Nowhere-zero k-flows of supergraphs

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Abstract

Let G be a 2-edge-connected graph with o vertices of odd degree. It is well-known that one should (and can) add $\frac{o}{2}$ edges to G in order to obtain a graph which admits a nowhere-zero 2-flow. We prove that one can add to G a set of $\leq \lfloor \frac{o}{4} \rfloor$, $\lceil \frac{1}{2} \lfloor \frac{o}{5} \rfloor \rceil$, and $\lceil \frac{1}{2} \lfloor \frac{o}{7} \rfloor \rceil$ edges such that the resulting graph admits a nowhere-zero 3-flow, 4-flow, and 5-flow, respectively.

1 Introduction

Graphs in this paper may contain multiple edges and loops. A vertex of G is *odd* (*even*) if its degree is odd (even). We denote by $o(G)$ the number of odd vertices of G. Let G be a graph such that no component of G is a cycle. Then there is a unique graph G' which is homeomorphic to G and has no vertices of degree 2. We say that G' is obtained from G by *suppressing* vertices of degree 2, and we denote this by $G' \propto G$.

Let Γ be an Abelian group, let D be an orientation of a graph G and $f : E(G) \to \Gamma$. The pair (D, f) is a Γ-*flow* in G if the following condition is satisfied at every vertex $v \in V(G)$:

$$\sum_{e \in E^+(v)} f(e) = \sum_{e \in E^-(v)} f(e),$$

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where $E^+(v)$ and $E^-(v)$ denote the sets of outgoing and ingoing edges (with respect to the orientation D) incident with v, respectively.

A flow (D, f) is *nowhere-zero* if $f(e) \neq 0$ for every $e \in E(G)$. If $\Gamma \cong \mathbb{Z}$ and $-k < f(e) < k$ then (D, f) is a *k-flow*. The concept of nowhere-zero flows was introduced and studied by Tutte [9]. For a 2-edge-connected graph G and a group Γ of order k, Tutte [8] proved that G admits a nowhere-zero k-flow if and only if it admits a nowhere-zero Γ-flow. Seymour [7] proved that every 2-edge-connected graph admits a nowhere-zero 6-flow. We refer to [10] for further results on flows in graphs.

Let $\phi_k^+(G)$ be the minimum number of edges whose addition to G gives rise to a graph which admits a nowhere-zero k-flow. Similarly, let $\phi_k^-(G)$ be the minimum number of edges whose deletion from G leaves a graph with a nowhere-zero k-flow. Clearly, $\phi_k^+(G) \leq \phi_k^-(G)$ since we achieve a similar effect by doubling an edge as we do by deleting it.

Let G be a 2-edge-connected graph with $o = o(G)$ vertices of odd degree. It is obvious that we should and that we can add $\frac{o}{2}$ edges to G in order to obtain an Eulerian graph, i.e. a graph which admits a nowhere-zero 2-flow. Thus, $\phi_2^+(G) = \frac{o}{2}$. We shall prove that $\phi_3^+(G) \leq \lfloor \frac{o}{4} \rfloor$, $\phi_4^+(G) \leq \lceil \frac{1}{2} \lfloor \frac{o}{5} \rfloor \rceil$, and $\phi_5^+(G) \leq \lceil \frac{1}{2} \lfloor \frac{o}{7} \rfloor \rceil$, respectively. It is also shown that upper bounds which are linear in $o(G)$ are best possible for 3-flows and 4-flows. They are also best possible for 5-flows if the Tutte 5-Flow-Conjecture is not true (otherwise $\phi_5(G) = 0$ for 2-edge-connected graphs). Some additional comments on the tightness and importance of these bounds are collected at the end of the paper.

We will use the following lemma of Fleischner [4] (see also [10]):

Lemma 1.1 (Splitting Lemma) *Let G be a 2-edge-connected graph, let v be a vertex of G of degree ≥ 4, and let e_0, e_1, e_2 be edges incident with v which do not form an edge-cut of G. Let G_i $(i = 1, 2)$ be the graph constructed from G by splitting v into vertices v_1 and v_2 such that v_1 is incident with e_0 and e_i and v_2 is incident with all other edges at v. Then one of G_1 and G_2 is 2-edge-connected.*

2 3-flows

Theorem 2.1 *Let G be a loopless cubic multigraph on n vertices. Then, $\phi_3^+(G) \leq \lfloor \frac{n}{4} \rfloor$.*

Proof. Suppose that the theorem is false and G is a counterexample with minimum number of vertices. Suppose that $G = G_1 \cup G_2$, where G_1 and G_2 are vertex disjoint graphs. Let $n_i = |V(G_i)|$, $i = 1, 2$. By the minimality

$$\phi_3^+(G) \leq \phi_3^+(G_1) + \phi_3^+(G_2) \leq \left\lfloor \frac{n_1}{4} \right\rfloor + \left\lfloor \frac{n_2}{4} \right\rfloor \leq \left\lfloor \frac{n_1 + n_2}{4} \right\rfloor.$$

This shows that G is connected. Let $C = v_1 v_2 \cdots v_k v_1$ be a shortest cycle in G. For $i = 1, \ldots, k$, denote by v_i' the neighbor of v_i distinct from v_{i+1} and v_{i-1}. (All indices are considered modulo k.) If v_i' does not exist, then $k = 2$ and G has two vertices only. In this case, the claim clearly holds.

If $k = 2$, choose C such that $v_1' \neq v_2'$ whenever possible. If $v_1' \neq v_2'$, let $G' = G - V(C) + v_1'v_2'$. By the induction hypothesis, there is a set F of at most $\lfloor \frac{n-2}{4} \rfloor$ edges such that $G_F' := G' + F$ has a nowhere-zero 3-flow. Then, clearly $G_F := G + F$ also has a nowhere-zero 3-flow.

In the sequel we shall apply the induction hypothesis several times. We shall always denote by G' the smaller graph and then use F, G_F', and G_F in the same way as above.

Suppose now that $k = 2$ and $v_1' = v_2'$. Let v_1'' be the neighbor of v_1' distinct from v_1 and v_2. Let G' be the cubic graph which is homeomorphic to $G - \{v_1, v_2, v_1'\}$. Then G' has $n - 4$ vertices. If it had a loop, then the two edges of $G - \{v_1, v_2, v_1'\}$ incident with v_1'' would be parallel edges, and we would choose them as the cycle C since their neighbors are distinct vertices. Therefore, G' is loopless, and we can apply the induction hypothesis to G'. It is easy to see that a nowhere-zero 3-flow of G_F' can be extended to a nowhere-zero 3-flow of $G_F + v_1''v_1$.

Suppose now that $k = 3$. If $v_1' = v_2' = v_3'$, then $G = K_4$ for which $\phi_3^+(K_4) = 1$. Assume now that $v_1' = v_2' \neq v_3'$. Let v_1'' be as above. If $v_1'' \neq v_3'$, let $G' \propto G - V(C) - v_1' + v_3'v_1''$. We apply the induction hypothesis to G' and get an edge set F, $|F| \leq \lfloor \frac{n-4}{4} \rfloor$, such that G_F' has a nowhere-zero 3-flow. Finally, the nowhere-zero 3-flow of G_F' can be extended to a nowhere-zero 3-flow of $G_F + v_1v_3'$. If $v_1'' = v_3'$, let $G' \propto G - V(C) - v_1' - v_1''$. Let v_1''' be the third neighbor of v_1''. It is easy to see that the flow of G_F' can be extended to a nowhere-zero 3-flow of $G_F + v_1'''v_1$.

The remaining case for $k = 3$ is when v_1', v_2', v_3' are all distinct. Here we let $G' \propto G - V(C) + v_2'v_3'$. Again, G' is a loopless cubic graph on $n - 4$ vertices and the 3-flow of G_F' can be extended to $G_F + v_1'v_2$.

From now on we assume that $k \geq 4$. First we deal with the case when $v_i' = v_j'$, for a pair of distinct indices i, j, $1 \leq i < j \leq k$. We may assume that $i = 1$ and $j \leq \lceil \frac{k+1}{2} \rceil$. Consider the cycle $C' = v_1v_2 \cdots v_jv_1'v_1$. Since its length is $j + 1 \geq k$, we get $k = 4$ and $j = 3$. We have two subcases. First, suppose that also $v_2' = v_4'$. Let $G' \propto G - V(C) - v_1' - v_2'$ (if $v_1'' \neq v_2''$), and $G' \propto G - V(C) - v_1' - v_2' - v_1''$ (if $v_1'' = v_2''$), respectively. If $v_1'' = v_2'$, then also $v_2'' = v_1'$ and $G = K_{3,3}$. Since $K_{3,3}$ has a nowhere-zero 3-flow, we may assume that $v_1'' \neq v_2'$ and that G' is nonempty. It has $n - 8$ vertices. It is easy to see that G' has no loops (otherwise it would have a cycle of length ≤ 3). Now, $G_F + v_1''v_2' + v_1'v_2''$ and $G_F + v_1'''v_1' + v_1''v_1'$ (respectively) admits an extension of the flow of G_F' to a nowhere-zero 3-flow.

The second subcase is when $v_1' = v_3'$ but $v_2' \neq v_4'$. We may assume that $v_1'' \neq v_2'$. Let $G' \propto G - V(C) - v_1' + v_1''v_2'$. By the induction hypothesis, there is an edge set F, $|F| \leq \lfloor \frac{n-6}{4} \rfloor$, such that G_F' has a nowhere-zero 3-flow. This flow can be extended to a nowhere-zero 3-flow in $G_F + v_2v_4'$.

From now on we may assume that $v_i' \neq v_j'$ if $i \neq j$. If k is even, put $G' \propto G - V(C)$. If k is odd, let $G' \propto G - V(C) + v_1'v_2'$. Suppose that G' has a loop. A loop in G' corresponds to a cycle C' of G such that precisely one vertex of C' has degree 3 in $G - V(C)$ (or $G - V(C) + v_1'v_2'$), and other vertices of C' have degree 2. Since C' has length $\geq k$, it contains a path P' of length $k - 2$ such that $V(P') \subseteq \{v_1', \ldots, v_k'\}$.

Suppose that $v_i'v_j' \in E(P')$. Then $v_iv_i'v_j'v_j$ and a segment of C form a cycle in G of length $\le \frac{k}{2}+3$. This implies that $\frac{k}{2}+3 \ge k$, i.e. $k \le 6$. If $k = 6$, then $i = j \pm 3$, so P' cannot exist. Similarly, if $k = 5$, then $P' = v'_i v'_{i+2} v'_{i+4} v'_{i+6}$ (indices modulo 5). In particular, $V(P')$ contains either v'_1 or v'_2. A contradiction, since v'_1 and v'_2 have degree 3 in $G - V(C) + v'_1 v'_2$. The remaining possibility is when $k = 4$. In that case, we let $G'' \propto G - V(C) + v'_1 v'_2 + v'_3 v'_4$ and apply the induction hypothesis on G''. The resulting nowhere-zero 3-flow in G''_F either gives rise to a nowhere-zero 3-flow in G_F or in $G_F + v_1 v_3$.

Now, we return to the general case where we may assume that G' is loopless. Observe that $n - |V(G')| = 4\lfloor \frac{k}{2} \rfloor$. So, after applying the induction hypothesis to G', we may add further $\lfloor \frac{k}{2} \rfloor$ edges to G_F in order to get a graph with a nowhere-zero 3-flow. If k is even, we add the edges $v'_1 v'_2, v'_3 v'_4, \ldots, v'_{k-1} v'_k$. If k is odd we add the edges $v'_3 v'_4, \ldots, v'_{k-2} v'_{k-1}$, and $v'_k v_1$. In both cases, it is easy to see that a nowhere-zero 3-flow of G'_F gives rise to a nowhere-zero 3-flow in G_F with the additional $\lfloor \frac{k}{2} \rfloor$ edges. □

By Lemma 1.1 and Theorem 2.1 we obtain the following result.

Corollary 2.2 *Let G be a 2-edge-connected multigraph with $o = o(G)$ odd vertices. Then we can add $\le \lfloor \frac{o}{4} \rfloor$ edges such that the new graph G admits a nowhere-zero 3-flow.*

3 4-flows

The next lemma known as Parity Lemma is due to Blanuša [2].

Lemma 3.1 (Parity Lemma) *Let G be a cubic graph and let $c : E(G) \to \{1, 2, 3\}$ be an edge-coloring of G. If a cutset T consists of n edges such that n_i edges of T are colored i $(i = 1, 2, 3)$, then*

$$n_1 \equiv n_2 \equiv n_3 \equiv n \pmod 2.$$

A *minimal 4-coloring* of a cubic graph G is an edge-coloring $c : E(G) \to \{1, 2, 3, 4\}$ such that $|c^{-1}(4)|$ is minimum. Let G be a cubic graph and let $c : E(G) \to \{1, 2, 3, 4\}$ be a minimal 4-coloring of G. Denote by E_i the set of all edges of color 4 which are incident with precisely two edges of color i. Since c is minimal, it is easy to see that $\{E_1, E_2, E_3\}$ is a partition of $c^{-1}(4)$.

The following lemma is a well known consequence of the Parity Lemma (see, e.g., [7]). For the sake of completeness, we include its proof.

Lemma 3.2 *Let c be a minimal 4-coloring of a cubic graph G. Then $|E_1| \equiv |E_2| \equiv |E_3|$ (mod 2).*

Proof. Delete from G the edges colored 4. Let G_1 and G_2 be two disjoint copies of this graph. Add an edge between every vertex from G_1 which is of degree two and the corresponding vertex from G_2. Finally, color each such edge with the free color $1, 2$, or 3. We obtain a cubic graph with an edge 3-coloring. There is a cutset of order $2(|E_1|+|E_2|+|E_3|)$ between G_1 and G_2. In this cutset, precisely $|E_{i+1}|+|E_{i+2}|$ (indices modulo 3) edges are colored i for $i = 1, 2, 3$. By Lemma 3.1, $|E_{i+1}| + |E_{i+2}| \equiv 2(|E_1| + |E_2| + |E_3|) \equiv 0$ (mod 2). It follows that $|E_{i+1}| \equiv |E_{i+2}| \pmod 2$. This completes the proof. □

Proposition 3.3 *Let G be a connected simple cubic graph of order n, and let c be a minimal 4-coloring of G. Then $|c^{-1}(4)| \leq \frac{1}{5}n$.*

Proof. Let $c' : E(G) \to \{1, 2, 3\}$ be a 3-coloring of G, which colors as many edges of G as possible. If c' cannot be extended to a 4-edge-coloring of G, then we have two incident uncolored edges, say vu and vw. Let the third neighbor of v be z. We may assume that $c'(vz) = 3$. Then both colors 1 and 2 are already used at the edges incident with u, and the same holds at w. Let P be the maximal path which contains the edge vz and whose edges are colored by colors 1 and 3. Note that the other endvertex of this path could be u or w. Now, change the color of every edge on P from 1 to 3, and vice versa. It is not hard to see that we can extend the resulting partial edge coloring of G to vu or vw, a contradiction.

So, c' can be extended to a 4-edge-coloring $\bar{c}$ of G. In particular, $\bar{c}$ is a minimal 4-coloring of G and $|c^{-1}(\{1, 2, 3\})| = |c'^{-1}(\{1, 2, 3\})|$. Albertson and Haas [1] proved that such a coloring colors at least $\frac{13}{15}$ of the edges of G. Since c' colors at least $\frac{13}{15}$ of the edges of G, $|c^{-1}(4)| \leq \frac{2}{15}|E(G)| = \frac{1}{5}n$. □

Theorem 3.4 *Let G be a 2-edge-connected graph with $o = o(G)$ odd vertices. Then we can add $\leq \lceil \frac{1}{2} \lfloor \frac{o}{5} \rfloor \rceil$ edges such that the new graph admits a nowhere-zero 4-flow.*

Proof. Suppose that the claim is false and G is a counterexample with $|E(G)| + |V(G)|$ as small as possible. Let $n = |V(G)|$.

We claim that G is a simple cubic graph. Since G is 2-edge-connected, there are no vertices of degree 1. It is easy to see that G has no vertices of degree 0 or 2. Otherwise, we obtain a smaller counterexample. Suppose now that v is a vertex in G of degree ≥ 4. By the Splitting Lemma, we can split this vertex such that the resulting graph is 2-edge-connected. Note that this graph has one or two vertices of degree 2. Let G^* be the graph obtained by suppressing the vertices of degree 2. Then, $|E(G^*)| + |V(G^*)| < |E(G)| + |V(G)|$ and $o(G) = o(G^*)$. It is easy to see that if we can add at most $\lceil \frac{1}{2} \lfloor \frac{o}{5} \rfloor \rceil$ edges to G^* in order to obtain a graph which admits a nowhere-zero 4-flow, then we can do it also in G. So, G^* contradicts the minimality of G. This shows that G is a cubic graph. Since G is 2-edge-connected, it has no loops. If it contains a double edge joining vertices u, v, we delete one of these edges and obtain a smaller counterexample. This completes the proof of the claim.

Since G is a cubic graph, $n = o$. Let c be a minimal 4-coloring of G. By Lemma 3.2, $|E_1| \equiv |E_2| \equiv |E_3| \pmod 2$. By Proposition 3.3, $|E_1| + |E_2| + |E_3| \leq \lfloor \frac{n}{5} \rfloor$.

Suppose first that the sets E_i are of even cardinality. Partition each E_i into pairs. Consider one of such pairs, $e_1 = u_1v_1 \in E_i$ and $e_2 = u_2v_2 \in E_i$, where the edges incident with u_j are colored i and $i+1$ (modulo 3), $j = 1, 2$. Then, we add the edge u_1u_2 and color it by color i. Recolor the edges e_1 and e_2 by color $i + 1$. We repeat the same procedure for all selected pairs. If we interpret colors 1, 2, 3 as the nonzero elements of $\mathbb{Z}_2 \times \mathbb{Z}_2$, we see that we constructed a graph with a nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2$-flow. We have added $\frac{1}{2}(|E_1| + |E_2| + |E_3|) \leq \frac{1}{2} \lfloor \frac{o}{5} \rfloor$ edges.

If E_1, E_2, E_3 have odd cardinalities, then we do the same procedure with pairs. At the end, we are left with three edges $e_i = u_i v_i \in E_i$ ($i = 1, 2, 3$). We may assume that edges incident with u_i are colored i and $i+1$ (modulo 3). So the colors at v_i are i and $i-1$ (modulo 3). Add two edges $v_1 v_2$ and $u_2 u_3$. Now, we color the edges e_3 and $v_1 v_2$ by color 1, the edge e_2 with 2, and color e_1 and $u_2 u_3$ by color 3. As above, we see that we thus constructed a graph with a nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2$-flow. The number of added edges is $\lceil \frac{|E_1|+|E_2|+|E_3|}{2} \rceil \leq \lceil \frac{1}{2} \lfloor \frac{o}{5} \rfloor \rceil$. □

4 5-flows

Theorem 4.1 *Let G be a 2-edge-connected graph with $o = o(G)$ odd vertices. Then we can add $\lceil \frac{1}{2} \lfloor \frac{o}{7} \rfloor \rceil$ or fewer edges such that the new graph admits a nowhere-zero 5-flow.*

Proof. Suppose that the claim is false and G is a counterexample with $|E(G)|+|V(G)|$ as small as possible. Let $n = |V(G)|$. By the similar arguing as in the proof of Theorem 3.4, we may assume that G is a simple cubic graph.

Now, we will prove that G is of girth ≥ 6. Let $C = x_0 x_1 \cdots x_{r-1} x_0$ be a cycle of G with r minimum. Suppose that $r \leq 5$. Let us contract the edges $x_{2i} x_{2i+1}$ for $i = 0, \ldots, \lfloor \frac{r}{2} \rfloor - 1$. If $r \leq 4$, let G' denote the resulting graph. Suppose now that $r = 5$. Then we first apply the Splitting Lemma at both new vertices of degree 4 such that $e_1 = x_0 x_4$ (resp. $e_1 = x_3 x_4$) and such that e_0 corresponds to the edge of $G - E(C)$ incident with x_0 (resp x_3). Denote the resulting graph by G'. Since G' is 2-edge-connected, there are only two possibilities (up to the obvious left-right symmetry) for the structure of G' locally at the vertices of C. See Figure 1(a) and (b).

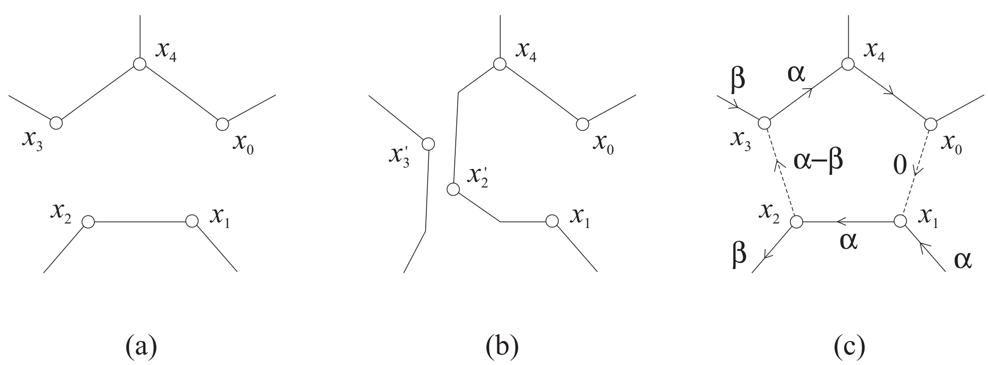

Figure 1: The two possibilities for G' when $r = 5$.

By the minimality of G, we can prove that G' admits a nowhere-zero 5-flow by adding a set F of at most $\lceil \frac{1}{2} \lfloor \frac{o-r}{7} \rfloor \rceil$ edges. Equivalently, G'_F admits a nowhere-zero $\mathbb{Z}_5$-flow (D', ϕ'), $\phi' : E(G') \cup F \to \mathbb{Z}_5$. If $r \leq 4$, then ϕ' determines a $\mathbb{Z}_5$-flow (D, ϕ_1) of G_F which agrees with (D', ϕ') on $E(G') \cup F$. Note that ϕ_1 is nonzero on $(E(G) \cup F) \setminus E(C)$. If $r = 5$, then we claim that ϕ' determines a $\mathbb{Z}_5$-flow (D, ϕ_1) of G_F which agrees with (D', ϕ') on

$(E(G) \cap E(G')) \cup F$, which is nonzero on $(E(G) \cup F) \setminus E(C)$, and such that ϕ_1 takes at most four distinct values on $E(C)$, where all edges of C are assumed to be oriented "clockwise". In the first case of Figure 1, the claim is obvious. We just set $\phi_1(e) = \phi'(e)$ for $e \in E(G') \cup F$ and set $\phi_1(x_0x_1) = \phi_1(x_2x_3) = 0$. In the second case of Figure 1, we consider the edges x_1x_2', $x_2'x_4$, and the two edges incident with x_3' as being the edges x_1x_2, x_3x_4, and edges incident with x_2 and x_3, respectively, as indicated in Figure 1(c). The flow condition may be violated at x_2 and x_3 but there is a unique value for $\phi_1(x_2x_3)$ such that we get a flow. (Also, we set $\phi_1(x_1x_5) = 0$.) All vertices of C except x_4 give rise to vertices of degree 2 in G'. Therefore, no edges of F are incident with them. This implies that the ϕ_1-flow on the edges x_1x_2 and x_3x_4 is the same as the ϕ'-flow through the vertex x_2' of G'. Consequently, ϕ_1 takes at most four distinct values on $E(C)$.

Returning to the general case $r \leq 5$, let $i \in \mathbb{Z}_5$ be a value which does not occur as a ϕ_1-value on $E(C)$. Recall that the orientation D orients C clockwise. So, there is a $\mathbb{Z}_5$-flow (D, ϕ_2) of G_F which is 0 on $(E(G) \setminus E(C)) \cup F$ and such that $\phi_2(e) = i$ for edges on C. Now, $(D, \phi_1 - \phi_2)$ is a nowhere-zero $\mathbb{Z}_5$-flow of G_F. This contradiction shows that $r \geq 6$.

Since G is a 2-edge-connected cubic graph, it has a 2-factor Q by the Petersen theorem. Since every cycle in Q is of length ≥ 6, we can color at least $\frac{6}{7}$ of the edges of Q using colors 1 and 2. Color every edge of the 1-factor $E(G) - E(Q)$ by color 3. Thus, we have a partial 3-edge-coloring of G, which colors at least $\frac{19}{21}$ of the edges of G. So, the number of uncolored edges is $\leq \lfloor \frac{2}{21}|E(G)| \rfloor = \lfloor \frac{o}{7} \rfloor$.

In a similar way as in Theorem 3.4, we can add at most $\lceil \frac{1}{2} \lfloor \frac{o}{7} \rfloor \rceil$ edges to G in order to a obtain a graph which admits a nowhere-zero 5-flow. (In fact, we even get a nowhere-zero 4-flow in this case.)

□

5 Concluding remarks

We will conclude the paper with the following remarks. First, in all results of the paper, we are restricted to 2-edge-connected graphs. It is not hard to construct graphs with cutedges for which bounds from the theorems are not valid.

Another obvious question is: "How good is the bound of Theorem 4.1." The 5-Flow-Conjecture of Tutte [9] namely asserts that $\phi_5^+ = 0$ for every 2-edge-connected graph. The following proposition answers this question.

Proposition 5.1 *Let $k \in \{3, 4, 5\}$. For every integer s there is a 2-edge-connected graph G with $o(G) \geq s$ such that*

(a) If $k = 3$, then $\phi_k^+(G) \geq \frac{1}{8}o(G)$.

(b) If $k = 4$, then $\phi_k^+(G) \geq \frac{1}{20}o(G)$.

(c) If $k = 5$, and the 5-Flow-Conjecture is false, then there is a constant $c > 0$ such that $\phi_k^+(G) \geq c \cdot o(G)$.

Proof. Let G be a 2-edge-connected graph without a nowhere-zero k-flow. Let $e = uv$ be an edge of G_0. Take s copies of $G_0 - e$ and form the graph G by joining the copy v_i of v in the i^{th} copy of $G_0 - e$ with the copy u_{i+1} of u in the $(i+1)^{\text{st}}$ copy of $G_0 - e$, $i = 1, 2, \ldots, s$ (indices modulo s). Then G is 2-edge-connected and $o(G) \geq s$. If $\phi_k^+(G) < \frac{s}{2}$, then there is an edge set F such that G_F has a nowhere-zero k-flow and there is a copy of $G_0 - e$ such that no edge of F is incident with its vertices. Then it is easy to see that the flow of G_F gives rise to a nowhere-zero k-flow of G_0, a contradiction. This shows that

$$\phi_k^+(G) \geq \frac{s}{2} \geq \frac{1}{2|V(G_0)|}\, o(G). \tag{1}$$

Finally, let $G_0 = K_4$ if $k = 3$, let G_0 be the Petersen graph if $k = 4$, and let G_0 be a hypothetical counterexample to the Tutte 5-Flow-Conjecture if $k = 5$. Then (1) implies the proposition. □

Let $k \in \{2, 3, 4, 5\}$. One can ask how hard it is to calculate $\phi_k^+(G)$ and $\phi_k^-(G)$ for a given graph G. As we already said, $\phi_2^+(G) = \frac{o(G)}{2}$. Calculating $\phi_2^-(G)$ is equivalent to finding a Chinese postman tour in G (see Lemma 8.1.4 in [10]). Edmonds and Johnson [3] proved that the Chinese postman problem is solvable by a polynomial time algorithm. The decision problem whether $\phi_4^+(G) = 0$ is an NP-complete problem. This follows by the fact that it is an NP-complete problem to decide whether a (cubic) graph is 3-edge-colorable.

The decision whether $\phi_5^+(G) = 0$ or not is either trivial (if the 5-Flow-Conjecture holds) or NP-complete, as proved by Kochol [5]. Similar conclusion holds for 3-flows, depending on the Tutte 3-Flow-Conjecture (cf. Kochol [5]).

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