Set-Systems with Restricted Multiple Intersections

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Abstract

We give a generalization for the Deza-Frankl-Singhi Theorem in case of multiple intersections. More exactly, we prove, that if $\mathcal{H}$ is a set-system, which satisfies that for some k, the k-wise intersections occupy only ℓ residue-classes modulo a p prime, while the sizes of the members of $\mathcal{H}$ are not in these residue classes, then the size of $\mathcal{H}$ is at most

$$(k-1)\sum_{i=0}^{\ell}\binom{n}{i}$$

This result considerably strengthens an upper bound of Füredi (1983), and gives partial answer to a question of T. Sós (1976).

As an application, we give a direct, explicit construction for coloring the k-subsets of an n element set with t colors, such that no monochromatic complete hypergraph on $\exp\left(c(\log m)^{1/t}(\log\log m)^{1/(t-1)}\right)$ vertices exists.

Keywords: set-systems, algorithmic constructions, explicit Ramsey-graphs, explicit Ramsey-hypergraphs

1 Introduction

We are interested in set-systems with restricted intersection-sizes. The famous Ray-Chaudhuri–Wilson [RCW75] and Frankl–Wilson [FW81] theorems give strong upper bounds for the size of set-systems with restricted pairwise intersection sizes. T. Sós asked in 1976 [Sós76], what happens if not the pairwise intersections, but the k-wise intersection-sizes are restricted.

Füredi [Für83], [Für91] showed (actually proving a much more general structure theorem) that for d-uniform set-systems over an n element universe, for very small d's, ($d = O(\log\log n)$), the order of magnitude of the largest set-systems, satisfying k-wise or just pairwise intersection restrictions are the same.

In the present paper we strengthen this result of Füredi [Für83]. More exactly, we prove the following k-wise version of the Deza-Frankl-Singhi theorem [DFS83]. Note, that no upper bounds for the sizes of sets in the set-system and no uniformity assumptions are made.

Theorem 1 *Let p be a prime, let $L \subset \{0, 1, \ldots, p-1\}$, and let $k \geq 2$ be an integer. Let $\mathcal{H}$ be a set-system over the n element universe, satisfying that*

- *(i)* $\forall H \in \mathcal{H}: \quad |H| \bmod p \notin L$,
- *(ii)* $\forall H_1, H_2, \ldots, H_k \in \mathcal{H}$, *where* $H_i \neq H_j$ *for* $i \neq j$*:*

$$|H_1 \cap H_2 \cap \ldots \cap H_k| \bmod p \in L,$$

Then

$$|\mathcal{H}| \leq (k-1) \sum_{i=0}^{|L|} \binom{n}{i}.$$

As well as in the original Deza-Frankl-Singhi theorem, the upper bound does not depend on p, so we can choose a large enough p for proving the non-modular version, $p > n$ certainly suffices.

Our main tool is substituting set-systems into multi-variate polynomials [Gro01]. This tool, together with the linear-algebraic proof of Theorem 9 implies our result.

In the seminal paper of Frankl and Wilson [FW81], the Frankl-Wilson upper bound to the size of a set-system was used for an explicit Ramsey-graph construction. Similarly, we can also use our Theorem 1 to an explicit construction of a t-coloring of the edges of the k-uniform complete *hypergraph*, such that no color class will contain a complete, monochromatic hypergraph on a vertex set of size $\exp(c(\log n \log\log n)^{1/t})$. Our explicit construction is similar to the explicit Ramsey-graph construction of [Gro00]. We note, that much better explicit Ramsey hypergraphs can be constructed using the Stepping-up Lemma of Erdős and Hajnal [GRS80]: from an explicit construction of k-uniform hypergraphs a (much larger) explicit construction of $k+1$-uniform hypergraphs follows, where $k \geq 3$. Another construction for 3-uniform hypergraphs from explicit Ramsey-graphs is due to A. Hajnal [Gyá].

Our present Ramsey-hypergraph construction is the best known for 3-uniform hypergraphs with more than 2 colors, and while it is weaker than the (recursive) constructions for $k > 3$ with the Stepping-up Lemma of Erdős and Hajnal [GRS80], it is at least direct: does not use constructions for $k-1$-uniform hypergraphs.

2 Preliminaries

Definition 2 ([Gro01]) *Let $A = \{a_{ij}\}$ and $B = \{b_{ij}\}$ two $u \times v$ matrices over a ring R. Their* Hadamard-product *is an $u \times v$ matrix $C = \{c_{ij}\}$, denoted by $A \odot B$, and is defined as $c_{ij} = a_{ij}b_{ij}$, for $1 \leq i \leq u$, $1 \leq j \leq v$.*

Lemma 3 *Suppose that R is commutative. Then the Hadamard-product is an associative, commutative and distributive operation:*

- *(i)* $(A \odot B) \odot C = A \odot (B \odot C)$,
- *(ii)* $A \odot B = B \odot A$,
- *(iii)* $(A + B) \odot C = A \odot C + B \odot C$.

And, for all $\lambda \in R$:

- *(iv)* $(\lambda A) \odot B = \lambda(A \odot B)$.

□

We make difference between hypergraphs and set systems over a universe V. A hypergraph is a collection of several subsets of V, where some subsets may be present with a multiplicity, greater than 1 (called multi-edges). A set system may, however, contain each subset of V at most once.

Definition 4 *Let $\mathcal{H} = \{H_1, H_2, \ldots, H_m\}$ be a hypergraph of m edges (sets) over an n element universe $V = \{v_1, v_2, \ldots, v_n\}$, and let $U = \{u_{ij}\}$ be the $n \times m$ 0-1* incidence-matrix *of hypergraph $\mathcal{H}$, that is, the columns of U correspond to the sets (edges) of $\mathcal{H}$, the rows of U correspond to the elements of V, and $u_{ij} = 1$ if and only if $v_i \in H_j$. The $n \times 1$ incidence-matrix of a single subset $A \subset V$ is called the* characteristic vector *of A.*

Note, that every member of a set system is different; so there are no identical columns in an incidence matrix of a set system, but there may be identical columns in an incidence matrix of a hypergraph in case of multi-edges. If U is a 0-1 matrix with no identical columns, then U is an incidence matrix of a set system.

2.1 Arithmetic operations on set systems

Definition 5 *Let $f(x_1, x_2, \ldots, x_n) = \sum_{I \subset \{1,2,\ldots,n\}} a_I x_I$ be a multi-linear polynomial, where $x_I = \prod_{i \in I} x_i$. Let $w(f) = |\{a_I : a_I \neq 0\}|$ and let $\mathrm{L}_1(f) = \sum_{I \subset \{1,2,\ldots,n\}} |a_I|$.*

We need the following definition from [Gro01]:

Definition 6 ([Gro01]) *Let $\mathcal{H}$ be a set-system on the n element universe $V = \{v_1, v_2, \ldots, v_n\}$ and with $n \times m$ incidence-matrix U, and let $f(x_1, x_2, \ldots, x_n) = \sum_{I \subset \{1,2,\ldots,n\}} a_I x_I$ be a multi-linear polynomial with non-negative integer coefficients. Then $f(\mathcal{H}_U)$ is a hypergraph on the $\mathrm{L}_1(f)$-element vertex-set, and its incidence-matrix is the $\mathrm{L}_1(f) \times m$ matrix W. The rows of W correspond to x_I's of f; there are a_I identical rows of W, corresponding to the same x_I. The row, corresponding to x_I is defined as the Hadamard-product of those rows of U, which correspond to $v_i, i \in I$.*

Let us remark, that W has rank at most $w(f)$. Also note, that if the coefficients of $x_1, x_2, \ldots, x_n$ are all non-zero, then $f(\mathcal{H}_U)$ is a set-system, since the rows of U is among the rows of the incidence-matrix of $f(\mathcal{H}_U)$.

The crucial property of this operation is given by the following Theorem (Theorem 11 of [Gro01]):

Theorem 7 ([Gro01]) *Let $\mathcal{H} = \{H_1, H_2, \ldots, H_m\}$ be a set-system, and let U be their $n \times m$ incidence-matrix. Let f be a multi-linear polynomial with non-negative integer coefficients, or from coefficients from $\mathbf{Z}_r$. Let $f(\mathcal{H}) = \{\hat{H}_1, \hat{H}_2, \ldots, \hat{H}_m\}$. Then, for any $1 \le k \le m$ and for any $1 \le i_1 < i_2 < \ldots < i_k \le m$:*

$$f(H_{i_1} \cap H_{i_2} \cap \ldots \cap H_{i_k}) = |\hat{H}_{i_1} \cap \hat{H}_{i_2} \cap \ldots \cap \hat{H}_{i_k}|. \tag{1}$$

We remark, that in (1) on the left-hand side, f is applied to the characteristic vector (a length-n 0-1 vector) of the set $H_{i_1} \cap H_{i_2} \cap \ldots \cap H_{i_k}$.

2.2 Multiple intersections

The proof of the original, pairwise version of the Deza-Frankl-Singhi theorem [DFS83] uses tools from linear algebra: the sets of the set-system $\mathcal{H}$ are associated with independent vectors in a vector space of known dimension; consequently, their number is bounded above by that dimension. Here we also use this idea with some natural modifications.

In the following theorems, the universe of the set-system or the hypergraph is $S = \{v_1, v_2, \ldots, v_n\}$. When we say hypergraph here, we allow hypergraphs with multi-edges also; consequently, if F, G are two edges of the hypergraph, then we allow that F is the same set, as G.

The first step is the following obvious theorem:

Theorem 8 *Let $\mathcal{H} = \{H_1, H_2, \ldots, H_m\}$ be a hypergraph on the n-element universe, satisfying $H_i \neq \emptyset$ for $i = 1, 2, \ldots m$. Suppose, that for some positive integer $k \ge 2$, every k-wise intersection is empty:*

$$\forall I \subset \{1, 2, \ldots, n\}, |I| = k : \bigcap_{i \in I} H_i = \emptyset \tag{2}$$

Then

$$|\mathcal{H}| \le (k-1)n.$$

Proof: Every element of the universe is in at most $k-1$ sets of $\mathcal{H}$. $\Box$

We remark, that the above theorem is sharp, as it is shown by $\mathcal{H} = \{H_1, H_2, \ldots, H_{(k-1)n}\}$, where $H_i = \{v_j\}$, for $i = (j-1)(k-1)+1, (j-1)(k-1)+2, \ldots, j(k-1)$ and $j = 1, 2, \ldots, n$.

We need the modular version of Theorem 8. The modular version is an easy exercise for $k=2$; for larger k's, we need an additional idea.

Theorem 9 *Let p be a prime, and let $\mathcal{H} = \{H_1, H_2, \ldots, H_m\}$ be a hypergraph on the n-element universe. Suppose, that $|H_i| \not\equiv 0 \pmod{p}$ for $i = 1, 2, \ldots, m$, and for some positive integer $k \geq 2$, every k-wise intersection-size is zero modulo p:*

$$\forall I \subset \{1, 2, \ldots, m\}, |I| = k : \bigcap_{i \in I} H_i \equiv 0 \pmod{p}. \tag{3}$$

Then

$$|\mathcal{H}| \leq (k-1)n_0 \leq (k-1)n,$$

if the incidence-vectors of the edges of the hypergraph $\mathcal{H}$ span an $n_0 \leq n$-dimensional subspace of the n-dimensional vector-space over GF(p).

Proof: For $i = 1$ through m, let $x^{(i)} \in \{0,1\}^n$ denote the characteristic vector of set H_i. In the case of $k=2$, it is easy to see that their dot-product, $x^{(i)} \cdot x^{(j)}$, is zero modulo p if $i \neq j$, and non-zero otherwise; thus vectors $x^{(i)}, i = 1, 2, \ldots, m$ are independent in an n_0-dimensional subspace, so $m \leq n_0$.

We generalize this proof for larger values of k. Obviously, $|H_i \cap H_j| = x^{(i)} \cdot x^{(j)}$. This can also be written as $|H_i \cap H_j| = (x^{(i)} \odot x^{(j)}) \cdot \mathbf{1}$, where $\mathbf{1}$ denotes the length-n all-1 vector, and $x^{(i)} \odot x^{(j)}$ is the characteristic vector of $H_i \cap H_j$. Now it is easy to see, that the characteristic vector of

$$\bigcap_{i \in I} H_i$$

is

$$\bigodot_{i \in I} x^{(i)},$$

consequently,

$$|\bigcap_{i \in I} H_i| = \bigodot_{i \in I} x^{(i)} \cdot \mathbf{1}.$$

Let $z^{(i)}$, for $i = 1, 2, \ldots, k$, n-dimensional vectors. Let us define

$$g(z^{(1)}, z^{(2)}, \ldots, z^{(k)}) = \left(\bigodot_{i=1}^{k} z^{(i)} \right) \cdot \mathbf{1}.$$

In particular,

$$g(x^{(i_1)}, x^{(i_2)}, \ldots, x^{(i_k)}) = |\bigcap_{j=1}^{k} H_{i_j}|.$$

Consequently, from our assumptions, if $i_s \neq i_t$ for $s \neq t$, then

$$g(x^{(i_1)}, x^{(i_2)}, \ldots, x^{(i_k)}) \equiv 0 \pmod{p} \tag{4}$$

while for all $i = 1, 2, \ldots, m$:

$$g(x^{(i)}, x^{(i)}, \ldots, x^{(i)}) \not\equiv 0 \pmod{p}. \tag{5}$$

From Lemma 3, g is a multi-linear function. We need the following Lemma to conclude the proof:

Lemma 10 *Let $U \subset V$, where V is a vector-space over the field F. Suppose, that vectors in U generates an n_0-dimensional subspace of V, also assume that $|U| \geq n_0(k-1)+1$. Then there exists an $u \in U$, such that u can be written k different ways as the linear combinations of vectors from U such that no vector appears in two of these linear combinations.*

In other words, the Lemma states that there exist pairwise disjoint subsets $W_1, W_2, \ldots, W_k \subset U$, such that

$$u = \sum_{v \in W_1} a_v v = \sum_{v \in W_2} a_v v = \cdots = \sum_{v \in W_k} a_v v,$$

for $a_v \in F$.

Proof: Let W_1 be a maximal linear independent vector-set from U, and for $j = 2, 3, \ldots, k-1$, let W_j be a maximal linear independent vector-set from $U - (W_1 \cup W_2 \cup \ldots \cup W_{j-1})$. Since $|W_i| \leq n_0$ for $i = 1, 2, \ldots, k-1$, there exists a u such that $u \in U - (W_1 \cup W_2 \cup \ldots \cup W_{k-1})$. Let us define $W_k = \{u\}$.

Now, for $i = 1, 2, \ldots, k-1$, set $W_i \cup \{u\}$ is dependent, while W_i is not, and we are done. □

Now we give an indirect proof for the theorem. Suppose, that $|\mathcal{H}| \geq (k-1)n_0 + 1$. Apply Lemma 10 to $U = \{x^{(1)}, x^{(2)}, \ldots, x^{((k-1)n_0+1)}\}$. Now, there exists a $u \in U$, such that u can be given as k linear combinations of disjoint vector-subsets of U. Since $u = x^{(i)}$, for some i, from (5),

$$g(u, u, \ldots, u) \not\equiv 0 \pmod{p}. \tag{6}$$

But, on the other hand, u can be given in k linear combinations, each containing vectors from pairwise disjoint vector sets. Consequently, by the multi-linearity of g, $g(u, u, \ldots, u) \not\equiv 0 \pmod{p}$ can be written as a linear combination of numbers $g(x^{(i_1)}, x^{(i_2)}, \ldots, x^{(i_k)})$, where $i_s \neq i_t$ for $s \neq t$. By (4), all of these numbers are 0 modulo p, so their linear combination is also zero modulo p, and this contradicts to (6). □

2.3 Proof of the main theorem

Now we have all the tools needed for the proof of Theorem 1. Certainly, $L \neq \emptyset$. Let

$$g(x) = \prod_{a \in L} (x - a).$$

Now let f be the unique multi-linear polynomial over $\mathrm{GF}(p)$, such that

$$f(x_1, x_2, \ldots, x_n) = g(x_1 + x_2 + \cdots + x_n).$$

The degree of f is at most $|L|$, so $\mathrm{L}_1(f) \le (p-1)\sum_{i=0}^{|L|}\binom{n}{i}$, and $w(f) \le \sum_{i=0}^{|L|}\binom{n}{i}$. Consider now hypergraph $f(\mathcal{H})$. The vertex-set of this hypergraph is of size $\mathrm{L}_1(f)$, and the incidence-vectors of the edges span a $w(f)$-dimensional subspace U of the $\mathrm{L}_1(f)$-dimensional vector space V. By Theorem 7, hypergraph $f(\mathcal{H})$ satisfies the assumptions of Theorem 9, so

$$|\mathcal{H}| = |f(\mathcal{H})| \le (k-1)\left(\sum_{i=0}^{|L|}\binom{n}{i}\right).$$

□

3 Set-systems with restricted k-wise intersections

In this section we give an explicit construction for a set-system with similar (but stronger) properties described in [Gro00].

It was conjectured (see [BF92]), that if $\mathcal{H}$ is a set-system over an n element universe, satisfying that $\forall H \in \mathcal{H}$: $|H| \equiv 0 \pmod 6$, but $\forall G, H \in \mathcal{H},\ G \ne H :\ |G \cap H| \not\equiv 0 \pmod 6$ has size polynomial in n. The conjecture was motivated by theorems of Frankl and Wilson, showing polynomial upper bounds for prime or prime-power moduli [FW81]. We have shown in [Gro00] that there exists an $\mathcal{H}$ with these properties and with super-polynomial size in n. (see the details in [Gro00].) In [Gro01] we gave this construction with the notions of Definition 6. Here we present a k-wise intersection-version, which will be useful for a Ramsey hypergraph construction. On the other hand, this construction will also show, that our Theorem 1 does *not* generalize to non-prime-power composite moduli.

Theorem 11 *Let $n, t \ge 2$ integers, and let $p_1, p_2, \ldots, p_t$ be pairwise different primes, and let $q = p_1p_2\cdots p_t$. There exists an explicitly constructible set-system $\mathcal{H} = \{H_1, H_2, \ldots, H_m\}$ on the n-element universe, such that*

(i) $|\mathcal{H}| = m \ge \exp\left(\frac{c(\log n)^t}{(\log\log n)^{t-1}}\right)$

(ii) $\forall H \in \mathcal{H},\ |H| \equiv 0 \pmod q$,

(iii) $\forall I \subset \{1, 2, \ldots, m\},\ 2 \le |I|,\ |\bigcap_{i\in I} H_i| \not\equiv 0 \pmod q$.

Proof:

Let s be a positive integer, and for $i = 1, 2, \ldots, t$ let α_i be the smallest integer that $s < p_i^{\alpha_i}$. By a result of Barrington, Beigel and Rudich [BBR94], for any $\ell \ge s$ there exists an explicitly constructible ℓ-variable, degree-$O(s)$ polynomial f, satisfying over $x = (x_1, x_2, \ldots, x_\ell) \in \{0,1\}^\ell$:

$$f(x) \equiv 0 \pmod{q} \iff \sum_{i=1}^{\ell} x_i \equiv 0 \pmod{p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_t^{\alpha_t}}.$$

Let $r = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_t^{\alpha_t}$, and let $\mathcal{G}_0$ denote the set-system of all $r-1$-element subsets of the $\ell - 1$-element universe. Let us take an additional element e outside this universe, and let us define set-system $\mathcal{G} = \{G \cup \{e\}\ G \in \mathcal{G}_0\}$. Indeed, for any $k \geq 2$, all k-wise intersections in $\mathcal{G}$ are non-empty, and of size less than r, while the size of any element of $\mathcal{G}$ is exactly r.

Then consider $\mathcal{H} = f(\mathcal{G})$. By Theorem 7, $\mathcal{H}$ satisfies (ii) and (iii), and since the f of Barrington, Beigel and Rudich [BBR94] contains all variable x_i with a non-zero coefficient, then $\mathcal{H}$ is a set-system. The size of $\mathcal{H}$ is the same as the size of $\mathcal{G}$:

$$\binom{\ell-1}{r-1}.$$

Now set $\ell = r^2$, then

$$|\mathcal{H}| = |\mathcal{G}| = \binom{r^2}{r-1} \geq r^r.$$

The size of the universe of $\mathcal{H} = f(\mathcal{G})$ is

$$n = \mathrm{L}_1(f) = \ell^{O(s)} = r^{O(r^{1/t})},$$

so

$$|\mathcal{H}| = \exp\left(\frac{c(\log n)^t}{(\log\log n)^{t-1}}\right),$$

for some positive constant c, depending only on q (or the primes $p_1, p_2, \ldots, p_t$).

□

4 An Explicit Ramsey-Hypergraph Construction

Theorem 12 *Let $m, k, t \geq 2$ integers. Let $\mathcal{F}$ denote the complete k-uniform set-system on the m-element universe S. Then there exists an explicitly constructible t-coloring of the sets of the k-uniform set-system $\mathcal{F}$ which does not contain monochromatic complete sub-system on*

$$\exp\left(c(\log m)^{1/t}(\log\log m)^{1/(t-1)}\right)$$

vertices.

Proof: First construct a set-system $\mathcal{H}$ with Theorem 11 with the first t primes: $p_1 = 2, p_2 = 3, \ldots, p_t$. Set $S = \mathcal{H}$. (If m is not exactly the size of $\mathcal{H}$, then generate the smallest $\mathcal{H}$ with at least m elements, and let $S \subset \mathcal{H}$.) Consequently, a member of our set-system $F \in \mathcal{F}$ corresponds to k sets of $\mathcal{H}$: $F = \{H_1, H_2, \ldots, H_k\}$.

Next we define the coloring of $\mathcal{F}$.

Color F to color c_v, $(1 \le v \le t)$ if v is the smallest number that p_v does not divide

$$\left|\bigcap_{i=1}^{k} H_i\right|.$$

Clearly, every F will have some color. If every k-set in $S' \subset S$ is of color c_v, then apply Theorem 1 with $p = p_v$, and get the upper bound.

□

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