j\}$. Its top row is $\{X_{i1k} \mid 1 \leq k \leq J\}$. Clearly each difference triangle is determined by its top row. The maximum difference in each difference triangle is its bottom element $X_{iJ1}$ which is the sum of the top row.

We can now formulate a linear program which gives a lower bound for $m = M(I, J)$. The LP variables will be $m$ and the top elements $\{X_{i1k} \mid 1 \leq i \leq I, 1 \leq k \leq J\}$ of the difference triangles. Minimizing $m$ will be the objective. Clearly since $m$ is the maximum difference we have:

$$m \geq X_{iJ1} = \sum_{k=1}^{J} X_{i1k} \qquad 1 \leq i \leq I \tag{1}$$

Also since the differences are all distinct positive integers we know that the sum of any n of them will be at least as large as the sum of the first $n$ positive integers or $n(n+1)/2$. So if S is a subset of the differences, we have:

$$\sum_{X_{ijk} \in S} X_{ijk} \geq n(n+1)/2 \qquad \text{where} \;\; |S| = n \tag{2}$$

(since $X_{ijk} = \sum_{h=k}^{j+k-1} X_{i1h}$ the inequalities (2) can be expressed in terms of our LP variables).

This is essentially the LP formulation used by Lorentzen and Nilsen [3] to bound $M(I, J)$. In practice when solving the LP, we use symmetry to reduce the $IJ$ variables $\{X_{i1k} \mid 1 \leq i \leq I, 1 \leq k \leq J\}$ to $\lfloor (J+1)/2 \rfloor$ variables (since the difference triangles can be permuted and reflected). Also we do not use all the inequalities (2). Instead we start with a small subset and solve the LP. Then we check if any of the unused inequalities are violated by the solution. If we find a violated inequality, we add it to the subset and resolve the LP repeating as necessary. If the number of inequalities becomes too large, we delete some of the non-binding ones. In practice this procedure rapidly produces a solution without having to solve large LP's.

Given a solution to a subset problem we can quickly check the inequalities (2) as follows. We use the solution to compute the remaining elements of the difference triangles. We next sort all $IJ(J+1)/2$ elements from smallest to largest. We then compute the partial sums and compare to the corresponding partial sums of the smallest $IJ(J+1)/2$ positive integers. If, for example, we find the sum of the $n$ smallest differences is less than $n(n+1)/2$, then we have found a violated inequality (2) which can be added to our subset LP. Otherwise we have shown the solution to the subset LP is also a solution to the entire LP. In practice the above procedure is modified to reflect the reduced number of LP variables due to symmetry.

The above produces the bounds on $M(I, J)$ in Lorentzen and Nilsen [3]. Using the following lemma we can improve these bounds when $I > 1$.

Lemma 1: Suppose we have a set, S, containing $n$ distinct integers with average $r$. Let $m$ be the maximum element of S. Then we have $m \geq r + (n-1)/2$.

Proof: Let $s$ be the sum of the elements of S. Clearly $s = nr$. Also since the elements of S are distinct with maximum $m$ we must have $s \leq m+(m-1)+\cdots+(m-n+1) = mn-\frac{(n-1)n}{2}$. So $nr \leq mn - \frac{(n-1)n}{2}$ or dividing by $n$ and rearranging $m \geq r + (n-1)/2$ as claimed. ■

Now we can use Lemma 1 to strengthen the inequalities (1) when $I > 1$ as follows. By symmetry the LP solution will have $m = X_{iJ1}$ $1 \leq i \leq I$. However the $X_{iJ1}$ are actually distinct positive integers and $m$ is their maximum. This means using Lemma 1 we may replace (1) with

$$m \geq \frac{1}{J} \sum_{i=1}^{J} X_{iJ1} + (I-1)/2 \tag{3}$$

Clearly this will improve the lower bound on $m$ by $(I-1)/2$. However we can do still better. (3) is just one of a family of inequalities for $m$ each corresponding to a subset T of the $IJ(J+1)/2$ differences and derived by use of Lemma 1. When T is the set of the $I$ bottom elements of the difference triangles, we obtain (3). In general, we have

$$m \geq \frac{1}{n} \sum_{X_{ijk} \in T} X_{ijk} + (n-1)/2 \qquad \text{where } \ |T| = n \tag{4}$$

Again as with (2) these inequalities can be rewritten in terms of our LP variables. Similarly it is easy to check given a solution of a LP containing a subset of the inequalities (4) whether any of the remaining inequalities (4) are violated. Simply compute and sort the differences $X_{ijk}$ and verify (4) for sets T consisting of the $n$ largest differences $1 \leq n \leq IJ(J+1)/2$. Again in practice this procedure is modified to reflect the reduced number of variables due to symmetry.

Remark 1: Let S and T be the entire set of differences in (2) and (4) respectively. Then we have $m \geq \frac{n+1}{2} + \frac{n-1}{2} = n$ where $n = IJ(J+1)/2$. So the bound given by our LP formulation is always at least as good as the trivial bound $M(I,J) \geq IJ(J+1)/2$ from the fact that all $IJ(J+1)/2$ differences must be distinct. This is not true for Lorentzen and Nilsen's formulation.

We have solved the LP with inequalities (2) and (4) using the above described iterative procedure for $1 \leq I \leq 15, 5 \leq J \leq 20$ (for $J \leq 4$ the solution gives the trivial bound $M(I,J) \geq IJ(J+1)/2$ mentioned in Remark 1 above). The resulting lower bounds on $M(I,J)$ are listed in Table 1. These bounds are generally improvements (compare [1], [3]). In most cases however there is still a considerable gap between these bounds and the best upper bounds known. For example the lower bound for $M(15,10)$ in [1] is 958, the lower bound using Lorentzen and Nilsen's formulation is 962 and the lower bound in this paper is 974. This remains far from the best upper bound known for $M(15,10)$, 1415 (from [1]). The fault may be more with the upper bound than with the lower bound however as the known ways of constructing $(I,J)$ difference triangle sets do not seem to be very good for large I. In a few cases the lower bound in this paper is known to be quite good. For example exhaustive search (see below) has shown $M(5,5) = 79$ (the claim that $M(5,5) = 83$ in [2] and propagated elsewhere is incorrect). The Lorentzen and Nilsen [3] lower bound for this case is 75, the lower bound in this paper is 77.

Remark 2: For small values of $J$ we can solve the LP for all values of $I$ obtaining:

$$M(I,5) \geq (91I+6)/6 \tag{5}$$
$$M(I,6) \geq (179I+9)/8 \tag{6}$$

We have also found some new exact values of $M(I,J)$ by computer search using a program similar to that described in [4]. We have $M(2,9) = 121$, $M(3,7) = 100$, $M(5,5) = 79$ and $M(9,4) = 91$. Examples obtaining these values are given in Table 2. The first two are unique, the third is probably not unique and the last is far from unique. The web page <http://www.research.ibm.com/people/s/shearer/dtsub.html> lists these values as well as numerous additional improvements on the results in [4].

**Table 1 - Lower Bounds for M(I,J)**

| J/I | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 5 | 17 | 32 | 47 | 62 | 77 | 92 | 108 | 123 | 138 | 153 | 168 | 183 | 199 | 214 | 229 |
| 6 | 24 | 46 | 69 | 91 | 113 | 136 | 158 | 181 | 203 | 225 | 248 | 270 | 292 | 315 | 337 |
| 7 | 32 | 63 | 94 | 124 | 155 | 186 | 217 | 247 | 278 | 309 | 339 | 370 | 401 | 431 | 462 |
| 8 | 43 | 83 | 124 | 164 | 205 | 245 | 286 | 327 | 367 | 408 | 448 | 489 | 529 | 570 | 610 |
| 9 | 54 | 106 | 158 | 209 | 261 | 313 | 365 | 417 | 469 | 521 | 572 | 624 | 676 | 728 | 780 |
| 10 | 67 | 132 | 197 | 261 | 326 | 391 | 456 | 520 | 585 | 650 | 715 | 780 | 844 | 909 | 974 |
| 11 | 82 | 161 | 240 | 320 | 399 | 478 | 558 | 637 | 716 | 796 | 875 | 954 | 1034 | 1113 | 1193 |
| 12 | 98 | 194 | 289 | 385 | 480 | 576 | 671 | 767 | 862 | 958 | 1054 | 1149 | 1245 | 1340 | 1436 |
| 13 | 116 | 229 | 343 | 456 | 569 | 683 | 796 | 909 | 1023 | 1136 | 1249 | 1363 | 1476 | 1589 | 1703 |
| 14 | 136 | 268 | 401 | 534 | 667 | 800 | 933 | 1066 | 1198 | 1331 | 1464 | 1597 | 1730 | 1863 | 1996 |
| 15 | 157 | 311 | 464 | 618 | 772 | 926 | 1080 | 1234 | 1388 | 1542 | 1696 | 1850 | 2003 | 2157 | 2311 |
| 16 | 180 | 356 | 533 | 709 | 886 | 1062 | 1239 | 1415 | 1592 | 1768 | 1945 | 2121 | 2298 | 2475 | 2651 |
| 17 | 204 | 405 | 606 | 807 | 1007 | 1208 | 1409 | 1610 | 1811 | 2012 | 2213 | 2413 | 2614 | 2815 | 3016 |
| 18 | 230 | 457 | 684 | 910 | 1137 | 1364 | 1591 | 1818 | 2044 | 2271 | 2498 | 2725 | 2952 | 3178 | 3405 |
| 19 | 258 | 512 | 767 | 1021 | 1276 | 1530 | 1784 | 2039 | 2293 | 2548 | 2802 | 3057 | 3311 | 3565 | 3820 |
| 20 | 287 | 571 | 855 | 1139 | 1422 | 1706 | 1990 | 2274 | 2557 | 2841 | 3125 | 3409 | 3692 | 3976 | 4260 |

**Table 2 - New Values for M(I,J)**

M(2,9) = 121

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 4 | 13 | 45 | 46 | 69 | 94 | 109 | 116 | 121 |
| 0 | 3 | 19 | 21 | 29 | 57 | 87 | 101 | 107 | 118 |

M(3,7) = 100

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0 | 12 | 15 | 31 | 55 | 87 | 88 | 93 |
| 0 | 7 | 30 | 41 | 51 | 77 | 90 | 99 |
| 0 | 2 | 29 | 37 | 54 | 82 | 96 | 100 |

M(5,5) = 79

| | | | | | |
|---|---|---|---|---|---|
| 0 | 12 | 20 | 41 | 72 | 75 |
| 0 | 6 | 25 | 49 | 62 | 79 |
| 0 | 4 | 15 | 51 | 65 | 74 |
| 0 | 2 | 18 | 44 | 66 | 71 |
| 0 | 1 | 33 | 40 | 68 | 78 |

M(9,4) = 91

| | | | | |
|---|---|---|---|---|
| 0 | 26 | 42 | 79 | 90 |
| 0 | 20 | 25 | 69 | 76 |
| 0 | 17 | 21 | 82 | 84 |
| 0 | 13 | 35 | 75 | 87 |
| 0 | 10 | 33 | 80 | 83 |
| 0 | 9 | 38 | 66 | 81 |
| 0 | 8 | 27 | 68 | 86 |
| 0 | 6 | 45 | 77 | 91 |
| 0 | 1 | 31 | 55 | 89 |

## References

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