Combinatorial Laplacian of the Matching Complex

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Abstract

A striking result of Bouc gives the decomposition of the representation of the symmetric group on the homology of the matching complex into irreducibles that are self-conjugate. We show how the combinatorial Laplacian can be used to give an elegant proof of this result. We also show that the spectrum of the Laplacian is integral.

1 Introduction

The matching complex of a graph G is the abstract simplicial complex whose vertex set is the set of edges of G and whose faces are sets of edges of G with no two edges meeting at a vertex. The matching complex of the complete graph (known simply as the matching complex) and the matching complex of the complete bipartite graph (known as the chessboard complex) have arisen in a number of contexts in the literature (see eg. [6] [16] [2] [19] [3] [4] [8] [12] [1] [9] [15] [13] [17] [18]). Closely related complexes have been considered in [7] and [14].

Let M_n denote the matching complex of the complete graph on node set $\{1, \ldots, n\}$. The symmetric group $\mathfrak{S}_n$ acts on the matching complex M_n by permuting the graph

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nodes. This induces a representation on the reduced simplicial homology $\tilde{H}_r(M_n;k)$, where throughout this paper k is a field of characteristic 0. The Betti numbers for the matching complex and the decomposition of the representation into irreducibles were computed by Bouc [3], and later independently by Karaguezian [8] and by Reiner and Roberts [12] as part of a more general study. They prove the following result.

Theorem 1.1 (Bouc [3]). *For all $r \geq 1$ and $n \geq 2$, the following isomorphism of $\mathfrak{S}_n$-modules holds*

$$\tilde{H}_{r-1}(M_n;k) \cong_{\mathfrak{S}_n} \bigoplus_{\substack{\lambda\,:\,\lambda\vdash n\\ \lambda=\lambda'\\ d(\lambda)=n-2r}} S^\lambda,$$

where S^λ denotes the Specht module indexed by λ, λ' denotes the conjugate of λ and $d(\lambda)$ denotes the size of the Durfee square of λ.

Józefiak and Weyman [7] and Sigg [14] independently obtained an equivalent result for a Koszul complex of $GL(n,k)$-modules (cf. [9]). They use this to give representation theoretic interpretations of the following classical symmetric function identity of Littlewood

$$\prod_{i\leq j}(1-x_ix_j)\prod_i(1+x_i)^{-1} = \sum_{\lambda=\lambda'}(-1)^{\frac{|\lambda|+d(\lambda)}{2}}s_\lambda.$$

Using Theorem 1.1 one can interpret Littlewood's formula as the Hopf trace formula for the matching complex. This interpretation is essentially equivalent to Sigg's interpretation.

A decomposition for the chessboard complex analogous to Theorem 1.1 was obtained independently by Friedman and Hanlon [4] and later by Reiner and Roberts [12] in greater generality. The method of Friedman and Hanlon [4] is particularly striking. It involves the combinatorial Laplacian which is an analogue of the Laplacian on differential forms for a Riemannian manifold. The analogue of Hodge theory states that the kernel of the combinatorial Laplacian is isomorphic to the homology of the complex. By analyzing the action of the Laplacian on oriented simplexes and applying results from symmetric function theory, Friedman and Hanlon are able to decompose all the eigenspaces of the Laplacian into irreducibles and thereby decompose the homology. They also show that the spectrum of the Laplacian is integral.

The aim of this note is to work out analogous decompositions for the combinatorial Laplacian on the matching complex. This results in an elegant proof of Theorem 1.1 which is given in Section 3. Our key observation is that the Laplacian operator behaves as multiplication by a certain element in the center of the group algebra, namely the sum of all transpositions in $\mathfrak{S}_n$. We also establish integrality of the spectrum of the Laplacian in Section 3.

Sigg [14] uses the Lie algebra Laplacian to obtain equivalent decompositions for the Lie algebra homology of the free two-step nilpotent complex Lie algebra. Sigg works within the framework of representation theory of the Lie algebra gl_n and expresses the Laplacian in terms of the Casimir operator. Our approach parallels his, but at a more elementary level, making use of readily available facts from symmetric group representation theory. In Section 4 we establish the equivalence of our results to Sigg's results.

Reiner and Roberts [12] generalize Theorem 1.1 to general bounded degree graph complexes by using techniques from commutative algebra. In [9] it is shown that one can derive the Reiner-Roberts result from Theorem 1.1 by taking weight spaces of the GL_n-modules considered by Józefiak and Weyman. Hence, although the Laplacian technique does not appear to be directly applicable to general bounded degree graph complexes, the Laplacian provides an indirect path to the Reiner-Roberts result (and the Józefiak-Weyman result) that is considerably simpler than the earlier approaches.

2 The Combinatorial Laplacian

Let Δ be a finite simplicial complex on which a group G acts simplicially. For $r \geq -1$, let $C_r(\Delta)$ be the rth chain space (with coefficients in k) of Δ. That is, $C_r(\Delta)$ is the k-vector space generated by oriented simplexes of dimension r. Two oriented simplexes are related by

$$(v_1, v_2, \ldots, v_{r+1}) = \operatorname{sgn}\sigma\,(v_{\sigma(1)}, v_{\sigma(2)}, \ldots, v_{\sigma(r+1)}),$$

where $\sigma \in \mathfrak{S}_{r+1}$. The simplicial action of G induces a representation of G on the vector space $C_r(\Delta)$.

The boundary map

$$\partial_r : C_r(\Delta) \to C_{r-1}(\Delta)$$

is defined on oriented simplexes by

$$\partial_r(v_1, \ldots, v_{r+1}) = \sum_{j=1}^{r+1} (-1)^j (v_1, \ldots, \hat{v}_j, \ldots, v_{r+1}).$$

Since ∂_* commutes with the action of G on $C_*(\Delta)$, $(C_r(\Delta), \partial_r)$ is a complex of G-modules. It follows that the (reduced) homology groups $\tilde{H}_r(\Delta; k)$ are G-modules.

The coboundary map

$$\delta_r : C_r(\Delta) \to C_{r+1}(\Delta)$$

is defined by

$$\langle \delta_r(\alpha), \beta \rangle = \langle \alpha, \partial_{r+1}(\beta) \rangle,$$

where $\alpha \in C_r(\Delta)$, $\beta \in C_{r+1}(\Delta)$ and $\langle , \rangle$ is the bilinear form on $\oplus_{r=-1}^{d} C_r(\Delta)$ for which any basis of oriented simplexes is orthonormal. Note that the action of G on $C_r(\Delta)$ respects the form $\langle , \rangle$ and commutes with the coboundary map. Hence $(C_r(\Delta), \delta_r)$ is a complex of G-modules.

The *combinatorial Laplacian* is the G-module homomorphism

$$\Lambda_r : C_r(\Delta) \to C_r(\Delta)$$

defined by

$$\Lambda_r = \delta_{r-1}\partial_r + \partial_{r+1}\delta_r.$$

Although the following analogue of Hodge theory is usually stated and easily proved for $k = \mathbb{R}$ or $k = \mathbb{C}$ (cf. [4, Proposition 1], [14, Proposition 9], [10]), the universal coefficient theorem enables one to prove it for general fields k of characteristic 0. Indeed, one uses the universal coefficient theorem first to derive the result for $k = \mathbb{Q}$ from the result for $k = \mathbb{C}$ and then to go from $k = \mathbb{Q}$ to general fields of characteristic 0.

Proposition 2.1. *For all r, the following kG-module isomorphism holds:*

$$\ker \Lambda_r \cong_G \tilde{H}_r(\Delta; k).$$

3 Spectrum of the Laplacian and Bouc's Theorem

The notation used here comes from [11]. The plethysm or composition product of a $\mathfrak{S}_m$-module V and a $\mathfrak{S}_n$-module U is the $\mathfrak{S}_{mn}$-module denoted by $V \circ U$. The induction product of U and V is the $\mathfrak{S}_{m+n}$-module denoted by $U.V$.

Proposition 3.1. *For all $r \geq 1$ and $n \geq 2$ we have the following isomorphism of $\mathfrak{S}_n$-modules*

$$C_{r-1}(M_n) \cong_{\mathfrak{S}_n} (S^{1^r} \circ S^2) \,.\, S^{n-2r}.$$

Proof. Straight forward observation. □

We say that a partition is *almost self-conjugate* if it is of the form $(\alpha_1 + 1, \dots, \alpha_d + 1 \mid \alpha_1, \dots, \alpha_d)$ in Frobenius notation.

Proposition 3.2 (Littlewood, cf. [11, I 5 Ex. 9b]). *For all $r \geq 1$,*

$$S^{1^r} \circ S^2 \cong_{\mathfrak{S}_{2r}} \bigoplus_{\lambda} S^{\lambda},$$

summed over all almost self-conjugate partitions $\lambda \vdash 2r$. □

Pieri's rule and the fact that the induction product is linear in each of its factors yields the following.

Proposition 3.3. *For all $r \geq 1$ and $n \geq 2$ we have*

$$C_{r-1}(M_n) \cong_{\mathfrak{S}_n} \bigoplus_{\substack{\lambda \in \mathcal{A} \\ |\lambda| = n}} a^r_{\lambda} S^{\lambda},$$

where

$$\mathcal{A} = \{(\alpha_1, \dots, \alpha_d \mid \beta_1, \dots, \beta_d) \mid d \geq 1, \;\; \alpha_i \geq \beta_i \; \forall i\}$$

and a^r_λ is the number of almost self-conjugate partitions $\mu \vdash 2r$ such that λ/μ is a horizontal strip. □

Proposition 3.4. *Let $\lambda \vdash n$ be self-conjugate. Then*

$$a_\lambda^r = \begin{cases} 1 & \text{if } d(\lambda) = n - 2r \\ 0 & \text{otherwise} \end{cases}$$

Proof. Straight forward observation. □

Propositions 3.1, 3.3 and 3.4 comprise the first steps of Bouc's proof of Theorem 1.1. At this point our proof departs from Bouc's and follows a path analogous to that of Friedman and Hanlon [4] for the chessboard complex.

Consider the element $T_n = \sum_{1 \le i < j \le n} (i, j)$ of $k\mathfrak{S}_n$, where (i, j) denotes a transposition in $\mathfrak{S}_n$. In any $\mathfrak{S}_n$-module M, left multiplication by T_n is an endomorphism of the $\mathfrak{S}_n$-module, since T_n is in the center of $k\mathfrak{S}_n$. We will denote this endomorphism by T_M. The Laplacian will be denoted by $\Lambda_{n,r} : C_r(M_n) \to C_r(M_n)$.

Lemma 3.5. *For all $r \ge 1$ and $n \ge 2$,*

$$\Lambda_{n,r-1} = T_{C_{r-1}(M_n)}.$$

Proof. It is a routine exercise to check that $\Lambda_{n,r-1}(\gamma) = T_n \cdot \gamma$ for the oriented $(r-1)$-simplex $\gamma = (\{1, 2\}, \ldots, \{2r - 1, 2r\})$ which generates the $\mathfrak{S}_n$-module $C_{r-1}(M_n)$. □

Proposition 3.6 (Friedman and Hanlon [4, Lemma 1]). *For all $\lambda \vdash n$,*

$$T_{S^\lambda} = c_\lambda \,\mathrm{id}_{S^\lambda},$$

where $c_\lambda = \sum (i - 1)\lambda'_i - \sum (i - 1)\lambda_i$.

Proof. This follows from Schur's lemma and from I7, Example 7 of [11]. □

Lemma 3.7. *For any partition $\lambda = (\alpha_1, \ldots, \alpha_d \mid \beta_1, \ldots, \beta_d)$ we have*

$$c_\lambda = \sum_{i=1}^{d} \left(\binom{\alpha_i + 1}{2} - \binom{\beta_i + 1}{2} \right).$$

Proof. Easy. □

Theorem 3.8. *All the eigenvalues of the Laplacian on the matching complex are non-negative integers. Moreover for each eigenvalue c, the c-eigenspace of $\Lambda_{n,r-1}$ decomposes into the following direct sum of irreducibles*

$$\bigoplus_{\substack{\lambda \in \mathcal{A} \\ |\lambda| = n \\ c_\lambda = c}} a_\lambda^r \, S^\lambda,$$

where $\mathcal{A}$ and a_λ^r are as in Proposition 3.3.

Proof. Proposition 3.3, Lemma 3.5 and Proposition 3.6 imply that each eigenvalue c is an integer and yield the decomposition of the c-eigenspace. It is immediate from Lemma 3.7 that $c_\lambda \geq 0$ for all $\lambda \in \mathcal{A}$. Hence there can be no negative eigenvalues. One can also conclude that the eigenvalues are nonnegative by using the fact that all rational eigenvalues of the Laplacian (over any field of characteristic 0) of any simplicial complex are nonnegative. This fact follows from the positive semidefinitness of the Laplacian over $\mathbb{C}$. □

Proof of Bouc's Theorem. It follows from Lemma 3.7 that if $\lambda \in \mathcal{A}$ then $c_\lambda = 0$ if and only if λ is self-conjugate. Hence Theorem 3.8 and Proposition 3.4 imply that the kernel of the Laplacian decomposes into

$$\ker \Lambda_{n,r-1} \cong_{\mathfrak{S}_n} \bigoplus_{\substack{\lambda :\ \lambda \vdash n \\ \lambda = \lambda' \\ d(\lambda) = n-2r}} S^\lambda.$$

Bouc's Theorem now follows from Proposition 2.1. □

Remark. Bouc's Theorem is stated and proved in [3] for fields of finite characteristic $p > n$ as well as for fields of characteristic 0. The characteristic $p > n$ case follows from the characteristic 0 case provided one knows that there is no p-torsion in integral homology. The lack of p-torsion for $p > n$ follows easily from a long exact sequence of Bouc [3, Lemme 7] which is the starting point of Bouc's proof.

4 Sigg's Lie Algebra Homology Theorem

In [14], Sigg decomposes the homology of the free two-step nilpotent complex Lie algebra of rank n into irreducible $GL(n, \mathbb{C})$-modules by using a Laplacian operator. In this section we will describe how his results relate to ours.

Let E be an n-dimensional complex vector space, where $n \geq 2$. Let $\wedge^r E$ denote the rth exterior power and $\wedge^* E$ denote the exterior algebra of E. The free two-step nilpotent complex Lie algebra of rank n is the vector space $\wedge^2 E \oplus E$ with Lie bracket defined on generators by

$$[x, y] = \begin{cases} x \wedge y & \text{if } x, y \in E \\ 0 & \text{if } x \in \wedge^2 E \text{ or } y \in \wedge^2 E. \end{cases}$$

For each $r \in \mathbb{N}$, form the $GL(E)$-submodule

$$V_r(E) = \wedge^r(\wedge^2 E) \otimes \wedge^* E$$

of the exterior algebra of $\wedge^2 E \oplus E$. The map $\partial_r^E : V_r(E) \to V_{r+1}(E)$ defined on generators by

$$\partial_r^E(f \otimes (e_1 \wedge \cdots \wedge e_t)) = \sum_{i<j} (-1)^{i+j} (f \wedge (e_i \wedge e_j)) \otimes (e_1 \wedge \cdots \wedge \hat{e}_i \wedge \cdots \wedge \hat{e}_j \wedge \cdots \wedge e_t),$$

where $f \in \wedge^r(\wedge^2 E)$ and $e_1, \ldots, e_t \in E$, is the standard Lie algebra homology differential for the Lie algebra $\wedge^2 E \oplus E$. The complex $(V_r(E), \partial_r^E)$ is a complex of $GL(E)$-modules.

The adjoint map $\delta_r^E : V_r(E) \to V_{r-1}(E)$ (with respect to the Hermitian form for which the standard basis of $V_r(E)$ is orthonormal) is defined on generators by

$$\delta_r^E \left(((e_1 \wedge e_2) \wedge \cdots \wedge (e_{2r-1} \wedge e_{2r})) \otimes e\right)$$

$$= \sum_{j=1}^{r} (-1)^{j+r+1} ((e_1 \wedge e_2) \wedge \cdots \wedge \widehat{(e_{2j-1} \wedge e_{2j})} \wedge \cdots \wedge (e_{2r-1} \wedge e_{2r})) \otimes (e_{2j-1} \wedge e_{2j} \wedge e),$$

where $e_1, \ldots, e_{2r} \in E$ and $e \in \wedge^* E$. Let $H_r(V(E))$ denote the homology of the $GL(E)$-complex $(V_r(E), \partial_r^E)$ and let $H^r(V(E))$ denote the homology of the $GL(E)$-complex $(V_r(E), \delta_r^E)$.

The Laplacian used by Sigg is the $GL(E)$-homomorphism $\Lambda_r^E : V_r(E) \to V_r(E)$ defined by

$$\Lambda_r^E = \delta_{r+1}^E \partial_r^E + \partial_{r-1}^E \delta_r^E.$$

It follows from the discrete version of Hodge theory that

$$H_r(V(E)) \cong_{GL(E)} H^r(V(E)) \cong_{GL(E)} \ker \Lambda_r^E.$$

For each partition λ, let E^λ be the irreducible polynomial representation of $GL(E)$ of highest weight λ if $\ell(\lambda) \leq \dim E$ and 0 otherwise.

Proposition 4.1 (Sigg [14]). *All eigenvalues of Λ_r^E are nonnegative integers. Moreover for each eigenvalue c, the c-eigenspace decomposes into the following direct sum of irreducibles*

$$\bigoplus_{\substack{\lambda \in \mathcal{A} \\ c_\lambda = c}} a_\lambda^r \, E^{\lambda'},$$

where $\mathcal{A}$ and a_λ^r are as in Proposition 3.3. Consequently,

$$H_r(V(E)) \ \cong_{GL(E)} \ H^r(V(E)) \ \cong_{GL(E)} \bigoplus_{\substack{\lambda \,:\, \lambda = \lambda' \\ d(\lambda) = |\lambda| - 2r}} E^\lambda.$$

Sigg proves this result by first switching to the derivative representation of the Lie algebra $gl(E)$ on $\oplus_r V_r(E)$, and then comparing the Casimir operator of the $gl(E)$-module to the Laplacian operator.

We will now describe how one can obtain Sigg's result from our Proposition 3.8 and vice-versa. Let $U_{n,r}$ be the 1^n-weight space of the $GL(E)$-module $V_r(E)$ (recall $n = \dim E$). (See [5] for information on weight spaces.) The weight space $U_{n,r}$ is a $\mathfrak{S}_n$-module. Since $\Lambda_r^E(U_{n,r}) \subseteq U_{n,r}$, by restricting the Laplacian we get a $\mathfrak{S}_n$-module homomorphism $\mathcal{L}_{n,r} : U_{n,r} \to U_{n,r}$. Now we take the Young dual. That is, we consider the map

$$\mathcal{L}_{n,r} \otimes \mathrm{id} : U_{n,r} \otimes \mathrm{sgn} \to U_{n,r} \otimes \mathrm{sgn},$$

where sgn denotes the sign representation of $\mathfrak{S}_n$.

Theorem 4.2. *There is an $\mathfrak{S}_n$-module isomorphism $\phi_r : U_{n,r} \otimes \operatorname{sgn} \to C_{r-1}(M_n)$ such that*

$$\phi_r \circ (\mathcal{L}_{n,r} \otimes \mathrm{id}) = \Lambda_{n,r-1} \circ \phi_r.$$

.

Proof. Consider the oriented $(r-1)$-simplex

$$c = (\{1,2\}, \ldots, \{2r-1, 2r\})$$

in $C_{r-1}(M_n)$ and the element

$$d = ((e_1 \wedge e_2) \wedge \cdots \wedge (e_{2r-1} \wedge e_{2r})) \otimes (e_{2r+1} \wedge \cdots \wedge e_n)$$

of $U_{n,r}$, where $e_1, \ldots, e_n$ is a fixed ordered basis for E. Then

$$\{\sigma \cdot c \mid \sigma \in \mathfrak{S}_n\} \text{ and } \{\sigma \cdot d \mid \sigma \in \mathfrak{S}_n\}$$

are spanning sets for the respective vector spaces $C_{r-1}(M_n)$ and $U_{n,r}$.

Let ξ be a generator of the one dimensional sgn representation. Define ϕ_r on the elements of the spanning set by

$$\phi_r(\sigma \cdot d \otimes \xi) = \operatorname{sgn}(\sigma)\, \sigma \cdot c.$$

It is easy to see that this determines a well-defined $\mathfrak{S}_n$-module isomorphism $\phi_r : U_{n,r} \otimes \operatorname{sgn} \to C_{r-1}(M_n)$ by checking that relations on the elements of the spanning set of the vector space $U_{n,r} \otimes \operatorname{sgn}$ correspond (under ϕ_r) to relations on the elements of the spanning set of the vector space $C_{r-1}(M_n)$.

One can also easily check that

$$\phi_{r+1}(\partial^E_r(d) \otimes \xi) = \delta_{n,r-1}(\phi_r(d \otimes \xi))$$

and

$$\phi_{r-1}(\delta^E_r(d) \otimes \xi) = \partial_{n,r-1}(\phi_r(d \otimes \xi)),$$

where $\partial_{n,r-1}$ and $\delta_{n,r-1}$ denote the $(r-1)$-boundary and $(r-1)$-coboundary maps, respectively, of the matching complex M_n. From this it follows that

$$\phi_r(\Lambda^E_r(d) \otimes \xi) = \Lambda_{n,r-1}(\phi_r(d \otimes \xi)).$$

So $\phi_r \circ (\mathcal{L}_{n,r} \otimes \mathrm{id})$ and $\Lambda_{n,r-1} \circ \phi_r$ agree on the generator $d \otimes \xi$ of the cyclic $\mathfrak{S}_n$-module, and they are therefore the same map. $\square$

Corollary 4.3. *The eigenvalues of $\Lambda_{n,r-1}$ and $\mathcal{L}_{n,r}$ are the same. Moreover, for each eigenvalue c and partition λ of n, the multiplicity of the irreducible S^λ in the c-eigenspace of $\Lambda_{n,r-1}$ equals the multiplicity of $S^{\lambda'}$ in the c-eigenspace of $\mathcal{L}_{n,r}$.*

The following proposition and Corollary 4.3 establish the equivalence of Proposition 4.1 and Theorem 3.8.

Proposition 4.4. *For all c and all partitions λ such that $\ell(\lambda) \leq \dim E$, the multiplicity of the irreducible E^λ in the c-eigenspace of Λ_r^E equals the multiplicity of S^λ in the c-eigenspace of $\mathcal{L}_{|\lambda|,r}$.*

Proof. Suppose $|\lambda| = \dim E$. Then the c-eigenspace of $\mathcal{L}_{|\lambda|,r}$ is the $1^{|\lambda|}$-weight space of the c-eigenspace of Λ_r^E. By taking the $1^{|\lambda|}$-weight space of each summand in the decomposition of the c-eigenspace of Λ_r^E into irreducible $GL(E)$-modules, we obtain a decomposition of the c-eigenspace of $\mathcal{L}_{|\lambda|,r}$ into irreducible $\mathfrak{S}_{|\lambda|}$-modules S^λ whose multiplicity is the same as that of E^λ in the c-eigenspace of Λ_r^E.

To obtain the result for general λ from the case that $|\lambda| = \dim E$, we need only observe the fact that if E_1 and E_2 are t and s dimensional vector spaces, respectively, and λ is a partition such that $\ell(\lambda) \leq t \leq s$, then the multiplicity of E_2^λ in the c-eigenspace of $\Lambda_r^{E_2}$ equals the multiplicity of E_1^λ in the c-eigenspace of $\Lambda_r^{E_1}$. To establish this fact, suppose E_1 has ordered basis $e_1, \ldots, e_t$ and E_2 has ordered basis $e_1, \ldots, e_s$. For any polynomial $GL(E_2)$-module V and sequence $\mu = (\mu_1, \ldots, \mu_s)$ of nonnegative integers let V_μ denote the μ-weight space of V. For $i = 1, 2$, let $W_i(c)$ be the c-eigenspace of $\Lambda_r^{E_i}$. Note that

$$W_1(c) = \bigoplus_\mu W_2(c)_\mu,$$

where μ ranges over all weights $(\mu_1, \ldots, \mu_s)$ such that $\mu_{t+1} = \cdots = \mu_s = 0$. Suppose $W_2(c)$ decomposes into $\bigoplus b_\lambda E_2^\lambda$. Then

$$W_2(c)_\mu = \bigoplus_\lambda b_\lambda (E_2^\lambda)_\mu.$$

It follows that

$$W_1(c) = \bigoplus_\mu \bigoplus_\lambda b_\lambda (E_2^\lambda)_\mu = \bigoplus_\lambda \bigoplus_\mu b_\lambda (E_2^\lambda)_\mu = \bigoplus_\lambda b_\lambda E_1^\lambda.$$

Hence the multiplicities of E_2^λ in $W_2(c)$ and E_1^λ in $W_1(c)$ are the same. $\square$

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