How to find overfull subgraphs in graphs with large maximum degree, II

Thomas Niessen
Institute of Statistics, RWTH Aachen,
52056 Aachen, Germany
niessen@stochastik.rwth-aachen.de

Submitted: November 9, 1999; Accepted: December 10, 2000.

Mathematical Subject Classification: 05C85.

Abstract

Let G be a simple graph with $3\Delta(G) > |V|$. The *Overfull Graph Conjecture* states that the chromatic index of G is equal to $\Delta(G)$, if G does not contain an induced overfull subgraph H with $\Delta(H) = \Delta(G)$, and otherwise it is equal to $\Delta(G) + 1$. We present an algorithm that determines these subgraphs in $O(n^{5/3}m)$ time, in general, and in $O(n^3)$ time, if G is regular. Moreover, it is shown that G can have at most three of these subgraphs. If $2\Delta(G) \geq |V|$, then G contains at most one of these subgraphs, and our former algorithm for this situation is improved to run in linear time.

1 Introduction

Let $V(G)$, $E(G)$, $\Delta(G)$ and $\chi'(G)$ denote the vertex set, edge set, maximum degree and chromatic index of a simple graph G, respectively. In unambiguous cases, we prefer to write V, E, Δ and χ'.

G is called *Class 1*, if $\chi' = \Delta$ holds, and otherwise, G is called *Class 2*. By Vizing's Theorem [10], $\chi' = \Delta + 1$ holds for every Class 2 graph G.

G is called *overfull*, if $|E| > \lfloor |V|/2 \rfloor \Delta$. Every overfull graph is Class 2, as well as every graph G having an overfull subgraph H with $\Delta(H) = \Delta(G)$. We call such a subgraph Δ-*overfull*. It is easy to see that a subgraph H of G is Δ-overfull if and only if $|E(H)| > \lfloor |V(H)|/2 \rfloor \Delta(G)$.

Holyer [6] proved that the problem of deciding whether a graph is Class 1 is NP-complete. However, for graphs with large maximum degree this problem seems to be easier. Chetwynd and Hilton [2] conjectured that a graph G with $2\Delta \geq |V|$ is Class 2 if and only if it has a Δ-overfull subgraph. This conjecture was known to be true for many special cases when we presented an algorithm finding all induced Δ-overfull subgraphs of a graph G with $2\Delta \geq |V|$ [8]. Recently, Perkovic and Reed [9] proved that regular graphs of even order satisfying $(2-\varepsilon)\Delta > |V|$ are Class 1, if their order is sufficiently large depending on $\varepsilon > 0$. In the same paper, they announce similar partial results for the following conjecture.

Overfull Graph Conjecture [3, 4]. *A graph G with $3\Delta > |V|$ is Class 2 if and only if it has a Δ-overfull subgraph.*

In the present literature, this conjecture replaces the former one. It is best possible in some sense, since the graph P^*, which is obtained from the Petersen graph by removing an arbitrary vertex, is Class 2, has no overfull subgraph, and satisfies $3\Delta(P^*) = |V(P^*)|$. In view of both conjectures, the attention can be restricted to induced subgraphs, since the vertex set of any Δ-overfull subgraph induces a Δ-overfull subgraph.

The aim of this paper is to extend our former results to the more general situation of the Overfull Graph Conjecture. Therefore, we modify our algorithm from [8] such that it determines every induced Δ-overfull subgraph H of an arbitrary graph G with $|V(H)| > |V(G)| - \Delta(G)$ in $O(n \log n + m)$ time (Algorithm 1). A variant of this algorithm finds all induced Δ-overfull subgraphs of every graph G with $2\Delta \geq |V|$ in $O(n+m)$ time. These results are presented in Section 3. Thereafter, we develop in Section 4 an algorithm (Algorithm 2) for the determination of all induced Δ-overfull subgraphs of a graph G with $3\Delta(G) > |V(G)|$. Algorithm 2 applies Algorithm 1 to three subgraphs of G, but its worst-case complexity is dominated by the amount needed to find a certain edge cut. We use two procedures for this problem, one for the general case and another one for regular graphs needing $O(n^{5/3}m)$ time and $O(n^3)$ time, respectively.

In [8] we showed that a graph G with $2\Delta \geq |V|$ cannot contain more than one induced Δ-overfull subgraph. This result has been used in [5], for example. In Section 3, we provide a generalization: every graph has at most one induced Δ-overfull subgraph H with $|V(H)| > |V(G)| - \Delta(G)$. Thus, every graph G with $3\Delta > |V|$ contains at most three induced Δ-overfull subgraphs.

2 Terminology and preliminary results

Let G be a graph and let $v \in V$. By $N_G(v)$ we denote the neighborhood of v and $d_G(v) = |N_G(v)|$ is the degree of v in G. We call vertices of maximum degree *major vertices* and let $d^*_G(v)$ be the number of major vertices in the neighborhood of v.

For disjoint sets $X, Y \subseteq V$, we use $e_G(X, Y)$ to denote the number of edges joining a vertex in X to a vertex in Y. For convenience, we write $e_G(X)$ instead of $e_G(X, V(G) \setminus X)$ and $d_G(X)$ instead of $\sum_{x \in X} d_G(x)$.

We start with three simple results. Proofs can be found in [8].

Lemma 2.1 *A graph H is overfull if and only if $|V|$ is odd and*

$$\sum_{v \in V} (\Delta - d_H(v)) \leq \Delta - 2 \quad \textit{holds.}$$

Note that, if $|V|$ is odd, then both sides of the above inequality have the same parity. So, if they are not equal, their difference is at least two. Thereby, some estimates and conditions below could be improved by 1 or -1, but no real improvement would be achieved.

Lemma 2.2 *For every vertex v of an overfull graph H*

$$d_H(v) \geq 2 + \sum_{u \in N_H(v)} (\Delta - d_H(u)) \quad \textit{and} \quad d_H^*(v) \geq 2 \ \textit{hold.}$$

Lemma 2.3 *Let G be a graph with a Δ-overfull subgraph H. Then*

$$e_G(V(H)) \leq \Delta(G) - 2 - \sum_{v \in V(H)} \Big(\Delta(G) - d_G(v)\Big) \leq \Delta(G) - 2.$$

We call a vertex u of the graph G a *proper major vertex* of G, if

$$d_G(N_G(u)) \geq \Delta^2 - \Delta + 2.$$

Every proper major vertex is a major vertex. The following result is proved in [8].

Lemma 2.4 *Let G be a graph with Δ-overfull subgraph H. Then every major vertex of H is a proper major vertex of G.*

Let $d_G^{**}(u)$ denote the number of proper major vertices in $N_G(u)$. If G has a Δ-overfull subgraph H, then $d_G^{**}(v) \geq d_H^*(v) \geq 2$ for every vertex v of H, by Lemma 2.4 and Lemma 2.2. This implies the following result.

Lemma 2.5 *Let G be a graph and let u be a vertex of G with $d_G^{**}(u) \leq 1$. Then u belongs to no Δ-overfull subgraph of G.*

A repeated application of this lemma is now used to define the *kernel* of a graph G. Let $G_0 = G$, $S_1 = \{u \in V(G_0) : d_{G_0}^{**}(u) \leq 1\}$ and $G_1 = G_0 - S_1$. If $\Delta(G_1) < \Delta(G)$ or $S_1 = \emptyset$, then the procedure stops. Otherwise, let $S_2 = \{u \in V(G_1) : d_{G_1}^{**}(u) \leq 1\}$ and $G_2 = G_1 - S_2$. Again the procedure stops, if $\Delta(G_2) < \Delta(G)$ or $S_2 = \emptyset$. Otherwise, continue with S_3 and G_3 and so on. Since at least one vertex is removed at each stage, the procedure stops with some G_j, $1 \leq j \leq |V|$. The *kernel* $ker(G)$ of G is defined to be G_j, if $\Delta(G_j) = \Delta(G)$, or the null graph, otherwise. Obviously, every vertex u of $ker(G)$ satisfies $d_{ker(G)}^{**}(u) \geq 2$.

Let $i \in \{1, 2, \ldots, j\}$. Given G_{i-1}, it is straightforward to see that S_i and G_i can be computed in $O(|V(G_{i-1})| + |E(G_{i-1})|)$ time. Since $j \leq |V(G)|$, the kernel can therefore be computed in $O(|V(G)| \cdot |E(G)|)$ time. By Lemma 2.5, every Δ-overfull subgraph of G_{i-1} is contained in G_i. Thus, every Δ-overfull subgraph of G is contained in $ker(G)$. For later reference, these statements are summarized in a lemma.

Lemma 2.6 *Let G be a graph with n vertices and m edges. The kernel $ker(G)$ of G can be computed in $O(nm)$ time. Every Δ-overfull subgraph of G is contained in $ker(G)$. Every vertex u of $ker(G)$ satisfies $d_{ker(G)}^{**}(u) \geq 2$.*

The common subject of the three final results are edge cuts of size less than Δ, which will play an important role (see also Lemma 2.3).

Lemma 2.7 *Let H be an overfull graph and let $U \subset V$ with $e_H(U) < \Delta$. Then $|U| \leq 1$ or $|U| \geq \Delta$.*

Proof. The proof is by contraposition. Suppose therefore $2 \leq |U| \leq \Delta - 1$. Then we have

$$\begin{aligned} \Delta(|V|-1)+2 &\leq 2|E| = d_H(U) + d_H(V \setminus U) \\ &\leq (|U|(|U|-1) + e_H(U)) + (|V| - |U|)\Delta, \end{aligned}$$

and thus

$$e_H(U) \geq \Delta(|U|-1) + 2 - |U|(|U|-1) = \Delta + (\Delta - 1 - |U|)(|U|-2) \geq \Delta,$$

as required. □

Corollary 2.8 *Let H be an overfull graph with $|V| < 2\Delta$ and let F be an edge cut of H with $|F| < \Delta$. Then F* cuts off *one vertex of H, i.e., $H - F$ has two components and one of them consists of exactly one vertex.*

Proof. Let C be a component of $H - F$ such that $|V(C)|$ is maximum. Since $e_H(V(C)) \leq |F| < \Delta$ holds, we have, by Lemma 2.7, $|V(C)| \geq \Delta$, since $|V(C)| = 1$ cannot occur, of course. So, $|V(H-C)| \leq |V| - \Delta \leq \Delta - 1$, and therefore $|V(H-C)| = 1$, again by Lemma 2.7. □

Lemma 2.9 *Let G be a graph and let $U \subset V$ with $|U| < \Delta$ and $e_G(U) < \Delta$. Then at most one proper major vertex of G belongs to U.*

Proof. Assume there are two proper major vertices $w_1, w_2 \in U$. Let $p_i = e_G(w_i, V \setminus U)$, for $i = 1, 2$, and let $q = e_G(U \setminus \{w_1, w_2\}, V \setminus U)$. We suppose that $p_1 \leq p_2$. Then we have $2p_1 + q \leq p_1 + p_2 + q = e_G(U) \leq \Delta - 1$. Let $\varepsilon = 1$, if w_1 and w_2 are adjacent, and $\varepsilon = 0$, otherwise. Now we obtain

$$\begin{aligned} \Delta^2 - \Delta + 2 &\leq d_G(N_G(w_1)) \\ &= \varepsilon d_G(w_2) + d_G(N_G(w_1) \setminus U) + d_G(N_G(w_1) \cap (U \setminus \{w_2\})) \\ &\leq \varepsilon\Delta + p_1\Delta + (\Delta - p_1 - \varepsilon)(|U|-1) + q \\ &\leq \varepsilon\Delta + p_1\Delta + (\Delta - p_1 - \varepsilon)(\Delta - 2) + q \\ &= \Delta^2 - 2\Delta + 2\varepsilon + 2p_1 + q \\ &\leq \Delta^2 - 2\Delta + 2 + \Delta - 1 = \Delta^2 - \Delta + 1, \end{aligned}$$

a contradiction. □

3 Algorithm 1

The cornerstone of Algorithm 1 is provided by the following lemma.

Lemma 3.1 *Let G be a graph and let $S = \{u \in V(G) : d_G^{**}(u) \leq 1\}$. If G has an Δ-overfull subgraph H with $|V(H)| > |V(G)| - \Delta(G)$ such that $V(G-S) \setminus V(H)$ is nonempty, then*

$$\min\{d_{G-S}(v) \;:\; v \in V(H)\} \;\geq\; \max\{d_{G-S}(w) + 2 \;:\; w \in V(G-S) \setminus V(H)\}.$$

Proof. By Lemma 2.5, H is a subgraph of $G - S$, and so the minimum is well defined. In particular, H is a Δ-overfull subgraph of $G - S$, and thus $\Delta(G - S) = \Delta(H) = \Delta(G)$.

Assume that there are vertices $v \in V(H)$ and $w \in V(G-S) \setminus V(H)$ with $d_{G-S}(v) \leq d_{G-S}(w) + 1$. With Lemma 2.3 we obtain

$$\begin{aligned} e_{G-S}(V(H)) \;&\leq\; \Delta(G-S) - 2 - \sum_{x \in V(H)} (\Delta(G-S) - d_{G-S}(x)) \\ &\leq\; \Delta(G) - 2 - (\Delta(G) - d_{G-S}(v)) \;=\; d_{G-S}(v) - 2, \end{aligned}$$

and so $d_{G-S}(w) \geq d_{G-S}(v) - 1 \geq e_{G-S}(V(H)) + 1$.

Next we will see that this is impossible. Let $U = V(G) \setminus V(H)$. Then $|U| = |V(G)| - |V(H)| < \Delta(G)$ and $e_G(U) = e_G(V(H)) \leq \Delta(G) - 2$, by Lemma 2.3. Hence, by Lemma 2.9, at most one proper major vertex of G belongs to U. We consider the vertices in $N_{G-S}(w) \cap (V(G-S) \setminus V(H))$. For every vertex u in this set we have $d_G^{**}(u) \geq 2$, since otherwise it would belong to S. So, $e_{G-S}(u, V(H)) \geq d_G^{**}(u) - 1 \geq 1$. Therefore,

$$\begin{aligned} e_{G-S}(V(H)) \;&\geq\; e_{G-S}(w, V(H)) + \\ &\quad\; e_{G-S}(N_{G-S}(w) \cap (V(G-S) \setminus V(H)), V(H)) \\ &\geq\; e_{G-S}(w, V(H)) + |N_{G-S}(w) \cap (V(G-S) \setminus V(H))| \\ &=\; d_{G-S}(w). \end{aligned}$$

This contradiction completes the proof. □

Theorem 3.2 *Algorithm 1 finds all induced Δ-overfull subgraphs H of a graph G satisfying $|V(H)| > |V(G)| - \Delta(G)$ in $O(n \log n + m)$ time, where n and m denote the order and the size of G, respectively.*

Proof. Consider Algorithm 1 in Figure 1. S can be computed in $O(n + m)$ time, and also G^* can be determined with this amount. By Lemma 2.5, G and G^* have the same Δ-overfull subgraphs. If $\Delta(G^*) < \Delta(G)$, then G has no Δ-overfull subgraph. Otherwise, the degrees of the vertices of G^* are determined in $O(n + m)$ time, and the sorting is done in $O(n \log n)$ time. Lemma 3.1 shows that only the vertex sets considered in the final phase of the algorithm can induce Δ-overfull subgraphs in G with more than $|V(G)| - \Delta(G)$ vertices. These vertex sets can be checked in $O(n + m)$ time by a successive removal of

Algorithm 1: *Input: a graph G.*

```
determine S = {u ∈ V(G) : d**_G(u) ≤ 1};
set G* = G − S;
if Δ(G*) < Δ(G) then stop;
sort the vertices of G* such that d_{G*}(v_1) ≥ d_{G*}(v_2) ≥ ... ≥ d_{G*}(v_r),
     where r = |V(G*)|;
test for every odd j satisfying |V(G)| − Δ(G) < j ≤ |V(G*)|
     such that d_{G*}(v_j) ≥ d_{G*}(v_{j+1}) + 2 or j = r whether {v_1, ..., v_j}
     induces a Δ-overfull subgraph of G*.
```

Figure 1.

pairs of vertices with largest indices. So, every induced Δ-overfull subgraph H of G with $|V(H)| > |V(G)| - \Delta(G)$ is found in $O(n \log n + m)$ time. □

The algorithm presented in [8] computes $ker(G)$ instead of $G - S$, and continues similarly thereafter. So, its running time is $O(nm)$ (see Lemma 2.6).

By the condition $|V(G)| - \Delta(G) < j$ in the final phase, Algorithm 1 does not find any induced Δ-overfull subgraph H with $|V(H)| \leq |V(G)| - \Delta(G)$. Without this condition it possibly finds such subgraphs, but it can fail to determine all of them. Let $p \geq 2$ be an integer. We obtain the graph G_p^1 from two disjoint complete graphs K_{2p+1} and K_{2p} by removing an edge $xy \in E(K_{2p+1})$ and joining x to u and y to v, where u and v are distinct vertices of K_{2p}. G_p^1 is overfull and this is detected by Algorithm 1. The subgraph H_p induced by $E(K_{2p+1})$ is another Δ-overfull subgraph of G_p^1. However, only if the vertices u and v receive the largest indices among all vertices of maximum degree during the sorting, Algorithm 1 detects H_p. Note that $|V(H_p)| = |V(G_p^1)| - \Delta(G_p^1)$ holds.

In [8] we proved that a graph G with $2\Delta \geq |V|$ has at most one induced Δ-overfull subgraph. The following theorem is more general.

Theorem 3.3 *Every graph G has at most one induced Δ-overfull subgraph H with $|V(H)| > |V(G)| - \Delta(G)$.*

Proof. Assume that G contains two distinct induced Δ-overfull subgraphs H_i with $|V(H_i)| > |V(G)| - \Delta(G)$ for $i = 1, 2$. Algorithm 1 shows that one of them is contained in the other one, say $V(H_1) \subset V(H_2)$. Since both have odd order, $|V(H_2) \setminus V(H_1)| \geq 2$ follows. Moreover, $|V(H_2) \setminus V(H_1)| \leq |V(G)| - |V(H_1)| < \Delta(G)$. Therefore Lemma 2.7 implies $e_{H_2}(V(H_2) \setminus V(H_1)) \geq \Delta(G)$, and thus $e_{H_2}(V(H_1)) \geq \Delta(G)$. This contradicts Lemma 2.3. □

The next theorem summarizes our results for graphs with $2\Delta \geq |V|$.

Theorem 3.4 *Let G be a graph with $2\Delta \geq |V|$. Then G has at most one induced Δ-overfull subgraph, which can be found in $O(n + m)$ time, where n and m denote the order and the size of G, respectively.*

Proof. If G has a Δ-overfull subgraph H, then $|V(H)| > \Delta(H) = \Delta(G) \geq |V(G)| - \Delta(G)$ and $m \geq |E(H)| > \lfloor |V(H)|/2 \rfloor \Delta(G) \geq \Delta(G)^2/2 \geq n^2/8$ hold.

The first estimate guarantees that Algorithm 1 determines all induced Δ-overfull subgraphs of G (see Theorem 3.2) and that G contains at most one of them (see Theorem 3.3).

The second estimate implies that G cannot contain a Δ-overfull subgraph, if $m \leq n^2/8$ holds. So, we can check this first in $O(n+m)$ time, and only if $m > n^2/8$ holds, we need to apply Algorithm 1 to G, which then terminates in $O(m)$ steps. $\square$

4 Algorithm 2

Let G be a graph and let $U \subseteq V$. We say that an edge cut F of G *separates* U, if U is not contained in one component of $G - F$.

We can distinguish, roughly, three phases of Algorithm 2. At the beginning the kernel of the graph is determined and Algorithm 1 is applied to it. Thereby, we find all induced Δ-overfull subgraphs of G with more than $|V(ker(G))| - \Delta(G)$ vertices. The second phase consists of finding a certain edge cut F of $ker(G)$ with $|F| \leq \Delta(G) - 2$. Let H be an induced Δ-overfull subgraph of G that has not been found so far. Below we will see that F possibly separates $V(H)$, but one component C of $ker(G) - F$ contains at least $|V(H)| - 1$ vertices of H. The following lemma is needed below to find a missing vertex of H in $V(ker(G)) \setminus V(C)$.

Lemma 4.1 *Let G be a graph and let H be a Δ-overfull subgraph of G with $|V(H)| < 2\Delta(G)$. Let F be an edge cut of G with $|F| < \Delta(G)$ that separates $V(H)$. Then there is a component C of $G - F$ with $|V(C) \cap V(H)| = |V(H)| - 1$. Let x denote the vertex in $V(H) \setminus V(C)$. If $e_G(H - x) \geq |F|$, then x is the unique vertex in $V(G) \setminus V(C)$ with $e_G(x, V(C)) = \max\{e_G(u, V(C)) \ : \ u \in V(G) \setminus V(C)\}$.*

Proof. Let $F_H = F \cap E(H)$. Since $|F_H| \leq |F| < \Delta(G) = \Delta(H)$, F_H cuts off one vertex x of H, by Corollary 2.8. So, there is a component C of $G - F$ with $|V(C) \cap V(H)| = |V(H)| - 1$. If $e_G(H - x) < |F|$, then we are done. So, we assume that $e_G(H - x) \geq |F|$. Let $u \in V(G) \setminus V(C)$ with $u \neq x$. First, we observe that

$$\begin{aligned}
2|E(H-x)| &= \sum_{w \in H-x} d_{H-x}(w) \\
&\leq |V(H-x)|\Delta(H) - e_G(V(H-x)) \\
&\leq (|V(H)|-1)\Delta(G) - e_G(V(C) \setminus V(H-x), V(H-x)) \\
&\quad -e_G(x, V(H-x)) - e_G(u, V(H-x)).
\end{aligned}$$

Next, we have

$$\begin{aligned}
2|E(H-x)| &= 2|E(H)| - 2e_G(x, V(H-x)) \\
&\geq (|V(H)|-1)\Delta(G) + 2 - 2e_G(x, V(H-x)).
\end{aligned}$$

Combining both estimates we obtain

$$e_G(V(C) \setminus V(H-x), V(H-x)) + e_G(u, V(H-x)) \leq e_G(x, V(H-x)) - 2. \quad (1)$$

We also have $e_G(V(G) \setminus V(C)) \leq |F| \leq e_G(V(H-x))$. Subtracting $e_G(V(G) \setminus V(C), V(H-x))$ on both sides yields $e_G(V(G) \setminus V(C), V(C) \setminus V(H-x)) \leq e_G(V(C) \setminus V(H-x), V(H-x))$. Using this and (1) we obtain

$$\begin{aligned} e_G(u, V(C)) &= e_G(u, V(C) \setminus V(H-x)) + e_G(u, V(H-x)) \\ &\leq e_G(V(G) \setminus V(C), V(C) \setminus V(H-x)) + e_G(u, V(H-x)) \\ &\leq e_G(V(C) \setminus V(H-x), V(H-x)) + e_G(u, V(H-x)) \\ &\leq e_G(x, V(H-x)) - 2 \leq e_G(x, V(C)) - 2, \end{aligned}$$

and so x is the unique vertex in $V(G) \setminus V(C)$ with $e_G(x, V(C)) = \max\{e_G(u, V(C)) : u \in V(G) \setminus V(C)\}$. □

In the third phase of Algorithm 2, we possibly apply Algorithm 1 to two subgraphs of $ker(G) - F$. The following lemma is needed to show that their order is at most 2Δ.

Lemma 4.2 *Let G be a kernel (i.e., $ker(G) = G$) and let F be a minimum edge cut separating the set of proper major vertices of G. If $|F| < \Delta$, then every component of G contains at least Δ vertices.*

Proof. Suppose that C is a component of $G - F$ with $|V(C)| < \Delta$. Since $e_G(V(C)) \leq |F| < \Delta$, $V(C)$ contains at most one proper major vertex of G, by Lemma 2.9. Therefore, $e_G(V(C)) > 0$, since G is a kernel. So, $V(C)$ contains a proper major vertex u of G, since otherwise F would not be a minimum edge cut separating the set of proper major vertices of G. Let $p = e_G(u, V(G) \setminus V(C))$. Then $\Delta - 1 \geq |F| \geq e_G(V(C)) \geq p + (\Delta - p) + 2(|V(C)| - 1 - (\Delta - p)) = 2|V(C)| - 2 - \Delta + 2p$, and so $|V(C)| \leq \Delta - p$. Now $\Delta = d_G(u) \leq p + (|V(C)| - 1) \leq p + \Delta - p - 1 = \Delta - 1$ yields a contradiction. □

Now we are in a position to prove the main results.

Theorem 4.3 *Algorithm 2 finds all induced Δ-overfull subgraphs of a graph G with $3\Delta(G) > |V(G)|$.*

Proof. First, we have to show that Algorithm 2 is correctly formulated, i.e., if line 7 is executed, then $G^* - F$ has exactly two components. Note therefore that in this situation G^* is a kernel with $\Delta(G^*) = \Delta(G)$, and that F is a minimum edge cut separating the set of proper major vertices of G^*. Lemma 4.2 shows that every component of $G^* - F$ has at least $\Delta(G)$ vertices, and so $|V(G^*)| \leq |V(G)| < 3\Delta(G)$ implies that $G^* - F$ has in fact only two components.

Let H be an induced Δ-overfull subgraph of the graph G. By Lemma 2.6, H is an induced subgraph of G^*. Therefore, $\Delta(G^*) = \Delta(H) = \Delta(G)$, and thus Algorithm 2 does not stop at line 2.

Algorithm 2: *Input: a graph G with $3\Delta(G) > |V(G)|$.*

1: set $G^* = ker(G)$;
2: **if** $\Delta(G^*) < \Delta(G)$ **then** stop;
3: apply Algorithm 1 to G^*;
4: find a minimum edge cut F of G^* separating
5: the set of proper major vertices of G^*;
6: **if** $|F| > \Delta(G) - 2$ **then** stop;
7: let C_1, C_2 be the components of $G^* - F$;
8: let $x_1 \in V(C_1)$ such that $e_{G^*}(x_1, V(C_2))$ is
9: maximum among all vertices of C_1;
10: let $x_2 \in V(C_2)$ such that $e_{G^*}(x_2, V(C_1))$ is
11: maximum among all vertices of C_2;
12: **if** $\Delta(G - (V(C_1) \setminus \{x_1\})) = \Delta(G)$ **then**
13: apply Algorithm 1 to $G - (V(C_1) \setminus \{x_1\})$;
14: **if** $\Delta(G - (V(C_2) \setminus \{x_2\})) = \Delta(G)$ **then**
15: apply Algorithm 1 to $G - (V(C_2) \setminus \{x_2\})$;
16: **end**.

Figure 2.

If $|V(H)| > |V(G^*)| - \Delta(G)$, H is detected, when Algorithm 1 is applied to G^* (see Theorem 3.2). So, suppose now $|V(H)| \leq |V(G^*)| - \Delta(G)$.

First, we show that $V(H)$ and $V(G^*) \setminus V(H)$ both contain at least two proper major vertices. By Lemma 2.2, H has at least three major vertices, and every major vertex of H is a proper major vertex if G^*, by Lemma 2.4. Suppose now that $V(G^*) \setminus V(H)$ contains at most one proper major vertex. Then, every $u \in V(G^*) \setminus V(H)$ has a neighbor in $V(H)$, since G^* is a kernel. Therefore $e_{G^*}(V(H)) \geq |V(G^*)| - |V(H)| \geq \Delta(G)$, contradicting Lemma 2.3.

F is chosen to be a minimum edge cut separating the set of proper major vertices of G^*. Hence $|F| \leq e_{G^*}(V(H)) \leq \Delta(G) - 2$, by Lemma 2.3, and therefore Algorithm 2 does not stop at line 6.

Next we verify the remaining hypotheses of Lemma 4.1. We have $|V(H)| \leq |V(G^*)| - \Delta(G) \leq |V(G)| - \Delta(G) < 3\Delta(G) - \Delta(G) = 2\Delta(G)$. Hence, by the first part of that lemma, one component, say C_1, of $G^* - F$ contains at least $|V(H)| - 1$ vertices of H. If there is a vertex $x \in V(H) \setminus V(C_1)$, then the set of edges leaving $V(H - x)$ separates the set of proper major vertices of G^*, since $V(H)$ and $V(G^*) \setminus V(H)$ both contain at least two proper major vertices. Thus, $e_{G^*}(V(H - x)) \geq |F|$. Now it follows from Lemma 4.1, that $x = x_2$ and so H is a subgraph of $G' = G^* - (V(C_2) \setminus \{x_2\})$. Therefore, in particular, $\Delta(G') = \Delta(G)$, and so Algorithm 1 is applied to G'. Hence, by Theorem 3.2, H is found or $|V(H)| \leq |V(G')| - \Delta(G')$. However, the latter case cannot occur, since, by Lemma 4.2, $|V(G')| - \Delta(G') = (|V(G^*)| - |V(C_2)| + 1) - \Delta(G) \leq |V(G)| - 2\Delta(G) + 1 \leq (3\Delta(G) - 1) - 2\Delta(G) + 1 = \Delta(G)$, and so $|V(H)| \leq |V(G')| - \Delta(G')$ would imply

$\Delta(H) < \Delta(G)$. □

Let us consider the worst-case complexity of Algorithm 2. The kernel of G can be found in $O(nm)$ time (see Lemma 2.6). C_1, C_2, x_1, and x_2 can all be determined in $O(n+m)$ time. Algorithm 1 is applied at most three times, which needs $O(n \log n + m)$ time (see Theorem 3.2). So, Algorithm 2 needs $O(nm + T(n,m))$ time, where $T(n,m)$ is the time needed to find the edge cut F.

In general, a minimum edge cut F_U separating an arbitrary set U of vertices can be found as follows. Choose a vertex $u_0 \in U$ and determine a minimum edge cut F_u separating u_0 and u for every $u \in U$, $u \neq u_0$. Then let F_U be a minimum edge cut among all these edge cuts. Every F_u can be found by means of a maximum flow algorithm in $O(n^{2/3}m)$ time (see [1], p. 254), and so the whole procedure can be performed in $O(n^{5/3}m)$ time.

Theorem 4.4 *All induced Δ-overfull subgraphs of a graph G with $3\Delta(G) > |V(G)|$ can be found in $O(n^{5/3}m)$ time, where n and m denote the order and size of G, respectively.*

Let G be a regular graph with $\Delta \geq 2$. Then every vertex of G is a proper major vertex, and thus $ker(G) = G$. So, every edge cut of G separates the set of proper major vertices. Since a minimum edge cut can be found in $O(nm)$ time [7], which is $O(n^3)$ time for regular graphs with $3\Delta > |V|$, we obtain the following theorem.

Theorem 4.5 *All induced Δ-overfull subgraphs of a regular graph G with $3\Delta(G) > |V(G)|$ can be found in $O(n^3)$ time, where n denotes the order of G.*

By Theorem 3.3, every application of Algorithm 1 within Algorithm 2 yields at most one induced Δ-overfull subgraph.

Corollary 4.6 *Let G be a graph with $3\Delta > |V|$. Then G has at most three induced Δ-overfull subgraphs.*

This corollary is best possible as the next family of graphs shows. Let K_{2p} be a complete graph of order $2p$, where $p \geq 3$ is an integer. Remove an edge uv from this graph, and add two edges xu, xv, where x is a new vertex (in other words, we insert a the vertex x into the edge uv). Let K^*_{2p} denote this graph. Take two vertex-disjoint copies of K^*_{2p} and identify the two vertices of degree two. The resulting graph G^2_p has three induced Δ-overfull subgraphs corresponding to the vertex sets of the two copies of K^*_{2p} and to $V(G^*_p)$. Moreover, the vertex sets of the copies of K^*_{2p} are not disjoint in G^*_p. So, we see the necessity of adding vertices to C_1 and C_2 in the final phase of Algorithm 2.

We end with a family of graphs showing that the condition $3\Delta > |V|$ is almost best possible for Algorithm 2. For an odd integer $p \geq 3$, let G^3_p be the graph resulting from three vertex disjoint complete graphs K_p, K_{p+1} and K'_{p+1} of order p and $p+1$, respectively, by removing one perfect matching from both graphs of order $p+1$. Note that $\Delta(G^3_p) = |V(G^3_p)| - 5$. The following four sets induce Δ-overfull subgraphs of G^3_p: $V(K_p)$, $V(K_p) \cup V(K_{p+1})$, $V(K_p) \cup V(K'_{p+1})$, and $V(G^3_p)$. Since Algorithm 2 can find at most three induced Δ-overfull subgraphs, it fails to find all these subgraphs of G^3_p.

Acknowledgment

The author gratefully acknowledges the useful suggestions of the referee.

References

[1] R.K. Ahuja, T.L. Magnanti and J.B. Orlin. *Network flows.* Prentice Hall (1993).

[2] A.G. Chetwynd and A.J.W. Hilton. *The chromatic index of graphs of even order with many edges.* J. Graph Theory **8** (1984), 463–470.

[3] A.G. Chetwynd and A.J.W. Hilton. *Star multigraphs with three vertices of maximum degree.* Math. Proc. Cambridge Philos. Soc. **100** (1986), 303–317.

[4] A.J.W. Hilton and P.D. Johnson. *Graphs which are vertex-critical with respect to the edge-chromatic number.* Math. Proc. Cambridge Philos. Soc. **102** (1987), 103–112.

[5] A.J.W. Hilton and C. Zhao. *Vertex-splitting and chromatic index critical graphs.* Discrete Applied Math. **76** (1997), 205–211.

[6] I. Holyer. *The NP-completeness of edge-colouring.* SIAM J. Comp. **10** (1981), 718–720.

[7] D.W. Matula. *Determining edge connectivity in $O(nm)$.* Proc. of the 28th Symposium on Foundations of Computer Sciences (1987), 249–251.

[8] T. Niessen. *How to find overfull subgraphs in graphs with large maximum degree.* Discrete Applied Math. **51** (1994), 117–125.

[9] L. Perkovic and B. Reed. *Edge colouring regular graphs of high degree.* Discrete Math. **165/166** (1997), 567–578.

[10] V. G. Vizing. *On an estimate of the chromatic class of a p-graph.* Diskret. Analiz. **3** (1964), 25–30 (Russian).