Aviezri Fraenkel's work in Number Theory

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As well as spanning a number of decades, Aviezri Fraenkel's mathematical work has spanned a number of areas. He is well known as an expert on combinatorial games and his contributions in that area are described by Richard Guy in the accompanying article. He has also made advances in number theory, combinatorics, and computer science.

His doctoral thesis, written under the guidance of Ernst G. Straus at UCLA and awarded in 1961, concerned a theorem of Ridout in the theory of Diophantine equations [4]. At this time he was also working in the fledgling field of computer science. His first published paper [3] was on a method for using Mersenne primes for the design of a high speed multiplier. This was followed by one on "a very high-speed digital number sieve" [2] which was a special-purpose sieving device which worked at the rate of 10^{10} numbers per minute, 1000 times faster than the then state of the art IBM7090. This was before people talked about computational complexity but his interest in the area remained, particularly applied to his work on games. Other early work dealt with questions about transcendental numbers, Diophantine approximation, and Diophantine equations.

Another early paper was with Joe Gillis [10] on the avoidability of repetitions in the DNA code. This foreshadowed his later work on molecular biology and sequences.

A few years later he published "The bracket function and complementary sets of integers"[6]. The bracket function is the integer part or floor function, at that time written $[x]$, now usually $\lfloor x \rfloor$. Two sets of integers are complementary if they are disjoint and their union is the set of positive integers $\mathbf{Z}^+$. An old result is that the sets $\{\lfloor n\alpha \rfloor : n \in \mathbf{Z}^+\}$ and $\{\lfloor n\beta \rfloor : n \in \mathbf{Z}^+\}$ are complementary if and only if α and β are irrational and

$$\frac{1}{\alpha} + \frac{1}{\beta} = 1.$$

In this paper he investigated the inhomogeneous case in which the sets have the form $\{\lfloor n\alpha + \gamma \rfloor : n \in \mathbf{Z}^+\}$ and $\{\lfloor n\beta + \delta \rfloor : n \in \mathbf{Z}^+\}$. This was the first of many papers connected with the floor function.

The set $S(\alpha, \beta) = \{\lfloor n\alpha + \beta \rfloor : n \in \mathbf{Z}^+\}$ is called a Beatty sequence. A disjoint covering system of Beatty sequences is a collection $\{S(\alpha_i, \beta_i) : i = 1, ..., t\}$ such that every integer belongs to exactly one of the sequences. In 1969 he made two famous conjectures about such systems [7],[16]. The first is:

(1) In any disjoint covering system of Beatty sequences with $t > 2$ we must have at least one of α_i being a multiple of another.

He later strengthened this:

(2) If the α_i in a disjoint covering system of Beatty sequences with $t > 2$ are distinct then they must make up the set $\{(2^t - 1)/2^i : i = 0, ..., t - 1\}$.

Both conjectures remain open despite a number of contributions towards their resolution by Aviezri and others. In particular Ron Graham[16] showed that no counterexample exists with any α_i irrational, Jamie Simpson [17] showed that the first conjecture holds if $\min_i\{\alpha_i\} \leq 2$ and the second if $\min_i\{\alpha_i\} \leq 3/2$ and recently Rob Tijdeman [18] showed that both conjectures hold for all $t \leq 6$.

Closely related to coverings by Beatty sequences are covering congruence systems. A set of congruences $\{a_i \pmod{m_i} : i = 1, ..., t\}$ is a covering system if every integer n satisfies

$$n \equiv a_i \pmod{m_i} \tag{1}$$

for at least one value of i. If the moduli m_i are distinct then the covering system is *incongruent* (or *regular*). If (1) is satisfied by exactly one value of i for each n then the system is *disjoint* (or *exact*). Two celebrated questions of Erdős are "Do there exist incongruent systems with all moduli odd?" and "Can we construct incongruent systems with the least modulus arbitrarily large?" In the 70's Aviezri wrote several papers on disjoint covering systems and a much-cited paper with Mushkin and Tassa "Determination of $[n\alpha]$ by its sequence of differences"[9]. Such sequences frequently turn up in computer science where this type of algorithm is fundamental.

In the mid 1980s he began a marathon sequence of some 15 papers with Mark Berger and Alexander Felzenbaum (starting with [1]) on covering systems in which many questions were answered and generalizations made. Some of this work is described in Doron Zeilberger's article in the present volume. The work on covering systems and Beatty sequences continued with other authors. A remarkable paper with Ron Holzman [11] concerned the structure of the intersection of two Beatty sequences. This may be regarded as a Beatty sequence version of the Chinese Remainder Theorem, a proof of which was published by Aviezri early in his career [5].

A major problem in molecular biology is to determine the mechanism of protein folding. In water, a protein folds into a shape which is believed to attain the minimum energy of any configuration; simulating this process on a computer is very slow. Aviezri in [8] and [12] showed that, in one model, finding this minimum energy configuration is an NP-hard problem, which nature, somehow, can solve very quickly. Other researchers have since shown that more refined models of protein folding are also NP-hard.

Some of his recent work has to do with problems in the combinatorics of words. A word is a sequence of symbols, usually taken from a finite alphabet and usually written without the customary commas. Thus *abcbca* is a word constructed from the alphabet $\{a, b, c\}$. A square in such a word is a pair of adjacent identical blocks, thus the word above contains the square *bcbc*. Binary words (from an alphabet of size 2) are of particular interest. He and I have studied binary words which contain many squares [14], few squares [13] and the number of squares in a special type of word called a Fibonacci word [15].

It is a privilege to know and work with Aviezri. I am grateful for his friendship, his advice, his imagination and his gentle sense of humour. I remember driving him to my university the day after he arrived in Australia. On the way we witnessed a minor traffic accident; one car ran into the back of another at some traffic lights. "Ah," said Aviezri, shaking his head sadly, "I expected something like this would happen. Everyone is driving

on the wrong side of the road."

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