A note on the number of (k, ℓ)-sum-free sets

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Abstract

A set $A \subseteq \mathbb{N}$ is (k, ℓ)-sum-free, for $k, \ell \in \mathbb{N}$, $k > \ell$, if it contains no solutions to the equation $x_1 + \cdots + x_k = y_1 + \cdots + y_\ell$. Let $\rho = \rho(k - \ell)$ be the smallest natural number not dividing $k - \ell$, and let $r = r_n$, $0 \le r < \rho$, be such that $r \equiv n \pmod{\rho}$. The main result of this note says that if $(k - \ell)/\ell$ is small in terms of ρ, then the number of (k, ℓ)-sum-free subsets of $[1, n]$ is equal to $(\varphi(\rho) + \varphi_r(\rho) + o(1))2^{\lfloor n/\rho \rfloor}$, where $\varphi_r(x)$ denotes the number of positive integers $m \le r$ relatively prime to x and $\varphi(x) = \varphi_x(x)$.

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A set A of positive integers is (k, ℓ)-sum-free for $k, \ell \in \mathbb{N}$, $k > \ell$, if there are no solutions to the equation $x_1 + \cdots + x_k = y_1 + \cdots + y_\ell$ in A. Denote by $\mathcal{SF}^n_{k,\ell}$ the number of (k, ℓ)-sum-free subsets of $[1, n]$. Since the set of odd numbers is $(2,1)$-sum-free we have $\mathcal{SF}^n_{2,1} \geq 2^{\lfloor (n+1)/2 \rfloor}$. In fact Erdős and Cameron [6] conjectured $\mathcal{SF}^n_{2,1} = O(2^{n/2})$. This conjecture is still open and the best upper bounds for $\mathcal{SF}^n_{2,1}$ given independently by Alon [1] and Calkin [3], say that, for $\ell \geq 1$,

$$\mathcal{SF}^n_{\ell+1,\ell} \leq \mathcal{SF}^n_{2,1} = O(2^{n/2+o(n)}) .$$

For $\ell \geq 3$ this bound was recently improved by Bilu [2] who proved that in this case $\mathcal{SF}^n_{\ell+1,\ell} = (1+o(1))2^{\lfloor (n+1)/2 \rfloor}$.

The case of k being much larger than ℓ was treated by Calkin and Taylor [4]. They showed that for some constant c_k the number of $(k,1)$-sum-free subsets of $[1,n]$ is at most $c_k 2^{\frac{k-1}{k}n}$, provided $k \geq 3$. Furthermore, Calkin and Thomson proved [5] that for every k and ℓ with $k \geq 4\ell - 1$

$$\mathcal{SF}^n_{k,\ell} \leq c_k 2^{(k-\ell)n/k}.$$

In order to study the behaviour of $\mathcal{SF}^n_{k,\ell}$ let us observe first that there are two natural examples of large (k,ℓ)-sum-free subsets of the interval $[1,n]$:

$$\{\lfloor \ell n/k \rfloor + 1, \ldots, n\}$$

and

$$\{m \in \{1,2,\ldots,n\} : m \equiv r \pmod{\rho}\} ,$$

where $\gcd(r,\rho) = 1$ and $\rho = \rho(k-\ell) = \min\{s \in \mathbb{N} : s \text{ does not divide } k-\ell\}$. Thus,

$$\mathcal{SF}^n_{k,\ell} \geq \max\left(2^{\lfloor n/\rho \rfloor},\ 2^{\lceil (k-\ell)n/k \rceil}\right).$$

In this note we study the case $k < \frac{\rho}{\rho-1}\ell$ so that $2^{\lfloor n/\rho \rfloor} > 2^{\lceil (k-\ell)n/k \rceil}$, and we may expect $\mathcal{SF}^n_{k,\ell}$ to be close to $2^{\lfloor n/\rho \rfloor}$. Indeed, we will prove as our main result that for fixed k and ℓ there exists a bounded function $\xi = \xi(n)$ such that

$$\mathcal{SF}^n_{k,\ell} = (\xi + o(1))2^{\lfloor n/\rho \rfloor}$$

provided $k < \left(1 - \frac{c-1}{c\rho-1}\right)\frac{\rho}{\rho-1}\ell$, where $c = \frac{1+\ln 2}{2\ln 2}$, and ℓ is sufficiently large.

For every natural numbers x, r let $\varphi_r(x)$ be the number of positive integers $m \leq r$ relatively prime to x and let $\varphi(x)$ abbreviate $\varphi_x(x)$. For a finite set A of integers A define:

$$\begin{array}{rclrcl} d(A) & = & \gcd(A), & d'(A) & = & d(A-A), \\ \Lambda(A) & = & \max A - \min A, & \Lambda'(A) & = & \Lambda(A)/d'(A). \end{array}$$

Furthermore, let

$$\kappa(A) = \frac{\Lambda'(A) - 1}{|A| - 2}, \quad \theta(A) = \frac{\max(A)}{\Lambda(A)},$$

$$T(A) = (|A| - 2)(\lfloor \kappa(A) \rfloor + 1 - \kappa(A)) + 1$$

and

$$hA = \{a_1 + \cdots + a_h : a_1, \ldots, a_h \in A\}.$$

For a specified set A, we simply write d, d', Λ, etc.

Our approach is based on a remarkable result of Lev [7]. Using an affine transformation of variables his theorem can be stated as follows.

Theorem 1. *Let A be a finite set of integers and let h be a positive integer satisfying $h > 2\kappa - 1$. Then there exists an integer s such that*

$$\{sd', \ldots, (s+t)d'\} \subseteq hA\ ,$$

for $t = (h - 2\lfloor\kappa\rfloor)\Lambda' + 2\lfloor\kappa\rfloor T$.

Lemma 1. *Let A be a finite set of integers and let h be a positive integer satisfying $h > 2\kappa - 1$. Then $\{0, d', \ldots, td'\} \subseteq hA - hA$, where $t \geq (h + 1 - 2\kappa)\Lambda'$.*

Proof. Theorem 1 implies that hA contains $t = (h - 2\lfloor\kappa\rfloor)\Lambda' + 2\lfloor\kappa\rfloor T + 1$ consecutive multiples of d', so that

$$\{0, \ldots, td'\} \subseteq hA - hA.$$

Furthermore,

$$t = (h - 2\lfloor\kappa\rfloor)\Lambda' + 2\lfloor\kappa\rfloor T = (h + 2 - 2\kappa - \tau)\Lambda' + \frac{2\lfloor\kappa\rfloor(\kappa - \lfloor\kappa\rfloor) + 2\lfloor\kappa\rfloor(\kappa - 1)}{\kappa},$$

where

$$\tau = \frac{2(\kappa - \lfloor\kappa\rfloor)(\lfloor\kappa\rfloor + 1 - \kappa)}{\kappa}.$$

Since $\tau \leq 1$ and $\kappa \geq 1$, the result follows. $\square$

Lemma 2. *Let $A \subseteq [1,n]$ be a (k,ℓ)-sum-free set, and let r be the residue class* $\bmod d'$ *containing A. Assume that either*

$$d' < \rho, \tag{1}$$

or

$$(k-\ell)r \equiv 0 \pmod{d'}. \tag{2}$$

Then

$$\kappa \geq \frac{k+1-(k-\ell)\theta}{2}. \tag{3}$$

Proof. We may assume that $\ell > 2\kappa - 1$, otherwise the assertion is obvious. By Lemma 1 we have

$$\{0, d', \ldots, td'\} \subseteq \ell A - \ell A,$$

where $t \geq (\ell + 1 - 2\kappa)\Lambda'$. Put $m = \min A$. Then any of the assumptions (1), (2) implies $d'|(k-\ell)m$. Since A is a (k,ℓ)-sum-free set, it follows that

$$(k-\ell)m > td' \geq (\ell + 1 - 2\kappa)\Lambda,$$

which gives the required inequality. □

Theorem 2. *Assume that $k > \ell \geq 3$ are positive integers satisfying*

$$\frac{k-\ell}{2} \cdot \max_{2 \leq x \leq \frac{\ell+1}{2}} \frac{\frac{\ln x}{x} + \frac{x-1}{x} \ln \frac{x}{x-1}}{\frac{k+1}{2} - x} < \frac{\ln 2}{\rho}. \tag{4}$$

Then

$$\mathcal{SF}^n_{k,\ell} = (\varphi + \varphi_r + o(1))2^{\lfloor n/\rho \rfloor}, \tag{5}$$

where $0 \leq r < \rho$ and $r \equiv n \pmod{\rho}$.

Proof. In order to obtain the lower bound let us observe that there are exactly φ maximal (k,ℓ)-sum-free arithmetic progressions with the difference ρ. Precisely φ_r of them have length $\lceil n/\rho \rceil$ and $\varphi - \varphi_r$ are of length $\lfloor n/\rho \rfloor$. Since these progressions are pairwise disjoint, there are at least

$$(\varphi + \varphi_r)2^{\lfloor n/\rho \rfloor}$$

(k,ℓ)-sum-free subsets of $[1,n]$.

Now we estimate $\mathcal{SF}_{k,\ell}^n$ from above. First consider (k,ℓ)-sum-free sets satisfying neither (1), nor (2). Plainly each of these is contained in a residue class $r \bmod d'$, where $d' \geq \rho$ and $(k-\ell)r \not\equiv 0 \bmod d'$. If $d' = \rho$, by the same argument as above, exactly $(\varphi + \varphi_r)2^{\lfloor n/\rho \rfloor}$ (k,ℓ)-sum-free subsets of $[1,n]$ are contained in arithmetic progression $r \bmod \rho$, where $(k-\ell)r \not\equiv 0 \bmod \rho$. If $d' > \rho$ then every progression $r \bmod d'$ consists of at most $\lceil n/(\rho+1) \rceil$ elements hence it contains no more than $2^{\lceil n/(\rho+1) \rceil}$ subsets. Furthermore we have less than n^2 possible choices for the pair (d', ρ), hence there are at most $2n^2 2^{n/(\rho+1)}$ such (k,ℓ)-sum-free sets. Thus, the number of (k,ℓ)-sum-free sets satisfying neither (1), nor (2) does not exceed

$$(\varphi + \varphi_r)2^{\lfloor n/\rho \rfloor} + 2n^2 2^{n/(\rho+1)}.$$

To complete the proof it is sufficient to show that the number of (k,ℓ)-sum-free subsets of $[1,n]$ satisfying either (1) or (2) is $o(2^{n/\rho})$. Denote by $\mathcal{B}$ the set of all such subsets, and let

$$\mathcal{B}(K,L,M) = \{A \in \mathcal{B} : |A| = K,\ \Lambda(A) = L,\ \max A = M\},$$

so that

$$\mathcal{B} = \bigcup_{1 \leq K \leq L+1 \leq M \leq n} \mathcal{B}(K,L,M).$$

We will prove that

$$\max_{1 \leq K \leq L+1 \leq M \leq n} |\mathcal{B}(K,L,M)| \leq e^{\mu n + O(\ln n)}, \tag{6}$$

where μ is the left-hand side of (4) which in turn implies that

$$|\mathcal{B}| = o(2^{n/\rho}). \tag{7}$$

Let us define the following decreasing function $x(t) = (k+1-(k-\ell)t)/2$. Note that $x(1) = (\ell+1)/2$, $x(t_2) = 2$ and $x(t_1) = 1$, where

$$t_2 = \frac{k-3}{k-\ell} \geq 1 \quad \text{and} \quad t_1 = \frac{k-1}{k-\ell}.$$

Furthermore, put

$$H(x) = \frac{\ln x}{x} + \frac{x-1}{x} \ln \frac{x}{x-1}.$$

Observe that H is increasing on $(1,2]$ and decreasing on $[2,\infty)$. Moreover

$$\mu = \max_{1\le t\le t_2} \frac{H(x(t))}{t}$$

and

$$\max_{\substack{1\le t\le t_1\\ x\ge x(t)}} \frac{H(x)}{t} = \mu. \tag{8}$$

Indeed, if $1 \le t \le t_2$ then $x \ge x(t) \ge 2$ and $H(x)/t \le H(x(t))/t \le \mu$. If $t_2 \le t \le t_1$ then $H(x)/t \le H(2)/t_2 = H(x(t_2))/t_2 \le \mu$.

Now we are ready to prove (7). For a fixed triple K, L, M with $1 \le K \le L+1 \le M \le n$ put

$$\theta = \frac{M}{L}, \quad \kappa = \frac{L-1}{K-2}.$$

Then $\kappa(A) \le \kappa$ and $\theta(A) = \theta$ for any $A \in \mathcal{B}(K,L,M)$. By Lemma 2 we have $\kappa \ge x(\theta)$. Since $\kappa \ge 1$ by definition, we infer that $H(\kappa)/\theta \le \mu$ by (8). Using Stirling's formula we obtain

$$\begin{aligned}
|\mathcal{B}(K,L,M)| &\le \binom{L-1}{K-2} \\
&= \exp(H(\kappa)L + O(\ln L)) \\
&= \exp\left(\frac{H(\kappa)}{\theta}M + O(\ln n)\right) \\
&\le \exp(\mu n + O(\ln n)).
\end{aligned}$$

Thus

$$|\mathcal{B}| \le n^3 \exp(\mu n + O(\ln n)),$$

which completes the proof of Theorem 2. $\square$

Corollary 1. *The estimate (5) holds, provided $k > \ell \ge 3$ and*

$$\frac{\max\left(\frac{1+\ln 2}{2}(k-\ell),\ 2(1+\ln\frac{\ell+1}{2})\right)}{\ell+1} < \frac{\ln 2}{\rho}. \tag{9}$$

Proof. We need to show that the left-hand side of (4) is not larger than the left-hand side of (9). Since $\ln(1+u) \le u$ for $u \ge 0$, we have

$$\frac{x-1}{x} \ln \frac{x}{x-1} \le \frac{1}{x}$$

for $x \ge 1$, so that

$$\mu \le \frac{k-l}{2} \max_{2 \le x \le \frac{\ell+1}{2}} \frac{1+\ln x}{x(\frac{k+1}{2}-x)}. \tag{10}$$

Furthermore,

$$\max_{2 \le x \le \frac{k-\ell}{2}} \frac{1+\ln x}{x(\frac{k+1}{2}-x)} \le \frac{2}{\ell+1} \max_{2 \le x \le \frac{\ell+1}{2}} \frac{1+\ln x}{x} = \frac{1+\ln 2}{\ell+1},$$

$$\max_{\frac{k-\ell}{2} \le x \le \frac{\ell+1}{2}} \frac{1+\ln x}{x(\frac{k+1}{2}-x)} \le \frac{1+\ln\frac{\ell+1}{2}}{\min\limits_{\frac{k-\ell}{2} \le x \le \frac{\ell+1}{2}} x(\frac{k+1}{2}-x)} = 4\frac{1+\ln\frac{\ell+1}{2}}{(k-\ell)(\ell+1)}.$$

Combining the above inequalities with (10), the result follows. □

Let us conclude this note with some further remarks on the range of k and ℓ satisfying (4). If $\frac{1+\ln 2}{2}(k-\ell) \le 2(1+\ln\frac{\ell+1}{2})$, that is $(k-\ell) \le \frac{4}{1+\ln 2}(1+\ln\frac{\ell+1}{2})$, then by Corollary 1 (4) holds, provided $\ell \ge \frac{2}{\ln 2}\Big(1+\ln\frac{\ell+1}{2}\Big)\rho(k-\ell)$. By the prime number theorem, $\rho(n) \le (1+o(1))\ln n$, hence the inequality $\ell \ge \frac{2}{\ln 2}\Big(1+\ln\frac{\ell+1}{2}\Big)\rho(k-\ell)$ is fulfilled for every sufficiently large ℓ. If $\frac{1+\ln 2}{2}(k-\ell) \ge 2(1+\ln\frac{\ell+1}{2})$ then (4) holds for every k and ℓ such that $\ell < k < \frac{c\rho}{c\rho-1}\ell = \big(1-\frac{c-1}{c\rho-1}\big)\frac{\rho}{\rho-1}\ell$, where $c = \frac{1+\ln 2}{2\ln 2}$. Thus, from Theorem 2, one can deduce that there exists an absolute constant ℓ_0 such that

$$\mathcal{SF}^n_{k,\ell} = (\varphi+\varphi_r+o(1))2^{\lfloor n/\rho \rfloor},$$

provided $\ell_0 < \ell < k < \big(1-\frac{c-1}{c\rho-1}\big)\frac{\rho}{\rho-1}\ell$.

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