On the number of permutations admitting an m-th root

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Abstract

Let m be a positive integer, and $p_n(m)$ the proportion of permutations of the symmetric group $\mathfrak{S}_n$ that admit an m-th root. Calculating the exponential generating function of these permutations, we show the following asymptotic formula

$$p_n(m) \underset{n\to+\infty}{\sim} \frac{\pi_m}{n^{1-\varphi(m)/m}},$$

where φ is the Euler function and π_m an explicit constant.

1. Introduction

The question consists in estimating the number of permutations of the symmetric group $\mathfrak{S}_n$ which admit an m-th root when n is large. Turán gave an upperbound when m is a prime number [Tu] and Blum found an asymptotically equivalent form for $m = 2$ [Bl]. In the general case, Bender applied a theorem of Hardy, Littlewood and Karamata to the exponential generating function of these permutations to obtain an asymptotic equivalent of the partial sums of the required numbers [Be]. In [BoMcLWh], it is shown that the sequence tends monotonically to zero in the case when m is prime.

Whether a permutation of $\mathfrak{S}_n$ admits an m-th root can be read on the partition of n determined by the lengths of the permutation's cycles, because the class of such

permutations is stable under conjugacy in $\mathfrak{S}_n$. This characterisation, already mentioned in [Be] is established in section **2**.

The computation of the exponential generating function (EGF) P_m of these permutations follows from the preceding result. This EGF splits in a natural way as a product of two others EGF:

$$P_m = C_m \times R_m.$$

Singularity analysis provides the asymptotics of the coefficients of $C_m = \sum_n c_n(m) X^n$ because C_m has a finite number of algebraic singularities on its circle of convergence. This asymptotics turns to be of the following form

$$c_n(m) \underset{n\to+\infty}{\sim} \frac{\kappa_m}{n^{1-\frac{\varphi(m)}{m}}},$$

where κ_m is an explicit constant and φ the Euler function. This formula was already established in [BoGl] only when m is a prime number.

On the contrary, the singularities of $R_m = \sum_n r_n(m) X^n$ form a dense subset of its circle of convergence; this prevents transfer theorems to apply to R_m and to the whole series P_m. Nevertheless, the series with positive coefficients $\sum_n r_n(m)$ converges. Now, since

$$\frac{p_n(m)}{c_n(m)} = \sum_{k=o}^{n} \frac{c_{n-k}(m)}{c_n(m)} r_k(m),$$

and since $c_{n-k}(m)/c_n(m)$ tends to 1 as n tends to infinity for every k, the asymptotics of the $p_n(m)$ will follow from an interchange of limits.

Lebesgue's dominated convergence theorem for the counting measure on the natural numbers does not directly apply because $c_{n-k}(m)/c_n(m)$ is too large when k is not far from n (if k equals n, its value is $n^{1-\varphi(m)/m}$ up to a positive factor). If the sequences $\big(c_{n-k}(m)/c_n(m)\big)_n$ were monotonic, the result would be a consequence of Lebesgue's monotonic convergence theorem (for the counting measure once again). Unfortunately, this is not the case. We approximate the $c_n(m)$ by the coefficients $d_n(m)$ of the expansion in power series of the principal part D_m of C_m in a neighbourhood of its dominant singularity 1. The sequences $\big(d_{n-k}(m)/d_n(m)\big)_n$ are this time monotonic, so that

$$\lim_{n\to+\infty} \sum_{k=0}^{n} \frac{d_{n-k}(m)}{d_n(m)} r_k(m) = \sum_{n\geq 0} r_n(m).$$

Now, the approximation of the $c_n(m)$ by the $d_n(m)$ is good enough to ensure the application of dominated convergence theorem; this last fact implies the announced result.

In an appendix, we give an explicit formula giving the number $c_n(m) \times n!$ of permutations of $\mathfrak{S}_n$ whose canonical decomposition has only cycles of length prime to m (these permutations are m-th powers).

2. What does an m-th power look like in $\mathfrak{S}_n$?

Every permutation has a canonical decomposition (unique up to order) as a product of cycles of disjoint supports. These cycles commute. Therefore, a permutation is an m-th power if and only if it is a product of m-th powers of cycles with disjoint supports. Then, it suffices to check what the m-th power of a cycle looks like.

Lemma. *The m-th power of a cycle of length l is a product of $\gcd(l,m)$ cycles of length $l/\gcd(l,m)$ with disjoint supports.*

In algebraic terms, this lemma can be understood in the following way: if c is a cycle of length l, the order of the element c^m in the symmetric group is $l/\gcd(l,m)$.

In order to establish the shape of an m-th power of $\mathfrak{S}_n$, let us introduce the notation $l^\infty \wedge m$: if l and m are integers, $\gcd(l^n, m)$ does not depend on n, provided n is large enough; $l^\infty \wedge m$ is defined as this common value of $\gcd(l^n, m)$, $n \gg 1$. In terms of decomposition in prime factors, $l^\infty \wedge m$ is the part of m having a common divisor with l: let $m = \pm \prod p^{v_p(m)}$ be the decomposition of m in primes, the products ranges over all primes numbers p, the valuations $v_p(m)$ are nonnegative integers, almost all of them are zero. Then, $l^\infty \wedge m = \prod p^{v_p(m)}$ where the product ranges over all primes p such that p divides l. At last, one can see the number $l^\infty \wedge m$ as the least positive divisor d of m such that l and m/d are coprimes.

Proposition. *A permutation $\sigma \in \mathfrak{S}_n$ has an m-th root if and only if for every positive integer l, the number of l-cycles in the canonical decomposition of σ is a multiple of $l^\infty \wedge m$.*

Proof. Let $\delta = l^\infty \wedge m$. Then δ divides m, and $\gcd(m/\delta, l) = 1$. For every positive integer k, with the help of the lemma, a product of $k\delta$ cycles with disjoint supports is the m-th power of a cycle of length $lk\delta$. Doing this for every l, one sees that the condition is sufficient. Now, let c be a cycle of length k. Then, thanks to the lemma, c^m is the product of $\gcd(k,m)$ cycles of length $l = k/\gcd(k,m)$. To catch the necessity of the condition, it is enough to show that $\gcd(k,m)$ is a multiple of δ, *i.e.* that for every prime p, one has $v_p(\gcd(k,m)) \geq v_p(\delta)$. It follows from the definition of $l^\infty \wedge m$ that

$$v_p(\delta) = \begin{cases} 0 & \text{if } p \text{ divides } \gcd(l,m) \\ v_p(m) & \text{if } p \text{ does not divide } \gcd(l,m). \end{cases}$$

Suppose that p is a prime divisor of $\gcd(l,m)$. In particular, $v_p(l) \neq 0$ Then, $v_p(m) < v_p(k)$ since $v_p(l) = v_p(k) - \min\{v_p(m), v_p(k)\}$. This implies that $v_p(\gcd(k,m)) = v_p(m) = v_p(\delta)$. On the other hand, if the prime p does not divide $\gcd(l,m)$, then $v_p(\delta) = 0 \leq v_p(\gcd(k,m))$ and the proof is complete.

Examples. 1: In the case where m is a prime number, the recipe to build an m-th power in $\mathfrak{S}_n$ is the following: compose arbitrarily cycles of length not divisible by m with groups of m cycles of same length divisible by m (all cycles with disjoint supports).

2: The notations for partitions are the standard ones. If the partition associated to a permutation σ is $(2^6, 3^{27}, 4^2, 5, 6^{18}, 7^2)$, then σ is the 18-th power of a permutation whose partition is $(4^3, 5, 7^2, 8, 27^3, 104)$. In general, a permutation admits many m-th roots, which do not have necessarily the same partition.

3. The exponential generating function of the m-th powers

We adopt the following notations :

$$P_m = \sum_{n\geq 0} p_n(m) X^n$$

$$C_m = \sum_{n\geq 0} c_n(m) X^n$$

$$R_m = \sum_{n\geq 0} r_n(m) X^n.$$

$P_m \in \mathbf{Q}[[X]]$ is the exponential generating function (EGF, formal series) of the m-th powers in the groups $\mathfrak{S}_n$. This means that the number of m-th powers in $\mathfrak{S}_n$ is $p_n(m) \times n!$ for each n. In the same way, C_m is the EGF of the permutations having only cycles of length prime to m in their canonical decomposition (they admit a m-th root) and R_m the EGF of the *rectangular m-th powers*, that is the m-th powers with only cycles of length having a common factor with m (the adjective rectangular is chosen because of the form of the Ferrers diagram associated to such a permutation : a sequence of rectangular blocks of height greater than 1).

Now, the standard way to compute these series [FlSe] leads to the following expressions, according to the previous proposition:

$$P_m = C_m \times R_m = \prod_{l\geq 1} e_{l^\infty \wedge m}\left(\frac{X^l}{l}\right). \tag{1}$$

In the last formula, $l^\infty \wedge m$ is defined in **2-** and e_d denotes the formal series (or the entire function) defined for $d \geq 1$ by

$$e_d(X) = \sum_{n\geq 0} \frac{X^{nd}}{(nd)!} = \frac{1}{d} \sum_{\zeta} \exp(\zeta X).$$

The last sum is extended to all d-th (complex) roots of 1. Note that for $d = 1$ this series is the exponential and for $d = 2$ the hyperbolic cosine.

3.1. Isolating the numbers l prime to m, one finds

$$C_m = \exp\left(\sum_{\substack{l\geq 1 \\ \gcd(l,m)=1}} \frac{X^l}{l} \right). \tag{2}$$

If the decomposition into prime numbers of m is $m = p_1^{\alpha_1} \dots p_r^{\alpha_r}$ with all α_i greater or equal to one, let $q(m) = p_1 \dots p_r$ be the *quadratfrei radical** of m (a positive integer is said to be quadratfrei if and only if it has no square factor). For conciseness, we shall write q in place of $q(m)$ if the situation is unambiguous. Formula (2) shows that

$$C(m) = C(q).$$

If m is the power of a prime number, $\gcd(k, m) = 1$ if and only if k is not divisible by the prime q, which gives the expression $C_m = \sqrt[q]{1 - X^q}/(1 - X)$. Furthermore, if p is a prime number and q a quadratfrei number prime to p, formula (2) shows that

$$C_{pq}(X) = C_q(X) \times C_q(X^p)^{1/p}. \tag{3}$$

We note μ the Möbius function on the positive integers, defined by $\mu(m) = 0$ if m has a square prime factor, and $\mu(q) = (-1)^r$ if q is a quadratfrei number with r prime factors (in particular, $\mu(1) = 1$). The function μ is multiplicative in the following sense : if m_1 and m_2 are coprime numbers, then $\mu(m_1 m_2) = \mu(m_1)\mu(m_2)$ (see [HaWr]).

Proposition. *For every positive m, the EGF of the permutations having only cycles of length prime to m in their canonical decomposition is*

$$C_m = \prod_{k|m} \left(1 - X^k\right)^{-\mu(k)/k}$$

Proof. Induction with formula (3).

Note that one can write the proposition with the product being extended only to all divisors of the quadratfrei radical q of m. Indeed, only the quadratfrei divisors of m have a non trivial contribution.

3.2. The contribution of the rectangular m-th powers to the series P_m is the product extended to the l which have a common factor with m, *i.e.*

$$R_m = \prod_{\substack{l \geq 1 \\ \gcd(l,m) \neq 1}} e_{l\infty \wedge m}\left(\frac{X^l}{l}\right). \tag{4}$$

* In terms of commutative algebra, the radical of an ideal I is the set of all elements of the ring some positive power of which belongs to I; in the present situation, $q(m)$ is the positive generator of the radical of the ideal of $\mathbf{Z}$ generated by m.

4. Main theorem

We now aim to calculate an asymptotic equivalent of the coefficients of $P_m = C_m R_m$. Singularity analysis will allow us to establish such an asymptotics for the coefficients of C_m, because the radius of convergence of the associated analytic function it defines is 1, with a finite number of algebraic singularities on the unit circle. Unfortunately, the series R_m admits the unit circle as a natural boundary: the singularities of R_m form a dense subset of the unit circle.

The argument given to reach the desired asymptotics uses the convergence of the series of coefficients of R_m, and a combination of monotonic and dominated convergences round C_m, together with a new occurence of singularity analysis.

4.1. Convergence of the series $\sum_n r_n(m)$

The infinite product

$$R_m(1) = \prod_{\substack{l\geq 1 \\ \gcd(l,m)\neq 1}} e_{l^\infty \wedge m}\left(\frac{1}{l}\right)$$

converges because its general term is $1 + \mathcal{O}(1/l^2)$ as l tends to infinity.

Moreover, $e_d(X^l/l) = 1 + \frac{1}{l^d d!}X^{ld} + \cdots$, which shows that just a finite number of factors of the infinite product R_m are enough to calculate the n-th coefficient $r_n(m)$ (roughly speaking, one needs less than the first $\lceil n/2 \rceil$ terms of the product).

If t is a positive integer, let $R_m^t = \sum r_n^t(m) X^n$ be the product of the first t terms of the product R_m. The series R_m^t has an infinite radius of convergence; in particular, the series $\sum_n r_n^t(m)$ converges to $R_m^t(1)$. Then, all terms being nonnegative, if t is greater than $\lceil n/2 \rceil$, one has successively

$$\sum_{k=0}^{n} r_k(m) = \sum_{k=0}^{n} r_k^t(m) \leq \sum_{k=0}^{+\infty} r_k^t(m) = R_m^t(1) \leq R_m(1).$$

The last inequality is due to the fact that the e_d are greater than 1 on the nonnegative real numbers. Since the terms $r_n(m)$ are all positive, the series $\sum_n r_n(m)$ converges and thanks to Abel's theorem*, one has at last

$$\sum_{n\geq 0} r_n(m) = R_m(1). \tag{5}$$

Remark. The series R_m admits the unit circle as a natural boundary. We illustrate this phenomenon on the particular case where $m = 2$. The general case, more complicated to write, is conceptually of the same kind.

* We refer to the following theorem of Abel: if the series $\sum a_n$ converges, then the power series $\sum a_n z^n$ is uniformly convergent on $[0, 1]$.

For $m = 2$, the series is

$$R_2 = \prod_{n\geq 1} \cosh\left(\frac{X^{2n}}{2n}\right) = \exp\left(\sum_{m\geq 1} \frac{(-1)^{m-1}\tau_{m-1}}{m2^{2m+1}} \mathrm{Li}_{2m}(X^{4m})\right), \tag{6}$$

where $\mathrm{Li}_n(X) = \sum X^k/k^n$ is the n-th polylogarithm and τ_m are the tangent numbers, defined by the expansion $\tan X = \sum \tau_m X^{2m+1}$. The n-th polylogarithm has a singularity at 1, with principal part $(1-z)^{n-1}\log 1/(1-z)$ up to a factor. Thus every primitive $4m$-th root of unity ζ is a singularity of R_2 with principal part $(1-z/\zeta)^{2m-1}\log 1/(1-z/\zeta)$ up to a factor, so that R_2 is singular at a dense subset of points on the unit circle.

4.2. Asymptotics of the $c_n(m)$

We use a restricted notion of order of a singularity: we will say that an analytic function f has order $\alpha \in \mathbf{R}\setminus\mathbf{Z}_-$ at its (isolated) singularity ζ if

$$f(z) = \frac{\mathrm{c}}{\left(1-\frac{z}{\zeta}\right)^{\alpha}}\big(1+\mathcal{O}(z-\zeta)\big)$$

in a neighbourhood of ζ which avoids the ray $[\zeta, +\infty[$, where c is a non zero constant (c is the value at ζ of the function $z \mapsto (1-\frac{z}{\zeta})^{\alpha} f(z)$).

All the singularities of C_m are on the unit circle : they are the q-th roots of unity, where q is the quadratfrei radical of m. The order of the singularity 1 is clearly $\sum \mu(k)/k$, where the sum extends to all divisors of q. Let φ be the Euler function, *i.e.* $\varphi(q)$ is the number of all positive integers less or equal to q and prime to q. Because of the Möbius inversion formula (see [HaWr]), since $q = \sum \varphi(k)$ where k ranges over all divisors of q, one finds $\sum \mu(k)/k = \varphi(q)/q$. An elementary calculation of the same kind, using the multiplicativity of the arithmetical functions φ and μ leads to the following result.

Lemma. *If ζ is a primitive k-th root of unity (where k divides q), then C_m has at ζ a singularity of order* $\dfrac{\mu(k)}{\varphi(k)}\dfrac{\varphi(q)}{q}$.

Note once more that one could state this result without the use of q, writing directly m instead of q. Indeed, μ is zero on non-quadratfrei numbers, and $\varphi(q)/q = \varphi(m)/m$.

Proposition. *For every positive integer m, the number $c_n(m)\times n!$ of permutations of $\mathfrak{S}_n$ having only cycles of length prime to m in their canonical decomposition satisfies*

$$c_n(m) \underset{n\to+\infty}{\sim} \frac{\kappa_m}{n^{1-\frac{\varphi(m)}{m}}},$$

where κ_m is the following constant depending only on the quadratfrei radical q of m

$$\kappa_m = \frac{1}{\Gamma\big(\frac{\varphi(m)}{m}\big)} \prod_{k|m} k^{-\frac{\mu(k)}{k}}.$$

Proof. C_m defines an analytic (single-valued) function in any simply connected domain that avoids its singularities. The lemma shows that the singularity of C_m at 1 determines alone the asymptotics of $c_n(m)$ *via* transfer theorem *. The constant $\kappa_m \times \Gamma\big(\frac{\varphi(m)}{m}\big)$ is the value at 1 of the function $z \mapsto (1-z)^{\varphi(m)/m} C_m(z)$.

For a formula giving the exact value of $c_n(m)$, see the appendix. Figure 1 shows the first thousand values of kappa, with m on the x-axis and κ_m on the y-axis.

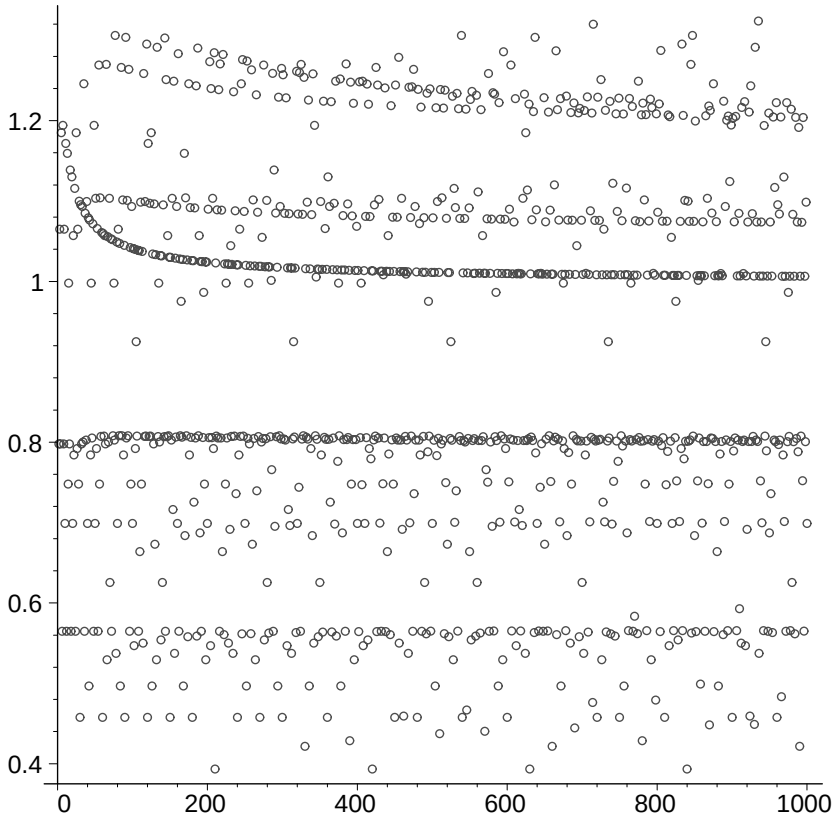

Figure 1: The function $m \mapsto \kappa_m$

4.3. Statement and proof of the main theorem

The situation is the following: we look for the asymptotics of the coefficients $p_n(m)$ of the formal series $P_m = C_m R_m$ where the coefficients $c_n(m)$ are equivalent to $n^{-1+\varphi(m)/m}$ up to a constant factor, and the series of coefficients $r_n(m)$ converges.

* By transfer theorem, we mean analysis of singularities that consists in deducing the asymptotics of the coefficients of a power series from the local analysis of its singularities when they involve only powers and logarithms. For a detailed study, see [FlSe].

Theorem. *Let m be a positive integer. The number $p_n(m) \times n!$ of permutations of $\mathfrak{S}_n$ which admit a m-th root satisfies*

$$p_n(m) \underset{n\to+\infty}{\sim} \frac{\pi_m}{n^{1-\frac{\varphi(m)}{m}}}$$

where π_m is the positive constant

$$\pi_m = \kappa_m R_m(1) = \frac{1}{\Gamma\left(\frac{\varphi(m)}{m}\right)} \prod_{k|m} k^{-\frac{\mu(k)}{k}} \prod_{\substack{l\geq 1 \\ \gcd(l,m)\neq 1}} e_{l^\infty \wedge m}\left(\frac{1}{l}\right).$$

Proof. For simplicity, we note $p_n = p_n(m)$, and similarly for c_n and r_n. We deduce from the formula $P_m = C_m R_m$ that $p_n = \sum c_{n-k} r_k$, where k ranges over $\{0, \ldots, n\}$. Since c_{n-k}/c_n tends to 1 as n tends to infinity for every k (see the asymptotics of c_n), it is enough to show that the following interchanging of limits is valid:

$$\lim_{n\to+\infty} \sum_{k=0}^{n} \frac{c_{n-k}}{c_n} r_k = \sum_{n\geq 0} r_n.$$

Let D_m be the series $D_m = \kappa_m \times \Gamma(\varphi(m)/m) \times (1-X)^{-\varphi(m)/m} = \sum_{n\geq 0} d_n X^n$, principal term of the series C_m in a neighbourhood of 1 (see proof of the previous proposition). For each integer k, the sequence $(d_{n-k}/d_n)_n$ decreases (compute it explicitely, d_n is a generalised binomial number up to a factor) and converges to one. Then, by monotonic convergence theorem,

$$\lim_{n\to+\infty} \sum_{k=0}^{n} \frac{d_{n-k}}{d_n} r_k = \sum_{n\geq 0} r_n.$$

On the other hand, the formal series $C_m - D_m$ defines a function analytic on the unit disk, whose singularities are those of C_m except 1 which becomes of order $\varphi(m)/m - 1$. If $m \neq 1$, the singularity that determines the asymptotics of its coefficient has order α strictly less than $\varphi(m)/m$ (the previous lemma gives α explicitely). As a consequence, $1 - d_n/c_n$ tends to zero as n tends to $+\infty$. In particular, there exist two positive constants A and B such that

$$\forall n \geq 0, \quad A \leq \frac{d_n}{c_n} \leq B.$$

Then, for all n and k (with $k \leq n$), one has

$$\frac{c_{n-k}}{c_n} \leq \frac{B}{A} \frac{d_{n-k}}{d_n}.$$

The conclusion follows now from the dominated convergence theorem.

Figure 2 shows the first thousand values of the function $m \mapsto \pi_m$, with m on the x-axis and π_m on the y-axis.

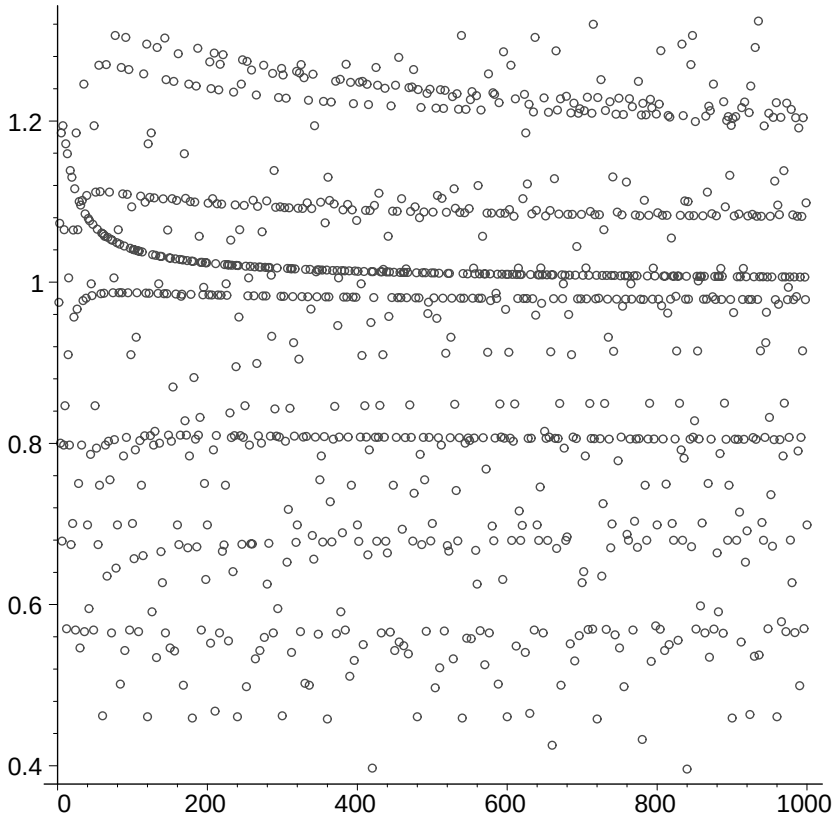

Figure 2: The function $m \mapsto \pi_m$

Remarks.
i) When m is the power of a prime number q, there is another way to catch the interchange of limits because one can explicitly write the coefficients $c_n(m) = c_n(q)$ as products and quotients of integers (see section **5-** : under this assumption, $b_n(q)$ equals $c_n(q)$). It is just a matter of elementary computation to see that for every k, the "congruence subsequences" of $c_{n-k}(q)/c_n(q)$ are monotonic :

$$\forall k \geq 0, \quad \forall r \in \{0, \ldots, q-1\}, \quad \text{the sequence } \left(\frac{c_{nq+r-k}(q)}{c_{nq+r}(q)} \right)_n \text{ is monotonic.}$$

Putting together the common asymptotics these congruence subsequences give is enough to prove the theorem.
ii) The expression of P_m with the help of polylogarithms such as in formula (6) would give an alternative proof of the theorem, and a way to obtain further asymptotics of the numbers $p_n(m)$, using a hybrid method of singular analysis and of Darboux's method as it is described in [FlGoPa].

5. Appendix

Let $b_n(m) \times n!$ be the number of permutations of $\mathfrak{S}_n$ which admit no cycle of length divisible by m in their canonical decomposition. Calculating the exponential generating function of these permutations leads to a recurrence formula for the $b_n(m)$; finally, one finds

$$b_n(m) = \prod_{\substack{1 \leq k \leq n \\ m|k}} \left(1 - \frac{1}{k}\right)$$

(see [BeGo]). One can calculate these numbers with the induction formula:

$$\begin{cases} b_n(m) & = & b_{n-1}(m) & \text{if } n \notin m\mathbf{N}^* \\ b_n(m) & = & b_{n-1}(m)(1 - \frac{1}{n}) & \text{if } n \in m\mathbf{N}^* \end{cases}$$

If $\mathcal{B}_m$ (*resp.* $\mathcal{C}_m$) denotes the set of all permutations (of any $\mathfrak{S}_n$) which admit no cycle of length divisible by m (*resp.* having only cycles of length prime to m) in their canonical decomposition, then $\mathcal{C}_m = \bigcup \mathcal{B}_d$, where the union is extended to all divisors d of q greater than or equal to 2. Once more, q denotes the quadratfrei radical of m. The sieve formula gives $\#(\mathcal{C}_m) = \sum -\mu(d)\#(\mathcal{B}_d)$, the sum being extended to the same d as before; μ is the Möbius function. This implies the following result.

Proposition. *The number $c_n(m) \times n!$ of permutations of $\mathfrak{S}_n$ having only cycles of length prime to m satisfies*

$$c_n(m) = \sum_{\substack{d \geq 2 \\ d|m}} - \mu(d) b_n(d) = \sum_{\substack{d \geq 2 \\ d|m}} - \mu(d) \sum_{\substack{k \in d\mathbf{Z} \\ 1 \leq k \leq n}} \left(1 - \frac{1}{k}\right).$$

6. Aknowledgements

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