

Aviezri Fraenkel and Combinatorial Games

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The subject of combinatorial games has, like combinatorics itself, been slow to find recognition with the mathematical establishment. Combinatorics is now on sure ground, and combinatorial games is well on its way. This is in no small part due to the energy, enthusiasm and insight of Aviezri Fraenkel, who has linked combinatorial games with much else in mathematics — graph theory [24], error-correcting codes [22], numeration systems [25, 28], continued fractions [17] and especially complexity theory.

The great variety in the difficulty of the vast range of combinatorial games has enabled him to exhibit the whole spectrum of complexity theory [18, 27, 55]. He has also ascertained the status of many particular combinatorial games [34], diophantine games [60], the Grundy function [14], chess [40], and checkers [33].

His interest in games may well have been sparked by a classic paper of Coxeter [6]: certainly he has long been fascinated by the relation between Beatty sequences, i.e. complementing sequences of integers, on the one hand, and Wythoff's Game [9, 39, 11, 1, 8] on the other. Wythoff's Game is played with two heaps from which players alternately take any number from one heap or equal numbers from each; several of Aviezri's papers have been concerned with generalizations of this game [3, 7, 15, 17, 32, 45, 61].

He has written on a variety of individual games, many of which are his own invention: Nimbi [36], Nimhoff games [43], and Nim itself [4]; Geography [50], Epidemiography [41, 42, 44], geodetic contraction games [35], Particles and Antiparticles [12], a deletion game [48], a new heap game [58], modular Nim (sometimes called Kotzig's Nim) [37], Multivision [26], partizan octal and subtraction games [38] and extensions of Conway's 'short games' [52].

He has been especially interested in annihilation games [10, 53, 54, 56] and in games using Cedric Smith's extension of the Sprague-Grundy theory to cover games in which there are possible draws through infinite play [2, 371–375]. He provided Conway with an early example [5]; other examples of his treatment of games with cycles are [46, 52] and the 'additional subtraction' games, where you may put as well as take [51].

A recent development is his adaptation of (one-player) cellular automata games to two players [29, 30, 31]. The motivation for this comes from his continuing interest in complexity theory, and the connections with linear error-correcting codes.

In addition to this remarkable output, he has served the combinatorial games fraternity in several important ways. In the organization of workshops, conferences and seminars — he displayed 11 different games at the 1986 Strens Memorial Conference in Calgary [21], and was on the organizing committee of both the 1994 and 2000 workshops at MSRI, Berkeley, and there are other examples in the past and in the future. In stimulating dozens of co-workers — the publications listed below include more than 30 coauthors, besides his score of solo papers.. In writing general surveys [13, 20, 23]. In circulating

problems, the life-blood of any mathematical discipline [16]. And, most importantly, in maintaining the definitive bibliography of the subject [19], which has also appeared in [59, 62] and will soon be updated in *More Games of No Chance*, the proceedings of the 2000 MSRI Workshop.

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