A $[k,\ k+1]$-Factor Containing A Given Hamiltonian Cycle

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Abstract

We prove the following best possible result. Let $k \geq 2$ be an integer and G be a graph of order n with minimum degree at least k. Assume $n \geq 8k - 16$ for even n and $n \geq 6k - 13$ for odd n. If the degree sum of each pair of nonadjacent vertices of G is at least n, then for any given Hamiltonian cycle C of G, G has a $[k,\ k+1]$-factor containing C.

Submitted: December 15, 1997; Accepted: November 27, 1998.

MR Subject Number: 05C75
Keywords: $[k,\ k+1]$-factor, Hamiltonian cycle, degree condition

1 Introduction

All graphs under consideration are undirected, finite and simple. A graph G consists of a non-empty set $V(G)$ of vertices and a set $E(G)$ of edges. For two vertices x and y of G, let xy and yx denote an edge joining x to y. Let X be a subset of $V(G)$.

*Research supported partially by the exchange program between Chinese Academy of Sciences and Japan Society for Promotion of Sciences and by National Natural Science Foundation of China.

We write $G[X]$ for the subgraph of G induced by X, and define $\overline{X} := V(G) \setminus X$. The subset X is said to be *independent* if no two vertices of X are adjacent in G. Sometimes x is used for a singleton $\{x\}$. For a vertex x of G, we denote by $d_G(x)$ the degree of x in G, that is, the number of edges of G incident with x. We denote by $\delta(G)$ the minimum degree of G. For integers a and b, $0 \le a \le b$, an $[a,\ b]$*-factor* of G is defined to be a spanning subgraph F of G such that

$$a \le d_F(x) \le b \quad \text{for all } x \in V(G),$$

and an $[a,\ a]$-factor is abbreviated to an *a-factor*. A subset M of $E(G)$ is called a *matching* if no two edges of M are adjacent in G. For two graphs H and K, the *union* $H \cup K$ is the graph with vertex set $V(H) \cup V(K)$ and edge set $E(H) \cup E(K)$, and the *join* $H + K$ is the graph with vertex set $V(H) \cup V(K)$ and edge set $E(H) \cup E(K) \cup \{xy \mid x \in V(H) \text{ and } y \in V(K)\}$. Other notation and definitions not defined here can be found in [1].

We first mention some known results concerning our theorem.

Theorem A ([9]) *Let G be a graph of order $n \ge 3$. If the degree sum of each pair of nonadjacent vertices is at least n, then G has a Hamilton cycle.*

Theorem B ([3]) *Let k be a positive integer and G be a graph of order n with $n \ge 4k - 5$, kn even, and $\delta(G) \ge k$. If the degree sum of each pair of nonadjacent vertices is at least n, then G has a k-factor.*

Combining the above two theorems, we can say that if a graph G satisfies the conditions in Theorem B, then G has a Hamilton cycle C together with a connected $[k,\ k+2]$-factor containing C, which is the union of C and a k-factor of G [4].

Theorem C ([8]) *Let $k \ge 3$ be an integer and G be a connected graph of order n with $n \ge 4k - 3$, kn even, and $\delta(G) \ge k$. If for each pair (x, y) of nonadjacent vertices of $V(G)$,*

$$\max\{d_G(x),\ d_G(y)\} \ge \frac{n}{2},$$

then G has a k-factor.

Theorem D ([2]) *Let $k \ge 3$ be an odd integer and G be a connected graph of odd order n with $n \ge 4k - 3$, and $\delta(G) \ge k$. If for each pair (x, y) of nonadjacent vertices of G,*

$$\max\{d_G(x),\ d_G(y)\} \ge \frac{n}{2},$$

then G has a connected $[k,\ k+1]$-factor.

Theorem E ([5]) *Let G be a connected graph of order n, let f and g be two positive integer functions defined on $V(G)$ which satisfy $2 \le f(v) \le g(v)$ for each vertex $v \in V(G)$. Let G have an $[f, g]$-factor F and put $\mu = min\{f(v) : v \in V(G)\}$. Suppose that among any three independent vertices of G there are (at least) two vertices with degree sum at least $n - \mu$. Then G has a matching M such that M and F are edge-disjoint and $M + F$ is a connected $[f, g+1]$-factor of G.*

The purpose of this paper is to extend "connected $[k,\ k+1]$-factor" in some of the above theorems to "$[k,\ k+1]$-factor containing a given Hamiltonian cycle", which is obviously a 2-connected $[k,\ k+1]$-factor.

Our main result is the following

Theorem 1 *Let $k \geq 2$ be an integer and G be a graph of order $n \geq 3$ with $\delta(G) \geq k$. Assume $n \geq 8k-16$ for even n and $n \geq 6k-13$ for odd n. If for each pair (x, y) of nonadjacent vertices of G,*

$$d_G(x) + d_G(y) \geq n, \tag{1}$$

then for any given Hamiltonian cycle C, G has a $[k,\ k+1]$-factor containing C.

Now we conclude this section with a new result concerning our theorem.

Theorem F [11] *Let $k \geq 2$ be an integer and G be a connected graph of order n such that $n \geq 8k-4$, kn is even and $\delta(G) \geq n/2$. Then G has a k-factor containing a Hamiltonian cycle.*

For a graph G of order n, the condition $\delta(G) \geq n/2$ does not guarantee the existence of a k-factor which contains a given Hamiltonian cycle of G. Let $n \geq 5$ and $k \geq 3$ be integers, and set

$$m = \begin{cases} \frac{n}{2} + 2 & \text{for even } n, \\ \frac{n+3}{2} & \text{for odd } n. \end{cases}$$

Let $C_m = (v_1 v_2 \ldots v_m)$ be a cycle of order m and $P_{n-m} = (v_{m+1} v_{m+2} \ldots v_n)$ a path of order $n-m$. Then the join $G := C_m + P_{n-m}$ has no k-factor containing the Hamiltonian cycle $(v_1 v_2 \ldots v_n)$ but satisfies $\delta(G) \geq n/2$.

2 Proof

Our proof depends on the following theorem, which is a special case of Lovász's (g, f)-factor theorem [7]([10]).

Theorem 2 *Let G be a graph and a and b be integers such that $1 \leq a < b$. Then G has an $[a, b]$-factor if and only if*

$$\gamma(S, T) := b|S| - a|T| + \sum_{x \in T} d_{G-S}(x) \geq 0$$

for all disjoint subsets $S,\ T \subseteq V(G)$.

Proof of Theorem 1 We may assume $k \geq 3$ since G has C for $k = 2$. Let

$$H := G - E(C), \quad U := \{x \in V(G) \mid d_G(x) \geq \frac{n}{2}\}, \quad W := V(G) \setminus U, \quad \rho := k-2.$$

Then $V(H) = V(G)$, $\rho \ge 1$,

$$d_H(x) = d_G(x) - 2 \ge \rho \quad \text{for all} \quad x \in V(H),$$

$n \ge 8\rho$ for even n and $n \ge 6\rho - 1$ for odd n. Moreover the induced subgraph $G[W]$ is a complete graph since $d_G(x) + d_G(y) < n$ for any two vertices x and y of W.

Obviously, G has a required factor if and only if H has a $[\rho, \rho+1]$-factor. Suppose, to the contrary, that H has no such factor. Then, by Theorem 2, there exist disjoint subsets S and T of $V(H)$ such that

$$\gamma(S,T) = (\rho+1)s - \rho t + \sum_{x \in T} d_{H-S}(x) < 0. \tag{2}$$

where $t = |T|$ and $s = |S|$.

If $d_{H-S}(v) \ge \rho$ for some $v \in T$, then $\gamma(S,T) \ge \gamma(S, T \setminus \{v\})$, and thus (2) is still holds for S and $T \setminus \{v\}$. Thus we may assume that

$$d_{H-S}(x) \le \rho - 1 \qquad \text{for all} \qquad x \in T. \tag{3}$$

If $S = \emptyset$, then $\gamma(\emptyset, T) = -\rho t + \sum_{x \in T} d_H(x) \ge 0$ as $d_H(x) \ge \rho$ for all $x \in V(H)$. Thus

$$s \ge 1. \tag{4}$$

If $t \le \rho + 1$, then we have

$$\begin{aligned} \gamma(S,T) &\ge (\rho+1)s - \rho t + \sum_{x \in T} (d_H(x) - s) \\ &\ge (\rho+1)s - \rho t + t(\rho - s) \\ &= s(\rho + 1 - t) \ge 0. \end{aligned}$$

This contradicts (2). Hence

$$t \ge \rho + 2. \tag{5}$$

We now prove the next Claim:

Claim 1. $s \le \frac{n}{2} - 3$ *if n is even, and* $s \le \frac{n-5}{2}$ *if n is odd.*

Assume that n is even and $s \ge (n/2) - 2$. Let $q := s - (n/2) + 2 \ge 0$ and $r := n - s - t \ge 0$. Then it follows from $\rho \ge 1$ and $n \ge 8\rho$ that

$$\begin{aligned} \gamma(S,T) &= (\rho+1)q + \rho(r+q) + \sum_{x \in T} d_{H-S}(x) + \frac{n}{2} - 4\rho - 2 \\ &\ge 2q + r + q + \sum_{x \in T} d_{H-S}(x) - 2. \end{aligned}$$

Hence we may assume $q = 0$ and $r \le 1$ since otherwise $\gamma(S,T) \ge 0$. If $r = 1$ and $\sum_{x \in T} d_{H-S}(x) \ge 1$, then $\gamma(S,T) \ge 0$. If $r = 0$ and $\sum_{x \in T} d_{H-S}(x) \ge 1$, then $V(H) = S \cup T$ and

$$\sum_{x \in T} d_{H-S}(x) = \sum_{x \in T} d_{H[T]}(x) = 2|E(H[T])| \equiv 0 \pmod 2,$$

and so $\gamma(S,T) \geq 0$. Therefore it suffices to show that $\sum_{x\in T} d_{H-S}(x) \geq 1$ under the assumption that $q=0$ and $0 \leq r \leq 1$.

Suppose that $\sum_{x\in T} d_{H-S}(x) = 0$, $q=0$ and $0 \leq r \leq 1$. Let $\overline{S} := V(G) \setminus S \supseteq T$, $X := \{x \in \overline{S} \mid d_G(x) \geq n/2\}$ and $Y := \overline{S} \setminus X$. Then a complete graph $G[Y]$ is contained in C, and it follows from $s = (n/2) - 2$ that for each vertex $x \in X$, there exist two edges of C which join x to two vertices in $\overline{S}$. Hence we have

$$|X|+|Y|-1 = |\overline{S}|-1 \geq |E(G[\overline{S}]) \cap E(C)| \geq |X|+1+|E(G[Y])| = |X|+1+\frac{|Y|(|Y|-1)}{2},$$

which implies $|Y| \geq 2 + |Y|(|Y|-1)/2$. Now we get a contradiction, because it is obvious that there is no nonnegative integral solution of $|Y|$ to this quadratic inequality. Therefore Claim 1 holds for even n.

We next assume that n is odd and $s \geq (n-3)/2$. Let $q := s - (n-3)/2 \geq 0$ and $r := n - s - t \geq 0$. Then it follows from $\rho \geq 1$ and $n \geq 6\rho - 1$ that

$$\begin{aligned}
\gamma(S,T) &= (\rho+1)q + \rho(r+q) + \sum_{x\in T} d_{H-S}(x) + \frac{n}{2} - 3\rho - \frac{3}{2} \\
&\geq 2q + r + q + \sum_{x\in T} d_{H-S}(x) - 2.
\end{aligned}$$

Hence, by the same argument as above, we may assume that $q=0$, $0 \leq r \leq 1$ and $\sum_{x\in T} d_{H-S}(x) = 0$. Let $X := \{x \in \overline{S} \mid d_G(x) \geq (n+1)/2\}$ and $Y := \overline{S} \setminus X$. Then we similarly obtain $|Y| \geq 2 + |Y|(|Y|-1)/2$, and derive a contradiction. Consequently Claim 1 also holds for odd n.

Claim 2. $T \cap U \neq \emptyset$.

Indeed, assume $T \subseteq W$. Then $G[T]$ is a complete graph and $|E(G[T])| = t(t-1)/2$. Since C is a Hamiltonian cycle, $|E(G[T]) \cap E(C)| \leq t-1$. Hence

$$\sum_{x\in T} d_{H-S}(x) \geq 2|E(G[T]) \setminus E(C)| \geq t(t-1) - 2(t-1) = (t-1)(t-2).$$

Thus

$$\begin{aligned}
\gamma(S,T) &\geq (\rho+1)s - \rho t + (t-1)(t-2) \\
&\geq (\rho+1)s - \rho t + (t-1)\rho && \text{(by (5))} \\
&= (\rho+1)s - \rho > 0. && \text{(by (4))}
\end{aligned}$$

This contradicts (2).

Claim 3. $T \cap W \neq \emptyset$.

Suppose $T \subseteq U$ and n is even. Then for every $x \in T$, we have by (3)

$$\frac{n}{2} \leq d_G(x) \leq d_{H-S}(x) + s + 2 \leq \rho + s + 1,$$

which implies $d_{H-S}(x) \geq (n/2) - s - 2$ and $\rho + s + 2 - n/2 \geq 1$. Hence

$$\begin{aligned}
\gamma(S,T) &\geq (\rho+1)s - \rho t + t(\frac{n}{2} - s - 2) \\
&= (\rho+1)s - t(\rho + s + 2 - \frac{n}{2}) \\
&\geq (\rho+1)s - (n-s)(\rho + s + 2 - \frac{n}{2}) \\
&= (\rho+1)s + (\frac{n}{2} - s - 3 + \frac{n}{2} + 3)(\frac{n}{2} - s - 3 - 2\rho + \rho + 1) \\
&= (\frac{n}{2} - s - 3)^2 + (\frac{n}{2} - s - 3)(\frac{n}{2} + 3 - 2\rho) + n - 6\rho \\
&\geq 0. \qquad \text{(by } n \geq 8\rho \text{ and Claim 1)}
\end{aligned}$$

This contradicts (2).

Next assume $T \subseteq U$ and n is odd. Then for every $x \in T$, we have

$$\frac{n+1}{2} \leq d_G(x) \leq d_{H-S}(x) + s + 2 \leq \rho + s + 1,$$

which implies $d_{H-S}(x) \geq (n/2) - s - (3/2)$ and $\rho + s + (3/2) - (n/2) \geq 1$. Hence

$$\begin{aligned}
\gamma(S,T) &\geq (\rho+1)s - \rho t + t(\frac{n}{2} - s - \frac{3}{2}) \\
&= (\rho+1)s - t(\rho + s + \frac{3}{2} - \frac{n}{2}) \\
&\geq (\rho+1)s - (n-s)(\rho + s + \frac{3}{2} - \frac{n}{2}) \\
&= (\frac{n}{2} - s - \frac{5}{2})^2 + (\frac{n}{2} - s - \frac{5}{2})(\frac{n}{2} + \frac{5}{2} - 2\rho) + n - 5\rho \\
&\geq 0. \qquad \text{(by } n \geq 6\rho - 1 \text{ and Claim 1)}
\end{aligned}$$

This contradicts (2). Therefore Claim 2 is proved.

Now put

$$T_1 := T \cap U, \quad T_2 := T \cap W, \quad t_1 = |T_1|, \quad t_2 := |T_2|.$$

By Claims 2 and 3, we have $t_1 \geq 1$ and $t_2 \geq 1$. It is clear that $d_{H-S}(x) \geq d_G(x) - s - 2$ for all $x \in T$, in particular, for every $y \in T_1$,

$$d_{H-S}(y) \geq \begin{cases} \frac{n}{2} - s - 2 & \text{if} \quad n \quad \text{is even} \\ \frac{n}{2} - s - \frac{3}{2} & \text{if} \quad n \quad \text{is odd.} \end{cases} \tag{6}$$

It follows from (3) that

$$\frac{n}{2} - \rho - s - 2 \leq -1 \quad \text{if } n \text{ is even,} \quad \text{and} \quad \frac{n}{2} - \rho - s - \frac{3}{2} \leq -1 \quad \text{if } n \text{ is odd.} \tag{7}$$

By Claim 1 and by the above inequalities, we have

$$\rho \geq 2. \tag{8}$$

For every $x \in T_2$, we have $d_{H-S}(x) \geq t_2 - 3$ by the fact that $G[W]$ is a complete graph, and obtain the following inequality from (3).

$$t_2 \leq \rho + 2. \tag{9}$$

In order to complete the proof, we consider two cases. Assume first n is even. By making use of $n \geq 8\rho$, (6), (7), (8), (9) and Claim 1, we have

$$\begin{aligned}
\gamma(S,T) &\geq (\rho+1)s - \rho(t_1+t_2) + t_1(\frac{n}{2} - s - 2) \\
&= (\rho+1)s - \rho t_2 + t_1(\frac{n}{2} - s - 2 - \rho) \\
&\geq (\rho+1)s - \rho t_2 + (n-s-t_2)(\frac{n}{2} - \rho - s - 2) \\
&= (\frac{n}{2} - s - 3)^2 + (\frac{n}{2} - s - 3)(\frac{n}{2} + 3 - 2\rho - t_2) \\
&\qquad + n - 6\rho - t_2 \\
&\geq 2\rho - t_2 \geq \rho + 2 - t_2 \geq 0.
\end{aligned}$$

This contradicts (2).

We next assume n is odd. Let $r := n - s - t$. It is easy to see that

$$\sum_{x \in T_2} d_{H-S}(x) \geq 2|E(G[T_2]) \setminus E(C)| \geq t_2(t_2-1) - 2(t_2-1) = (t_2-1)(t_2-2). \tag{10}$$

By using $n \geq 6\rho - 1$, (6), (7), (8) (9) and (10), we have

$$\begin{aligned}
\gamma(S,T) &\geq (\rho+1)s - \rho(t_1+t_2) + t_1(\frac{n}{2} - s - \frac{3}{2}) + (t_2-1)(t_2-2) \\
&= (\rho+1)s + t_1(\frac{n}{2} - \rho - s - \frac{3}{2}) - \rho t_2 + (t_2-1)(t_2-2) \\
&\geq (\rho+1)s + (n-s-t_2-r)(\frac{n}{2} - \rho - s - \frac{3}{2}) - \rho t_2 + (t_2-1)(t_2-2) \\
&= (\frac{n}{2} - s - \frac{5}{2})^2 + (\frac{n}{2} - s - \frac{5}{2})(\frac{n}{2} + \frac{5}{2} - t_2 - 2\rho) \\
&\qquad + n - 5\rho + (t_2-1)(t_2-2) - t_2 + r(\rho + s + \frac{3}{2} - \frac{n}{2}) \\
&= (\frac{n}{2} - s - \frac{5}{2})^2 + \rho - 1 + (t_2-1)(t_2-2) - t_2 + r.
\end{aligned}$$

Since $(t_2-1)(t_2-2) - t_2 \geq -2$ with equality only when $t_2 = 2$, we have $\rho - 1 + (t_2 - 1)(t_2-2) - t_2 + r \geq \rho - 1 - 2 + r = \rho - 2 + r - 1 \geq r - 1$ and thus $\gamma(S,T) \geq 0$ unless $s = (n-5)/2$, $t_2 = 2$ $r = 0$, $\rho = 2$ and (10) holds with equality. Since $t_2 = 2$ and (10) holds with equality,

$$|E(G[T_2])| = |E(G[T_2]) \cap E(C)| = 1.$$

Since $s = (n+1)/2 - 3$ and $\rho = 2$, it follows from (3) and (6) that

$$d_{H-S}(x) = 1 \quad \text{and} \quad d_G(x) = \frac{n+1}{2} \quad \text{for all} \quad x \in T_1.$$

This implies that all the edges of C incident with vertices in T_1 are contained in $E(G[T]) \setminus E(G[T_2])$, and thus the number of such edges is at least $t_1 + 1$. Therefore $|E(G[T]) \cap C| \geq t_1 + 1 + 1 = t$, contradicting the fact that C is a Hamiltonian cycle of G. Consequently the theorem is proved.

Remark. The condition that $n \geq 8k - 16$ for even n and $n \geq 6k - 13$ for odd n in Theorem 1 are best possible. To see this, either let n be an even integer such that $2k \leq n < 8k - 16$ and put $m = (n/2) + 2$, or let n be an odd integer such that $2k - 1 \leq n < 6k - 13$ and put $m = (n + 3)/2$. Let $C_m = (v_1 v_2 \dots v_m)$ be a cycle of order m and $P_{n-m} = (v_{m+1} v_{m+2} \dots v_n)$ a path of order $n - m$. Then the join $G := C_m + P_{n-m}$ has no $[k, k+1]$-factor containing Hamiltonian cycle $(v_1 v_2 \dots v_n)$ but satisfies $\delta(G) \geq k$ and $d_G(x) + d_G(y) \geq n$ for all nonadjacent vertices x and y of G.

We explain why G has no such factor when n is even. By setting $S = \{v_{m+1}, \dots, v_n\}$ and $T = \{v_1, \dots, v_m\}$ in (2), we obtain $\gamma(S,T) = (k-1)(n/2-2) - (k-2)(n/2+2) + 2 < 0$, which implies G has no such factor.

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