A Proof of the Two-path Conjecture

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Abstract

Let G be a connected graph that is the edge-disjoint union of two paths of length n, where $n \geq 2$. Using a result of Thomason on decompositions of 4-regular graphs into pairs of Hamiltonian cycles, we prove that G has a third path of length n.

The "two-path conjecture" states that if a graph G is the edge-disjoint union of two paths of length n with at least one common vertex, then the graph has a third subgraph that is also a path of length n. For example, the complete graph K_4 is an edge-disjoint union of two paths of length 3, each path meeting the other in four vertices. The cycle C_6 is the edge-disjoint union of two paths of length 3 with common endpoints. In the first case, the graph has twelve paths of length 3; in the second there are six such paths.

The two-path conjecture arose in a problem on randomly decomposable graphs. An H-decomposition of a graph G is a family of edge disjoint H–subgraphs of G whose union is G. An H-decomposable graph G is *randomly* H–decomposable if any edge disjoint family of H–subgraphs of G can be extended to an H–decomposition of G. (This concept was introduced by Ruiz in [7].)

Randomly P_n-decomposable graphs were studied in [1, 5, 6, 4]. In attempting to classify randomly P_n-decomposable graphs, in [5] and [6] it was necessary to know whether the edge-disjoint union of two copies of P_n could have a unique P_n-decomposition. The two-path conjecture is stated as an unproved lemma in [3].

Our notation follows [2]. A path of length n is a trail with distinct vertices $x_0, \ldots, x_n$, ([2], p. 5). We say that G *decomposes* into subgraphs X and Y when G is the edge-disjoint union of X and Y.

Theorem. If G decomposes into two paths X and Y, each of length n with $n \geq 2$, and X and Y have least one common vertex, then G has a path of length n distinct from X and Y.

Proof. Label the vertices of X as $x_0, x_1, \ldots, x_n$, with x_{i-1} adjacent to x_i for $1 \leq i \leq n$. Similarly, label the vertices of Y as $y_0, y_1, \ldots, y_n$. Let s be the number of common vertices; thus G has $2n + 2 - s$ vertices.

If $s = 1$, then we may assume by symmetry that $x_i = y_j$ with $i \geq j$ and $i \geq 1$ and $j < n$. In this case, the vertices $x_0, \ldots, x_i, y_{j+1}, \ldots y_n$ form a path of length at least n having a subpath of length n different from X and Y.

Similarly, if $s = 2$, then we may let the common vertices be x_{i_1}, x_{i_2} and y_{i_1}, y_{i_2} with $x_{i_1} = y_{j_1}$ and $x_{i_2} = y_{j_2}$. Using symmetry again, we may assume that $i_1 < i_2$, $j_1 < j_2$, and $i_1 \geq j_1$. With this labeling, again the vertices $x_0, \ldots, x_{i_1}, y_{j_1+1}, \ldots y_n$ form a path with a subpath of length n different from X and Y.

Hence we may assume that $s \geq 3$. The approach above no longer works, since now the points of intersection need not occur in the same order on X and Y. Suppose first that the intersection contains an endpoint of one of the paths. We may assume that $x_0 = y_k$ for some k with $k < n$. Now we consider two cases. If y_{k+1} is not a vertex of X, then we replace the edge $x_{n-1}x_n$ with the edge $y_{k+1}x_0$ to create a third path of length n. If $y_{k+1} = x_i$ for some i, then we replace the edge $x_i x_{i-1}$ with the edge $y_{k+1}x_0$ to create a new path of length n.

Therefore, we may assume that $s \geq 3$ and that none of $\{x_0, x_n, y_0, y_n\}$ is among the s shared vertices. We apply a result of Thomason ([8], Theorem 2.1, pages 263-4): If H is a regular multigraph of degree 4 with at least 3 vertices, then for any two edges e and f

there are an even number of decompositions of H into two Hamiltonian cycles C_1 and C_2 with e in C_1 and f in C_2.

From the given graph G, we construct a 4-regular multigraph H. We first add the edges $e_0 = x_0x_n$ and $f_0 = y_0y_n$. We then "smooth out" all vertices of degree 2; that is, we iteratively contract edges incident to vertices of degree 2 until no such vertices remain. Since every vertex of $G \cup \{e_0, f_0\}$ has degree 2 or degree 4, the resulting multigraph H is regular of degree 4. Since $s \geq 3$, H has at least three vertices.

In H, the edge e_0 is absorbed into an edge e, and f_0 is absorbed into an edge f. The cycles $X \cup \{e_0\}$ and $Y \cup \{f_0\}$ have been contracted to become Hamiltonian cycles in H. Together they decompose H. By the theorem of Thomason, there is another Hamiltonian decomposition C_1, C_2 of H with e in C_1 and f in C_2.

Now we reverse our steps. Restore the vertices of degree 2 and remove the edges e_0 and f_0. The cycle C_1 becomes a path from x_0 to x_n, and C_2 becomes a path from y_0 to y_n. Neither of these paths is the original X or Y. Since G has $2n$ edges and is the edge-disjoint union of these two paths, one of the paths has length at least n. It contains a new path of length n.

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