The arc-width of a graph

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Abstract

The arc-representation of a graph is a mapping from the set of vertices to the arcs of a circle such that adjacent vertices are mapped to intersecting arcs. The width of such a representation is the maximum number of arcs having a point in common. The arc-width(aw) of a graph is the minimum width of its arc-representations. We show how arc-width is related to path-width and vertex-width. We prove that $aw(K_{s,s}) = s$.

1 Introduction

The notation and terminology of the paper follows [2].

In the Graph Minors project Robertson and Seymour (often with other co-authors) introduced several minor-monotone graph parameters. We recall their first such parameter. Our definition is a dual to the original one appearing in [3]. About the equivalence, see e.g. Exercises 24, 25 in Chapter 12 of [2].

Definition 1 *The interval-representation of a graph G is a mapping ϕ from its vertex set to the intervals of a base line, such that adjacent vertices are mapped to intersecting intervals. The width (in a representation) of a point P of the base line is the number of intervals containing P. The width of ϕ is the maximum width*

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of the points of the base line. (This is equal to the maximum number of pairwise intersecting intervals.) The interval-width *of a graph G is the minimal possible width of such interval-representations, $pw^*(G)$ in notation.*

Actually, Robertson and Seymour defined path-width(pw), which is one less than the above defined interval-width. They extended the notion of path-width by changing the base line to a tree, and substituting intervals with subtrees. (This is still the dual, not the original definition.) The new parameter thus obtained is the tree-width of a graph. We take another route for extension. We substitute the base line with a base circle:

Definition 2 *The* arc-representation *of a graph G is a mapping ϱ from the vertex set $V(G)$ to the set of arcs of a base circle, such that adjacent vertices of G are mapped to intersecting arcs. The* width *(in a representation) of a point P of the base circle is the number of representing arcs containing P. The* width *of ϱ is the maximum width of the points of the base circle. (This is not the maximum number of pairwise intersecting arcs.) The* arc-width *of a graph G is the minimal possible width of such arc-representations, $aw(G)$ in notation.*

Our base circle will be the unit circle on the coordinate-plane, i.e. the set of points $(\cos x, \sin x)$, where $x \in [0, 2\pi)$. An arc is a connected subset of the circle. It is easy to see that the notion of arc-width is not changed if we insist on representing vertices with closed arcs. We also assume that all the representing arcs are proper subsets of the circle. Fixing the clockwise orientation of the circle, each arc has a unique starting and ending point. Let $arc(x_1, x_2)$ denote the closed arc starting at $(\cos x_1, \sin x_1)$ (the left endpoint of the arc) and ending at $(\cos x_2, \sin x_2)$ (the right endpoint of the arc). In this way, points on the circle and their corresponding angles are naturally identified. For an arc-representation ϱ of G, the left endpoint of the arc representing u is denoted by $l(u)$, and the right endpoint is denoted by $r(u)$, i.e. $\varrho(u) = arc(l(u), r(u))$.

It is easy to prove the minor-monotonicity of the newly introduced graph parameter.

Lemma 3 *Arc-width is minor-monotone.* □

2 Connections to other graph parameters

Robertson and Seymour proved that any graph without H as a minor can be obtained by clique sums from graphs that can be "nearly drawn" in a surface $\Sigma - k_H$ (where Σ is any closed surface such that H can not be drawn in it, and k_H is a suitable integer). This theorem extends their earlier results, when they established similar structural theorems in the cases when H was a tree, respectively a planar graph. This extension is crucial in the proof of the Graph Minor Theorem. To measure how nearly a graph is drawn in $\Sigma - k_H$, they introduced the notion of vortex-decomposition in [4], see also [5]. Using their previous results on path-width, Robertson and Seymour did not need to investigate this new parameter in depth (in spite of its inherent naturalness).

Definition 4 [5] *Let G be a graph, and let U be a cyclic ordering of a subset of its vertices. We say that $(X_u)_{u \in U}$ is a* vortex-decomposition *of the pair (G, U) if*

$(V1)$ $u \in X_u$ for every $u \in U$,

$(V2)$ $\bigcup_{u \in U} X_u = V(G)$, and every edge of G has both ends in some X_u,

$(V3)$ for every vertex $v \in V(G)$, the set of all $u \in U$ with $v \in X_u$ is a contiguous interval (the empty set, or the whole U are possibilities).

We say that $(X_u)_{u \in U}$ has width k, if $|X_u| \leq k$ for every $u \in U$.

The vortex-width *of G (denoted by $vw(G)$) is the minimum width taken over all vortex-decompositions of G.*

Lemma 5 *$aw(G) = vw(G)$.*

Proof: We construct a vortex-decomposition of G from an arc-representation of G with the same width, and vice versa.

Assume first that there is a given arc-representation ϱ of G of width k. Let L be the set of the points of the base circle being a left endpoint of some representing arc. For each element $\ell \in L$, choose a vertex v such that the corresponding arc $\varrho(v)$ starts at ℓ. Let U be the set of chosen vertices. Let $X_u := \{v : \varrho(v) \cap l(u) \neq \emptyset\}$.

(V1) and (V3) are satisfied. An edge of G corresponds to two intersecting arcs in ϱ. Moreover, if two arcs intersect, then they also intersect above a left endpoint. Hence (V2) holds. The width is k by the definition of X_u.

Assume now that a vortex-decomposition (G, U) of width k is given. To each element $u \in U$ associate a point P_u of the base circle, inheriting the cyclic order of the vertices in U. By (V3) we can associate an arc to every vertex v as follows: if $v \in X_{u_{i_1}}, \ldots, X_{u_{i_s}} \in U$ (following the cyclic order), then $l(v) := P_{u_{i_1}}$ and $r(v) := P_{u_{i_s}}$ in ϱ. In this way an arc-representation ϱ arises. The width of ϱ is k. $\square$

Arc-width is a natural modification of interval-width (path-width). There is a quantitative connection, not just a formal one. The next lemma shows that the two measures are within a factor of 2.

Definition 6 *Let w_{min} and w_{max} denote the minimum and maximum width of the points in an arc-representation ϱ. Then ϱ is called a* (u, w)-representation *if and only if $u = w_{min}(\varrho)$ and $w = w_{max}(\varrho)$.*

Observe that $w_{max}(\varrho) \gneq w_{min}(\varrho)$.

Lemma 7 *Let ϱ be an arc-representation of G. Then*

(i) $pw^*(G) \leq w_{max}(\varrho) + w_{min}(\varrho)$ and

(ii) $\left\lceil \frac{1}{2}(pw^*(G) + 1) \right\rceil \leq aw(G) \leq pw^*(G)$.

Proof: (i) Let $i := w_{min}(\varrho)$, $j := w_{max}(\varrho)$. There is a point x of the base circle where the width is precisely i. Cut the base circle at x, and strengthen it to obtain a line. In this way, the arcs not containing x become intervals. Substitute the arcs containing x (there are i of them) with the whole line as a representing interval. We produced an interval representation of G of width $i + j$.

(ii) The second inequality is trivial. The first one follows from (i). In the next section, we exhibit examples showing that both bounds are sharp. $\square$

3 Special graph classes

Extending the base line to a base circle (Definition 1 → Definition 2) does not always help in the representation.

Lemma 8 *If T is a tree, then $aw(T) = pw^*(T)$.*

Proof: If C denotes the base circle, and ℓ is a covering line of C, then let $p : \ell \to C$ be the corresponding continuous projection. We prove by induction that we can construct an interval-representation λ of T from an arc-representation ϱ of T, with the following properties:
– every arc $\varrho(v)$ of C has a corresponding interval $\lambda(v)$ on ℓ such that $p(\lambda(v)) = \varrho(v)$,
– λ is an interval-representation of T,
– the width of λ at a point P of ℓ is at most the width of ϱ at the point $p(P)$ of C.

Let us call such an interval-representation 'good'.

If T is a single vertex, then we are easily done.

Assume now that the statement is true for any tree with at most $k-1$ vertices. Consider a tree T with k vertices. Delete a leaf v of T getting a graph T' with $k-1$ vertices. Let u be the only neighbor of v. Let ϱ be an arbitrary arc-representation of T, inducing an arc-representation $\varrho' = \varrho|_{T'}$ of T'. By the induction hypothesis we can obtain a good interval-representation λ' of T' based on ϱ'. Let $P \in \varrho(u) \cap \varrho(v)$. There exists $\widehat{P} \in \lambda(u)$ such that $p(\widehat{P}) = P$. Let $\varrho(v)$ be an interval I of ℓ, containing $\widehat{P}$, and intersecting none of the other intervals $\lambda(w)$. Extending λ' with I as $\lambda(v)$, we obtain the desired good interval-representation of T. □

The condition on T is not necessary. There are other graphs satisfying $aw(T) = pw^*(T)$. In general one expects $aw(G) < pw^*(G)$, e.g. we can easily see the following.

Lemma 9 *$aw(C_n) = 2$ and $pw^*(C_n) = 3$, where C_n is the cycle with $n \geq 3$ vertices.* □

The complete graphs exhibit the other extreme: their arc-width and their path-width are as far as possible from each other. This is not difficult to prove, so we omit the details.

Lemma 10 $aw(K_n) = \left[\frac{n}{2}\right] + 1$ □

The complete bipartite graph has much less edges. But surprisingly, its arc-width is almost the same. We first give an arc-representation of $K_{s,s}$ of width s. Then we prove that this arc-width is best possible for $K_{s,s}$.

Lemma 11 *$aw(K_{s,s}) \leq s$.*

Proof: Let the two color-classes of $K_{s,s}$ be $\{x_1, \ldots, x_s\}$ and $\{y_1, \ldots, y_s\}$. Consider the following arc-representation of $K_{s,s}$ (ε denotes an arbitrarily small positive real number):

$$a_1 = arc\left(-\varepsilon, (s-2)\frac{2\pi}{s} + \varepsilon\right)$$

$$b_1 = arc\left(\varepsilon, \frac{2\pi}{s} - \varepsilon\right)$$

Let ϕ denote the clockwise rotation by $\frac{2\pi}{s}$, and ϕ^k that the rotation ϕ is repeated k times. Let $\varrho(x_i) = \phi^i a_1$, and $\varrho(y_i) = \phi^i b_1$. $\varrho(x_i) \cap \varrho(y_j) \neq \emptyset$, so ϱ is an arc-representation of $K_{s,s}$. Counting the width of the points of the base circle we get:

$width\left(\frac{2\pi}{s} \pm \varepsilon\right) = s$, otherwise $width(x) = s - 1$. □

Theorem 12 $aw(K_{s,s}) = s$.

Proof: We have to prove $w_{max}(\varrho(K_{s,s})) \geq s$ for any arc-representation ϱ of $K_{s,s}$. Let the two color-classes G and B be called green and blue. $\mathcal{F} := \{\varrho(v) : v \in G\}$; this is a system, not necessarily a set. We call the elements of $\mathcal{F}$ green arcs. Let the set of left respectively right endpoints of the arcs in $\mathcal{F}$ be denoted by L and R.

1. We may assume that the $2s$ endpoints are different, because this can be reached with a small movement of the endpoints (without increasing the width).

2. We may assume that $\bigcap_{I \in \mathcal{F}} I = \emptyset$. If this were false, then the width would already be at least s.

3. We may assume that if $I \neq J \in \mathcal{F}$, then $I \not\subset J$. If there were two arcs $I \subset J$, then the endpoints would satisfy $l(J) < l(I) < r(I) < r(J)$. Substitute I by $I' := (l(J), r(I))$ and J by $J' := (l(I), r(J))$. The width did not change, and now $I' \not\subset J'$. Moreover, if any arc A intersects I (and hence J), then A intersects both I' and J', for $I \subset I'$ and $I \subset J'$. Hence we still have a representation of $K_{s,s}$. We repeat this operation as long as we find two green arcs, one containing the other. Our process will terminate, because the length of the longer arc after the change is strictly less than the length of the longer arc before, and the possible length of the arcs are determined by the possible $2s$ endpoints (hence finite).

4. Consider now the blue arcs. We may assume that they are minimal, i.e. we cannot decrease them by maintaining the necessary intersections. Hence there are only finitely many possible blue arcs (depending on the fixed green arcs). We call these *candidates*. For an arc $I \in \mathcal{F}$, consider its candidate complementary arc I' which is defined as follows: $l(I') := r(I)$, and $r(I') := l(J)$, where $J \in \mathcal{F}$ is the arc such that if I' intersects J, then I' intersects every arc of $\mathcal{F}$. In this way we have defined a set system of arcs: $\mathcal{F}' := \{I' : I \in \mathcal{F}\}$. The correspondence $I \leftrightarrow I'$ described above is a bijection. Moreover, the $2s$ endpoints of the arcs of $\mathcal{F}'$ are exactly the same as the endpoints of the arcs of $\mathcal{F}$. The left endpoints of the arcs in $\mathcal{F}'$ are different. Hence we only have to show that two right endpoints cannot coincide. Assume that $l(I'_1) < l(I'_2)$ and $r(I'_1) = r(I'_2)$. Then by 3 and the definition of I'_2, $r(I'_2) > l(I'_1)$, contradicting the definition of $r(I'_1)$.

5. We consider now a special case, when the blue arcs are exactly the elements of $\mathcal{F}'$. Let us take all the arcs in $\mathcal{F}$ or in $\mathcal{F}'$ as disjoint arcs. Glue them together at the common endpoints. This 'snake' covers the base circle $s - 1$ times. To see this, consider a point, $l(I)$ say, $I \in \mathcal{F}$. Cut the base circle at $l(I)$ to get the non-negative real line with $l(I) = 0$, and the natural $<$ relation. $l(I) \in J'$ if $r(I) < r(J)$; $l(I) \in J$ if $l(I) < r(J) < r(I)$, and for every $J \in \mathcal{F}$, $J \neq I$, exactly one case occurs.

In this special case we are done: each endpoint is covered by s arcs.

6. From now on we consider the general case, when the blue arcs constitute an arbitrary system of n arcs from $\mathcal{F}'$. Let $A \in \mathcal{F}'$. Let $\mu(A)$ be the multiplicity of A among the blue arcs.

The points of L and R divide the base circle into $2s$ open arcs. Let us call them *elementary arcs*. We consider the width over an elementary arc, e say. There are some arcs representing green vertices, which contain e. Moreover, there are some candidate arcs $A_1, \dots, A_k$ of $\mathcal{F}'$ containing e. There are multiplicities associated with these arcs, say $\mu(A_1), \dots, \mu(A_k)$ respectively. Assume there are q green arcs containing e. Let $\mathcal{F}'_e$ denote the set of blue arcs containing e. From 5 we know that $q + |\mathcal{F}'_e| = s - 1$. If $q + \sum_{A \in \mathcal{F}'_e} \mu(A) > s - 1$, then we are done. Otherwise we get the following inequality:

$$\sum_{A \in \mathcal{F}'_e} \mu(A) \leq |\mathcal{F}'_e|.$$

Observe that the number on the right-hand side is the number of terms on the left-hand side.

7. For every elementary arc e, we define its *successor* e^* as follows. Cutting the base circle at $r(e)$, consider the first arc $I_e \in \mathcal{F}$, which is completely after $r(e)$. (Observe that 'first' is well-defined. If an arc starts first, it also has to end first by 3.) There is an elementary arc beginning at $r(I_e)$, which is defined to be e^*. Formally, $l(e^*) := \min_{I \in \mathcal{F}: r(e) \notin I} r(I)$.

8. Let us define a directed graph D. The elementary arcs are the vertices of D, and the edges of D are of the form (e, e^*). More precisely, we define the edges to be geometric objects, namely (e, e^*) is the arc of the base circle (in clockwise order) from the middle of e to the middle of e^*. Every elementary arc has out-degree one in D, hence there is a directed circuit C in D.

9. The edges of C (glued together as in 5) cover the base circle homogeneously t times. If we consider an arbitrary point P of the base circle, then there exist exactly t edges of D going over P. Let $I' \in \mathcal{F}'$ be the first arc which is completely after P. Every edge f over P has a tail e such that e is an elementary arc disjoint from I'. This is a consequence of the definition in 7. Also vice versa: if an edge $f \in D$ is not over P, then the tail e of f is an elementary arc which intersects I'. We can interpret this result in another way: whenever we consider an arbitrary arc $I' \in \mathcal{F}'$, then the number of elementary arcs intersecting I' is a constant positive number, c say.

10. Consider now the inequalities of 6 only for the vertices of C. Let $V(C) = \{e_1, \dots, e_p\}$. Summing up all of these inequalities:

$$\sum_{I \in \mathcal{F}'_{e_1}} \mu(I) \leq |\mathcal{F}'_{e_1}|$$

$$\vdots$$

$$\sum_{I \in \mathcal{F}'_{e_p}} \mu(I) \leq |\mathcal{F}'_{e_p}|$$

$$\sum_{e\in C}\sum_{I\in\mathcal{F}'_e}\mu(I)\leq\sum_{e\in C}|\mathcal{F}'_e|.$$

By 9 the left-hand side is $c\sum_{I'\in\mathcal{F}}\mu(I')$. Hence by 6, the right-hand side is $c\cdot s$. We obtain the following:

$$c\sum_{I'\in\mathcal{F}}\mu(I')\leq c\cdot s.$$

c is positive, so simplification gives:

$$\sum_{I'\in\mathcal{F}}\mu(I')\leq s.$$

But we know that here equality holds. This is only possible if equality holds everywhere in the above inequalities. Hence the width of the representation over an elementary arc $e\in C$ is exactly $s-1$. Hence at the endpoint of an elementary arc the width is at least s. □

4 Concluding remarks

We pointed out a natural graph parameter which had been neglected so far. Considering the arc-width we found challenging problems (like determining the arc-width of $K_{s,s}$). We hope that our work might lead to a better understanding of some phenomena connected to the Graph Minor Theorem.

The results of Lemma 10 and Theorem 12 show that $K_{s,s}$ has much fewer edges than K_{2s}, but its arc-width is less only by one. Hence we ask the following:

Problem 13 *How many edges of K_{2s} can be deleted without decreasing the arc-width?*

Both inequalities of Lemma 7(ii) are sharp by Lemma 8 and Lemma 10. A natural question is the determination of all extremal graphs:

Problem 14 *Characterize the graphs G with property $aw(G)=pw^*(G)$, respectively $2aw(G)=pw^*(G)+1$.*

Arc-width is minor-monotone, hence we can also ask the excluded minors of the class $aw(G)\leq k$. Such a list is similar to the one we get for path-width. By Lemma 8 some of the elements of the two parallel lists are the same. It is known that the number of excluded trees for $pw(G)\leq k$ grows superexponentially with k. So finding complete lists is an enormous task even for small k. However the following comparison is of interest:

Problem 15 *Is the number of excluded minors for arc-width k always greater than the number of excluded minors for interval-width k?*

We have seen in the proof of Lemma 7(i) that knowing the arc-width of a graph is not sufficient for some arguments. The minimum width of the representation is also necessary to be taken into account. The minimum-maximum width pair (mM) is a refinement of arc-width which carries more information. This parameter is still minor-monotone. An interesting property of mM is that disjoint union of graphs appear among the excluded minors. For further results see [1].

We also wonder if there is any real-life usage of arc-width, similar to the application of circular-arc graphs.

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