Applying balanced generalized weighing matrices to construct block designs

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Submitted: October 10, 2000; Accepted: January 27, 2001.
MR Subject Classifications: 05B05

Abstract

Balanced generalized weighing matrices are applied for constructing a family of symmetric designs with parameters $(1 + qr(r^{m+1} - 1)/(r - 1), r^m, r^{m-1}(r - 1)/q)$, where m is any positive integer and q and $r = (q^d - 1)/(q - 1)$ are prime powers, and a family of non-embeddable quasi-residual $2-((r+1)(r^{m+1}-1)/(r-1), r^m(r+1)/2, r^m(r-1)/2)$ designs, where m is any positive integer and $r = 2^d - 1$, $3 \cdot 2^d - 1$ or $5 \cdot 2^d - 1$ is a prime power, $r \geq 11$.

1 Introduction

A *balanced incomplete block design* (BIBD) with parameters (v, b, r, k, λ) or a 2-(v, k, λ) design is a pair $D = (V, \mathcal{B})$, where V is a set (of points) of cardinality v and $\mathcal{B}$ is a collection of b k-subsets of V (blocks) such that each point is contained in exactly r blocks and each 2-subset of V is contained in exactly λ blocks. If $V = \{x_1, x_2, \ldots, x_v\}$ and $\mathcal{B} = \{B_1, B_2, \ldots, B_b\}$, then the $v \times b$ matrix, whose (i, j)-entry is equal to 1 if $x_i \in B_j$ and is equal to 0 otherwise, is the *incidence matrix* of the design. A $(0, 1)$ matrix X of size $v \times b$ is the incidence matrix of a (v, b, r, k, λ) BIBD if and only if $XX^t = (r - \lambda)I_v + \lambda J_v$ and $J_v X = kJ_{v\times b}$, where I_v, J_v, and $J_{v\times b}$ are the identity matrix of order v, the $v \times v$ all-one matrix, and the $v \times b$ all-one matrix, respectively.

It is admissible that two distinct blocks of a BIBD consist of the same points. In particular, repeating s times each block of a (v, b, r, k, λ) BIBD yields its *s-fold multiple* whose parameters are $(v, sb, sr, k, s\lambda)$.

The parameters of a (v, b, r, k, λ) BIBD satisfy equations $vr = bk$ and $(v - 1)\lambda = r(k - 1)$. If $v = b$ (or equivalently $r = k$), the BIBD is called a *symmetric (or square)* (v, k, λ)*-design*. Any two distinct blocks of a symmetric (v, k, λ)-design meet in exactly λ

points. The most celebrated symmetric design, PG(d, q), is formed by the one-dimensional and d-dimensional subspaces of the $(d+1)$-dimensional vector space over the finite field GF(q). It has parameters $((q^{d+1}-1)/(q-1), (q^d-1)/(q-1), (q^{d-1}-1)/(q-1))$. Another famous symmetric design, a *Hadamard 2-design*, has parameters $(4n-1, 2n-1, n-1)$, where $4n$ is the order of a Hadamard matrix.

Replacing each block of a (v, b, r, k, λ) BIBD by its complement yields the *complementary* $(v, b, b-r, v-k, b-2r+\lambda)$ BIBD. Another standard construction produces BIBDs from a symmetric design. If $D = (V, \mathcal{B})$ is a symmetric (v, k, λ)-design and $A \in \mathcal{B}$, then define $\mathcal{B}_A = \{B \cap A \colon B \in \mathcal{B}, B \neq A\}$ and $\mathcal{B}^A = \{B \setminus A \colon B \in \mathcal{B}, B \neq A\}$. Then $D_A = (A, \mathcal{B}_A)$ is a $(k, v-1, k-1, \lambda, \lambda-1)$ BIBD called a *derived design of* D and $D^A = (V \setminus A, \mathcal{B}^A)$ is a $(v-k, v-1, k, k-\lambda, \lambda)$ BIBD called a *residual design of* D. Any derived design of PG(d, q) is a q-fold multiple of PG$(d-1, q)$. Any residual design of PG(d, q) is isomorphic to the design AG(d, q) which formed by the points and hyperplanes of the d-dimensional vector space over GF(q). Its parameters are $(q^d, q(q^d-1)/(q-1), (q^d-1)/(q-1), q^{d-1}, (q^{d-1}-1)/(q-1))$.

The parameters (v, b, r, k, λ) of a residual design satisfy the equation $r = k + \lambda$. Any (v, b, r, k, λ) BIBD with $r = k + \lambda$ (or equivalently $b = v + r - 1$) is called *quasi-residual.* Any (v, b, r, k, λ) BIBD with $k = \lambda + 1$ (or equivalently $v = r + 1$) is called *quasi-derived.* The complement of a quasi-residual design is quasi-derived and vice versa. If a quasi-residual (quasi-derived) design is not a residual (derived) design of a symmetric design, it is called *non-embeddable.*

For further references on BIBDs see [1].

The goal of this paper is to construct parametrically new symmetric designs and non-embeddable quasi-residual designs. The main tool in both constructions are balanced generalized weighing matrices.

A *balanced generalized weighing matrix* BGW(w, l, μ) over a multiplicatively written finite group G is a matrix $W = [\omega_{ij}]$ of order w with entries from the set $G \cup \{0\}$ such that (i) each row of W contains exactly l nonzero entries and (ii) for any distinct $i, h \in \{1, 2, \ldots, w\}$, the multiset

$$\{\omega_{hj}^{-1}\omega_{ij} \colon 1 \leq j \leq w, \omega_{ij} \neq 0, \omega_{hj} \neq 0\} \tag{1}$$

contains exactly $\mu/|G|$ copies of every element of G. If $\mathcal{M}$ is a set of $v \times b$ incidence matrices of (v, b, r, k, λ) BIBDs and G is a group of bijections $\mathcal{M} \to \mathcal{M}$, then, for any $X \in \mathcal{M}$, $W \otimes X$ is the $(wv) \times (wb)$ block-matrix obtained by replacing every nonzero entry ω_{ij} of W by the matrix $\omega_{ij}X$ and every zero entry of W by the $v \times b$ zero matrix. In Theorem 2.4, we give a sufficient condition for the matrix $W \otimes X$ to be the incidence matrix of a $(vw, bw, rl, kl, \lambda l)$ BIBD. (This condition was originally proved in the author's paper [4].) In the papers [5, 6, 7], the author applied this technique to a square matrix X to obtain a large symmetric design from a smaller one. In the current paper, we will use balanced generalized weighing matrices to obtain a large quasi-residual design from a smaller one.

In Section 4 we apply this technique to non-embeddable quasi-residual designs with parameters $(r+1, 2r, r, (r+1)/2, (r-1)/2)$, where $r \geq 11$ is of the form 2^d-1, $3 \cdot 2^d - 1$

or $5 \cdot 2^d - 1$ (these designs were recently found by Mackenzie-Fleming [10, 11]), and, if r is a prime power, we obtain (Theorem 4.5) for any positive integer m a non-embeddable quasi-residual design with parameters $((r+1)(r^m-1)/(r-1), 2r(r^m-1)/(r-1), r^m, (r+1)r^{m-1}/2, (r-1)r^{m-1}/2)$.

A balanced generalized weighing matrix without zero entries, i.e., a matrix BGW (w, w, w) over a group G is called a *generalized Hadamard matrix* over G. If $|G| = g$, then the matrix BGW(w, w, w) is denoted by GH(g, s), where $s = w/g$, so the multiset (1) contains exactly s copies of each element of G. If $G = \{\pm 1\}$, then matrices GH$(2, s)$ over G are precisely Hadamard matrices of order $2s$.

In the paper [12], Rajkundlia showed how generalized Hadamard matrices can be used to obtain a large quasi-derived design from a smaller one. In Section 5, we combine ours and Rajkundlia's methods and, starting with a symmetric (v, r, λ)-design with r a prime power, construct a quasi-residual design with parameters $((v-r)(r^m-1)/(r-1), (v-1)(r^m-1)/(r-1), r^m, (r-\lambda)r^{m-1}, \lambda r^{m-1})$ and a quasi-derived design with parameters $((r^m, (v-1)(r^m-1)/(r-1), r^m-1, \lambda r^{m-1}, \lambda r^{m-1}-1)$. Though the parameters of both designs are those of a residual and a derived design of a symmetric $(1+(v-1)(r^m-1)/(r-1), r^m, \lambda r^{m-1})$-design, these constructions do not imply that such a symmetric design exists. In Theorem 5.1, we give a sufficient condition for this symmetric design to exist. We then demonstrate (Theorems 5.3, 5.8, and 6.4) several possible realizations of this condition.

The first realization (Corollary 5.4) yields a new family of symmetric designs with parameters $(1+qr(r^{m+1}-1)/(r-1), r^m, r^{m-1}(r-1)/q)$, where m is any positive integer and q and $r = (q^d-1)/(q-1)$ are prime powers. If $m = 2$, this is precisely the Rajkundlia–Mitchell family (Family 10 in [3]). Designs with $q = 8$ and $m = 3$ were obtained by the author in [4] by a different method.

The second realization yields the Wilson–Brouwer family of symmetric designs (Family 11 in [3]).

The third realization (Theorem 5.8 and Remark 5.9) yields a family of symmetric designs that the author constructed in [4].

Theorem 6.4 shows that certain residual designs, which admit a cyclic automorphism group on the point-set, would also lead to infinite families of symmetric designs though we were not able to obtain new symmetric designs on this way.

Throughout the paper, I, J, and O denote identity, all-one, and zero matrices of suitable orders.

For any matrix M and any positive integer m, $m \times M$ will denote the matrix obtained by repeating m times consecutively each row of M.

We will use angular brackets $\langle, \rangle$ for the inner product of rows of matrices.

2 Balanced generalized weighing matrices

Definition 2.1 *A* balanced generalized weighing matrix *$BGW(w, l, \mu)$ over a (multiplicatively written) group G is a matrix $W = [\omega_{ij}]$ of order w with entries from the set $G \cup \{0\}$*

such that (i) each row of W contains exactly l non-zero entries and (ii) for any distinct $i, h \in \{1, 2, \ldots, w\}$, the multiset

$$\{\omega_{hj}^{-1}\omega_{ij} \colon 1 \leq j \leq w, \omega_{ij} \neq 0, \omega_{hj} \neq 0\}$$

contains exactly $\mu/|G|$ copies of every element of G. A balanced generalized weighing matrix $BGW(w, w, w)$ over G is called a generalized Hadamard matrix *over G and is denoted by $GH(g, s)$, where $g = |G|$ and $s = w/|G|$.*

Remark 2.2 *Replacing by 1 every nonzero entry of a $BGW(w, l, \mu)$ yields the incidence matrix of a symmetric (w, l, μ)-design. This implies that every column of a $BGW(w, l, \mu)$ has exactly l nonzero entries.*

Remark 2.3 *Any matrix obtained from a balanced generalized weighing matrix BGW (w, l, μ) over G by a permutation of rows or a permutation of columns or by multiplying all entries in a row or a column by the same element of G is a balanced generalized weighing matrix over G with the same parameters. In particular, one can make all $(i, 1)$ and $(1, i)$ entries with $w - l + 1 \leq i \leq w$ equal to the identity element of group G. Such a BGW matrix is called* normalized.

If $\mathcal{M}$ is a set of $v \times b$ matrices, G is a group of bijections $\mathcal{M} \to \mathcal{M}$, and W is a BGW(w, l, μ) over G, then, for any $X \in \mathcal{M}$, $W \otimes X$ will denote the $(wv) \times (wb)$ matrix obtained by replacing every nonzero entry ω_{ij} in W by the matrix $\omega_{ij}X \in \mathcal{M}$ and every zero entry in W by the $v \times b$ zero matrix.

In this paper we will use balance generalized weighing matrices

$$\mathrm{BGW}\left(\frac{q^{m+1} - 1}{q - 1}, q^m, q^m - q^{m-1}\right) \text{ over } \mathbf{Z}_s, \tag{2}$$

where q is a prime power, s is a divisor of $q - 1$, m is a positive integer, and $\mathbf{Z}_s$ is a cyclic group of order s. Different constructions of these matrices can be found in [4, 8, 9]. We will also use in the sequel generalized Hadamard matrices $\mathrm{GH}(q, q^{m-1})$ over $\mathrm{EA}(q)$, where q is a prime power, m is a positive integer, and $\mathrm{EA}(q)$ is an elementary abelian group of order q. A construction of these matrices can be found in [1, Corollary VIII.3.12].

The construction of quasi-residual designs in the current paper will be based on the following theorem.

Theorem 2.4 *Let $\mathcal{M}$ be a set of v by b incidence matrices of (v, b, r, k, λ) BIBDs. Let G be a finite group of bijections $\mathcal{M} \to \mathcal{M}$ satisfying conditions (i) $(\sigma X)(\sigma Z)^t = XZ^t$ for all $X, Z \in \mathcal{M}$ and all $\sigma \in G$ and (ii) $\sum_{\sigma \in G} \sigma X = \frac{k|G|}{v} J$ for all $X \in \mathcal{M}$. Let W be a balanced generalized weighing matrix $BGW(w, l, \mu)$ over G with $kr\mu = v\lambda l$. Then, for any $X \in \mathcal{M}$, $W \otimes X$ is the incidence matrix of a BIBD with parameters $(vw, bw, rl, kl, \lambda l)$.*

Proof. Let $X \in \mathcal{M}$. Since every column of $W = [\omega_{ij}]$ has exactly l nonzero entries, the column sum of $W \otimes X$ is equal to kl. For $i, h = 1, 2, \dots, w$, let

$$P_{ih} = \sum_{j=1}^{w} (\omega_{ij} X)(\omega_{hj} X)^t.$$

It suffices to show that

$$P_{ih} = \begin{cases} (rl - \lambda l)I + \lambda l J & \text{if } i = h, \\ \lambda l J & \text{if } i \neq h. \end{cases} \tag{3}$$

Since each row of W has exactly l nonzero entries, we have, for some $\sigma_j \in G$,

$$P_{ii} = \sum_{j=1}^{l} (\sigma_j X)(\sigma_j X)^t = lXX^t = (rl - \lambda l)I + \lambda l J.$$

If $i \neq h$, then, for some $\sigma_j, \tau_j \in G$,

$$P_{ih} = \sum_{j=1}^{\mu} (\sigma_j X)(\tau_j X)^t = \sum_{j=1}^{\mu} (\tau_j^{-1} \sigma_j X)X^t = \frac{\mu}{|G|} \sum_{\sigma \in G} (\sigma X) X^t = \frac{k\mu}{v} J X^t = \frac{kr\mu}{v} J = \lambda l J.$$

□

Remark 2.5 *Under the conditions of Theorem 2.4, if X is the incidence matrix of a quasi-residual design, then so is $W \otimes X$.*

Remark 2.6 *If the (v, b, r, k, λ) designs in Theorem 2.4 are quasi-residual and W is a matrix (2), then the equality $kr\mu = v\lambda l$ required by Theorem 2.4 is equivalent to $r = q$.*

The construction of quasi-derived designs in the current paper will be based on the following theorem by Rajkundlia [12].

Theorem 2.7 *Let Y be the incidence matrix of a (v, b, r, k, λ) BIBD with $v > 1$ and $k > 1$ and let G be a sharply transitive group of permutations of rows of Y. For each positive integer m, let H_m be a generalized Hadamard matrix $GH(v, v^{m-1})$ over G. Put $Y_0 = Y$ and define inductively for $m \geq 1$ block-matrices $Y_m = [H_m \otimes Y \quad v \times Y_{m-1}]$. Then Y_m is the incidence matrix of a $(v^{m+1}, b_m, r_m, v^m k, \lambda_m)$ BIBD with $b_m = b(v^{m+1} - 1)/(v - 1)$, $r_m = \lambda(v^{m+1} - 1)/(k - 1)$, and $\lambda_m = \lambda(v^m k - 1)/(k - 1)$.*

Proof. The statement is true for $m = 0$, so let $m \geq 1$ and let Y_{m-1} be the incidence matrix of a $(v^m, b_{m-1}, r_{m-1}, v^{m-1}k, \lambda_{m-1})$ BIBD. Then the column sum of Y_m is $v^m k$ and the row sum is $v^{m+1} r + r_{m-1} = r_m$. We will split the rows of Y_m into v^m groups of v consecutive rows each and denote by y_{ij} the j^{th} row of the i^{th} group. Then

$$\langle y_{ij}, y_{hl} \rangle = \begin{cases} v^m \lambda + r_{m-1} & \text{if } i = h \text{ and } j \neq l, \\ v^{m-1} r + (v^m - v^{m-1})\lambda + \lambda_{m-1} & \text{if } i \neq h. \end{cases}$$

Since $v^m \lambda + r_{m-1} = v^{m-1} r + (v^m - v^{m-1})\lambda + \lambda_{m-1} = \lambda_m$, the proof is now complete. □

Remark 2.8 *Any group of order v can be regarded as a sharply transitive group of permutations of rows of a $v \times b$ matrix if the rows of the matrix are indexed by the elements of the group and the action of the group is given by $\sigma(\tau^{th}$ row$) = (\sigma\tau)^{th}$ row.*

Remark 2.9 *If Y is the incidence matrix of a quasi-derived design, then so is each Y_m.*

Remark 2.10 *The proof of Theorem 2.7 shows that each row of Y_m can be represented as $[y_1\, y_2 \ldots y_s]$, where each y_j is a row of Y and $s = (v^{m+1} - 1)/(v - 1)$.*

Remark 2.11 *It will not be important in the sequel how derived designs Y_m are constructed as far as they satisfy the property stated in Remark 2.10. It is possible to construct such designs in the case of r a prime power by a method proposed by S.S. Shrikhande and Raghavarao in the paper [13]*

3 Resolvability in 2-designs

Definition 3.1 *Let $D = (V, \mathcal{B})$ be a $2-(v, k, \lambda)$ design. A non-empty subset $\mathcal{C}$ of $\mathcal{B}$ is called a* resolution class *if there exists an integer $\alpha(\mathcal{C})$ such that every point $x \in V$ is contained in exactly $\alpha(\mathcal{C})$ blocks from $\mathcal{C}$.*

Counting in two ways yields

Lemma 3.2 *If $\mathcal{C}$ is a resolution class of a $2-(v, k, \lambda)$ design, then $v\alpha(\mathcal{C}) = k|\mathcal{C}|$.*

Definition 3.3 *Let $D = (V, \mathcal{B})$ be a $2-(v, k, \lambda)$ design. A partition of $\mathcal{B}$ into resolution classes is called a* resolution *of the design D. If D admits a resolution R such that $\alpha(\mathcal{C})$ has the same value α for all $\mathcal{C} \in \mathsf{R}$, the design D is called* α-resolvable*. If the cardinality of the intersection of any two distinct blocks in an α-resolvable design depends only on whether or not the blocks belong to the same resolution class, the design is called* affine α-resolvable*; α-resolvable and affine α-resolvable designs with $\alpha = 1$ are called* resolvable *and* affine resolvable*, respectively. Resolution classes of resolvable designs are called* parallel classes*.*

Remark 3.4 *All known affine resolvable designs are residual designs of either a $PG(d, q)$ or a Hadamard 2-design.*

Remark 3.5 *The designs constructed in Theorem 2.7 are r-resolvable.*

The following two propositions explain our interest in resolvability.

Proposition 3.6 *Let X be the incidence matrix of a $2-(v, k, \lambda)$ design D and let G be a group of permutations on the set of columns of X. For each $\sigma \in G$, let σX be the matrix obtained by applying σ to the set of columns of X. If*

$$\sum_{\sigma \in G} \sigma X = \frac{k|G|}{v} J, \tag{4}$$

then the set of blocks of D corresponding to the columns of a G-orbit is a resolution class.

Proof. Let a set C of columns of X be a G-orbit and let $H = \{\sigma \in G \colon \sigma(C) = C\}$. Let x be a point of the design D and let α be the number of blocks which contain x and correspond to columns from C. Then (4) implies that $\alpha|G|/|H| = k|G|/v$, so $\alpha = k|H|/v$ is the same for all points x. □

Proposition 3.7 *Let X be the incidence matrix of a $2-(v, k, \lambda)$ design D which admits a resolution $\mathsf{R} = \{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_s\}$. There exists a cyclic group G of permutations of columns of X such that the order of G is the least common multiple of $|\mathcal{C}_1|, |\mathcal{C}_2|, \ldots, |\mathcal{C}_s|$ and (4) is satisfied.*

Proof. For $i = 1, 2, \ldots, s$, let C_i be the set of columns of X corresponding to blocks from $\mathcal{C}_i$ and let σ_i cyclically permute C_i. Let G be the group generated by $\sigma_1\sigma_2\ldots\sigma_s$. Since $\alpha(\mathcal{C}_i)|G|/|\mathcal{C}_i| = k|G|/v$, (4) is satisfied. □

4 Quasi-residual designs

In order to apply Theorem 2.4 to constructing quasi-residual designs one has to deal with two obstacles. Firstly, balanced generalized weighing matrices are relatively rare. Most BGW matrices, which are not generalized Hadamard matrices, are given by (2). If these matrices are applied to quasi-residual (v, b, r, k, λ) BIBDs in Theorem 2.4, then the equality $kr\mu = v\lambda l$ required by the theorem is equivalent to $q = r$. Secondly, we have to find a cyclic group G satisfying conditions (i) and (ii) of Theorem 2.4. The condition (i) is always satisfied if G is a group of permutations of columns of the given matrices. In this case, we can start with the incidence matrix X of a (v, b, r, k, λ) BIBD and define $\mathcal{M} = \{\sigma X \colon \sigma \in G\}$. Then, as Propositions 3.6 and 3.7 show, condition (ii) of Theorem 2.4 is satisfied if and only if the design with the incidence matrix X admits a resolution with the cardinality of each resolution class dividing $r - 1$.

These considerations lead to the following

Theorem 4.1 *Let r be a prime power. Suppose there exists a quasi-residual (v, b, r, k, λ) BIBD which admits a resolution with the cardinality of each resolution class dividing $r-1$. Then, for any positive integer m, there exists a quasi-residual design with parameters*

$$\left(\frac{v(r^m-1)}{r-1}, \frac{b(r^m-1)}{r-1}, r^m, kr^{m-1}, \lambda r^{m-1}\right).$$

Proof. Let X be the incidence matrix of a quasi-residual (v, b, r, k, λ) BIBD which admits a resolution with the cardinality of each resolution class dividing $r - 1$. Consider the same group G as in Proposition 3.7 and define $\mathcal{M} = \{\sigma X \colon \sigma \in G\}$. For $m \geq 2$, let W be a BGW(w, l, μ) over G with $w = (r^m - 1)/(r - 1)$, $l = r^{m-1}$, and $\mu = r^{m-1} - r^{m-2}$. Then, by Theorem 2.4, $W \otimes X$ is the incidence matrix of a BIBD with the required parameters. □

Affine resolvable designs AG(d, q) satisfy the conditions of Theorem 4.1 whenever $r = (q^d - 1)/(q - 1)$ is a prime power. Thus we obtain

Corollary 4.2 *If q and $r = (q^d - 1)/(q - 1)$ are prime powers, then, for any positive integer m, there exists a quasi-residual design with parameters*

$$\left(\frac{q^d(r^m-1)}{r-1}, \frac{qr(r^m-1)}{r-1}, r^m, q^{d-1}r^{m-1}, \frac{(q^{d-1}-1)r^{m-1}}{q-1}\right). \tag{5}$$

The group G in Theorem 2.4 does not have to be a group of permutations of columns of a matrix X. A different group will work for quasi-residual designs with parameters

$$\left(r+1, 2r, r, \frac{r+1}{2}, \frac{r-1}{2}\right). \tag{6}$$

Observe that the complement of such a design is a quasi-residual design with the same parameters. If X is the incidence matrix of a design with parameters (6), then $(J - X)(J - X)^t = X(J - X)^t = (J - X)X^t = XX^t$. Therefore, if $\mathcal{M} = \{X, J - X\}$ and σ is the transposition of X and $J - X$, then the group G of order 2 generated by σ satisfies conditions (i) and (ii) of Theorem 2.4. Thus we obtain

Theorem 4.3 *If there exists a BIBD with parameters* (6)*, where r is an odd prime power, then, for any positive integer m, there exists a quasi-residual design with parameters*

$$\left(\frac{(r+1)(r^m-1)}{r-1}, \frac{2r(r^m-1)}{r-1}, r^m, \frac{(r+1)r^{m-1}}{2}, \frac{(r-1)r^{m-1}}{2}\right). \tag{7}$$

We shall show that some of the designs (7) are non-embeddable. We start with the following sufficient condition for non-embeddability.

Proposition 4.4 *If a quasi-residual (v, b, r, k, λ) BIBD has three distinct blocks, B_1, B_2, and B_3, such that*

$$|B_1 \cap B_2| + |B_1 \cap B_3| + |B_2 \cap B_3| > r, \tag{8}$$

then the design is non-embeddable.

Proof. Suppose that a quasi-residual (v, b, r, k, λ) BIBD has distinct blocks B_1, B_2, and B_3 satisfying (8). Suppose that this design is embeddable in a symmetric $(v+r, r, \lambda)$-design D. For $i = 1, 2, 3$, let A_i be the block of D that contains B_i and let $C_i = A_i \setminus B_i$. Then, for $i \neq j$, $|C_i \cap C_j| = \lambda - |B_i \cap B_j|$. Therefore,

$$r \geq |C_1 \cup C_2 \cup C_3| \geq |C_1| + |C_2| + |C_3| - (|C_1 \cap C_2| + |C_1 \cap C_3| + |C_2 \cap C_3|)$$

$$= 3(r-k) - 3\lambda + (|B_1 \cap B_2| + |B_1 \cap B_3| + |B_2 \cap B_3|) = |B_1 \cap B_2| + |B_1 \cap B_3| + |B_2 \cap B_3| > r,$$

a contradiction. □

Let X be the incidence matrix of a quasi-residual design (6) with r an odd prime power. Suppose further that this design has three distinct blocks, B_1, B_2, and B_3, satisfying (8), and let these blocks correspond to the first three columns of X. Suppose that, for $m \geq 2$,

W is a normalized BGW$((r^m-1)/(r-1), r^{m-1}, r^{m-1}-r^{m-2})$ matrix (Remark 2.3) such that $W\otimes X$ is the incidence matrix of the design (7) that is obtained in Theorem 4.3. Let B_1', B_2', and B_3' be the blocks of this design corresponding to the first three columns of $W\otimes X$. Then, for $i,j=1,2,3$, $|B_i'\cap B_j'| = r^{m-1}|B_i\cap B_j|$. Therefore, the blocks B_1', B_2', and B_3' satisfy (8), and the design with the incidence matrix $W\otimes X$ is non-embeddable.

In the recent papers [10, 11] Mackenzie-Fleming showed that for every positive integer d there exist designs (6) with $r=2^d-1\geq 15$, $r=3\cdot 2^d-1\geq 11$, and $r=5\cdot 2^d-1\geq 19$ having three distinct blocks satisfying (8). Therefore,

Theorem 4.5 *Let m and d be positive integers and let $r=2^d-1$ or $r=3\cdot 2^d-1$ or $r=5\cdot 2^d-1$. If r is a prime power and $r\geq 11$, then there exists a non-embeddable quasi-residual design with parameters* (7).

5 Three families of symmetric designs

Let X be the incidence matrix of a quasi-residual $(v-r, v-1, r, r-\lambda, \lambda)$ BIBD and Y the incidence matrix of a quasi-derived $(r, v-1, r-1, \lambda, \lambda-1)$ BIBD. If r is a prime power, then, for any positive integer m, there exist matrices BGW$((r^{m+1}-1)/(r-1), r^m, r^m-r^{m-1})$ and GH(r, r^{m-1}). Suppose that the conditions of Theorem 2.4 are satisfied. Then we can obtain a quasi-residual design with parameters

$$\left(\frac{(v-r)(r^{m+1}-1)}{r-1}, \frac{(v-1)(r^{m+1}-1)}{r-1}, r^{m+1}, (r-\lambda)r^m, \lambda r^m\right). \tag{9}$$

Theorem 2.7 yields a quasi-derived design with parameters

$$\left(r^{m+1}, \frac{(v-1)(r^{m+1}-1)}{r-1}, r^{m+1}-1, \lambda r^m, \lambda r^m-1\right). \tag{10}$$

These designs could be a residual design and a derived design of a symmetric design with parameters

$$\left(1+\frac{(v-1)(r^{m+1}-1)}{r-1}, r^{m+1}, \lambda r^m\right) \tag{11}$$

if such a symmetric design exists. Of course, existence of designs (9) and (10) does not automatically imply existence of a symmetric design (11). The next theorem gives a sufficient condition for combining designs (9) and (10) into a symmetric design.

Theorem 5.1 *Let X and Y be incidence matrices of a $(v-r, v-1, r, r-\lambda, \lambda)$ BIBD and a $(r, v-1, r-1, \lambda, \lambda-1)$ BIBD, respectively. Suppose there exists a set $\mathcal{M}$ of matrices containing X, a finite group G of bijections $\mathcal{M}\to\mathcal{M}$, and, for every positive integer m, a BGW$((r^{m+1}-1)/(r-1), r^m, r^m-r^{m-1})$ over G, which satisfy the conditions of Theorem 2.4. Suppose further that, for every positive integer m, there exists a GH(r, r^{m-1}) over a group of order r. If $(\sigma X)Y^t=\lambda J$ for all $\sigma\in G$, then there exists a symmetric design with parameters* (11).

Proof. Let $\mathcal{M}$ be a set of incidence matrices of $(v-r, v-1, r, r-\lambda, \lambda)$ BIBDs, containing X, G a group of bijections $\mathcal{M} \to \mathcal{M}$, and W a BGW$((r^{m+1}-1)/(r-1), r^m, r^m - r^{m-1})$ over G which satisfy Theorem 2.4. Then $W \otimes X$ is the incidence matrix of a BIBD with parameters (9). Let H be a GH(r, r^{m-1}) over a group of order r whose elements index the rows of Y. Then, in the notation of Theorem 2.7, Y_m is a BIBD with parameters (10). Let matrix S be defined by

$$S = \begin{bmatrix} W \otimes X & \mathbf{0} \\ Y_m & \mathbf{1} \end{bmatrix}, \tag{12}$$

where $\mathbf{0}$ and $\mathbf{1}$ are the all-zero and the all-one column, respectively. We claim that S is the incidence matrix of a symmetric design (11). It suffices to show that $(W \otimes X)Y_m^t = \lambda r^m J$.

Each row x of $W \otimes X$ can be represented as $x = [x_1\, x_2 \ldots x_s]$, where $s = (r^{m+1} - 1)/(r-1)$ and, for r^m values of j, x_j is a row of σX for some $\sigma \in G$, while for the remaining values of j, x_j is the row of $v-1$ zeros. We will represent a row y of Y_m as $y = [y_1\, y_2 \ldots y_s]$ where each y_j is a row of Y (Remark 2.10). Since $(\sigma X)Y^t = \lambda J$ for all $\sigma \in G$, we obtain that $\langle x, y \rangle = \lambda r^m$. □

Remark 5.2 *Applying the equality $(\sigma X)Y^t = \lambda J$ required by Theorem 5.1 to the identity element of group G yields $XY^t = \lambda J$, which means that X and Y are the incidence matrices of a residual design and the corresponding derived design of a symmetric (v, r, λ)-design.*

The following theorem presents two cases when the conditions of Theorem 5.1 are satisfied.

Theorem 5.3 *Let r be a prime power and D a symmetric (v, r, λ)-design. If D is (i) a PG(d, q) or (ii) a Hadamard 2-design, then, for any nonnegative integer m, there exists a symmetric design with parameters* (11).

Proof. Since r is a prime power, then, for any positive integer m, there exists a BGW$(r^{m+1}-1)/(r-1), r^m, r^m - r^{m-1})$ over any cyclic group G with $|G|$ dividing $r-1$ and there exists a GH(r, r^{m-1}) over an elementary abelian group of order r.

Let M be the incidence matrix of D. Remove a column from M and split the obtained matrix into two matrices, X and Y, corresponding to a residual design and a derived design of D.

(i) Suppose D is a PG(d, q). Then the residual design is AG(d, q) and the derived design is a q-fold multiple of PG$(d-1, q)$. The columns of X are partitioned into r classes corresponding to parallel classes of AG(d, q). The i^{th} column of X and the j^{th} column of X are in the same class if and only if the i^{th} column of Y and the j^{th} column of Y are equal to each other. Therefore, if ρ is a permutation of columns of X which permutes the columns of each parallel class without changing the classes, then $(\rho X)Y^t = XY^t = \lambda J$ with $\lambda = (q^{d-1}-1)/(q-1)$. Thus, if we assume that ρ cyclically permutes the columns of each class and denote by G the cyclic group of order q generated by ρ, then G and

$\mathcal{M} = \{\sigma X : \sigma \in G\}$ satisfy the conditions of Theorem 5.1 which yields a symmetric design with parameters (11).

(ii) Suppose D is a symmetric $(2r+1, r, (r-1)/2)$-design (this is a Hadamard 2-design and it exists for any odd prime power r). Then X is the incidence matrix of a $(r+1, 2r, r, (r+1)/2, (r-1)/2)$ BIBD, Y is the incidence matrix of a $(r, 2r, r-1, (r-1)/2, (r-3)/2)$ BIBD, and $XY^t = \frac{r-1}{2}J$. Therefore, $(J-X)Y^t = (r-1)J - XY^t = \frac{r-1}{2}J$. Let $\mathcal{M} = \{X, J-X\}$ and let G be the group of order 2 generated by the transposition τ acting on $\mathcal{M}$. Then again Theorem 5.1 yields a symmetric design with parameters (11). $\square$

For case (i), we obtain

Corollary 5.4 *Let q and $r = (q^d-1)/(q-1)$ be prime powers. Then, for any nonnegative integer m, there exists a symmetric design with parameters*

$$\left(1 + \frac{qr(r^{m+1}-1)}{r-1}, r^{m+1}, \frac{r^m(r-1)}{q}\right). \tag{13}$$

Remark 5.5 *For $d = 2$ and $m \geq 1$, the parameters* (13) *are precisely the parameters of the Rajkundlia–Mitchell family (Family 10 in the CRC Handbook [3]). For $q = 2$ and $m \geq 1$, the designs* (13) *are contained in the Wilson–Brouwer family (Family 11 in [3]). The family with $q = 8$, $d = 3$, and $m \geq 1$ had been obtained by the author in [4] by a different method. With these exceptions, the parameters* (13) *for $m \geq 1$ were previously undecided.*

Remark 5.6 *The designs of case (ii) are precisely the designs of the Wilson–Brouwer family of symmetric designs (Family 11 in [3]) constructed by Brouwer in [2].*

The smallest previously unknown design in family (13) is a symmetric $(547, 169, 52)$-design corresponding to $q = 3$, $d = 3$, $r = 13$, and $m = 1$. The incidence matrix of this design is the matrix (12), where $W = [\omega_{ij}]$ is a $\mathrm{BGW}(14, 13, 12)$ over $\mathbf{Z}_3$, X is the incidence matrix of the design $\mathrm{AG}(3,3)$ whose blocks are so ordered that the blocks of each parallel class correspond to consecutive columns of X, $Y_m = Y_1 = [\mathrm{GH}(13,1) \otimes Y \quad 13 \times Y]$, and Y is the incidence matrix of the 3-fold multiple of $\mathrm{PG}(2,3)$ whose blocks are so ordered that equal blocks correspond to consecutive columns of Y. Let $D = (V, \mathcal{B})$ be the complementary $(547, 378, 261)$-design and let A be the block of D corresponding to the last column of the incidence matrix $J - S$. Then $\mathcal{B} \setminus \{A\}$ is partitioned into classes $\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_{14}$ of cardinality 39 so that each point $a \in V \setminus A$ is contained in 27 blocks of each class $\mathcal{C}_i$, i.e., the design D^A is 27-resolvable. Each class $\mathcal{C}_i$ is in turn partitioned into subclasses $\mathcal{C}_{i1}, \mathcal{C}_{i2}, \dots, \mathcal{C}_{i,13}$ of cardinality 3 so that, for each $a \in V$,

$$|\{B \in \mathcal{C}_{ij} : a \in \mathcal{B}\}| = \begin{cases} 0 \text{ or } 3 & \text{if } a \notin A, \\ 3 & \text{if } a \in A \text{ and } \omega_{ij} = 0, \\ 2 & \text{if } a \in A \text{ and } \omega_{ij} \neq 0. \end{cases} \tag{14}$$

The following theorem shows that a symmetric design that admits an α-resolvable residual design and satisfies a condition similar to (14) and certain restrictions of arithmetical nature, starts an infinite family of symmetric designs.

Theorem 5.7 *Let $D = (V, \mathcal{B})$ be a symmetric (v, r, λ)-design and let $A \in \mathcal{B}$. Suppose that the residual design D^A is α-resolvable with resolution classes $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_s$. Suppose further that each $\mathcal{C}_i$ admits a partition $\{\mathcal{C}_{i1}, \mathcal{C}_{i2}, \ldots, \mathcal{C}_{it}\}$ into classes of the same cardinality q so that, for each $a \in V$, for $i = 1, 2, \ldots, s$, and for $j = 1, 2, \ldots, t$,*

$$|\{B \in \mathcal{B} \colon B \setminus A \in \mathcal{C}_{ij}, a \in B\}| = \begin{cases} 0 \text{ or } q & \text{if } a \notin A, \\ n_i(a) & \text{if } a \in A, \end{cases}$$

where $n_i(a)$ is an integer depending on i and a but not on j. If r is a prime power and t divides $r - 1$, then, for any positive integer m, there exists a symmetric design with parameters (11).

Proof. We assume that the blocks of D are so ordered that if B_1 precedes B_2, $B_1 \setminus A \in \mathcal{C}_{ij}$, $B_2 \setminus A \in \mathcal{C}_{kl}$, then either $i < k$ or $i = k$ and $j < l$.

Let X and Y be the incidence matrices of the designs D^A and D_A, respectively. Let $1, 2, \ldots, v-1$ be the indices of consecutive columns of X and let σ be a permutation on the set of columns of X such that, for $l = 1, 2, \ldots, v-1$, $\sigma(l) \equiv l + q \pmod{qt}$ and l and $\sigma(l)$ correspond to blocks from the same resolution class. In other words, σ permutes cyclically subclasses $\mathcal{C}_{i1}, \mathcal{C}_{i2}, \ldots, \mathcal{C}_{it}$, for each i, without changing the order of blocks within each subclass $\mathcal{C}_{ij}$. Let G be the cyclic group generated by σ. Then $|G| = t$. We claim that $(\sigma^m X)Y^t = XY^t$ for $m = 0, 1, \ldots, t-1$.

We will represent any row x of length $v-1$ as $x = [x_1\ x_2 \ldots\ x_s]$, where each x_i is a row of length $(v-1)/s$, and then represent each x_i as $x_i = [x_{i1}\ x_{i2} \ldots\ x_{it}]$, where each x_{ij} is a row of length q.

Let x and y be a row of X and a row of Y, respectively. Let $a \in A$ be the point of D corresponding to y, and let x' be the row of $\sigma^m X$ corresponding to x. We have to show that $\langle x, y \rangle = \langle x', y \rangle$.

Each x_{ij} as well as each x'_{ij} is the all-one or the all-zero row of length q. Since the row sum of both x_i and x'_i is equal to α, the number of all-one rows x_{ij} for a fixed i as well as the number of the all-one rows x'_{ij} is equal to α/q. Since the row sum of each y_{ij} is equal to $n_i(a)/t$, we obtain that

$$\langle x_i, y_i \rangle = \sum_{j=1}^{t} \langle x_{ij}, y_{ij} \rangle = \frac{\alpha n_i(a)}{qt}$$

and

$$\langle x'_i, y_i \rangle = \sum_{j=1}^{t} \langle x'_{ij}, y_{ij} \rangle = \frac{\alpha n_i(a)}{qt}.$$

This proves that $(\sigma^m X)Y^t = XY^t$. We also have

$$\sum_{m=1}^{t} \sigma^m X = \frac{\alpha}{q} J.$$

Since D^A is α-resolvable, we have $\alpha = r/s$, which implies $\alpha/q = (r-\lambda)t/(v-r)$. Since r is a prime power and t divides $r-1$, there exists, for any m, a BGW$((r^{m+1}-1)/(r-1), r^m, r^m - r^{m-1})$ over G. Thus all the conditions of Theorem 5.1 are satisfied and we obtain a family of symmetric designs with parameters (11). $\square$

Let D be the complement of the symmetric design obtained in Corollary 5.4. Then D is a symmetric (v,k,λ)-design with $v = 1+qr(r^{m+1}-1)/(r-1)$, $k = q^d(r^{m+1}-1)/(r-1)$, and $\lambda = qr(q^{d-1}r^m - 1)/(r-1)$, where q and $r = (q^d-1)/(q-1)$ are prime powers. If A is the block corresponding to the last column of the incidence matrix $J - S$, where S is the matrix (12), then the residual design D^A is q^d-resolvable. If each resolution class is divided into $t = r$ subclasses, each formed by q blocks corresponding to consecutive columns of the incidence matrix, then all the structural conditions of Theorem 5.7 are satisfied. Since $k \equiv q^d \pmod{t}$, t divides $k-1$. Therefore, we obtain

Theorem 5.8 *Let d and e be positive integers. If q, $p = (q^d-1)/(q-1)$, and $r = q^d(p^{e+1}-1)/(p-1)$ are prime powers, then, for any nonnegative integer m, there exists a symmetric design with parameters* (11)*, where $v = 1 + qp(p^{e+1}-1)/(p-1)$ and $\lambda = qp(q^{d-1}p^e - 1)/(p-1)$.*

Remark 5.9 *The only realization of the conditions of Theorem 5.8 that we are aware of is $q = 2$, $p = 2^d - 1$ is a Mersenne prime, and $e = 1$, so $r = 2^{2d}$. The designs that the theorem yields in this case were constructed by the author in [4].*

6 Cyclic designs

In this section, we will describe another approach to satisfying the conditions of Theorem 5.1 and thus obtaining an infinite family of symmetric designs. Though we do not have an example when this approach leads to new symmetric designs, we hope that such examples can be found.

Definition 6.1 *Let $D = (V, \mathcal{B})$ be a BIBD. A bijection $\sigma\colon V \to V$ is called an* automorphism *of D if $\sigma(B) \in \mathcal{B}$ for all $B \in \mathcal{B}$. A group G of automorphisms of D is called* regular *if it acts transitively and faithfully on points, i.e., for any $x, y \in V$, there is a unique $\sigma \in G$ such that $\sigma(x) = y$. If a BIBD has a regular cyclic group of automorphisms, it is called* cyclic*.*

Remark 6.2 *The order of a regular automorphism group of a (v,b,r,k,λ) BIBD is v.*

Remark 6.3 *It is immediate that the complement of a cyclic BIBD is cyclic and multiples of a cyclic BIBD are cyclic. A classical example of a cyclic design is a symmetric design generated by a cyclic difference set. Thus, for any prime power q and any positive integer d, there exists a cyclic $PG(d,q)$. An extensive list of cyclic BIBDs can be found in the section on difference families in [3].*

Theorem 6.4 *Let D be a symmetric (v,r,λ)-design with r a prime power and $v-r$ dividing $r-1$. If D has a cyclic residual design, then, for any positive integer m, there exists a symmetric design with parameters* (11).

Proof. Let D^A be a residual design of D having a cyclic regular automorphism group G and let X and Y be the incidence matrices of D^A and D_A, respectively, corresponding to the same order of blocks. Since r is a prime power and $|G| = v - r$ divides $r-1$, there exists a BGW with parameters (2) over G. For each $\sigma \in G$, σX can be obtained from X by a permutation of rows. Since each row of σX is a row of X, we have $(\sigma X)Y^t = \lambda J$ for all $\sigma \in G$. and we apply Theorem 5.1. □

Remark 6.5 *If D is the complement of $PG(d,q)$ with $d \geq 2$, then any residual design D^A is a q-fold multiple of the complement of $PG(d-1,q)$. Therefore, D can be selected so that the conditions of Theorem 6.4 are satisfied. The design that Theorem 6.4 yields in this case is the complement of $PG((m+1)d,q)$.*

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