Colouring 4-cycle systems with specified block colour patterns: the case of embedding P_3-designs *

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Abstract

A *colouring* of a 4-cycle system $(V, \mathcal{B})$ is a surjective mapping $\phi : V \rightarrow \Gamma$. The elements of Γ are *colours*. If $|\Gamma| = m$, we have an *m-colouring* of $(V, \mathcal{B})$. For every $B \in \mathcal{B}$, let $\phi(B) = \{\phi(x) | x \in B\}$. There are seven distinct colouring patterns in which a 4-cycle can be coloured: type a ($\times\times\times\times$, monochromatic), type b ($\times\times\times\Box$, two-coloured of pattern $3 + 1$), type c ($\times\times\Box\Box$, two-coloured of pattern $2 + 2$), type d ($\times\Box\times\Box$, mixed two-coloured), type e ($\times\times\Box\triangle$, three-coloured of pattern $2 + 1 + 1$), type f ($\times\Box\times\triangle$, mixed three-coloured), type g ($\times\Box\triangle\diamondsuit$, four-coloured or polychromatic).

Let S be a subset of $\{a, b, c, d, e, f, g\}$. An m-colouring ϕ of $(V, \mathcal{B})$ is said *of type* S if the type of every 4-cycle of $\mathcal{B}$ is in S. A type S colouring is said to be *proper* if for every type $\alpha \in S$ there is at least one 4-cycle of $\mathcal{B}$ having colour type α.

We say that a $P(v, 3, 1)$, $(W, \mathcal{P})$, *is embedded* in a 4-cycle system of order n, $(V, \mathcal{B})$, if every path $p = [a_1, a_2, a_3] \in \mathcal{P}$ occurs in a 4-cycle $(a_1, a_2, a_3, x) \in \mathcal{B}$ such that $x \notin W$.

In this paper we consider the following spectrum problem: given an integer m and a set $S \subseteq \{b, d, f\}$, determine the set of integers n such that there exists a 4-cycle system of order n with a proper m-colouring of type S (note that each colour class of a such colouration is the point set of a P_3-design *embedded* in the 4-cycle system).

We give a complete answer to the above problem except when $S = \{b\}$. In this case the problem is completely solved only for $m = 2$.

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1 Introduction

Let G be a subgraph of K_v, the complete undirected graph on v vertices. A *G-design* of K_v is a pair $(V, \mathcal{B})$, where V is the vertex set of K_v and $\mathcal{B}$ is an edge-disjoint decomposition of K_v into copies of the graph G. Usually we say that B is a block of the G-design if $B \in \mathcal{B}$, and $\mathcal{B}$ is called the block-set.

A *path design* $P(v, k, 1)$ [4] is a P_k-design of K_v, where P_k is the simple path with $k-1$ edges (k vertices) $[a_1, a_2, \ldots, a_k] = \{\{a_1, a_2\}, \{a_2, a_3\}, \ldots, \{a_{k-1}, a_k\}\}$.

M. Tarsi [11] proved that the necessary conditions for the existence of a $P(v, k, 1)$, $v \geq k$ (if $v > 1$) and $v(v-1) \equiv 0 \pmod{2(k-1)}$, are also sufficient. Therefore a $P(v, 3, 1)$ exists if and only if $v \equiv 0$ or $1 \pmod 4$.

An *m-cycle system* of order n is a C_m-design of K_n, where C_m is the m-cycle (cycle of length m) $(a_1, a_2, \ldots, a_m) = \{\{a_1, a_2\}, \{a_2, a_3\}, \ldots, \{a_{m-1}, a_m\}, \{a_1, a_m\}\}$.

It is well-known that the spectrum for 4-cycle system is precisely the set of all $n \equiv 1 \pmod 8$ (see for example [5]).

We say that a $P(v, 3, 1)$, $(\Omega, \mathcal{P})$, *is embedded* in a 4-cycle system of order n, $(W, \mathcal{C})$, if every path $p = [a_1, a_2, a_3] \in \mathcal{P}$ occurs in a 4-cycle $(a_1, a_2, a_3, x) \in \mathcal{C}$ such that $x \notin \Omega$, see [9].

Example 1. Let $\Omega_1 = \{a_0, a_1, a_2, a_3\}$, $W_1 = \Omega_1 \cup \{b_0, b_1, b_2, b_3, b_4\}$, $\mathcal{P}_1 = \{[a_0, a_1, a_2],$ $[a_0, a_3, a_1]$, $[a_0, a_2, a_3]\}$, $\mathcal{S}_1 = \{(a_0, a_1, a_2, b_0)$, (a_0, a_3, a_1, b_1), (a_0, a_2, a_3, b_2), (a_0, b_4, b_0, b_3), (a_1, b_0, a_3, b_3), (a_2, b_1, b_0, b_2), (a_2, b_4, b_2, b_3), (a_3, b_1, b_3, b_4), $(a_1, b_4, b_1, b_2)\}$. It is easy to see that $(\Omega_1, \mathcal{P}_1)$ is a $P(4, 3, 1)$ *embedded* in the 4-cycle system $(W_1, \mathcal{S}_1)$ of order 9.

A *colouring* of a G-design $(V, \mathcal{B})$ is a surjective mapping $\phi : V \to \Gamma$. The elements of Γ are *colours*. If $|\Gamma| = m$, we have an *m-colouring* of $(V, \mathcal{B})$. For each $c \in \Gamma$, the set $\phi^{-1}(c) = \{x \ : \ \phi(x) = c\}$ is a *colour class*. A colouring ϕ of $(V, \mathcal{B})$ is *weak* (*strong*) if for all $B \in \mathcal{B}$, $|\phi(B)| > 1$ ($|\phi(B)| = k$, where k is the number of vertices of the subgraph G, respectively), where $\phi(B) = \{\phi(x) | x \in B\}$. In a weak colouring, no block is monochromatic (i.e., no block has all its elements of the same colour), while in a strong colouring, the elements of every block B get $|B|$ distinct colours. There exists an extensive literature on subject of colourings (for a survey, see [2]). Most of the existing papers are devoted to the case of *weak* colourings. However, recently other types of colouring started to be investigated, mainly in connection with the notion of the upper chromatic number of a hypergraph [12] (see, e.g., [1], [6], [7]). Most of them satisfy the inequalities $1 < |\phi(B)| < k$, i.e. are strict colourings in the sense of Voloshin [12] in which the blocks are both edges and co-edges. A step further is given by Milici, Rosa and Voloshin [8] where the authors consider some types of colouring of $S(2, 3, v)$ and $S(2, 4, v)$ (K_3-designs and K_4-designs in our terminology) in which only specified block colouring patterns are allowed. In this paper we want to consider strict colouring in the sense of Voloshin of 4-cycle systems in which only specified block colouring patterns are allowed.

There are seven distinct colouring patterns in which a 4-cycle can be coloured: type

a ($\times\times\times\times$, monochromatic), type b ($\times\times\times\square$, two-coloured of pattern $3+1$), type c ($\times\times\square\square$, two-coloured of pattern $2+2$), type d ($\times\square\times\square$, mixed two-coloured), type e ($\times\times\square\triangle$, three-coloured of pattern $2+1+1$), type f ($\times\square\times\triangle$, mixed three-coloured), type g ($\times\square\triangle\diamondsuit$, four-coloured or polychromatic).

Let S be a subset of $\{a,b,c,d,e,f,g\}$ and let $(V,\mathcal{B})$ be a 4-cycle system. An m-colouring ϕ of $(V,\mathcal{B})$ is said *of type* S if the type of every 4-cycle of $\mathcal{B}$ is in S.

A type S colouring is said to be *proper* if for every type $\alpha \in S$ there is at least one 4-cycle of $\mathcal{B}$ having colour type α.

Since we are looking for 4-cycle systems having a proper strict colouring in the sense of Voloshin in which the blocks are both edges and co-edges, it is $a, g \notin S$. There are 31 distinct nonempty subsets S of $\{b,c,d,e,f\}$. Then 31 distinct types of strict colourings of a 4-cycle system are possible. We deal here with some of these types; it is hoped that the remaining types will be dealt with in a future paper by the author. More precisely we are looking for proper strict colouring of a 4-cycle system having the property that each colour class is the point set of a P_3-design *embedded* into the given cycle system [9]. In other words, we consider the following spectrum problem: given an integer m and a set $S \subseteq \{b,d,f\}$, determine the set of integers n such that there exist a 4-cycle system of order n having an m-colouring of type S. It is clear that a such colouring must contain b. [Here and in what follows, all braces and commas are omitted for the sake of brevity.] For types *bdf*, *bf* and *bd*, a complete answer is obtained. The spectrum problem for type b colouring seems to be the most interesting but also very difficult (at least for the author). In this paper only the case $m=2$ is completely settled. Remark that the analogous problem for 3-cycle systems (or Steiner triple systems) is also very hard. This problem has been considered and partially solved by Colbourn, Dinitz and Rosa [1] and Dinitz and Stinson [3].

2 Colouring of type *bdf* and *bf*

It is trivial to see that the necessary condition for the existence of an m-colouring of type *bdf* of a 4-cycle system of order n is $m \in \{2,3,\dots,\frac{n+3}{4}\}$. In this section we will prove the sufficiency.

Lemma 2.1 (D. Sotteau [10]). *The complete bipartite graph $K_{X,Y}$ can be decomposed into edge disjoint cycles of length $2k$ if and only if (1) $|X|=x$ and $|Y|=y$ are even, (2) $x \geq k$ and $y \geq k$, and (3) $2k$ divides xy.*

Theorem 2.1 *For every $n \equiv 1 \pmod 8$, $n \geq 9$, there is a 4-cycle system of order n with a proper $(\frac{n+3}{4})$-colouring of type bdf.*

Proof. Put $n = 1+8k$, $k \geq 1$. Let $\Omega_i = \{x_0^i, x_1^i, x_2^i, x_3^i\}$, $i = 0,1,\dots,2k-1$, and $\Omega_{2k} = \{\infty\}$ be the colour classes. Define the following set $\mathcal{B}$ of 4-cycles.

(I) For $j = 0,1,\dots,k-1$, put in $\mathcal{B}$ the cycles of a proper type *bdf* 3-coloured 4-cycle system on point set $\Omega_{2k} \cup \Omega_{2j} \cup \Omega_{2j+1}$:

$(x_0^{2j}, x_1^{2j}, x_2^{2j}, x_0^{2j+1})$, $(x_0^{2j}, x_2^{2j}, x_3^{2j}, x_1^{2j+1})$, $(x_0^{2j}, x_3^{2j}, x_1^{2j}, \infty)$, $(x_0^{2j+1}, x_1^{2j+1}, x_2^{2j+1}, x_1^{2j})$, $(x_0^{2j+1}, x_2^{2j+1}, x_3^{2j+1}, x_3^{2j})$, $(x_0^{2j+1}, x_3^{2j+1}, x_1^{2j+1}, \infty)$, $(x_1^{2j}, x_3^{2j+1}, x_2^{2j}, x_1^{2j+1})$, $(x_2^{2j}, \infty, x_3^{2j}, x_2^{2j+1})$, $(x_2^{2j+1}, \infty, x_3^{2j+1}, x_0^{2j})$

(II) For $j, t = 0, 1, \ldots, k-1$, $j < t$, and $\alpha = 0, 1$, put in $\mathcal{B}$ the cycles: $(x_0^{2j+\alpha}, x_0^{2t}, x_1^{2j+\alpha}, x_0^{2t+1})$, $(x_2^{2j+\alpha}, x_0^{2t}, x_3^{2j+\alpha}, x_0^{2t+1})$, $(x_0^{2j+\alpha}, x_1^{2t}, x_1^{2j+\alpha}, x_1^{2t+1})$, $(x_2^{2j+\alpha}, x_1^{2t}, x_3^{2j+\alpha}, x_1^{2t+1})$, $(x_0^{2j+\alpha}, x_2^{2t}, x_1^{2j+\alpha}, x_2^{2t+1})$, $(x_2^{2j+\alpha}, x_2^{2t}, x_3^{2j+\alpha}, x_2^{2t+1})$, $(x_0^{2j+\alpha}, x_3^{2t}, x_1^{2j+\alpha}, x_3^{2t+1})$, $(x_2^{2j+\alpha}, x_3^{2t}, x_3^{2j+\alpha}, x_3^{2t+1})$.

Let $V = \cup_{i=1}^{2k} \Omega_i$, then $(V, \mathcal{B})$ is the required $2k+1$-coloured 4-cycle system of order $n = 8k+1$. $\square$

Lemma 2.2 *For every $n \equiv 1 \pmod 8$, $n \geq 9$, there is a 4-cycle system of order n with a proper 2-colouring of type bd.*

Proof. Put $n = 1 + 8k$, $k \geq 1$. Let $\Omega_1 = \cup_{i=0}^{k-1} \{x_0^i, x_1^i, x_2^i, x_3^i\}$ and $\Omega_2 = \{\infty\} \cup (\cup_{i=0}^{k-1} \{y_0^i, y_1^i, y_2^i, y_3^i\})$ be the colour classes. Define the following set $\mathcal{B}$ of 4-cycles.

(I) For $i = 0, 1, \ldots, k-1$, put in $\mathcal{B}$ the cycles $(x_0^i, x_1^i, x_2^i, y_0^i)$, $(x_0^i, x_3^i, x_1^i, y_1^i)$, $(x_0^i, x_2^i, x_3^i, y_2^i)$, $(y_0^i, y_1^i, y_3^i, x_3^i)$, $(y_1^i, y_2^i, \infty, x_3^i)$, $(y_2^i, y_3^i, y_0^i, x_1^i)$, $(y_3^i, \infty, y_1^i, x_2^i)$ and $(\infty, y_0^i, y_2^i, x_2^i)$.

(II) If $k \geq 2$, then for $i = 0, 1, \ldots, k-2$ and $j = i+1, i+2, \ldots, k-1$ put in $\mathcal{B}$ the cycles $(x_0^i, x_0^j, x_1^i, y_2^j)$, $(x_0^i, x_1^j, x_1^i, y_3^j)$, $(x_2^i, x_2^j, x_3^i, y_0^j)$, $(x_2^i, x_3^j, x_3^i, y_1^j)$, $(x_0^j, x_2^i, x_1^j, y_2^i)$, $(x_0^j, x_3^i, x_1^j, y_3^i)$, $(x_2^j, x_0^i, x_3^j, y_0^i)$, $(x_2^j, x_1^i, x_3^j, y_1^i)$, $(y_0^i, y_0^j, y_1^i, x_0^j)$, $(y_0^i, y_1^j, y_1^i, x_1^j)$, $(y_2^i, y_2^j, y_3^i, x_2^j)$, $(y_2^i, y_3^j, y_3^i, x_3^j)$, $(y_0^j, y_2^i, y_1^j, x_0^i)$, $(y_0^j, y_3^i, y_1^j, x_1^i)$, $(y_2^j, y_0^i, y_3^j, x_2^i)$ and $(y_2^j, y_1^i, y_3^j, x_3^i)$.

(III) For $i = 0, 1, \ldots, k-1$, put in $\mathcal{B}$ the cycles $(x_0^i, y_3^i, x_1^i, \infty)$.

Let $V = \Omega_1 \cup \Omega_2$, then $(V, \mathcal{B})$ is the required 2-coloured 4-cycle system of order n. Note that the cycles of colour type b are those given in (I) and (II). $\square$

Lemma 2.3 *If there is a 4-cycle system $(W, \mathcal{D})$ of order n having a proper m-colouring of type S, $S \subseteq \{bd, bdf\}$, then there is a 4-cycle system $(V, \mathcal{B})$ of order $n+8$ having a proper $(m+1)$-colouring of type bdf.*

Proof. Put $n = 1 + 8k$, $k \geq 1$. Let $W = \{0, 1, \ldots, 8k\}$. Suppose that the points 1 and 2 have different colours. Put $X = \{x_0, x_1, \ldots, x_7\}$ and $V = W \cup X$. Put in $\mathcal{B}$ the cycles of $\mathcal{D}$ and the following ones.

(I) The following 4-cycles cover the edges of both K_X and $K_{X,\{0,1,\ldots,6\}}$: $(x_0, x_1, x_3, 6)$, $(x_1, x_2, x_4, 5)$, $(x_2, x_3, x_5, 1)$, $(x_3, x_4, x_6, 2)$, $(x_4, x_5, x_0, 3)$, $(x_5, x_6, x_1, 4)$, $(x_6, x_0, x_2, 5)$, $(x_0, x_3, x_7, 0)$, $(x_1, x_4, x_7, 1)$, $(x_2, x_5, x_7, 2)$, $(x_3, x_6, x_7, 3)$, $(x_4, x_0, x_7, 4)$, $(x_5, x_1, x_7, 5)$,

$(x_6, x_2, x_7, 6)$, $(1, x_0, 2, x_4)$, $(4, x_0, 5, x_3)$, $(0, x_3, 1, x_6)$, $(3, x_2, 4, x_6)$, $(0, x_2, 6, x_5)$, $(2, x_1, 3, x_5)$ and $(0, x_1, 6, x_4)$.

(II) By Lemma 2.1 decompose the complete bipartite graph $K_{X,\{7,8,\ldots,2k\}}$ into edge disjoint 4-cycles.

Clearly $(V, \mathcal{B})$ is a 4-cycle system of order $9 + 8k$. Colour the elements of X with a new colour. □

Theorem 2.2 *For every $n \equiv 1 \pmod 8$, $n \geq 9$, and for every $m \in \{3, 4, \ldots, \frac{n+3}{4}\}$ there is a 4-cycle system of order n with a proper m-colouring of type bdf.*

Proof. Starting from a proper $m - coloured$ 4-cycle system of order 9 and type S, $S \subseteq \{bd, bdf\}$, and using repeatedly Lemmas 2.2 and 2.3, we get the proof. □

Theorem 2.3 *For every $n \equiv 1 \pmod 8$, $n \geq 9$, there is a 4-cycle system of order n with a proper 3-colouring of type bf.*

Proof. Put $n = 1 + 8k$, $k \geq 1$. Let $\Omega_1 = \{\infty\}$, $\Omega_2 = \cup_{i=0}^{k-1}\{x_0^i, x_1^i, x_2^i, x_3^i\}$ and $\Omega_3 = \cup_{i=0}^{k-1}\{y_0^i, y_1^i, y_2^i, y_3^i\}$ be the colour classes. Let $\mathcal{B}$ be the set of 4-cycles constructed using Lemma 2.2. Remove from $\mathcal{B}$ the 4-cycles $(y_0^i, y_1^i, y_3^i, x_3^i)$, $(y_1^i, y_2^i, \infty, x_3^i)$, $(y_3^i, \infty, y_1^i, x_2^i)$, $(\infty, y_0^i, y_2^i, x_2^i)$, and put on it the following ones $(y_0^i, y_1^i, y_3^i, \infty)$, $(y_1^i, x_2^i, y_2^i, \infty)$, $(y_0^i, y_2^i, y_1^i, x_3^i)$, $(y_3^i, x_2^i, \infty, x_3^i)$. Let $V = \Omega_1 \cup \Omega_2 \cup \Omega_3$, then $(V, \mathcal{B})$ is the required 3-coloured 4-cycle system of order n. □

Theorem 2.4 *For every $n \equiv 1 \pmod 8$, $n \geq 9$, there is a 4-cycle system of order n with a proper $(\frac{n+3}{4})$-colouring of type bf.*

Proof. Put $n = 1 + 8k$, $k \geq 1$. Let $\Omega_i = \{x_0^i, x_1^i, x_2^i, x_3^i\}$, $i = 0, 1, \ldots, 2k - 1$, and $\Omega_{2k} = \{\infty\}$ be the colour classes. Define the set $\mathcal{B}$ of 4-cycles by putting on it the cycles (II) of Theorem 2.1 and the following ones.

For $j = 0, 1, \ldots, k - 1$, put in $\mathcal{B}$ the cycles of a proper type bf 3-coloured 4-cycle system on point set $\Omega_{2k} \cup \Omega_{2j} \cup \Omega_{2j+1}$: $(x_0^{2j}, x_1^{2j}, x_2^{2j}, x_0^{2j+1})$, $(x_0^{2j}, x_2^{2j}, x_3^{2j}, x_2^{2j+1})$, $(x_0^{2j}, x_3^{2j}, x_1^{2j}, x_3^{2j+1})$, $(x_0^{2j+1}, x_1^{2j+1}, x_2^{2j+1}, \infty)$, $(x_0^{2j+1}, x_2^{2j+1}, x_3^{2j+1}, x_3^{2j})$, $(x_0^{2j+1}, x_3^{2j+1}, x_1^{2j+1}, x_1^{2j})$, $(x_0^{2j}, \infty, x_3^{2j}, x_1^{2j+1})$, $(x_2^{2j}, \infty, x_1^{2j}, x_2^{2j+1})$, $(x_3^{2j+1}, \infty, x_1^{2j+1}, x_2^{2j})$.

Let $V = \cup_{i=1}^{2k}\Omega_i$, then $(V, \mathcal{B})$ is the required $2k + 1$-coloured 4-cycle system of order $n = 8k + 1$. □

Lemma 2.4 *Suppose there is a type bf m-coloured 4-cycle system of order $n = 1 + 8k$, $(W, \mathcal{D})$, whose colour classes Ω_i, $i = 1, 2, \ldots, m$, have the following cardinalities:*
(1) If $3 \leq m \leq k + 2$, then $|\Omega_1| = 1$, $|\Omega_2| = |\Omega_3| = 4k - 4(m - 3)$, and (if $m \geq 4$) $|\Omega_4| = |\Omega_5| = \ldots = |\Omega_m| = 8$.

(2) If $k+3 \leq m \leq 2k+1$, then $|\Omega_1| = 1$, $|\Omega_2| = |\Omega_3| = \ldots = |\Omega_{2m-2k-1}| = 4$, and (if $m \leq 2k$) $|\Omega_{2m-2k}| = |\Omega_{2m-2k+1}| = \ldots = |\Omega_m| = 8$.
Then there is a type bf $(m+1)$-coloured 4-cycle system of order $9+8k$.

Proof. Put $W = \{0, 1, \ldots, 8k\}$, $X = \{x_0, x_1, \ldots, x_7\}$ and $V = W \cup X$. We now construct a $(m+1)$-coloured 4-cycle system of order $9+8k$, $(V, \mathcal{B})$. Let $\Omega_1 = \{6\}$, $0, 2, 4 \in \Omega_t$ and $1, 3, 5 \in \Omega_{t+1}$, where either $t = 2$ for odd m or $t = m-1$ for even m. Then it is easy to see that it is possible to partition the set $\{7, 8, \ldots, 8k\}$ into no monochromatic pairs $\{\alpha_j, \beta_j\}$, $j = 1, 2, \ldots, 4k-3$.

Define $\mathcal{B}$ by putting on it the following 4-cycles:
(a) the cycles of $\mathcal{D}$;
(b) the cycles (I) of Theorem 2.2;
(c) for each pair $\{\alpha_j, \beta_j\}$, the cycles $(x_i, \alpha_j, x_{2i+1}, \beta_j)$, $i = 0, 1, 2, 3$. Colour the elements of X with a new colour. □

Remark 1. The above Lemma 2.4 gets 4-cycle systems of order $9+8k$ satisfying the hypotheses of same Lemma 2.4 (where it is $n = 1 + 8(k+1)$). Theorems 2.3 and 2.4 get 4-cycle systems satisfying the hypotheses of Lemma 2.4 (where it is $n = 1 + 8k$).

Theorem 2.5 *For every $n \equiv 1 \pmod 8$, $n \geq 9$, and for every $m \in \{3, 4, \ldots, \frac{n+3}{4}\}$ there is a 4-cycle system of order n with a proper m-colouring of type bf.*

Proof. The cases $m = 3$ and $m = \frac{n+3}{4}$ are proved by using Theorem 2.3 and Theorem 2.4 respectively.

Starting from the 3-coloured 4-cycle system of order 9 constructed by using Theorem 2.3, a recursive use of Lemma 2.4 gets the proof. □

3 Colouring of type *bd*

Let $(V, \mathcal{B})$ be a 4-cycle system of order n, $n \geq 9$, having an m-colouring of type bd. Clearly $m \leq \frac{n-1}{4}$. Let ω_i be the cardinality of the colour class Ω_i, $i = 1, 2, \ldots, m$. Since Ω_i is the point set of a P_3-design embedded in $(V, \mathcal{B})$, $\omega_i \equiv 0$ or $1 \pmod 4$.

By definition $\{\Omega_i \mid i = 1, 2, \ldots, m\}$ is a partition of V, then at least one ω_i is odd. W.l.o.g. suppose that ω_1 is odd. If there is some other index $i \in \{2, 3, \ldots, m\}$ such that ω_i is odd, then the cardinality of the edge set of the complete bipartite graph K_{Ω_1, Ω_i} is odd. But this is impossible because each $B \in \mathcal{B}$ covers a nonnegative even number of edges of K_{Ω_1, Ω_i}. From now on we will denote by ω_1 the only odd integer of $\{\omega_i \mid i = 1, 2, \ldots, m\}$.

Lemma 3.1 *If $m \geq \frac{n+15}{8}$ then $\omega_1 \geq 5$.*

Proof. Let $\omega_1 = 1$. Since each cycle has no colour type f, it is $\omega_i \geq 8$ for each $i = 2, 3, \ldots, m$. □

Lemma 3.2 *Let* $\omega_1 \geq 5$*, and let*

$$\chi(\omega_1) = \begin{cases} 1 + 9\mu + 12\mu^2 & \text{if } \omega_1 = 5 + 12\mu \\ 6 + 17\mu + 12\mu^2 & \text{if } \omega_1 = 9 + 12\mu \\ 13 + 25\mu + 12\mu^2 & \text{if } \omega_1 = 13 + 12\mu \end{cases}$$

Then $|\{i \mid \omega_i = 4\}| \leq \chi(\omega_1)$*.*

Proof. Suppose $\omega_j = 4$ for some $j \in \{2, 3, \ldots, m\}$. Let $(\Omega_1, \mathcal{P}_1)$ and $(\Omega_j, \mathcal{P}_j)$ be the two P_3-designs of order ω_1 and 4 respectively, embedded in $(V, \mathcal{B})$. Put $\Omega_1 = \{1, 2, \ldots, \omega_1\}$, $\Omega_j = \{a_0, a_1, a_2, a_3\}$, $\mathcal{P}_j = \{[a_0, a_2, a_1], [a_0, a_3, a_2], [a_0, a_1, a_3]\}$,
$\mathcal{F} = \{(a_0, a_2, a_1, x), (a_0, a_3, a_2, y), (a_0, a_1, a_3, z)\} \subseteq \mathcal{B}$.

Let $\mathcal{D}(\Omega_j) = \{B_1, B_2, \ldots, B_\theta\}$ be the set of 4-cycles B of $\mathcal{B}$ meeting both Ω_j and Ω_1. Clearly it is $B \subseteq \Omega_j \cup \Omega_1$ for every $B \in \mathcal{D}(\Omega_j)$.

Let M be the $4 \times \theta$ array on symbol set $\mathcal{D}(\Omega_j)$ (with rows indexed by the elements of Ω_j and columns indexed by the elements of Ω_1) defined by $M(a_i, \alpha) = B_\sigma$ if and only if $\{a_i, \alpha\}$ is an edge of B_σ. The inclusion $\mathcal{F} \subseteq \mathcal{D}(\Omega_j)$ follows easily by the fact that the cardinality of the edge set of the complete bipartite graph $K_{\Omega_1, \{a_i\}}$ is odd, $i = 0, 1, 2, 3$, and each 4-cycle $B \notin \mathcal{F}$ covers a nonnegative even number of edges of $K_{\Omega_1, \{a_i\}}$.

Put $B_1 = (a_0, a_2, a_1, 1), B_2 = (a_0, a_3, a_2, 2), B_3 = (a_0, a_1, a_3, 3)$. Then $M(a_0, i) = M(a_i, i) = B_i$, $i = 1, 2, 3$. For $\beta = 1, 2$ let $\mathcal{D}_\beta(\Omega_j)$ denote the set of $B_\sigma \in \mathcal{D}(\Omega_j)$ such that $|B_\sigma \cap \Omega_j| = \beta\}$. Each $B_\sigma \in \mathcal{D}_2(\Omega_j)$ gets a 2×2 subsquare of M with all entries filled by the same symbol B_σ. Thus the number of entries of M containing a symbol of $\mathcal{D}_2(\Omega_j)$ is a multiple of four. Then $4\omega_1 = 6 + 2|\mathcal{D}_1(\Omega_j)| + 4|\mathcal{D}_2(\Omega_j)|$ and $|\mathcal{D}_1(\Omega_j)|$ must be odd.

Let $|\mathcal{D}_1(\Omega_j)| = 1$ and suppose $\mathcal{D}_1(\Omega_j) = \{B_4 = (\alpha_1, \alpha_3, \alpha_2, a_t)\}$, $t \in \{0, 1, 2, 3\}$ and $\alpha_1, \alpha_2, \alpha_3 \in \{1, 2, \ldots, \omega_1\}$. It follows $M(a_t, \alpha_1) = M(a_t, \alpha_2) = B_4$, $\alpha_1, \alpha_2 \geq 4$, and the remaining cells of columns α_1 and α_2 are filled by a symbol of $\mathcal{D}_2(\Omega_j)$. Since this is impossible, $|\mathcal{D}_1(\Omega_j)| \geq 3$.

By repeating this argument for each colour class Ω_j whose cardinality is four, we obtain $|\{i \mid \omega_i = 4\}| \leq \frac{1}{3}|\mathcal{P}_1| = \chi(\omega_1)$. $\square$

The upper bound for the number of colour classes is found in next theorem.

Theorem 3.1 *Let* $n \equiv 1 \pmod 8$*,* $n \geq 9$*, and let*

$$\omega(n) = \begin{cases} 5 + 12\mu & \text{if } 9 + 16\mu + 48\mu^2 \leq n \leq 9 + 48\mu + 48\mu^2 \\ 9 + 12\mu & \text{if } 17 + 48\mu + 48\mu^2 \leq n \leq 33 + 80\mu + 48\mu^2 \\ 13 + 12\mu & \text{if } 41 + 80\mu + 48\mu^2 \leq n \leq 65 + 112\mu + 48\mu^2 \end{cases}$$

Then $m \leq 1 + \frac{n - \omega(n)}{4}$*.*

Proof. For $m < \frac{n+15}{8}$ the proof is trivial. Suppose $m \geq \frac{n+15}{8}$. By Lemma 3.1 it is $\omega_1 \geq 5$.

If $\omega_1 \geq \omega(n)$ then $m \leq 1 + \frac{n - \omega_1}{4} \leq 1 + \frac{n - \omega(n)}{4}$.

Let $\omega_1 < \omega(n)$. Then, by Lemma 3.2

$$m \le 1 + \gamma + \frac{n - \omega_1 - 4\gamma}{8} \le 1 + \chi(\omega_1) + \frac{n - \omega_1 - 4\chi(\omega_1)}{8},$$

where $\gamma = |\{i \mid \omega_i = 4\}|$.

To complete the proof it is sufficient to prove that

$$n \ge 4\chi(\omega_1) - \omega_1 + 2\omega(n) \qquad (1)$$

We prove (1) only for $9+16\mu+48\mu^2 \le n \le 9+48\mu+48\mu^2$, leaving to the reader to check the remaining two cases. For $\mu = 0$, (1) is trivial. Let $\mu \ge 1$. If $\omega_1 = 5+12\rho$ then $\rho \le \mu-1$ and thus it is $n \ge 9+16\mu+48\mu^2 \ge 4(1+9\rho+12\rho^2)-(5+12\rho)+2(5+12\mu) = 4\chi(\omega_1)-\omega_1+2\omega(n)$. Similarly it is possible to check (1) for $\omega_1 \equiv 9$ or $13 \pmod{12}$. □

In order to prove that for every m such that $2 \le m \le 1 + \frac{n-\omega(n)}{4}$, there exists a 4-cycle system $(V, \mathcal{B})$ having an m-colouring of of type *bd*, we need to construct some classes of path designs $P(\omega_1, 3, 1)$, $\omega_1 \equiv 1 \pmod 4$, decomposable into the *special configurations.*

Let $(\Omega_1, \mathcal{P}_1)$ be a $P(\omega_1, 3, 1)$ and let $P_i = [x_0^i, x_1^i, x_2^i] \in \mathcal{P}_1$, $i = 1, 2, 3$. The set $\{P_1, P_2, P_3\}$ is said to be a *configuration of type 1* if there are three distinct elements γ_0, γ_1, $\gamma_2 \in \Omega_1$ such that $x_0^1 = x_0^2 = \gamma_0$, $x_0^3 = x_2^1 = \gamma_1$ and $x_2^2 = x_2^3 = \gamma_2$. We will denote by $\mathcal{L}_1(\gamma_0, \gamma_1, \gamma_2)$ a configuration of type 1 whose paths have endpoints $\gamma_0, \gamma_1, \gamma_2$.

Note that both a bowtie and a 6-cycle will provide a type 1 configuration.

Let γ_i, $i = 0, 1, \ldots, 7$ be eight mutually distinct elements of Ω_1 and let $\mathcal{L}_1(\gamma_0, \gamma_1, \gamma_2)$, $\mathcal{L}_1(\gamma_3, \gamma_4, \gamma_5)$ and $\mathcal{L}_1(\gamma_6, \gamma_4, \gamma_7)$ be three configurations of type 1. The configuration $\mathcal{L}_2(\gamma_0, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \gamma_7) = \mathcal{L}_1(\gamma_0, \gamma_1, \gamma_2) \cup \mathcal{L}_1(\gamma_3, \gamma_4, \gamma_5) \cup \mathcal{L}_1(\gamma_6, \gamma_4, \gamma_7)$ is said to be a *configuration of type 2*.

We say that a $(\Omega_1, \mathcal{P}_1)$ is $\mathcal{L}_1$-decomposable if either the path set $\mathcal{P}_1$ (if $\omega_1 \equiv 1$ or 9 (mod 12)), or the path set $\mathcal{P}_1$ from which two paths having the same endpoints have been deleted (if $\omega_1 \equiv 5 \pmod{12}$), is decomposable into configurations of type 1.

Example 2. Let $\Omega_1 = \{0, 1, \ldots, 4\}$ and let $\mathcal{L}_1(0, 2, 4) = \{[0, 1, 2], [0, 3, 4], [2, 0, 4]\}$. Put $\mathcal{P}_1 = \mathcal{L}_1 \cup \{[3, 1, 4], [3, 2, 4]\}$. Then $(\Omega_1, \mathcal{P}_1)$ is $\mathcal{L}_1$-decomposable.

Example 3. Let $\Omega_1 = \{0, 1, \ldots, 8\}$. A decomposition of $\mathcal{P}_1$ into 6 configurations of type 1 is the following
$\mathcal{L}_1(1, 3, 7) = \{[1, 2, 3], [1, 4, 7], [3, 1, 7]\}$, $\mathcal{L}_1(4, 8, 6) = \{[4, 3, 8], [4, 5, 6], [8, 4, 6]\}$,
$\mathcal{L}_1(0, 8, 2) = \{[0, 7, 8], [0, 4, 2], [8, 0, 2]\}$, $\mathcal{L}_1(3, 0, 7) = \{[3, 6, 0], [3, 5, 7], [0, 3, 7]\}$,
$\mathcal{L}_1(1, 8, 5) = \{[1, 6, 8], [1, 0, 5], [8, 1, 5]\}$, $\mathcal{L}_1(2, 8, 6) = \{[2, 5, 8], [2, 7, 6], [8, 2, 6]\}$.
Note that $\mathcal{L}_1(1, 3, 7) \cup \mathcal{L}_1(4, 8, 6) \cup \mathcal{L}_1(0, 8, 2)$, and $\mathcal{L}_1(3, 0, 7) \cup \mathcal{L}_1(1, 8, 5) \cup \mathcal{L}_1(2, 8, 6)\}$ are two configurations of type 2.

Example 4. Let $\Omega_1 = \{0, 1, \ldots, 12\}$. A decomposition of $\mathcal{P}_1$ into 13 configurations of type 1 is the following
$\mathcal{L}_1(0, 4, 7) = \{[0, 1, 4], [0, 5, 7], [4, 0, 7]\}$,

$\mathcal{L}_1(1,5,6) = \{[1,2,5],[1,8,6],[5,1,6]\}$,
$\mathcal{L}_1(2,6,9) = \{[2,3,6],[2,7,9],[6,2,9]\}$,
$\mathcal{L}_1(6,10,0) = \{[6,7,10],[6,11,0],[10,6,0]\}$,
$\mathcal{L}_1(4,8,9) = \{[4,5,8],[4,11,9],[8,4,9]\}$,
$\mathcal{L}_1(5,9,12) = \{[5,6,9],[5,10,12],[9,5,12]\}$,
$\mathcal{L}_1(9,0,3) = \{[9,10,0],[9,1,3],[0,9,3]\}$,
$\mathcal{L}_1(7,11,12) = \{[7,8,11],[7,1,12],[11,7,12]\}$,
$\mathcal{L}_1(8,12,2) = \{[8,9,12],[8,0,2],[12,8,2]\}$,
$\mathcal{L}_1(12,3,6) = \{[12,0,3],[12,4,6],[3,12,6]\}$,
$\mathcal{L}_1(10,1,2) = \{[10,11,1],[10,4,2],[1,10,2]\}$,
$\mathcal{L}_1(11,2,5) = \{[11,12,2],[11,3,5],[2,11,5]\}$,
$\mathcal{L}_1(3,7,10) = \{[3,4,7],[3,8,10],[7,3,10]\}$.
Note that the first 12 configurations of type 1 get 4 mutually disjoint type 2 configurations.

In order to prove Theorem 3.3 we need to construct $\mathcal{L}_1$-decomposable path designs having a *sufficient number* of disjoint decomposition of type 2 as specified by the following theorem.

Theorem 3.2 *Let $\omega_1 \geq 5$ and let*

$$\tau(\omega_1) = \begin{cases} -1 + 2\mu + 3\mu^2 & \text{if } \omega_1 = 1 + 12\mu \\ 4\mu + 3\mu^2 & \text{if } \omega_1 = 5 + 12\mu \\ 2 + 4\mu + 3\mu^2 & \text{if } \omega_1 = 9 + 12\mu \end{cases}$$

Then for each γ, $0 \leq \gamma \leq \tau(\omega_1)$, there is a $\mathcal{L}_1$-decomposable $P(\omega_1, 3, 1)$ having γ mutually disjoint configurations of type 2.

Proof. Since every configuration of type 2 is decomposable into 3 configurations of type 1, then it is sufficient to prove the theorem for $\gamma = \tau(\omega_1)$.

Suppose $\omega_1 = 1 + 12\mu$, $\mu \geq 1$. For $\mu = 1$ the proof follows by Example 4. Let $\mu \geq 2$. It is sufficient to prove that the existence of a $\mathcal{L}_1$-decomposable $P(\omega_1, 3, 1)$, $(\Omega_1, \mathcal{P}_1)$, containing $\tau(\omega_1)$ disjoint type 2 configurations implies the one of a $\mathcal{L}_1$-decomposable $P(\omega_1 + 12, 3, 1)$ with $\tau(\omega_1) + 5 + 6\mu$ disjoint type 2 configurations. Put $\Omega_1 = \{\alpha_0, \alpha_1, \ldots, \alpha_{12\mu}\}$. Let $(\Gamma, \mathcal{Q})$ be a copy of the $\mathcal{L}_1$-decomposable $P(13, 3, 1)$ given in Example 4 based on point set $\Gamma = \{\alpha_{12\mu}\} \cup \{1, 2, \ldots, 12\}$. We emphasize that the 4 disjoint configurations of type 2 of $(\Gamma, \mathcal{Q})$ do not contain $\mathcal{L}_1(3,7,10) = \{[3,4,7],[3,8,10],[7,3,10]\}$.

Now we construct the required $P(\omega_1 + 12, 3, 1)$, $(\Omega_1 \cup \Gamma, \mathcal{P})$. Put in $\mathcal{P}$ the paths of $\mathcal{P}_1 \cup \mathcal{Q}$ and the following ones.

(I) For $i = 0, 1, \ldots, 3\mu - 1$ put in $\mathcal{P}$ the paths of following type 2 configurations:
$\mathcal{L}_2^i(1,2,3,5,6,7,8,9) = \{[1,\alpha_{4i},2],[1,\alpha_{4i+1},3],[2,\alpha_{4i+2},3]\} \cup$
$\{[5,\alpha_{4i},6],[5,\alpha_{4i+2},7],[6,\alpha_{4i+3},7]\} \cup \{[8,\alpha_{4i},7],[8,\alpha_{4i+2},9],[7,\alpha_{4i+1},9]\}$,
$\mathcal{L}_2^i(3,4,5,9,10,11,12,1) = \{[3,\alpha_{4i},4],[3,\alpha_{4i+3},5],[4,\alpha_{4i+1},5]\} \cup$
$\{[9,\alpha_{4i},10],[9,\alpha_{4i+3},11],[10,\alpha_{4i+1},11]\} \cup \{[12,\alpha_{4i},11],[12,\alpha_{4i+3},1],[11,\alpha_{4i+2},1]\}$.

(II) For $i = 0, 1, \ldots, 3\mu - 1$ put in $\mathcal{P}$ the paths of following type 1 configurations:
$\mathcal{L}_1^i(2,4,6) = \{[2, \alpha_{4i+3}, 4], [2, \alpha_{4i+1}, 6], [4, \alpha_{4i+2}, 6]\}$,
$\mathcal{L}_1^i(8,10,12) = \{[8, \alpha_{4i+3}, 10], [8, \alpha_{4i+1}, 12], [10, \alpha_{4i+2}, 12]\}$.

Use $\mathcal{L}_1(3,7,10) = \{[3,4,7],[3,8,10],[7,3,10]\}$, $\mathcal{L}_1^0(2,4,6)$ and $\mathcal{L}_1^0(8,10,12)$ to form a further configuration of type 2.

It is easy to see that at least $\tau(\omega_1) + 4 + 2(3\mu) + 1$ disjoint configurations of type 2 appear in $\mathcal{P}$.

By similar arguments it is possible to prove the theorem for $\omega_1 = 5 + 12\mu, 9 + 12\mu$ (note that cases $\omega_1 = 5$ and $\omega_1 = 9$ are given in Example 2 and Example 3 respectively). □

Remark 2. Let $(\Omega_1, \mathcal{P}_1)$ be the $\mathcal{L}_1$-decomposable $P(\omega_1, 3, 1)$ constructed using Theorem 3.2 with $\omega_1 = 5 + 12\mu$. . Then $\mathcal{P}_1$ contains the block set $\mathcal{Q}$ of a P(5,3,1) isomorphic to the one given in Example 2. Moreover $\mathcal{P}_1 - \mathcal{Q}$ is decomposable into configurations of type 1.

Theorem 3.3 *Let $\bar{m} = 1 + \frac{n-\omega(n)}{4}$, $n \equiv 1 \pmod 8$, $n \geq 9$, where $\omega(n)$ is defined as in Theorem 3.1. Then there is a 4-cycle system of order n having a proper $\bar{m}$-colouring of type bd.*

Proof. Suppose
$$9 + 16\mu + 48\mu^2 \leq n \leq 9 + 48\mu + 48\mu^2 \tag{2}$$
Put $\omega_1 = \omega(n) = 5 + 12\mu$ and $\lambda = \frac{1}{3}\left[\frac{\omega_1(\omega_1 - 1)}{4} - 2\right] = 1 + 9\mu + 12\mu^2$. By (2) it is
$$1 + \mu + 12\mu^2 \leq \frac{n - \omega_1}{4} \leq 1 + 9\mu + 12\mu^2 \tag{3}$$
and
$$0 \leq \lambda - \frac{n - \omega_1}{4} \leq 8\mu \tag{4}$$
It is easy to see that $\rho = \lambda - \frac{n-\omega_1}{4}$ is even. Then $0 \leq \frac{\rho}{2} \leq 4\mu < \tau(5 + 12\mu)$. Using Theorem 3.2 it is possible to construct a $\mathcal{L}_1$-decomposable $P(\omega_1, 3, 1)$, $(\Omega_1, \mathcal{P}_1)$, containing $\frac{\rho}{2}$ configurations of type 2, say $\mathcal{L}_2^i$ $i = 1, 2, \ldots \frac{\rho}{2}$.

Let $\delta = \lambda - 3\frac{\rho}{2} = \frac{n - \omega_1 - 2\rho}{4}$. Denote by $\mathcal{L}_1^j$ $j = 1, 2, \ldots, \delta$, the type 1 configurations contained in $(\Omega_1, \mathcal{P}_1)$ not occuring in $\mathcal{L}_2^i$ for some $i \in \{1, 2, \ldots \frac{\rho}{2}\}$.

Let $(\Gamma, \mathcal{Q})$ be the $P(5,3,1)$ embedded in $(\Omega_1, \mathcal{P}_1)$. Suppose that $\mathcal{L}_1^1 \subseteq \mathcal{Q}$ (see above Remark 2).

Put $\Omega_1 = \{\alpha_0, \alpha_1, \ldots, \alpha_{4+12\mu}\}$, $A_i = \{a_0^i, a_1^i, a_2^i, a_3^i\}$, $i = 1, 2, \ldots, \frac{n-\omega_1}{4}$.

Now we construct a 4-cycle system $(V, \mathcal{B})$ of order n having a $\bar{m}$-colouring of type bd. Let $V = \Omega_1 \cup \left(\cup_{i=1}^{\frac{n-\omega_1}{4}} A_i\right)$. Let $\mathcal{B}$ be the following set of 4-cycles.

(I) Let $\Gamma = \{\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4\}$. Put in $\mathcal{B}$ the 4-cycles:
$(\alpha_1, \alpha_0, \alpha_2, a_2^1)$, $(\alpha_1, \alpha_3, \alpha_4, a_3^1)$, $(\alpha_2, \alpha_1, \alpha_4, a_1^1)$, $(\alpha_3, \alpha_0, \alpha_4, a_0^1)$, $(\alpha_3, \alpha_2, \alpha_4, a_2^1)$,
$(a_0^1, a_2^1, a_1^1, \alpha_1)$, $(a_0^1, a_3^1, a_2^1, \alpha_0)$, $(a_0^1, a_1^1, a_3^1, \alpha_2)$ and $(\alpha_0, a_3^1, \alpha_3, a_1^1)$.

If $n = 9$ ($\mu = 0$) then the proof is completed. If $\mu \geq 1$ then using Lemma 2.1 decompose the complete bipartite graph $K_{\Omega_1 - \Gamma, A_1}$ into edge disjoint 4-cycles and put them in $\mathcal{B}$. Moreover put in $\mathcal{B}$ the following ones.

(II). Let $j \in \{2, 3, \ldots, \delta\}$. We can suppose that $\mathcal{L}_1^j = \{[y_0, y_3, y_1], [y_0, y_4, y_2], [y_1, y_5, y_2]\}$, where $y_0, y_1, \ldots, y_5$ are elements of Ω_1 such that $y_0 \neq y_1 \neq y_2 \neq y_0$ and $y_3 \neq y_4 \neq y_5 \neq y_3$.

Put in $\mathcal{B}$ the 4-cycles (y_0, y_3, y_1, a_3^j), (y_0, y_4, y_2, a_2^j), (y_1, y_5, y_2, a_1^j), $(a_0^j, a_2^j, a_1^j, y_0)$, $(a_0^j, a_3^j, a_2^j, y_1)$ and $(a_0^j, a_1^j, a_3^j, y_2)$.

Decompose the complete bipartite graph $K_{\Omega_1 - \{y_0, y_1, y_2\}, A_j}$ into edge disjoint 4-cycles and put them in $\mathcal{B}$.

(III). Let $i \in \{1 + \delta, 2 + \delta, \ldots, \frac{\varrho}{2} + \delta\}$. We can suppose that $\mathcal{L}_2^{i-\delta} = \{[y_0, y_8, y_1], [y_0, y_9, y_2], [y_1, y_{10}, y_2]\} \cup \{[y_3, y_{11}, y_4], [y_3, y_{12}, y_5], [y_4, y_{13}, y_5]\} \cup \{[y_6, y_{14}, y_4], [y_6, y_{15}, y_7], [y_4, y_{16}, y_7]\}$, where $y_0, y_1, \ldots, y_{16}$ are elements of Ω_1 such that $|\{y_0, y_1, \ldots, y_7\}| = 8$.

Put in $\mathcal{B}$ the 4-cycles (y_0, y_8, y_1, a_3^i), (y_0, y_9, y_2, a_2^i), $(y_1, y_{10}, y_2, a_1^i)$, $(y_3, y_{11}, y_4, a_1^i)$, $(y_3, y_{12}, y_5, a_0^i)$, $(y_4, y_{13}, y_5, a_2^i)$, $(y_6, y_{14}, y_4, a_0^i)$, $(y_6, y_{15}, y_7, a_2^i)$, $(y_4, y_{16}, y_7, a_3^i)$, $(a_0^i, a_2^i, a_1^i, y_0)$, $(a_0^i, a_3^i, a_2^i, y_1)$, $(a_0^i, a_1^i, a_3^i, y_2)$, $(a_2^i, y_3, a_3^i, \bar{y})$, (a_1^i, y_5, a_3^i, y_6), $(a_0^i, y_7, a_1^i, \bar{y})$, where $\bar{y} \in \Omega_1$ and $\bar{y} \neq y_i$ for $i = 0, 1, \ldots, 7$.

Decompose the complete bipartite graph $K_{\Omega_1 - \{\bar{y}, y_0, y_1, \ldots, y_7\}, A_i}$ into edge disjoint 4-cycles and put them in $\mathcal{B}$.

(IV). Decompose the complete bipartite graph K_{A_i, A_j}, $i \neq j$, into edge disjoint 4-cycles and put them in $\mathcal{B}$.

It is easy to see that the above constructed $(V, \mathcal{B})$ is a 4-cycle system of order n having a proper $\bar{m}$-colouring of type *bd* (the colour classes are $\Omega_1, A_1, A_2, \ldots, A_{\frac{n-\omega_1}{4}}$).

Similarly it is possible to prove the theorem in the remaining cases $17 + 48\mu + 48\mu^2 \leq n \leq 33 + 80\mu + 48\mu^2$ and $33 + 80\mu + 48\mu^2 \leq n \leq 65 + 112\mu + 48\mu^2$. □

Theorem 3.4 *For every $n \equiv 1 \pmod 8$, $n \geq 9$, and for every $m \in \{2, 3, \ldots, 1 + \frac{n-\omega(n)}{4}\}$ there is a 4-cycle system of order n with a proper m-colouring of type bd.*

Proof. The cases $m = 2$ and $m = 1 + \frac{n-\omega(n)}{4}$ are proved by Lemma 2.2 and Theorem 3.3 respectively. As in Theorem 2.2 it is possible to prove that the existence of a 4-cycle system of order n having an m-colouring of type *bd*, implies the one of a 4-cycle system of order $n + 8$ having an $(m + 1)$-colouring of type *bd*. □

4 2-Colouring of type b

In this section we deal with the spectrum problem for 4-cycle systems having a 2-colouring of type b. This problem is equivalent to find a 4-cycle system $(V, \mathcal{B})$ having two P_3-designs

$(\Omega_i, \mathcal{P}_i)$, $i = 1, 2$, embedded on it and such that each 4-cycle of $\mathcal{B}$ contains exactly one path of $\mathcal{P}_1 \cup \mathcal{P}_2$, i.e. $|\mathcal{B}| = |\mathcal{P}_1| + |\mathcal{P}_2|$.

Theorem 4.1 *Let $(V, \mathcal{B})$ be a 4-cycle system of order n having a 2-colouring of type b, and let Ω_i, $|\Omega_i| = \omega_i$ $i = 1, 2$, be the two colour classes. Then either*
(1) $\omega_1 = 21 + 52\mu + 32\mu^2$ and $\omega_2 = 28 + 60\mu + 32\mu^2$, $\mu \geq 0$, *or*
(2) $\omega_1 = 4\mu + 32\mu^2$ and $\omega_2 = 1 + 12\mu + 32\mu^2$, $\mu \geq 1$.

Proof. Let $(\Omega_i, \mathcal{P}_i)$, $i = 1, 2$, be the two P_3-designs embedded in $(V, \mathcal{B})$. By $|\mathcal{B}| = |\mathcal{P}_1| + |\mathcal{P}_2|$ it is

$$(\omega_1 - \omega_2)^2 - (\omega_1 + \omega_2) = 0. \tag{5}$$

By (5), $\omega_1 \neq \omega_2$. Suppose $\omega_1 < \omega_2$ and put $t = \omega_2 - \omega_1$. Since $t^2 = \omega_2 + \omega_1$, then $\omega_1 = \frac{t^2-t}{2}$ and $\omega_2 = \frac{t^2+t}{2}$. So we obtain $t^2 - 1 \equiv 0 \pmod 8$, $\frac{t^2-t}{2} \equiv 0$ or $1 \pmod 4$ and $\frac{t^2+t}{2} \equiv 0$ or $1 \pmod 4$. It follows that $t \equiv 1$ or $7 \pmod 8$. Putting either $t = 1 + 8\mu$ or $t = 7 + 8\mu$ we complete the proof. □

Theorem 4.2 *For each nonnegative integer μ there is a 4-cycle system of order $\bar{n} = 49 + 112\mu + 64\mu^2$ having a 2-colouring of type b and colour classes Ω_1, Ω_2 of cardinality $\omega_1 = 21 + 52\mu + 32\mu^2$, $\omega_2 = 28 + 60\mu + 32\mu^2$ respectively.*

Proof. Let $n = \bar{n} - 8(1 + \mu)$, $\delta = 4 + 13\mu + 8\mu^2$. Put $X_i = \{x_0^i, x_1^i, x_2^i, x_3^i\}$, $Y_i = \{y_0^i, y_1^i, y_2^i, y_3^i,\}$, $A_j = \{a_0^j, a_1^j, \ldots, a_7^j\}$, $X = \cup_{i=0}^{\delta} X_i$ ($|X| = \omega_2 - 8(1 + \mu)$), $Y = \cup_{i=0}^{\delta} Y_i$, $\Omega_1 = \{\infty\} \cup Y$, $A = \cup_{j=0}^{\mu} A_j$ and $\Omega_2 = X \cup A$. Let $(W, \mathcal{D})$, $W = \Omega_1 \cup X$, be the 4-cycle system of order n having a 2-colouring of type bd constructed by using Lemma 2.2. Let $\mathcal{D}_1 = \{(x_0^i, y_3^i, x_1^i, \infty) \mid i = 0, 1, \ldots, \delta\}$ be the set of cycles of $\mathcal{D}$ having colour type bd. Let $V = \Omega_1 \cup \Omega_2$. Our aim is to produce a 4-cycle system of order $\bar{n}$ on vertex set V, having a 2-colouring of type b with colour classes Ω_1 and Ω_2. To do this at first we embed $(W, \mathcal{D})$ in a 4-cycle system $(V, \mathcal{D} \cup \mathcal{C})$, then we replace the cycles whose colour type is not b with type b cycles covering the same edge-set of the previous ones.

For $i = 1, 2, \ldots, 9$ let $\mathcal{C}_i$ be the cycle-set given in Appendix 1. Put $\mathcal{C} = \cup_{i=1}^{9} \mathcal{C}_i$. In order to prove that $(V, \mathcal{D} \cup \mathcal{C})$ is a 4-cycle system it is sufficient to verify that the cycles in $\mathcal{C}$ cover the edges of $K_A \cup K_{A, \{\infty\} \cup X \cup Y}$. Clearly $|\mathcal{C}_1| = 14(\mu + 1)$, $|\mathcal{C}_2| = 16\mu(\mu + 1)$, $|\mathcal{C}_3| = 30(\mu + 1) + 8(\mu + 1)^2 + 40\mu(\mu + 1)$, $|\mathcal{C}_4| = 16(2\mu + 2)(\mu + 1)$, $|\mathcal{C}_5| = 5(\mu + 1)$, $|\mathcal{C}_6| = 32(\mu + 1)\mu^2 + 24\mu(\mu + 1)$, $|\mathcal{C}_7| = |\mathcal{C}_6|$, $|\mathcal{C}_8| = 64\mu(\mu + 1)^2$ and $|\mathcal{C}_9| = 8\mu(\mu + 1)$. It follows that $\mathcal{C}$ covers the same number of edges of $K_A \cup K_{A, \{\infty\} \cup X \cup Y}$. Then it is sufficient to verify that every edge of $K_A \cup K_{A, \{\infty\} \cup X \cup Y}$ is covered by some cycle in $\mathcal{C}$. In the following we show how to check this:
– for $i = 0, 1, \ldots, \mu$, the edges of K_{A_i} are covered by cycles in $\mathcal{C}_1$;
– for $i = 0, 1, \ldots, \mu$, the edges of $K_{A_i, \{\infty\}}$ are covered by cycles in $\mathcal{C}_1$;
– if $\mu \geq 1$, then for $i = 0, 1, \ldots, \mu - 1$, $j = i + 1, i + 2, \ldots, \mu$ the edges of K_{A_i, A_j} are covered by cycles in $\mathcal{C}_2$;
– for $i = 0, 1, \ldots, 3\mu + 2$, the edges of K_{A, Y_i} are covered by cycles in $\mathcal{C}_1 \cup \mathcal{C}_3$;
– for $i = 3\mu + 3, 3\mu + 4, \ldots, 5\mu + 4$, the edges of K_{A, Y_i} are covered by cycles in $\mathcal{C}_4$;

– for $i = 5\mu+5, 5\mu+6, \dots, \delta$, the edges of K_{A,Y_i} are covered by cycles in $\mathcal{C}_2 \cup \mathcal{C}_6 \cup \mathcal{C}_7 \cup \mathcal{C}_8$;
– for $i = 0, 1, \dots, 5\mu+4$, the edges of K_{A,X_i} are covered by cycles in $\mathcal{C}_3 \cup \mathcal{C}_4 \cup \mathcal{C}_5$;
– for $i = 5\mu+5, 5\mu+6, \dots, \delta$, the edges of K_{A,X_i} are covered by cycles in $\mathcal{C}_6 \cup \mathcal{C}_7 \cup \mathcal{C}_8 \cup \mathcal{C}_9$.

Remark that the colour classes are Ω_1 and Ω_2. Then the cycles of $\mathcal{C}_5 \cup \mathcal{C}_9$ are monochromatic whereas the ones of $\mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3 \cup \mathcal{C}_4 \cup \mathcal{C}_6 \cup \mathcal{C}_7 \cup \mathcal{C}_8$ are of colour type b. Let $\mathcal{B}_1$ be the set of cycles, of colour type b, given in Appendix 1. It is easy to verify that $\mathcal{B}_1$ and $\mathcal{C}_5 \cup \mathcal{C}_9 \cup \mathcal{D}_1$ cover the same edges.

Put $\mathcal{B} = (\mathcal{D} - \mathcal{D}_1) \cup (\mathcal{C} - (\mathcal{C}_5 \cup \mathcal{C}_9)) \cup \mathcal{B}_1$. Then $(V, \mathcal{B})$ is the required 4-cycle system of order $\bar{n}$ having a 2-colouring of type b. □

Theorem 4.3 *For each $\mu \geq 1$ there is a 4-cycle system of order $\bar{n} = 1 + 16\mu + 64\mu^2$ having a 2-colouring of type b and colour classes Ω_1, Ω_2 of cardinality $\omega_1 = 4\mu + 32\mu^2$, $\omega_2 = 1 + 12\mu + 32\mu^2$ respectively.*

Proof. Let $n = \bar{n} - 8\mu$, $\delta = 8\mu^2 + \mu - 1$. Put $X_i = \{x_0^i, x_1^i, x_2^i, x_3^i\}$, $Y_i = \{y_0^i, y_1^i, y_2^i, y_3^i\}$, $A_j = \{a_0^j, a_1^j, \dots, a_7^j\}$, $\Omega_1 = \cup_{i=0}^{\delta} X_i$, $Y = \cup_{i=0}^{\delta} Y_i$, $A = \cup_{j=0}^{\mu-1} A_j$ and $\Omega_2 = \{\infty\} \cup Y \cup A$.

Let (I), (II) and (III) be the cycle-sets constructed in Lemma 2.2. Change y_0^i with ∞ in cycles of (I) and (III) and leave unchanged those of (II). Then we obtain a 4-cycle system of order n $(W, \mathcal{D})$, $W = \Omega_1 \cup Y \cup \{\infty\}$, having a 2-colouring of type bd, with colour classes Ω_1 and $Y \cup \{\infty\}$, and such that the set of cycles of colour type bd is $\mathcal{D}_1 = \{(x_0^i, y_3^i, x_1^i, y_0^i) \mid i = 0, 1, \dots, \delta\}$.

Let $V = \Omega_1 \cup \Omega_2$. For $i = 1, 2, \dots, 6$ let $\mathcal{C}_i$ be the cycle-set given in Appendix 2 (where the suffices of x and y are $\pmod 4$, and the suffices of a are $\pmod 8$).

Put $\mathcal{C} = \cup_{i=1}^{6} \mathcal{C}_i$ and $\mathcal{B} = \mathcal{C} \cup (\mathcal{D} - \mathcal{D}_1)$. In order to prove that $(V, \mathcal{B})$ is the required 4-cycle system of order $\bar{n}$ having a 2-colouring of type b, it is sufficient to verify that the cycles in $\mathcal{C}$ cover the edges of $K_A \cup K_{A,\{\infty\} \cup X \cup Y}$ and $\mathcal{D}_1$.

Clearly $|\mathcal{C}_1| = 14\mu$, $|\mathcal{C}_2| = 16\mu(\mu - 1)$, $|\mathcal{C}_3| = 9(4\mu^2 - 2\mu) + 108\mu^2$, $|\mathcal{C}_4| = 16\mu(8\mu^2 - 8\mu) - 16\mu(\mu-1) - 4\mu$, $|\mathcal{C}_5| = 24\mu$ and $|\mathcal{C}_6| = 16\mu(\mu-1)$. It follows that $\mathcal{C}$ covers the same number of edges of $\mathcal{D}_1$ and $K_A \cup K_{A,\{\infty\} \cup X \cup Y}$. Then it is sufficient to verify that every edge of $\mathcal{D}_1$ and $K_A \cup K_{A,\{\infty\} \cup X \cup Y}$ is covered by some cycle in $\mathcal{C}$. In the following we show how to check this:
– for $i = 0, 1, \dots, \mu - 1$, the edges of K_{A_i} are covered by cycles in $\mathcal{C}_1$;
– if $\mu \geq 2$, then for $i = 0, 1, \dots, \mu - 2$, $j = i + 1, i + 2, \dots, \mu - 1$ the edges of K_{A_i, A_j} are covered by cycles in $\mathcal{C}_2$;
– for $i = 0, 1, \dots, \mu - 1$, the edges of $K_{A_i, \{\infty\}}$ are covered by cycles in $\mathcal{C}_5$;
– for $i = 0, 1, \dots, 9\mu - 1$, the edges of K_{A,X_i} are covered by cycles in $\mathcal{C}_1 \cup \mathcal{C}_3 \cup \mathcal{C}_5$;
– for $i = 9\mu, 9\mu + 1, \dots, \delta$, the edges of K_{A,X_i} are covered by cycles in $\mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_4$;
– for $i = 0, 1, \dots, 9\mu - 1$, the edges of K_{A,Y_i} are covered by cycles in $\mathcal{C}_3 \cup \mathcal{C}_5$;
– for $i = 9\mu, 9\mu + 1, \dots, \delta$, the edges of K_{A,Y_i} are covered by cycles in $\mathcal{C}_4 \cup \mathcal{C}_5 \cup \mathcal{C}_6$;
– the edges of $\mathcal{D}_1$ are covered by cycles in $\mathcal{C}_5 \cup \mathcal{C}_6$. □

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Appendix 1

$\mathcal{C}_1$	
$i=0,1,\ldots,\mu$	$(a_0^i,a_7^i,a_1^i,\infty)$, $(a_2^i,a_4^i,a_3^i,\infty)$, $(a_4^i,a_6^i,a_5^i,\infty)$, $(a_6^i,a_2^i,a_7^i,\infty)$, $(a_0^i,a_1^i,a_5^i,y_0^{3i})$, $(a_0^i,a_2^i,a_5^i,y_1^{3i})$, $(a_2^i,a_1^i,a_6^i,y_2^{3i})$, $(a_2^i,a_3^i,a_6^i,y_3^{3i})$, $(a_4^i,a_1^i,a_3^i,y_0^{1+3i})$, $(a_4^i,a_0^i,a_3^i,y_1^{1+3i})$, $(a_0^i,a_5^i,a_7^i,y_2^{1+3i})$, $(a_0^i,a_6^i,a_7^i,y_3^{1+3i})$, $(a_4^i,a_5^i,a_3^i,y_0^{2+3i})$, $(a_4^i,a_7^i,a_3^i,y_1^{2+3i})$.

$\mathcal{C}_2$	
$\mu\geq 1$ $i=0,1,\ldots,\mu-1$ $j=i+1,i+2,\ldots,\mu$ $\tau=5+5\mu+$ $+16(j-\frac{i(i+1)}{2}+i\mu-1)$	$(a_0^i,a_0^j,a_1^i,y_0^{\tau})$, $(a_0^i,a_1^j,a_1^i,y_1^{\tau})$, $(a_0^i,a_2^j,a_1^i,y_0^{\tau+1})$, $(a_0^i,a_3^j,a_1^i,y_1^{\tau+1})$, $(a_2^i,a_4^j,a_3^i,y_0^{\tau+2})$, $(a_2^i,a_5^j,a_3^i,y_1^{\tau+2})$, $(a_2^i,a_6^j,a_3^i,y_0^{\tau+3})$, $(a_2^i,a_7^j,a_3^i,y_1^{\tau+3})$, $(a_4^i,a_0^j,a_5^i,y_0^{\tau+4})$, $(a_4^i,a_1^j,a_5^i,y_1^{\tau+4})$, $(a_4^i,a_2^j,a_5^i,y_0^{\tau+5})$, $(a_4^i,a_3^j,a_5^i,y_1^{\tau+5})$, $(a_6^i,a_4^j,a_7^i,y_0^{\tau+6})$, $(a_6^i,a_5^j,a_7^i,y_1^{\tau+6})$, $(a_6^i,a_6^j,a_7^i,y_0^{\tau+7})$, $(a_6^i,a_7^j,a_7^i,y_1^{\tau+7})$, $(a_0^j,a_2^i,a_1^j,y_0^{\tau+8})$, $(a_0^j,a_3^i,a_1^j,y_1^{\tau+8})$, $(a_0^j,a_6^i,a_1^j,y_0^{\tau+9})$, $(a_0^j,a_7^i,a_1^j,y_1^{\tau+9})$, $(a_2^j,a_2^i,a_3^j,y_0^{\tau+10})$, $(a_2^j,a_3^i,a_3^j,y_1^{\tau+10})$, $(a_2^j,a_6^i,a_3^j,y_0^{\tau+11})$, $(a_2^j,a_7^i,a_3^j,y_1^{\tau+11})$, $(a_4^j,a_0^i,a_5^j,y_0^{\tau+12})$, $(a_4^j,a_1^i,a_5^j,y_1^{\tau+12})$, $(a_4^j,a_4^i,a_5^j,y_0^{\tau+13})$, $(a_4^j,a_5^i,a_5^j,y_1^{\tau+13})$, $(a_6^j,a_0^i,a_7^j,y_0^{\tau+14})$, $(a_6^j,a_1^i,a_7^j,y_1^{\tau+14})$, $(a_6^j,a_4^i,a_7^j,y_0^{\tau+15})$, $(a_6^j,a_5^i,a_7^j,y_1^{\tau+15})$.

$\mathcal{C}_3$	
$i=0,1,\ldots,\mu$	$(a_1^i,x_0^{5i},a_4^i,y_0^{3i})$, $(a_2^i,x_0^{5i},a_6^i,y_0^{3i})$, $(a_3^i,x_0^{5i},a_7^i,y_0^{3i})$, $(a_1^i,x_1^{5i},a_4^i,y_1^{3i})$, $(a_2^i,x_1^{5i},a_6^i,y_1^{3i})$, $(a_3^i,x_1^{5i},a_7^i,y_1^{3i})$, $(a_0^i,x_0^{5i+1},a_4^i,y_2^{3i})$, $(a_1^i,x_0^{5i+1},a_5^i,y_2^{3i})$, $(a_3^i,x_0^{5i+1},a_7^i,y_2^{3i})$, $(a_0^i,x_1^{5i+1},a_4^i,y_3^{3i})$, $(a_1^i,x_1^{5i+1},a_5^i,y_3^{3i})$, $(a_3^i,x_1^{5i+1},a_7^i,y_3^{3i})$, $(a_0^i,x_0^{5i+2},a_5^i,y_0^{3i+1})$, $(a_1^i,x_0^{5i+2},a_6^i,y_0^{3i+1})$, $(a_2^i,x_0^{5i+2},a_7^i,y_0^{3i+1})$, $(a_0^i,x_1^{5i+2},a_5^i,y_1^{3i+1})$, $(a_1^i,x_1^{5i+2},a_6^i,y_1^{3i+1})$, $(a_2^i,x_1^{5i+2},a_7^i,y_1^{3i+1})$, $(a_1^i,x_0^{5i+3},a_4^i,y_2^{3i+1})$, $(a_2^i,x_0^{5i+3},a_5^i,y_2^{3i+1})$, $(a_3^i,x_0^{5i+3},a_6^i,y_2^{3i+1})$, $(a_1^i,x_1^{5i+3},a_4^i,y_3^{3i+1})$, $(a_2^i,x_1^{5i+3},a_5^i,y_3^{3i+1})$, $(a_3^i,x_1^{5i+3},a_6^i,y_3^{3i+1})$, $(a_0^i,x_0^{5i+4},a_5^i,y_0^{3i+2})$, $(a_1^i,x_0^{5i+4},a_6^i,y_0^{3i+2})$, $(a_2^i,x_0^{5i+4},a_7^i,y_0^{3i+2})$, $(a_0^i,x_1^{5i+4},a_5^i,y_1^{3i+2})$, $(a_1^i,x_1^{5i+4},a_6^i,y_1^{3i+2})$, $(a_2^i,x_1^{5i+4},a_7^i,y_1^{3i+2})$,
$i,j=0,1,\ldots,\mu$ $\sigma=0,1,2,3$	$(a_{2\sigma}^j,x_2^i,a_{2\sigma+1}^j,y_2^{3i+2})$, $(a_{2\sigma}^j,x_3^i,a_{2\sigma+1}^j,y_3^{3i+2})$,
$\mu\geq 1$ $i,\rho=0,1,\ldots,\mu$ $\rho\neq i$ $\sigma=0,1,2,3$	$(a_{2\sigma}^{\rho},x_0^{5i},a_{2\sigma+1}^{\rho},y_0^{3i})$, $(a_{2\sigma}^{\rho},x_1^{5i},a_{2\sigma+1}^{\rho},y_1^{3i})$, $(a_{2\sigma}^{\rho},x_0^{5i+1},a_{2\sigma+1}^{\rho},y_2^{3i})$, $(a_{2\sigma}^{\rho},x_1^{5i+1},a_{2\sigma+1}^{\rho},y_3^{3i})$, $(a_{2\sigma}^{\rho},x_0^{5i+2},a_{2\sigma+1}^{\rho},y_0^{3i+1})$, $(a_{2\sigma}^{\rho},x_1^{5i+2},a_{2\sigma+1}^{\rho},y_1^{3i+1})$, $(a_{2\sigma}^{\rho},x_0^{5i+3},a_{2\sigma+1}^{\rho},y_2^{3i+1})$, $(a_{2\sigma}^{\rho},x_1^{5i+3},a_{2\sigma+1}^{\rho},y_3^{3i+1})$, $(a_{2\sigma}^{\rho},x_0^{5i+4},a_{2\sigma+1}^{\rho},y_0^{3i+2})$, $(a_{2\sigma}^{\rho},x_1^{5i+4},a_{2\sigma+1}^{\rho},y_1^{3i+2})$.

$\mathcal{C}_4$	
$i=0,1,\ldots,2\mu+1$ $j=0,1,\ldots,\mu$ $\sigma=0,1,2,3$	$(a^j_{2\sigma},x_2^{2i+\mu+1},a^j_{2\sigma+1},y_0^{i+3\mu+3})$, $(a^j_{2\sigma},x_3^{2i+\mu+1},a^j_{2\sigma+1},y_1^{i+3\mu+3})$, $(a^j_{2\sigma},x_2^{2i+\mu+2},a^j_{2\sigma+1},y_2^{i+3\mu+3})$, $(a^j_{2\sigma},x_3^{2i+\mu+2},a^j_{2\sigma+1},y_3^{i+3\mu+3})$.

$\mathcal{C}_5$	
$i=0,1,\ldots,\mu$	$(a^i_0,x_0^{5i},a^i_5,x_1^{5i})$, $(a^i_2,x_0^{5i+1},a^i_6,x_1^{5i+1})$, $(a^i_3,x_0^{5i+2},a^i_4,x_1^{5i+2})$, $(a^i_0,x_0^{5i+3},a^i_7,x_1^{5i+3})$, $(a^i_3,x_0^{5i+4},a^i_4,x_1^{5i+4})$.

$\mathcal{C}_6$	
$\mu\geq 1$ $i=0,1,\ldots,\mu-1$ $j=i+1,i+2,\ldots,\mu$ $\tau=5+5\mu+$ $+16(j-\frac{i(i+1)}{2}+i\mu-1)$ $\rho=0,1,\ldots,\mu$ $\rho\neq i$ $\sigma=0,1,2,3$	$(a^\rho_{2\sigma},x_0^{\tau},a^\rho_{1+2\sigma},y_0^{\tau})$, $(a^\rho_{2\sigma},x_1^{\tau},a^\rho_{1+2\sigma},y_1^{\tau})$, $(a^\rho_{2\sigma},x_0^{\tau+1},a^\rho_{1+2\sigma},y_0^{\tau+1})$, $(a^\rho_{2\sigma},x_1^{\tau+1},a^\rho_{1+2\sigma},y_1^{\tau+1})$, $(a^\rho_{2\sigma},x_0^{\tau+2},a^\rho_{1+2\sigma},y_0^{\tau+2})$, $(a^\rho_{2\sigma},x_1^{\tau+2},a^\rho_{1+2\sigma},y_1^{\tau+2})$, $(a^\rho_{2\sigma},x_0^{\tau+3},a^\rho_{1+2\sigma},y_0^{\tau+3})$, $(a^\rho_{2\sigma},x_1^{\tau+3},a^\rho_{1+2\sigma},y_1^{\tau+3})$, $(a^\rho_{2\sigma},x_0^{\tau+4},a^\rho_{1+2\sigma},y_0^{\tau+4})$, $(a^\rho_{2\sigma},x_1^{\tau+4},a^\rho_{1+2\sigma},y_1^{\tau+4})$, $(a^\rho_{2\sigma},x_0^{\tau+5},a^\rho_{1+2\sigma},y_0^{\tau+5})$, $(a^\rho_{2\sigma},x_1^{\tau+5},a^\rho_{1+2\sigma},y_1^{\tau+5})$, $(a^\rho_{2\sigma},x_0^{\tau+6},a^\rho_{1+2\sigma},y_0^{\tau+6})$, $(a^\rho_{2\sigma},x_1^{\tau+6},a^\rho_{1+2\sigma},y_1^{\tau+6})$, $(a^\rho_{2\sigma},x_0^{\tau+7},a^\rho_{1+2\sigma},y_0^{\tau+7})$, $(a^\rho_{2\sigma},x_1^{\tau+7},a^\rho_{1+2\sigma},y_1^{\tau+7})$,
$\chi=1,2,3$ i, j, μ, τ as above	$(a^i_{2\chi},x_0^{\tau},a^i_{1+2\chi},y_0^{\tau})$, $(a^i_{2\chi},x_1^{\tau},a^i_{1+2\chi},y_1^{\tau})$, $(a^i_{2\chi},x_0^{\tau+1},a^i_{1+2\chi},y_0^{\tau+1})$, $(a^i_{2\chi},x_1^{\tau+1},a^i_{1+2\chi},y_1^{\tau+1})$,
$\chi=0,2,3$ i, j, μ, τ as above	$(a^i_{2\chi},x_0^{\tau+2},a^i_{1+2\chi},y_0^{\tau+2})$, $(a^i_{2\chi},x_1^{\tau+2},a^i_{1+2\chi},y_1^{\tau+2})$, $(a^i_{2\chi},x_0^{\tau+3},a^i_{1+2\chi},y_0^{\tau+3})$, $(a^i_{2\chi},x_1^{\tau+3},a^i_{1+2\chi},y_1^{\tau+3})$,
$\chi=0,1,3$ i, j, μ, τ as above	$(a^i_{2\chi},x_0^{\tau+4},a^i_{1+2\chi},y_0^{\tau+4})$, $(a^i_{2\chi},x_1^{\tau+4},a^i_{1+2\chi},y_1^{\tau+4})$, $(a^i_{2\chi},x_0^{\tau+5},a^i_{1+2\chi},y_0^{\tau+5})$, $(a^i_{2\chi},x_1^{\tau+5},a^i_{1+2\chi},y_1^{\tau+5})$,
$\chi=0,1,2$ i, j, μ, τ as above	$(a^i_{2\chi},x_0^{\tau+6},a^i_{1+2\chi},y_0^{\tau+6})$, $(a^i_{2\chi},x_1^{\tau+6},a^i_{1+2\chi},y_1^{\tau+6})$, $(a^i_{2\chi},x_0^{\tau+7},a^i_{1+2\chi},y_0^{\tau+7})$, $(a^i_{2\chi},x_1^{\tau+7},a^i_{1+2\chi},y_1^{\tau+7})$.

$\mathcal{C}_7$	
$\mu \geq 1$ $i = 0, 1, \ldots, \mu - 1$ $j = i + 1, i + 2, \ldots, \mu$ $\tau = 5 + 5\mu +$ $+16(j - \frac{i(i+1)}{2} + i\mu - 1)$ $\rho = 0, 1, \ldots, \mu$ $\rho \neq j$ $\sigma = 0, 1, 2, 3$	$(a^\rho_{2\sigma}, x^{\tau+8}_0, a^\rho_{1+2\sigma}, y^{\tau+8}_0),\ (a^\rho_{2\sigma}, x^{\tau+8}_1, a^\rho_{1+2\sigma}, y^{\tau+8}_1),$ $(a^\rho_{2\sigma}, x^{\tau+9}_0, a^\rho_{1+2\sigma}, y^{\tau+9}_0),\ (a^\rho_{2\sigma}, x^{\tau+9}_1, a^\rho_{1+2\sigma}, y^{\tau+9}_1)$ $(a^\rho_{2\sigma}, x^{\tau+10}_0, a^\rho_{1+2\sigma}, y^{\tau+10}_0),\ (a^\rho_{2\sigma}, x^{\tau+10}_1, a^\rho_{1+2\sigma}, y^{\tau+10}_1),$ $(a^\rho_{2\sigma}, x^{\tau+11}_0, a^\rho_{1+2\sigma}, y^{\tau+11}_0),\ (a^\rho_{2\sigma}, x^{\tau+11}_1, a^\rho_{1+2\sigma}, y^{\tau+11}_1),$ $(a^\rho_{2\sigma}, x^{\tau+12}_0, a^\rho_{1+2\sigma}, y^{\tau+12}_0),\ (a^\rho_{2\sigma}, x^{\tau+12}_1, a^\rho_{1+2\sigma}, y^{\tau+12}_1),$ $(a^\rho_{2\sigma}, x^{\tau+13}_0, a^\rho_{1+2\sigma}, y^{\tau+13}_0),\ (a^\rho_{2\sigma}, x^{\tau+13}_1, a^\rho_{1+2\sigma}, y^{\tau+13}_1)$ $(a^\rho_{2\sigma}, x^{\tau+14}_0, a^\rho_{1+2\sigma}, y^{\tau+14}_0),\ (a^\rho_{2\sigma}, x^{\tau+14}_1, a^\rho_{1+2\sigma}, y^{\tau+14}_1),$ $(a^\rho_{2\sigma}, x^{\tau+15}_0, a^\rho_{1+2\sigma}, y^{\tau+15}_0),\ (a^\rho_{2\sigma}, x^{\tau+15}_1, a^\rho_{1+2\sigma}, y^{\tau+15}_1),$
$\chi = 1, 2, 3$ i, j, μ, τ as above	$(a^j_{2\chi}, x^{\tau+8}_0, a^j_{1+2\chi}, y^{\tau+8}_0),\ (a^j_{2\chi}, x^{\tau+8}_1, a^j_{1+2\chi}, y^{\tau+8}_1),$ $(a^j_{2\chi}, x^{\tau+9}_0, a^j_{1+2\chi}, y^{\tau+9}_0),\ (a^j_{2\chi}, x^{\tau+9}_1, a^j_{1+2\chi}, y^{\tau+9}_1),$
$\chi = 0, 2, 3$ i, j, μ, τ as above	$(a^j_{2\chi}, x^{\tau+10}_0, a^j_{1+2\chi}, y^{\tau+10}_0),\ (a^j_{2\chi}, x^{\tau+10}_1, a^j_{1+2\chi}, y^{\tau+10}_1),$ $(a^j_{2\chi}, x^{\tau+11}_0, a^j_{1+2\chi}, y^{\tau+11}_0),\ (a^j_{2\chi}, x^{\tau+11}_1, a^j_{1+2\chi}, y^{\tau+11}_1),$
$\chi = 0, 1, 3$ i, j, μ, τ as above	$(a^j_{2\chi}, x^{\tau+12}_0, a^j_{1+2\chi}, y^{\tau+12}_0),\ (a^j_{2\chi}, x^{\tau+12}_1, a^j_{1+2\chi}, y^{\tau+12}_1),$ $(a^j_{2\chi}, x^{\tau+13}_0, a^j_{1+2\chi}, y^{\tau+13}_0),\ (a^j_{2\chi}, x^{\tau+13}_1, a^j_{1+2\chi}, y^{\tau+13}_1),$
$\chi = 0, 1, 2$ i, j, μ, τ as above	$(a^j_{2\chi}, x^{\tau+14}_0, a^j_{1+2\chi}, y^{\tau+14}_0),\ (a^j_{2\chi}, x^{\tau+14}_1, a^j_{1+2\chi}, y^{\tau+14}_1),$ $(a^j_{2\chi}, x^{\tau+15}_0, a^j_{1+2\chi}, y^{\tau+15}_0),\ (a^j_{2\chi}, x^{\tau+15}_1, a^j_{1+2\chi}, y^{\tau+15}_1).$

$\mathcal{C}_8$	
$\mu \geq 1$, $i = 0, 1, \ldots, \mu - 1$, $j = i + 1, i + 2, \ldots, \mu$ $\tau = 5 + 5\mu + 16(j - \frac{i(i+1)}{2} + i\mu - 1)$ $\gamma = 0, 1, \ldots, \mu$, $\alpha = 0, 1, \ldots, 15$, $\sigma = 0, 1, 2, 3$	$(a^\gamma_{2\sigma}, x^{\tau+\alpha}_2, a^\gamma_{1+2\sigma}, y^{\tau+\alpha}_2),$ $(a^\gamma_{2\sigma}, x^{\tau+\alpha}_3, a^\gamma_{1+2\sigma}, y^{\tau+\alpha}_3).$

$\mathcal{C}_9$	
$\mu \geq 1$, $i = 0, 1, \ldots, \mu - 1$ $j = i + 1, i + 2, \ldots, \mu$ $\tau = 5 + 5\mu + 16(j - \frac{i(i+1)}{2} + i\mu - 1)$ $\sigma = 0, 1, 2, 3$	$(a^i_{2\sigma}, x^{\tau+2\sigma}_0, a^i_{2\sigma+1}, x^{\tau+2\sigma}_1),$ $(a^i_{2\sigma}, x^{\tau+2\sigma+1}_0, a^i_{2\sigma+1}, x^{\tau+2\sigma+1}_1),$ $(a^j_{2\sigma}, x^{\tau+2\sigma+8}_0, a^j_{2\sigma+1}, x^{\tau+2\sigma+8}_1),$ $(a^j_{2\sigma}, x^{\tau+2\sigma+9}_0, a^j_{2\sigma+1}, x^{\tau+2\sigma+9}_1).$

$\mathcal{B}_1$	
$i=0,1,\ldots,\mu$	$(x_0^{5i},a_0^i,x_1^{5i},\infty)$, $(x_0^{5i},a_5^i,x_1^{5i},y_3^{5i})$, $(x_0^{5i+1},a_2^i,x_1^{5i+1},\infty)$, $(x_0^{5i+1},a_6^i,x_1^{5i+1},y_3^{5i+1})$, $(x_0^{5i+2},a_3^i,x_1^{5i+2},\infty)$, $(x_0^{5i+2},a_4^i,x_1^{5i+2},y_3^{5i+2})$, $(x_0^{5i+3},a_0^i,x_1^{5i+3},\infty)$, $(x_0^{5i+3},a_7^i,x_1^{5i+3},y_3^{5i+3})$, $(x_0^{5i+4},a_3^i,x_1^{5i+4},\infty)$, $(x_0^{5i+4},a_4^i,x_1^{5i+4},y_3^{5i+4})$.
$\mu\geq 1$, $\sigma=0,1,2,3$ $i=0,1,\ldots,\mu-1$ $j=i+1,i+2,\ldots,\mu$ $\tau=5+5\mu+$ $+16(j-\frac{i(i+1)}{2}+i\mu-1)$	$(x_0^{\tau+2\sigma},a_{2\sigma}^i,x_1^{\tau+2\sigma},\infty)$, $(x_0^{\tau+2\sigma},a_{2\sigma+1}^i,x_1^{\tau+2\sigma},y_3^{\tau+2\sigma})$, $(x_0^{\tau+2\sigma+1},a_{2\sigma}^i,x_1^{\tau+2\sigma+1},\infty)$, $(x_0^{\tau+2\sigma+1},a_{2\sigma+1}^i,x_1^{\tau+2\sigma+1},y_3^{\tau+2\sigma+1})$, $(x_0^{\tau+2\sigma+8},a_{2\sigma}^j,x_1^{\tau+2\sigma+8},\infty)$, $(x_0^{\tau+2\sigma+8},a_{2\sigma+1}^j,x_1^{\tau+2\sigma+8},y_3^{\tau+2\sigma+8})$, $(x_0^{\tau+2\sigma+9},a_{2\sigma}^j,x_1^{\tau+2\sigma+9},\infty)$, $(x_0^{\tau+2\sigma+9},a_{2\sigma+1}^j,x_1^{\tau+2\sigma+9},y_3^{\tau+2\sigma+9})$.

Appendix 2

$\mathcal{C}_1$	
$i = 0, 1, \ldots, \mu - 1$	$(a_0^i, a_7^i, a_1^i, x_3^\delta)$, $(a_2^i, a_4^i, a_3^i, x_3^\delta)$, $(a_4^i, a_6^i, a_5^i, x_3^\delta)$, $(a_6^i, a_2^i, a_7^i, x_3^\delta)$, $(a_0^i, a_1^i, a_5^i, x_0^{9i})$, $(a_0^i, a_2^i, a_5^i, x_1^{9i})$, $(a_2^i, a_1^i, a_6^i, x_0^{9i+1})$, $(a_2^i, a_3^i, a_6^i, x_1^{9i+1})$, $(a_4^i, a_1^i, a_3^i, x_0^{9i+2})$, $(a_4^i, a_0^i, a_3^i, x_1^{9i+2})$, $(a_0^i, a_5^i, a_7^i, x_0^{9i+3})$, $(a_0^i, a_6^i, a_7^i, x_1^{9i+3})$, $(a_4^i, a_5^i, a_3^i, x_0^{9i+4})$, $(a_4^i, a_7^i, a_3^i, x_1^{9i+4})$.

$\mathcal{C}_2$	
$\mu \geq 2$ $i = 0, 1, \ldots, \mu - 2$ $j = i+1, i+2, \ldots, \mu - 1$ $\tau = 9\mu + 16\big[i(\mu-1) -$ $- \frac{i(i+1)}{2} + j - 1\big]$ $\sigma = 0, 1, 2, 3$	$(a_{2\sigma}^i, a_{4\sigma}^j, a_{1+2\sigma}^i, x_0^{\tau+2\sigma})$, $(a_{2\sigma}^i, a_{1+4\sigma}^j, a_{1+2\sigma}^i, x_1^{\tau+2\sigma})$, $(a_{2\sigma}^i, a_{2+4\sigma}^j, a_{1+2\sigma}^i, x_0^{1+\tau+2\sigma})$, $(a_{2\sigma}^i, a_{3+4\sigma}^j, a_{1+2\sigma}^i, x_1^{1+\tau+2\sigma})$, $(a_0^j, a_2^i, a_1^j, x_0^{\tau+8})$, $(a_0^j, a_3^i, a_1^j, x_1^{\tau+8})$, $(a_0^j, a_6^i, a_1^j, x_0^{\tau+9})$, $(a_0^j, a_7^i, a_1^j, x_1^{\tau+9})$, $(a_2^j, a_2^i, a_3^j, x_0^{\tau+10})$, $(a_2^j, a_3^i, a_3^j, x_1^{\tau+10})$, $(a_2^j, a_6^i, a_3^j, x_0^{\tau+11})$, $(a_2^j, a_7^i, a_3^j, x_1^{\tau+11})$, $(a_4^j, a_0^i, a_5^j, x_0^{\tau+12})$, $(a_4^j, a_1^i, a_5^j, x_1^{\tau+12})$, $(a_4^j, a_4^i, a_5^j, x_0^{\tau+13})$, $(a_4^j, a_5^i, a_5^j, x_1^{\tau+13})$, $(a_6^j, a_0^i, a_7^j, x_0^{\tau+14})$, $(a_6^j, a_1^i, a_7^j, x_1^{\tau+14})$, $(a_6^j, a_4^i, a_7^j, x_0^{\tau+15})$, $(a_6^j, a_5^i, a_7^j, x_1^{\tau+15})$.

$\mathcal{C}_3$	
$i = 0, 1, \ldots, \mu - 1$ $j = 0, 1, \ldots, \mu - 1$ $\alpha = 0, 1, 2, 3$	$(a_0^j, y_\alpha^{9i}, a_5^j, x_{\alpha+1}^{9i})$ (missing $(a_0^i, y_\alpha^{9i}, a_5^i, x_{\alpha+1}^{9i})$, $\alpha = 0, 3$) $(a_1^j, y_\alpha^{9i}, a_2^j, x_{\alpha+1}^{9i})$, $(a_3^j, y_\alpha^{9i}, a_4^j, x_{\alpha+1}^{9i})$, $(a_6^j, y_\alpha^{9i}, a_7^j, x_{\alpha+1}^{9i})$,
	$(a_2^j, y_\alpha^{1+9i}, a_6^j, x_{\alpha+1}^{1+9i})$ (missing $(a_2^i, y_\alpha^{1+9i}, a_6^i, x_{\alpha+1}^{1+9i})$, $\alpha = 0, 3$) $(a_0^j, y_\alpha^{1+9i}, a_1^j, x_{\alpha+1}^{1+9i})$, $(a_3^j, y_\alpha^{1+9i}, a_4^j, x_{\alpha+1}^{1+9i})$, $(a_5^j, y_\alpha^{1+9i}, a_7^j, x_{\alpha+1}^{1+9i})$,
	$(a_4^j, y_\alpha^{2+9i}, a_3^j, x_{\alpha+1}^{2+9i})$ (missing $(a_4^i, y_\alpha^{2+9i}, a_3^i, x_{\alpha+1}^{2+9i})$, $\alpha = 0, 3$) $(a_0^j, y_\alpha^{2+9i}, a_1^j, x_{\alpha+1}^{2+9i})$, $(a_2^j, y_\alpha^{2+9i}, a_5^j, x_{\alpha+1}^{2+9i})$, $(a_6^j, y_\alpha^{2+9i}, a_7^j, x_{\alpha+1}^{2+9i})$,
	$(a_0^j, y_\alpha^{3+9i}, a_7^j, x_{\alpha+1}^{3+9i})$ (missing $(a_0^i, y_\alpha^{3+9i}, a_7^i, x_{\alpha+1}^{3+9i})$, $\alpha = 0, 3$) $(a_1^j, y_\alpha^{3+9i}, a_2^j, x_{\alpha+1}^{3+9i})$, $(a_3^j, y_\alpha^{3+9i}, a_4^j, x_{\alpha+1}^{3+9i})$, $(a_5^j, y_\alpha^{3+9i}, a_6^j, x_{\alpha+1}^{3+9i})$,
	$(a_4^j, y_\alpha^{4+9i}, a_3^j, x_{\alpha+1}^{4+9i})$ (missing $(a_4^i, y_\alpha^{4+9i}, a_3^i, x_{\alpha+1}^{4+9i})$, $\alpha = 0, 3$) $(a_0^j, y_\alpha^{4+9i}, a_1^j, x_{\alpha+1}^{4+9i})$, $(a_2^j, y_\alpha^{4+9i}, a_5^j, x_{\alpha+1}^{4+9i})$, $(a_6^j, y_\alpha^{4+9i}, a_7^j, x_{\alpha+1}^{4+9i})$,
	$(a_0^j, y_\alpha^{5+9i}, a_1^j, x_{\alpha+1}^{5+9i})$ (missing $(a_0^i, y_\alpha^{5+9i}, a_1^i, x_{\alpha+1}^{5+9i})$, $\alpha = 0, 3$) $(a_2^j, y_\alpha^{5+9i}, a_3^j, x_{\alpha+1}^{5+9i})$, $(a_4^j, y_\alpha^{5+9i}, a_5^j, x_{\alpha+1}^{5+9i})$, $(a_6^j, y_\alpha^{5+9i}, a_7^j, x_{\alpha+1}^{5+9i})$,
	$(a_2^j, y_\alpha^{6+9i}, a_3^j, x_{\alpha+1}^{6+9i})$ (missing $(a_2^i, y_\alpha^{6+9i}, a_3^i, x_{\alpha+1}^{6+9i})$, $\alpha = 0, 3$) $(a_0^j, y_\alpha^{6+9i}, a_1^j, x_{\alpha+1}^{6+9i})$, $(a_4^j, y_\alpha^{6+9i}, a_5^j, x_{\alpha+1}^{6+9i})$, $(a_6^j, y_\alpha^{6+9i}, a_7^j, x_{\alpha+1}^{6+9i})$,
	$(a_4^j, y_\alpha^{7+9i}, a_5^j, x_{\alpha+1}^{7+9i})$ (missing $(a_4^i, y_\alpha^{7+9i}, a_5^i, x_{\alpha+1}^{7+9i})$, $\alpha = 0, 3$) $(a_0^j, y_\alpha^{7+9i}, a_1^j, x_{\alpha+1}^{7+9i})$, $(a_2^j, y_\alpha^{7+9i}, a_3^j, x_{\alpha+1}^{7+9i})$, $(a_6^j, y_\alpha^{7+9i}, a_7^j, x_{\alpha+1}^{7+9i})$,
	$(a_6^j, y_\alpha^{8+9i}, a_7^j, x_{\alpha+1}^{8+9i})$ (missing $(a_6^i, y_\alpha^{8+9i}, a_7^i, x_{\alpha+1}^{8+9i})$, $\alpha = 0, 3$) $(a_0^j, y_\alpha^{8+9i}, a_1^j, x_{\alpha+1}^{8+9i})$, $(a_2^j, y_\alpha^{8+9i}, a_3^j, x_{\alpha+1}^{8+9i})$, $(a_4^j, y_\alpha^{8+9i}, a_5^j, x_{\alpha+1}^{8+9i})$.

$\mathcal{C}_4$	
$\mu \geq 2$ $j = 0, 1, \ldots, \mu - 1$ $\gamma = 9\mu, 9\mu + 1, \ldots, 8\mu^2 + \mu - 1$ $\alpha, \sigma = 0, 1, 2, 3$	$(a^j_{2\sigma}, y^\gamma_\alpha, a^j_{1+2\sigma}, x^\gamma_{1+\alpha})$ missing the following cycles: (a) For $j = 0, 1, \ldots, \mu - 2$, $\rho = j + 1, j + 2, \ldots, \mu - 1$, $\beta = 0, 3$, $\tau = 9\mu + 16\big[j(\mu - 1) - \frac{j(j+1)}{2} + \rho - 1\big]$, $(a^j_0, y^\tau_\beta, a^j_1, x^\tau_{1+\beta})$, $(a^j_0, y^{1+\tau}_\beta, a^j_1, x^{1+\tau}_{1+\beta})$, $(a^j_2, y^{2+\tau}_\beta, a^j_3, x^{2+\tau}_{1+\beta})$, $(a^j_2, y^{3+\tau}_\beta, a^j_3, x^{3+\tau}_{1+\beta})$, $(a^j_4, y^{4+\tau}_\beta, a^j_5, x^{4+\tau}_{1+\beta})$, $(a^j_4, y^{5+\tau}_\beta, a^j_5, x^{5+\tau}_{1+\beta})$, $(a^j_6, y^{6+\tau}_\beta, a^j_7, x^{6+\tau}_{1+\beta})$, $(a^j_6, y^{7+\tau}_\beta, a^j_7, x^{7+\tau}_{1+\beta})$, $(a^\rho_0, y^{8+\tau}_\beta, a^\rho_1, x^{8+\tau}_{1+\beta})$, $(a^\rho_0, y^{9+\tau}_\beta, a^\rho_1, x^{9+\tau}_{1+\beta})$, $(a^\rho_2, y^{10+\tau}_\beta, a^\rho_3, x^{10+\tau}_{1+\beta})$, $(a^\rho_2, y^{11+\tau}_\beta, a^\rho_3, x^{11+\tau}_{1+\beta})$, $(a^\rho_4, y^{12+\tau}_\beta, a^\rho_5, x^{12+\tau}_{1+\beta})$, $(a^\rho_4, y^{13+\tau}_\beta, a^\rho_5, x^{13+\tau}_{1+\beta})$, $(a^\rho_6, y^{14+\tau}_\beta, a^\rho_7, x^{14+\tau}_{1+\beta})$, $(a^\rho_6, y^{15+\tau}_\beta, a^\rho_7, x^{15+\tau}_{1+\beta})$. (b) For $j = 0, 1, \ldots, \mu - 1$ and $\sigma = 0, 1, 2, 3$, $(a^j_{2\sigma}, y^\delta_2, a^j_{1+2\sigma}, x^\delta_3)$.

$\mathcal{C}_5$	
$i = 0, 1, \ldots, \mu - 1$ $\sigma = 0, 1, 2, 3$	$(a^i_{2\sigma}, \infty, a^i_{1+2\sigma}, x^{5+\sigma+9i}_1)$, $(a^i_{2\sigma}, y^\delta_2, a^i_{1+2\sigma}, x^{5+\sigma+9i}_0)$, $(x^{9i}_0, y^{9i}_3, a^i_0, y^{9i}_0)$, $(x^{9i}_1, y^{9i}_3, a^i_5, y^{9i}_0)$, $(x^{1+9i}_0, y^{1+9i}_3, a^i_2, y^{1+9i}_0)$, $(x^{1+9i}_1, y^{1+9i}_3, a^i_6, y^{1+9i}_0)$, $(x^{2+9i}_0, y^{2+9i}_3, a^i_4, y^{2+9i}_0)$, $(x^{2+9i}_1, y^{2+9i}_3, a^i_3, y^{2+9i}_0)$, $(x^{3+9i}_0, y^{3+9i}_3, a^i_0, y^{3+9i}_0)$, $(x^{3+9i}_1, y^{3+9i}_3, a^i_7, y^{3+9i}_0)$, $(x^{4+9i}_0, y^{4+9i}_3, a^i_4, y^{4+9i}_0)$, $(x^{4+9i}_1, y^{4+9i}_3, a^i_3, y^{4+9i}_0)$, $(x^{5+9i}_0, y^{5+9i}_3, a^i_0, y^{5+9i}_0)$, $(x^{5+9i}_1, y^{5+9i}_3, a^i_1, y^{5+9i}_0)$, $(x^{6+9i}_0, y^{6+9i}_3, a^i_2, y^{6+9i}_0)$, $(x^{6+9i}_1, y^{6+9i}_3, a^i_3, y^{6+9i}_0)$, $(x^{7+9i}_0, y^{7+9i}_3, a^i_4, y^{7+9i}_0)$, $(x^{7+9i}_1, y^{7+9i}_3, a^i_5, y^{7+9i}_0)$, $(x^{8+9i}_0, y^{8+9i}_3, a^i_6, y^{8+9i}_0)$, $(x^{8+9i}_1, y^{8+9i}_3, a^i_7, y^{8+9i}_0)$.

$\mathcal{C}_6$	
$\mu \geq 2$ $j = 0, 1, \ldots, \mu - 2$ $\rho = j + 1, j + 2, \ldots, \mu - 1$ $\tau = 9\mu + 16\big[j(\mu - 1) -$ $- \frac{j(j+1)}{2} + \rho - 1\big]$ $\sigma = 0, 1, 2, 3$	$(x^{\tau+2\sigma}_0, y^{\tau+2\sigma}_3, a^j_{2\sigma}, y^{\tau+2\sigma}_0)$, $(x^{\tau+2\sigma}_1, y^{\tau+2\sigma}_3, a^j_{1+2\sigma}, y^{\tau+2\sigma}_0)$, $(x^{1+\tau+2\sigma}_0, y^{1+\tau+2\sigma}_3, a^j_{2\sigma}, y^{1+\tau+2\sigma}_0)$, $(x^{1+\tau+2\sigma}_1, y^{1+\tau+2\sigma}_3, a^j_{1+2\sigma}, y^{1+\tau+2\sigma}_0)$, $(x^{8+\tau+2\sigma}_0, y^{8+\tau+2\sigma}_3, a^j_{2\sigma}, y^{8+\tau+2\sigma}_0)$, $(x^{8+\tau+2\sigma}_1, y^{8+\tau+2\sigma}_3, a^j_{1+2\sigma}, y^{8+\tau+2\sigma}_0)$, $(x^{9+\tau+2\sigma}_0, y^{9+\tau+2\sigma}_3, a^j_{2\sigma}, y^{9+\tau+2\sigma}_0)$, $(x^{9+\tau+2\sigma}_1, y^{9\tau+2\sigma}_3, a^j_{1+2\sigma}, y^{9+\tau+2\sigma}_0)$.