All regular multigraphs of even order and high degree are 1-factorable

Michael J. Plantholt
Shailesh K. Tipnis

Department of Mathematics
Illinois State University
Normal, IL 61790-4520, USA
mikep@math.ilstu.edu, tipnis@math.ilstu.edu

Submitted: March 13, 2001; Accepted: December 8, 2001.
Mathematical Reviews Subject Classification (2000): 05C15, 05C70.

Abstract

Plantholt and Tipnis (1991) proved that for any even integer r, a regular multigraph G with even order n, multiplicity $\mu(G) \leq r$ and degree high relative to n and r is 1-factorable. Here we extend this result to include the case when r is any odd integer. Häggkvist and Perković and Reed (1997) proved that the One-factorization Conjecture for simple graphs is asymptotically true. Our techniques yield an extension of this asymptotic result on simple graphs to a corresponding asymptotic result on multigraphs.

1 Introduction

Let G be a multigraph with vertex set $V(G)$ and edge set $E(G)$. We denote the maximum degree of G by $\Delta(G)$, the minimum degree of G by $\delta(G)$ and the multiplicity of G, that is, the maximum number of parallel edges between any pair of vertices of G by $\mu(G)$. G is said to be *simple* if $\mu(G) = 1$. We say that G is *1-factorable* if the edges of G can be partitioned into 1-factors of G. We denote by $\mathrm{simp}(G)$, the simple graph underlying G, i.e. $\mathrm{simp}(G)$ is the graph obtained by replacing all edges of G with multiplicity greater than one by single edges. In this paper, a *decomposition* of G into edge-disjoint subgraphs $H_1, H_2, \ldots, H_k$ of G means a partition of $E(G)$ into the union of the edge sets of $H_1, H_2, \ldots, H_k$, and we abuse the notation and write $G = H_1 \cup H_2 \cup \ldots \cup H_k$ instead of $E(G) = E(H_1) \cup E(H_2) \cup \ldots \cup E(H_k)$. The reader is referred to Bondy and Murty [2] for all terminology undefined in this paper. The following long-standing conjecture whose origin is unclear claims that any regular, simple graph of even order and with degree at least half the number of vertices is 1-factorizable (see [10]).

One-factorization Conjecture *Let G be a Δ-regular simple graph with even order n. If $\Delta \geq \frac{1}{2}n$ then G is 1-factorable.*

This conjecture is best possible as indicated by the example when G consists of two disjoint copies of K_3. An example of a connected graph to illustrate that Conjecture 1 is best possible is obtained by taking two disjoint copies of $K_5 - e$ where e is any edge of K_5 and joining the corresponding end-vertices of e in the two copies of $K_5 - e$ by edges. Chetwynd and Hilton [3] proved that Conjecture 1 is true if we replace the condition that $\Delta(G) \geq \frac{1}{2}n$ in Conjecture 1 by the stronger condition that $\Delta(G) \geq \frac{\sqrt{7}-1}{2}n$.

Theorem 1 (Chetwynd and Hilton [3]) *Let G be a simple graph with even order n. If G is Δ-regular with $\Delta \geq \frac{\sqrt{7}-1}{2}n$ then G is 1-factorable.*

Hägkvist [6] and Perković and Reed [7] proved that Conjecture 1 is asymptotically true.

Theorem 2 (Häggkvist [6], Perković and Reed [7]) *For every $\epsilon > 0$, there exists $N(\epsilon)$ such that if G is a simple graph that is Δ-regular with even order $n > N(\epsilon)$ and with $\Delta \geq (\frac{1}{2} + \epsilon)n$, then G is 1-factorable.*

We offer the following natural extension of the One-factorization conjecture to multigraphs.

Multigraph One-factorization Conjecture *Let G be a Δ-regular multigraph with even order n and multiplicity $\mu(G) \leq r$. If $\Delta \geq \frac{1}{2}rn$ then G is 1-factorable.*

In this paper we prove extensions of Theorems 1 and 2 to multigraphs as given in Theorems 3 and 4 below.

Theorem 3 *Let G be a Δ-regular multigraph with even order n and multiplicity $\mu(G) \leq r$.*

(i) If r is even and $\Delta \geq \lceil \frac{\sqrt{7}-1}{2}n + 1\rceil r$, then G is 1-factorable.

(ii) If r is odd and $\Delta \geq \lceil \frac{\sqrt{7}-1}{2}n + 2\rceil r + 1$, then G is 1-factorable.

Theorem 4 *For every $\epsilon > 0$, there exists $N^*(\epsilon)$ such that if G is a Δ-regular multigraph with $\mu(G) \leq r$ and even order $n > N^*(\epsilon)$, then G is 1-factorizable if*

(i) r is even and $\Delta \geq (\frac{1}{2} + \epsilon)rn$, or

(ii) r is odd and $\Delta \geq (\frac{1}{2} + \frac{1}{2r} + \epsilon)rn$.

The proof of part (i) of Theorem 3 appeared in [8]. The approach taken in this proof was to decompose the edges of the multigraph G with even order n and multiplicity $\mu(G) \leq r$ (where r is even) into a relatively small number of 1-factors of G and a number of regular, simple graphs, each with degree high relative to n. Theorem 1 was then applied to each of the simple graphs in the decomposition to yield a 1-factorization of the original multigraph G. In Section 2 of this paper we use this decomposition result for the case when r is even and Tutte's f-factor theorem [9] to obtain a similar decomposition of the edges of G with even order n and multiplicity $\mu(G) \leq r$, where r is odd. In Section 3 we use our decomposition result from Section 2 to prove Theorem 3 and Theorem 4.

2 Decomposition of regular multigraphs into regular simple graphs

The following decomposition result for regular multigraphs G with even order n, multiplicity $\mu(G) \leq r$, where r is even, and with degree high relative to n and r was proved in [8]. Many similar results on decompositions of multigraphs into simple graphs were obtained in [5].

Theorem 5 (Plantholt and Tipnis [8]) *Let G be a Δ-regular multigraph with even order n and multiplicity $\mu(G) \leq r$, where r is an even integer. If $\Delta = kr + r$ for some integer $k \geq \frac{n}{2}$, then the edges of G can be decomposed into r 1-factors of G and r k-regular simple graphs.*

We will prove the following theorem that extends Theorem 5 to the case when $r > 1$ is an odd integer.

Theorem 6 *Let G be a Δ-regular multigraph with even order n and multiplicity $\mu(G) \leq r$, where $r > 1$ is an odd integer. If $\Delta = kr + 2r + 1$ for some integer $k \geq \frac{n}{2} + \frac{n}{2r}$, then the edges of G can be decomposed into $2r$ 1-factors of G, a $(k+1)$-regular simple graph, and $(r-1)$ k-regular simple graphs.*

In order to prove Theorem 6 we will need Theorem 7 and Theorem 8 stated below. Theorem 7 is a classic result of Dirac [4] giving a sufficient condition for the existence of a Hamilton cycle in a simple graph and Theorem 8 is a classic result of Tutte [9] giving a necessary and sufficient condition for the existence of an f-factor in a multigraph G.

Theorem 7 (Dirac [4]) *Let G be a simple graph with order $n \geq 3$. If $\delta(G) \geq \frac{1}{2}n$ then G contains a Hamilton cycle.*

We now define some terminology needed to state Tutte's f-factor theorem. See Bollobás [1] for most of this terminology and the statement of Theorem 8. We will denote the degree of vertex $v \in V(G)$ by $\deg_G(v)$. Let G be a multigraph and suppose that each $v \in V(G)$ is assigned a positive integer $f(v)$. An *f-factor* of G is a spanning subgraph F of G such that $\deg_F(v) = f(v)$ for each $v \in V(G)$. For $X, Y \subseteq V(G)$ we denote by $(X, Y; G)$ the set of edges of G that have one end-vertex in X and the other end-vertex in Y. For disjoint subsets $D, S \subseteq V(G)$ and a component C of $G - D - S$, we define $\rho(D, S; C) = |(C, S; G)| + \sum_{x \in C} f(x)$. Component C is said to be an *odd* or *even* component of $G - D - S$ with respect to D and S according as $\rho(D, S; C)$ is odd or even. The number of all odd components of $G - D - S$ is denoted by $q[D, S; G]$.

Theorem 8 (Tutte [9]) *Let G be a multigraph and suppose that each $v \in V(G)$ is assigned a positive integer $f(v)$. Then, G has an f-factor if and only if*

$$q[D, S; G] + \sum_{x \in S} f(x) \leq \sum_{x \in S} \deg_{G-D}(x) + \sum_{x \in D} f(x)$$

for all disjoint subsets $D, S \subseteq V(G)$.

We mention here that in proving Theorem 5, we will only use the sufficiency of a condition stronger than the condition in Theorem 8 to guarantee an f-factor in a certain multigraph. We need two Lemmas before we turn to the proof of Theorem 6.

Lemma 1 *Let G be a Δ-regular multigraph with maximum multiplicity $\mu(G) \leq r$ and suppose that $\Delta = rs$. Suppose that G contains $\lceil \frac{r}{2} \rceil$ edge-disjoint Hamilton cycles such that for all $u, v \in V(G)$, if t of these Hamilton cycles contain an edge of the form (u, v), then the multiplicity of the edge uv in G is at most $r - t$. Then, G contains a simple s-factor F such that $\mu(G - F) \leq r - 1$.*

Proof. Let G' be the graph obtained from G by deleting all sets of r parallel edges. Note that since $\deg_G(v)$ is a multiple of r for each $v \in V(G)$, $\deg_{G'}(v)$ is also a multiple of r for each $v \in V(G')$. Moreover, since G' contains all edges from the $\lceil \frac{r}{2} \rceil$ Hamilton cycles in G, $\deg_{G'}(v) > 0$ for each $v \in V(G')$. Define $f(v) = \frac{1}{r}\deg_{G'}(v)$ for each $v \in V(G')$. Then, it is clear that G contains a simple s-factor F such that $\mu(G - F) \leq r - 1$ if and only if G' has a simple f-factor.

From Theorem 8, to show that G' has a simple f-factor it suffices to show that

$$q[D, S; \mathrm{simp}(G')] + \sum_{x \in S} f(x) \leq \sum_{x \in S} \deg_{\mathrm{simp}(G'-D)}(x) + \sum_{x \in D} f(x) \tag{1}$$

for all disjoint subsets $D, S \subseteq V(\mathrm{simp}(G'))$.

Let $D, S \subseteq V(\mathrm{simp}(G'))$ be disjoint subsets. It is easy to check that each term in inequality (1) is zero if $D = \emptyset$ and $S = \emptyset$. So, for the rest of the proof, assume that $D \cup S \neq \emptyset$. Let C denote the multigraph $\mathrm{simp}(G') - D - S$ and suppose that the multigraph C consists of k components. We examine in turn, the three summations in inequality (1). First, by the definition of f, we have that

$$\sum_{x \in S} f(x) = \frac{1}{r} \sum_{x \in S} \deg_{G'}(x) = \frac{1}{r} \sum_{x \in S} \deg_{(G'-D)}(x) + \frac{1}{r}|(S, D; G')|. \tag{2}$$

To examine the second sum in inequality (1), let G'_+ be the multigraph whose underlying simple graph is $\mathrm{simp}(G')$ and the multiplicity of each of whose edges is r. Let l denote the number of edges (including multiplicity) from the $\lceil \frac{r}{2} \rceil$ edge-disjoint Hamilton cycles of G (as in the statement of Lemma 1) that are also in $(C, S; G)$. The definition of G'_+ implies that

$$\sum_{x \in S} \deg_{\mathrm{simp}(G'-D)}(x) = \sum_{x \in S} \deg_{\mathrm{simp}(G'_+-D)}(x) = \frac{1}{r} \sum_{x \in S} \deg_{(G'_+-D)}(x).$$

Now, since for all $u, v \in V(G)$, if t of the $\lceil \frac{r}{2} \rceil$ edge-disjoint Hamilton cycles of G (as in the statement of Lemma 1) contain an edge of the form (u, v), then the multiplicity of the edge uv in G is at most $r - t$, we have that

$$\sum_{x \in S} \deg_{\mathrm{simp}(G'-D)}(x) = \frac{1}{r} \sum_{x \in S} \deg_{(G'_+-D)}(x) \geq \frac{l}{r} + \frac{1}{r} \sum_{x \in S} \deg_{(G'-D)}(x). \tag{3}$$

Finally, for the third sum in inequality (1), the definition of f implies

$$\sum_{x \in D} f(x) = \frac{1}{r} \sum_{x \in D} \deg_{G'}(x) \geq \frac{1}{r} |(D, S; G')| + \frac{1}{r} |(D, C; G')|.$$

Note that since G contains $\lceil \frac{r}{2} \rceil$ Hamilton cycles and none of the edges in these Hamilton cycles have multiplicity r, we have that $|(C, D \bigcup S; G')| \geq \lceil \frac{r}{2} \rceil 2k$. Hence we have that,

$$\sum_{x \in D} f(x) \geq \frac{1}{r} |(D, S; G')| + \frac{1}{r} |(D, C; G')| \geq \frac{1}{r} |(D, S; G')| + \frac{1}{r} (rk - l). \tag{4}$$

Now, combining the fact that $q[D, S; G'] \leq k$ with equation (2) and inequalities (3) and (4) easily yields the desired inequality (1).

Lemma 2 *Let G be a Δ-regular multigraph with even order n and multiplicity $\mu(G) \leq r$, where $r > 1$ is an odd integer. If $\Delta = kr + 2r + 1$ for some integer $k \geq \frac{n}{2} + \frac{n}{2r}$, then G contains $\lceil \frac{r}{2} \rceil$ identical pairs of edge-disjoint Hamilton cycles.*

Proof. For a multigraph H, denote by $H2$ the spanning subgraph of H whose edge set consists of all edges of H with multiplicity at least two. Suppose that H is a Δ-regular multigraph with even order n and multiplicity $\mu(H) \leq r$, where $r > 1$ is an odd integer. If $\Delta \geq \frac{nr}{2} + \frac{n}{2}$ then $\deg_{\mathrm{simp}(H2)}(v) \geq \frac{n}{2}$ for each $v \in V(H)$, because $\deg_{\mathrm{simp}(H2)}(v) < \frac{n}{2}$ for some $v \in V(H)$ implies that $\deg_H(v) < \frac{rn}{2} + (n-1) - \frac{n}{2} = \frac{rn}{2} + \frac{n}{2} - 1$, a contradiction. Now let G be a Δ-regular multigraph with even order n and multiplicity $\mu(G) \leq r$, where $r > 1$ is an odd integer, and $\Delta(G) = kr + 2r + 1$ for some integer $k \geq \frac{n}{2} + \frac{n}{2r}$. Then, $\Delta(G) \geq \frac{nr}{2} + \frac{n}{2} + 2r + 1$ and so, $\deg_{\mathrm{simp}(G2)}(v) \geq \frac{n}{2}$ for each $v \in V(\mathrm{simp}(G2))$. Hence, Theorem 7 implies that $\mathrm{simp}(G2)$ contains a Hamilton cycle which in turn implies that G contains a pair of identical Hamilton cycles. We remove this pair of identical Hamilton cycles from G and claim that we can iterate this procedure $\lceil \frac{r}{2} \rceil$ times. This claim is justified because iterating the procedure i times leaves a regular multigraph G_i

with even order n and multiplicity $\mu(G_i) \le r$, where $r > 1$ is an odd integer, and with $\Delta(G_i) \ge \frac{nr}{2} + \frac{n}{2} + 2r + 1 - 4i \ge \frac{nr}{2} + \frac{n}{2}$ if $i \le (\lceil \frac{r}{2} \rceil - 1)$.

We now use the results in Lemma 1 and Lemma 2 to prove Theorem 6.

Theorem 6 *Let G be a Δ-regular multigraph with even order n and multiplicity $\mu(G) \le r$, where $r > 1$ is an odd integer. If $\Delta = kr + 2r + 1$ for some integer $k \ge \frac{n}{2} + \frac{n}{2r}$, then the edges of G can be decomposed into $2r$ 1-factors of G, a $(k+1)$-regular simple graph, and $(r-1)$ k-regular simple graphs.*

Proof. Let G be any Δ-regular multigraph with even order n and multiplicity $\mu(G) \le r$, where $r > 1$ is an odd integer, and suppose that $\Delta = kr + 2r + 1 = (k+1)r + (r+1)$ for some integer $k \ge \frac{n}{2} + \frac{n}{2r}$. Lemma 2 above implies that G contains $\lceil \frac{r}{2} \rceil$ identical pairs of edge-disjoint Hamilton cycles. Denote these identical pairs of Hamilton cycles by $(H_{i,A}, H_{i,B})$ for $i = 1, 2, \ldots, \lceil \frac{r}{2} \rceil$. Let $G' = G - H_{1,A} - H_{2,A} - \ldots - H_{\lceil \frac{r}{2} \rceil, A}$. Clearly, G' is an $r(k+1)$-regular multigraph with maximum multiplicity $\mu(G) \le r$. Also, G' contains $\lceil \frac{r}{2} \rceil$ edge-disjoint Hamilton cycles, $H_{1,B}, H_{2,B}, \ldots, H_{\lceil \frac{r}{2} \rceil, B}$, such that for all $u, v \in V(G')$, if t of these Hamilton cycles contain an edge of the form (u, v), then the multiplicity of the edge uv in G' is at most $r - t$. Now, Lemma 1 implies that G' contains a simple $(k+1)$-factor F such that $\mu(G' - F) \le r - 1$. Let $G'' = G' - F$. Clearly, G'' is a $(k(r-1) + (r-1))$-regular multigraph with even order n and with $k \ge \frac{n}{2} + \frac{n}{2r}$. Since $(r-1)$ is even, Theorem 5 implies that the edges of G'' can be decomposed into $(r-1)$ 1-factors, $F_1, F_2, \ldots, F_{(r-1)}$, of G'', and $(r-1)$ k-regular simple graphs $S_1, S_2, \ldots, S_{(r-1)}$. Overall we have that

$$G = (H_{1,A} \cup H_{2,A} \ldots \cup H_{\lceil \frac{r}{2} \rceil, A}) \cup F \cup (F_1 \cup F_2 \ldots \cup F_{(r-1)}) \cup (S_1 \cup S_2 \ldots \cup S_{(r-1)}),$$

where $(H_{i,A}, H_{i,B})$ for $i = 1, 2, \ldots, \lceil \frac{r}{2} \rceil$ are Hamilton cycles of G, $F_1, F_2, \ldots, F_{(r-1)}$ are 1-factors of G, F is a simple $(k+1)$-factor of G, and $S_1, S_2, \ldots, S_{(r-1)}$ are k-regular simple subgraphs of G. Since n is even, each of the Hamilton cycles $H_{i,A}$ for $i = 1, 2, \ldots, \lceil \frac{r}{2} \rceil$ give two 1-factors of G. This gives a decomposition of the edges of G into $2r$ 1-factors of G, a $(k+1)$-regular simple graph, and $(r-1)$ k-regular simple graphs.

3 1-factorization of regular multigraphs of even order and high degree

In this section we use our decomposition result in Theorem 6 of Section 2 and Theorems 1 and 2 on simple graphs in the Introduction to prove Theorems 3 and 4 on multigraphs in the Introduction.

Theorem 3 *Let G be a Δ-regular multigraph with even order n and multiplicity $\mu(G) \le r$.*

(i) If r is even and $\Delta \ge \lceil \frac{\sqrt{7}-1}{2} n + 1 \rceil r$, then G is 1-factorable.

(ii) If r is odd and $\Delta \ge \lceil \frac{\sqrt{7}-1}{2} n + 2 \rceil r + 1$, then G is 1-factorable.

Proof. If r is even and $\Delta \geq \lceil \frac{\sqrt{7}-1}{2}n+1\rceil r$, then it is clear that by repeated application of Theorem 7 we can remove 1-factors of G till we are left with a multigraph G' that is Δ'-regular with even order n, multiplicity $\mu(G) \leq r$, and where $\Delta' = kr + r$ for some integer $k \geq \frac{\sqrt{7}-1}{2}n$. Now, Theorem 5 implies that the edges of G' can be decomposed into r 1-factors and r k-regular simple graphs. Applying Theorem 1 from the Introduction to each of these k-regular simple graphs in the decomposition of G' yields a 1-factorization of the edges of G. If r is odd and $\Delta \geq \lceil \frac{\sqrt{7}-1}{2}n+2\rceil r+1$, similar applications of Theorem 7, followed by an application of the decomposition result in Theorem 6, and finally followed by several applications of Theorem 1 yields a 1-factorization of the edges of G.

Theorem 4 *For every $\epsilon > 0$, there exists $N^*(\epsilon)$ such that if G is a Δ-regular multigraph with $\mu(G) \leq r$ and even order $n > N^*(\epsilon)$, then G is 1-factorable if*

(i) r is even and $\Delta \geq (\frac{1}{2}+\epsilon)rn$, or

(ii) r is odd and $\Delta \geq (\frac{1}{2}+\frac{1}{2r}+\epsilon)rn$.

Proof. Let $\epsilon > 0$ be given. Theorem 2 of the Introduction implies that there exists $N(\frac{\epsilon}{3})$ such that if G is a simple graph that is Δ-regular with even order $n > N(\frac{\epsilon}{3})$ and with $\Delta \geq (\frac{1}{2}+\frac{\epsilon}{3})n$, then G is 1-factorizable. Let $M^*(\epsilon) = \max\{N(\frac{\epsilon}{3}), \lceil\frac{3}{\epsilon}\rceil\}$. Now, suppose that G is a Δ-regular multigraph with even order $n > M^*(\epsilon)$, with multiplicity $\mu(G') \leq r$, where r is even, and with $\Delta \geq (\frac{1}{2}+\epsilon)rn$. Then, we have that $\Delta \geq (\frac{1}{2}+\epsilon)rn = (\frac{1}{2}+\frac{\epsilon}{3})rn + \frac{2\epsilon}{3}rn > (\frac{1}{2}+\frac{\epsilon}{3})rn + 2r$. Now by repeated application of Theorem 7 to G, remove at most $(r-1)$ 1-factors of G to get a multigraph G' that is Δ^*-regular with even order $n > M^*(\epsilon)$, with multiplicity $\mu(G) \leq r$, where r is even, and with $\Delta^* = rs$ for some integer $s > (\frac{1}{2}+\frac{\epsilon}{3})n+1$. Theorem 5 implies that the edges of G' can be decomposed into r 1-factors of G' and r simple graphs that are regular with degree $s-1 > (\frac{1}{2}+\frac{\epsilon}{3})n$. Theorem 2 implies that each of these r $(s-1)$-regular simple graphs are 1-factorable. This in turn yields a 1-factorization of G' and hence a 1-factorization of G.

Let $L^*(\epsilon) = \max\{N(\frac{\epsilon}{2}), \lceil\frac{6}{\epsilon}\rceil\}$. Now, suppose that G is a Δ-regular multigraph with even order $n > L^*(\epsilon)$, with multiplicity $\mu(G) \leq r$, where $r > 1$ is odd, and with $\Delta \geq (\frac{1}{2}+\frac{1}{2r}+\epsilon)rn$. Then, we have that $\Delta \geq (\frac{1}{2}+\frac{1}{2r}+\epsilon)rn = (\frac{1}{2}+\frac{1}{2r}+\frac{\epsilon}{2})rn + \frac{\epsilon}{2}rn > (\frac{1}{2}+\frac{1}{2r}+\frac{\epsilon}{2})rn + 3r$. Now by repeated application of Theorem 7 to G, remove at most $(r-1)$ 1-factors of G to get a multigraph G' that is Δ^*-regular with even order $n > L^*(\epsilon)$, with multiplicity $\mu(G) \leq r$, where $r > 1$ is odd, and with $\Delta^* = rs+1$ for some integer $s > (\frac{1}{2}+\frac{1}{2r}+\frac{\epsilon}{2})n+2$. Theorem 6 implies that the edges of G' can be decomposed into $2r$ 1-factors of G', one $(s-1)$-regular simple graph, and $(r-1)$ simple graphs that are regular with degree $s-2 > (\frac{1}{2}+\frac{1}{r}+\frac{\epsilon}{2})n$. Theorem 2 implies that each of these $(r-1)$ $(s-2)$-regular simple graphs are 1-factorizable. This in turn yields a 1-factorization of G' and hence a 1-factorization of G.

Finally, taking $N^*(\epsilon) = \max\{M^*(\epsilon), L^*(\epsilon)\}$ proves the theorem.

We note that the weakest result is obtained in Theorem 4 when $r = 3$. This implies the following Corollary of Theorem 4.

Corollary *For every $\epsilon > 0$, there exists $N^*(\epsilon)$ such that if G is a Δ-regular multigraph with multiplicity $\mu(G) \leq r$, even order $n > N^*(\epsilon)$, and with $\Delta \geq (\frac{2}{3} + \epsilon)rn$, then G is 1-factorable.*

References

[1] Bollobás, *Extremal Graph Theory,* Academic Press, New York (1978).

[2] J.A. Bondy and U.S.R. Murty, *Graph Theory with Applications,* Macmillan, London (1976).

[3] A.G. Chetwynd and A.J.W. Hilton, 1-factorizing regular graphs with high degree: an improved bound. *Discrete Math.* **75** (1989) 103-112.

[4] J.A. Dirac, Some theorems on abstract graphs. *Proc. London Math. Soc.* **2** (1952) 69-81.

[5] S.I. El-Zanati, M.J. Plantholt and S.K. Tipnis, Factorization of regular multigraphs into regular simple graphs. *J. Graph Theory,* Vol. 19, No. 1 (1995), 93-105.

[6] R. Häggkvist, unpublished.

[7] L. Perković and B. Reed, Edge coloring regular graphs of high degree. *Discrete Math.,***165/166** (1997) 567-578.

[8] M.J. Plantholt and S.K. Tipnis, Regular multigraphs of high degree are 1-factorizable. *Proc. London Math. Soc. (2)* **44** (1991) 393-400.

[9] W.T. Tutte, A short proof of the factor theorem for finite graphs. *Canad. J. Math.*, **6** (1954) 347-352.

[10] W.D. Wallis, *One-Factorizations*, Kluwer Academic Publishers, 1997.